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Introduction

Mathematical model is indispensable in the simulation of physical systems or the design of control systems. The so-called natural science is, in fact, the systematic and categorized study of various kinds of physical, chemical, and other natural phenomena. Various models are used to describe and reproduce the observed phenomena. The models used in engineering are usually described as differential equations, difference equations, or statistical data. It may be said that the design of modern control systems is essentially based on the models of systems.

However, a mathematical model cannot describe the physical phenomenon of a system perfectly. Even if it could, the model would be unnecessarily complicated which makes it difficult to capture the main characteristic of the system. In particular, this is often the case in engineering practice. In almost all cases, a system is not isolated from the outside world. Instead, it is constantly influenced by the surrounding environment. It is difficult to describe the external influence by models. This means that there is inevitably a gap between the actual system and its mathematical model. This gap is called as the *model uncertainty*. The purpose of robust control is to extract the characteristics of model uncertainty and apply this information to the design of control system, so as to enhance the performance of the actual control system to the limit.

1.1 Engineering Background of Robust Control

Here, we give some specific examples to illustrate the model uncertainty.

Example 1.1 *Hard disk (refer to Figure 1.1(a)) is commonly used as the data storage device for computers. The frequency response of a hard disk drive (HDD in short hereafter) is shown in Figure 1.1(b) as the dotted line. The approximate model only considers the rigid body and is described as a double integrator (the solid line). Since the arm is very thin, the drive has numerous resonant modes in the high-frequency band, and these resonant modes vary with the manufacturing error in mass production. So it is difficult to obtain the exact model. Therefore, we have to design the control system based on the model of rigid body.*

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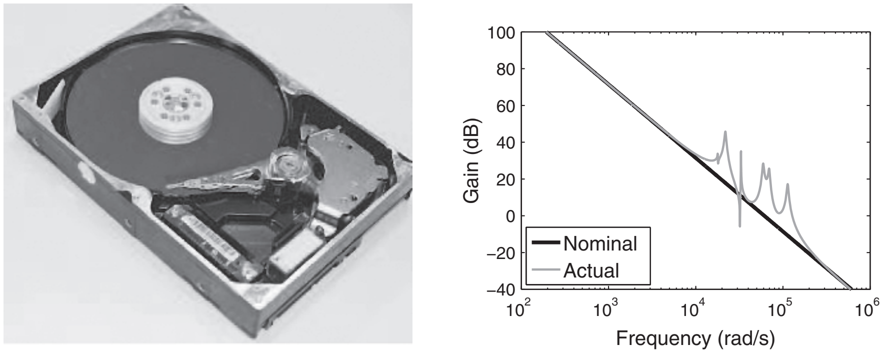
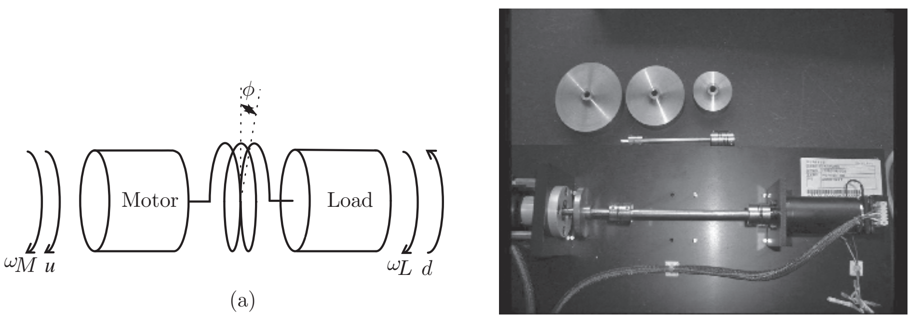


(a) (b)

Figure 1.1 Hard disk drive and its frequency response (a) Photo, (b) Bode plot



(b)

Figure 1.2 Motor drive system (a) Two-mass–spring system, (b) Motor drive

Example 1.2 *The system in Figure 1.2(a) is called* two-mass–spring system. *As shown in Figure 1.2(b), this is essentially a motor drive system in which the motor and load are connected through a shaft. The purpose is to control the load speed indirectly by controlling the motor inertia moment. There are many such systems around us. Typical examples are the DVD drive in home electronics, the rolling mill in steel factory, and so on.*

Let J_M be the inertia moment of the motor, k the spring constant of the shaft, J_L the inertia moment of the load, and D_M and D_L the viscous friction coefficients of the motor and the load. In addition, the motor speed is ω_M, the load speed is ω_L, the torsional angle of the shaft is ϕ, and the motor torque is u. As the direct measurement of load speed is difficult, we usually measure the motor speed ω_M only. The equations of moment balance as well as the speed equation are

$$J_L\dot{\omega}_L + D_L\omega_L = k\phi + d$$

$$\dot{\phi} = \omega_M - \omega_L$$

$$J_M\dot{\omega}_M + D_M\omega_M + k\phi = u.$$

Here, d denotes the torque disturbance acting on the load. Select the state vector as $x = [\omega_L \ \ \phi \ \ \omega_M]^T$, the following state equation is obtained easily:

$$\dot{x} = \begin{bmatrix} -\dfrac{D_L}{J_L} & \dfrac{k}{J_L} & 0 \\ -1 & 0 & 1 \\ 0 & -\dfrac{k}{J_M} & -\dfrac{D_M}{J_M} \end{bmatrix} x + \begin{bmatrix} \dfrac{1}{J_L} \\ 0 \\ 0 \end{bmatrix} d + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{J_M} \end{bmatrix} u \tag{1.1}$$

$$y = [0 \quad 0 \quad 1]x. \tag{1.2}$$

However, in practice the motor load is rather diverse. Specifically, the load inertia moment J_L and the spring constant k of the shaft change over a wide range. This will undoubtedly affect the performance of the motor drive system.

As illustrated by these examples, the mathematical model of a system always contains some uncertain part. Nevertheless, we still hope that a control system, designed based on a model with uncertainty, can run normally and has high performance. To achieve this goal, it is obvious that we should make use of all available information about the uncertainty. For example, in the loop shaping design of the classical control, a very important principle is to ensure that the high-frequency gain of controller rolls off sufficiently to avoid exciting the unmodeled high-frequency dynamics of the plant, so that the controller can be successfully applied to the actual system. In addition, the open-loop transfer function needs also to have sufficient gain margin and phase margin so as to ensure that the closed-loop system is insensitive to the model uncertainty in the low- and middle-frequency domain. This implies that the information about model uncertainty has already been used *indirectly* in system designs based on the classical control methods. Of course, indirect usage of model uncertainty has only a limited effect. So, the mission of robust control theory is to develop an effective, high-performance design method that makes use of the uncertainty information *fully and directly* for systems with uncertainty.

Next, we use an example to show the influence of model uncertainty. It will be illustrated that if the model uncertainty is neglected in the system design, the controller may have trouble when applied to the actual system. In the worst case, even the stability of the closed-loop system may be lost.

Example 1.3 *Consider a system containing both rigid body and a resonant mode (a simplified dynamics of HDD)*

$$\tilde{P}(s) = \frac{1}{s^2} - \frac{0.1 \times \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta = 0.02, \omega_n = 5.$$

In the control design, only the rigid body model $P(s) = 1/s^2$ is considered, and the resonant mode $-0.1 \times \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ is ignored. In the stabilization design of the rigid body model, we hope that the closed-loop system response is fast enough while the overshoot is as small as possible. So the damping ratio of the closed-loop system is set as $1/2$ and the natural

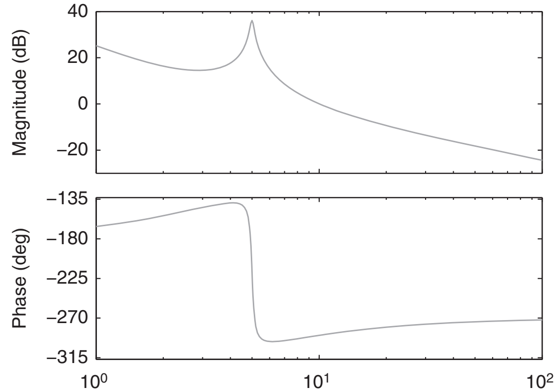


Figure 1.3 Bode plot of open-loop transfer function

frequency set as 4 [rad/s] (slightly lower than the natural frequency of the resonant mode). The corresponding characteristic polynomial is

$$p(s) = s^2 + 4s + 16.$$

The following proportional derivative (PD) compensator is able to achieve the goal:

$$K(s) = 4(s + 4).$$

However, when this controller is applied on the actual system $\tilde{P}(s)$, we find that the characteristic polynomial of the actual closed-loop becomes

$$\tilde{p}(s) = s^4 - 5.8s^3 + 1.8s^2 + 103.2s + 400$$

which obviously has unstable roots. In fact, the closed-loop poles are xx

$$5.2768 \pm j3.8875, \quad -2.3768 \pm j1.9357,$$

in which two of them are unstable. That is, the closed-loop system not only fails to achieve performance improvement but also losses the stability. This phenomenon is known as spillover in vibration control. The reason for the spillover is clear from Figure 1.3, the Bode plot of the open-loop transfer function $\tilde{P}K$. Since the response speed of the closed loop is designed too fast for the rigid body model, the roll-off of the controller gain is not sufficient near $\omega_n = 5$ rad/s, the natural frequency of the resonant mode, which excites the resonant mode.

1.2 Methodologies of Robust Control

In this section, various descriptions of model uncertainty and the corresponding robust control methods are briefly illustrated for single-input single-output systems.

1.2.1 Small-Gain Approach

Suppose that the system model is $P(s)$ while the actual system is $\tilde{P}(s)$. If no information is known about the structure of the actual system $\tilde{P}(s)$, the simplest method is to regard the difference between them as the model uncertainty $\Delta(s)$, that is,

$$\tilde{P}(s) = P(s) + \Delta(s). \tag{1.3}$$

Then, the closed-loop system can be transformed equivalently into the system of Figure 1.4 in which

$$M(s) = \frac{K}{1 + PK}$$

is the closed-loop transfer function about $P(s)$ and should be stable.

As for the description of the characteristics of $\Delta(s)$, a straightforward method is to use the frequency response. In general, the frequency response of $\Delta(s)$ varies within a certain range, such as

$$0 \le |\Delta(j\omega)| \le |W(j\omega)| \ \ \forall\omega. \tag{1.4}$$

Here $W(s)$ is a known transfer function whose gain specifies the boundary of the uncertainty $\Delta(s)$. When $\Delta(s)$ is a stable transfer function,

$$|M(j\omega)\Delta(j\omega)| < 1 \ \ \forall\omega \quad \Leftrightarrow \quad |M(j\omega)W(j\omega)| < 1 \ \ \forall\omega \tag{1.5}$$

guarantees the stability of the closed-loop system for all uncertainties $\Delta(s)$. This condition is known as the *small-gain condition*. The reason for the stability of the closed-loop system is clear. The open-loop transfer function in Figure 1.4 is $L(s) = M(s)\Delta(s)$ and is stable. $L(j\omega)$ does not encircle the critical point $(-1, j0)$ when the small-gain condition is satisfied, so the Nyquist stability condition holds. Here, the information of uncertainty gain is used.

Chapter 12 describes in detail the robust analysis based on the small-gain condition, while Chapters 16 and 17 provide the corresponding robust design methods.

1.2.2 Positive Real Method

On the contrary, in (1.3) if the phase angle $\arg\Delta(j\omega)$ of the uncertainty $\Delta(s)$ changes in a finite range, particularly when

$$-90^\circ \le \arg\Delta(j\omega) \le 90^\circ \ \ \forall\omega \quad \Leftrightarrow \quad \Re[\Delta(j\omega)] \ge 0 \ \ \forall\omega, \tag{1.6}$$

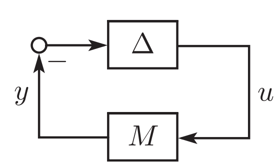


Figure 1.4 Closed-loop system with uncertainty

the following condition guarantees the stability of the closed-loop system for all the uncertainties $\Delta(s)$:

$$-90^\circ < \arg M(j\omega) < 90^\circ \ \ \forall\omega \ \ \Leftrightarrow \ \ \Re[M(j\omega)] > 0 \ \ \forall\omega. \tag{1.7}$$

This is because the open-loop transfer function $L(s) = M(s)\Delta(s)$ is stable and the phase angle is never equal to $\pm 180^\circ$. As such, $L(j\omega)$ does not encircle the critical point $(-1, j0)$, and the Nyquist stability condition is met.

For all frequencies ω, a transfer function satisfying $\Re[G(j\omega)] \geq 0$ is called a *positive real function*. So the previous condition is named as the *positive real condition*. General positive real condition is described detailedly in Section 13.4, and the robust design based on positive real condition is discussed in Chapter 21.

1.2.3 Lyapunov Method

For systems with known dynamics but uncertain parameters, such as the two-mass–spring system, methods based on quadratic Lyapunov function are quite effective. For example, consider an autonomous system

$$\dot{x} = Ax, \quad x(0) \neq 0.$$

If the quadratic Lyapunov function

$$V(x) = x^T Px, \quad P > 0$$

satisfies the strictly decreasing condition

$$\dot{V}(x) < 0 \quad \forall x \neq 0,$$

then $V(x)$ converges to zero eventually since it is bounded below. As $V(x)$ is positive definite, $V(x) = 0$ implies that $x = 0$. Further, the strictly decreasing condition is equivalent to the inequality on the matrix A:

$$A^T P + PA < 0.$$

Even if some parameters of the matrix A take value in a certain range, the stability of the system is secured so long as there exists a matrix $P > 0$ satisfying this inequality.

Refer to Chapter 13 for the analysis on such uncertain systems and Section 18.3, Chapter 19, and Chapter 20 for the design methods.

1.2.4 Robust Regional Pole Placement

For a system with parameter uncertainty, we can place the poles of the closed-loop system in some region of the complex plane, such as the region shown in Figure 1.5, so as to guarantee the quality of transient response. Detailed design conditions and design methods are given in Chapter 19.

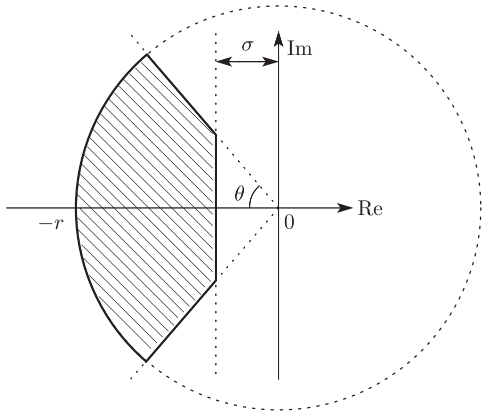


Figure 1.5 Regional pole placement

1.2.5 Gain Scheduling

Many nonlinear systems can be described as linear systems with time-varying parameters, in which the time-varying parameters are bounded functions of some states. Such a system is called as a *linear parameter-varying* (LPV) system . When the states contained in the LPV system can be measured online, the time-varying parameters can be computed. Then, if the controller parameters vary in accordance with the time-varying parameters, it is possible to obtain a better control performance. This method is called as *gain scheduling*. The details can be found in Chapter 20.

Example 1.4 *Let us look at a very simple example: a one-link arm. Assume that the angle between the arm and the vertical line is θ, the mass and inertia moment are m, J, the distance between the center of gravity and the joint is l, and the control torque is u. Then the motion equation is*

$$J\ddot{\theta} + mgl\ \sin\theta = u.$$

If we set $p(t) = \sin\theta/\theta$, the motion equation can be rewritten as

$$J\ddot{\theta}(t) + mglp(t)\theta(t) = u(t).$$

This can be regarded as an LPV system with a time-varying parameter $p(t)$. Clearly, the time-varying parameter satisfies

$$|p(t)| = \left|\frac{\sin\theta(t)}{\theta(t)}\right| \leq 1$$

and is bounded. In the control design, a better performance may be obtained by using a controller containing $p(t)$, or the control design is made easier. For example, when we use the control input

$$u(t) = mglp(t)\theta(t) - 2\zeta\omega_n J\dot{\theta}(t) - \omega_n^2 J\theta(t),$$

the closed-loop system becomes

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0.$$

By adjusting the damping ratio ζ and the natural frequency ω_n, we can easily achieve a high control performance. In this designed input, the first term about the time-varying parameter is in fact a nonlinear term $mgl\sin\theta$.

1.3 A Brief History of Robust Control

As mentioned earlier, in the classical control design, the model uncertainty in the high-frequency domain is taken into account, and effective feedback control is realized by tuning controller parameters online. However, model uncertainty is excluded from the modern control theory. So, it is rare that a controller designed by pole placement or least quadratic Gaussian (LQG) optimal control theory can be directly applied to the actual system. This is because the actual control effect is quite different from the design specification and simulation. In successful applications of modern control, filters are always used to filter out the high-frequency component of the output signal before feeding it to the controller. This in fact considers the neglected high-frequency characteristic of the plant indirectly.

For example, although the state feedback least quadratic control has a gain margin ranging from $1/2$ to infinity and a phase margin of $\pm 60°$ [82], Doyle [23] proved that in the output feedback of LQG control, even a small perturbation in the gain may destabilize the closed-loop system. Facing this flaw of the modern control theory, researchers, mainly from North America, began to consider how to deal with model uncertainty. First of all, inspired by the *perturbation theory* in mathematics, in 1964 Cruz *et al*. [19] analyzed the rate of relative change of the closed-loop transfer function $H(s) = L(s)/(1 + L(s))$, that is,

$$\frac{\Delta H/H}{\Delta L/L} = \frac{\dfrac{L+\Delta L}{1+L+\Delta L} - \dfrac{L}{1+L}}{\Delta L}\frac{L}{H}$$

when the open-loop transfer function $L(s)$ has a small perturbation. It is easy to know that when the perturbation ΔL approaches zero,

$$\lim_{\Delta L \to 0} \frac{\Delta H/H}{\Delta L/L} = \frac{1}{1+L} := S(s)$$

holds. This is the reason why the transfer function $S(s)$ is called as the *sensitivity*. Unfortunately, this relationship is valid only for very small perturbations and has nothing to do with the dynamic property of uncertainty. So, it makes no sense in handling uncertainty.

In the 1980s, Zames [99] and Doyle-Stein [28] for the first time challenged the issue of model uncertainty and discussed how to introduce the information of model uncertainty into the feedback control design. Both of them believe that the model uncertainty should be described by the range of the gain of its frequency response. The former stressed that the disturbance should be treated as a set and the control performance of the disturbance control measured by the $\mathcal{H}_\infty$ norm of the closed-loop transfer function. Meanwhile, the latter proposed the small-gain principle. In the following decade, robust control was developed along this line. Because the uncertainty and the control performance are characterized in the form of frequency response, robust control research commenced in the frequency domain, and the main mathematical tools were operator theory and Nevanlinna–Pick interpolation theory [48]. Later, Doyle advocated the use of state space in $\mathcal{H}_\infty$ control problems and completed the Riccati equation solution [36] with Glover in 1988. Particularly, the famous paper published in 1989, known as DGKF paper [26], has an immeasurable impact on the robust control research thereafter. This is because it reveals that, to solve the $\mathcal{H}_\infty$ control problem, one does not need to use advanced mathematical tools in function spaces which is formidable for engineers and the familiar state-space tool is sufficient.

In a roughly same period, Boyd [9, 10] advocated the application of numerical method and optimization approach to robust control. In particular, the book [10] had a great impact. It happened in the end of 1980s that Russian mathematicians Nesterov and Nemirovski [72] completed the numerical approach, the interior point method, to convex programming, thus paving the way for solving the robust problems by optimization methods. Since 1990s, numerous robust control methods based on LMI tools have sprung up. In particular, the French school, represented by Gahinet, not only proposed the LMI solution to $\mathcal{H}_\infty$ control but also solved a series of robust control problems, such as regional pole placement and gain scheduling [33, 16, 3]. Moreover, they developed the LMI toolbox [35] which helped promoting the LMI approach. So far, the mainstream of robust control is LMI. This book presents the major contents of robust control mainly based on the LMI approach.

The aforementioned studies are all robust control methods that use the information of uncertainty gain or parameter uncertainty. When the phase information of uncertainty is known and its range is not big, it is effective to use the phase information to design robust control system. Haddad-Berstein [37] and Tits [88] have done some works in this direction. More new results on this approach will be shown in this book.