1

Preliminaries

1.1. Sets

(1a) $A = \{-1, 4\}$

(1b) $B = \{-5, -3, -\frac{3}{2}, 1\}$

(1c) $C = \{-3, -2, -1, 0, 1, 2\}$

(1d) $D = \{-1\}$

(1e) $E = \{-3, -2, 0, 1, 2\}$

(1f) $F = \{1, 3, 5, 7, 9\}$

(2a) $B = \{x \in \mathbb{R} \mid x^2 - 3 = 0\}$

(2b) $A = \{x \mid x = 2k, \text{ where } k \in \mathbb{Z} \text{ and } 1 \leq k \leq 4\}$

(2c) $A = \{x \mid x = 2k, \text{ where } k \in \mathbb{Z}^+\}$

(2d) $A = \{x \mid x = 3k + 2, \text{ where } k \in \mathbb{Z}\}$

(2e) $A = \{x \mid x = n^3, \text{ where } n \in \mathbb{N}\}$

(2f) $B = \{x \in \mathbb{Z} \mid -3 \leq x \leq 6\}$

(2g) $B = \{x \in \mathbb{Z} \mid x \leq -3 \vee x \geq 5\}$

(3) Only a, d e, g, h and j are true.

(4) Only a, g, j and l are true.

(5a) $A \cap B = \{2, 7\}$

(5b) $A \cup B = \{0, 1, 2, 4, 5, 6, 7, 9\}$

(5c) $A - B = \{1, 4\}$

(5d) $B - A = \{0, 5, 6, 9\}$

(5e) $A - (A - B) = \{2, 7\}$

(5f) $B - (A - B) = B$

(5g) $B' = \{1, 3, 4, 8\}$

(5h) $(A \cap B)' = \{0, 1, 3, 4, 5, 6, 8, 9\}$

(5i) $A \cap B' = \{1, 4\}$

(5j) $(A \cup B)' = \{3, 8\}$

(5k) $(A \cup B)' - (B - A)' = \varnothing$

(6a) $A \cap B = \{0\}$

(6b) $A \cup B = \{0, A\}$

(6c) $B - A = \{A\}$

(6d) $A \cap (B \cup C) = \{0\}$

(6e) $(B \cup C) - A = \{A, \{0\}, \{A\}\}$

(6f) $(A \cap B) \cup (A \cap C) = \{0\}$

(6g) $(C - B) - A = \{\{0\}, \{A\}\}$

(6h) $A \cap (C - A) = \varnothing$

Logic and Discrete Mathematics: A Concise Introduction, Solutions Manual, First Edition.
Willem Conradie, Valentin Goranko and Claudette Robinson.

Companion Website: www.wiley.com/go/conradie/logic

(7) $A \times B = \{(1,0),(1,1),(2,0),(2,1),(3,0),(3,1),(4,0),(4,1)\}$
$B \times A = \{(0,1),(1,1),(0,2),(1,2),(0,3),(1,3),(0,4),(1,4)\}$

(8a) 30 (8c) 5 (8e) 42

(8b) 0 (8d) 0 (8f) 1

(9) Only a and e are graphs of functions.

(10) The right-to-left direction is obvious. For the converse direction, assume $(a_1, a_2) = (b_1, b_2)$. Then $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$. Hence,

$$\bigcap\{\{a_1\}, \{a_1, a_2\}\} = \bigcap\{\{b_1\}, \{b_1, b_2\}\},$$

and so $\{a_1\} = \{b_1\}$. This means that $a_1 = b_1$. Similarly,

$$\bigcup\{\{a_1\}, \{a_1, a_2\}\} = \bigcup\{\{b_1\}, \{b_1, b_2\}\},$$

which gives $\{a_1, a_2\} = \{b_1, b_2\}$. Thus, since $a_1 = b_1$, $a_2 = b_2$.

(11) Given the objects $a_1, a_2, \dots, a_n$, the ordered n-tuple $(a_1, a_2, \dots, a_n)$ is defined by

$$(a_1, (a_2, \dots, (a_{n-1}, a_n))).$$

1.2. Basics of logical connectives and expressions

(1) a, b, e, f, and h are propositions.

(2a) $A \wedge (\neg B \vee C) = \mathrm{T} \wedge (\neg \mathrm{T} \vee \mathrm{F}) = \mathrm{T} \wedge (\mathrm{F} \vee \mathrm{F}) = \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$

(2e) $C \to (A \to D) = \mathrm{F} \to (\mathrm{T} \to \mathrm{F}) = \mathrm{F} \to \mathrm{F} = \mathrm{T}$

(3a) Denote “The sun is hot.” by p, “The earth is larger than Jupiter.” by q, and “There is life on Jupiter.” by r. Then the composite proposition in symbolic form is $p \wedge (q \to r)$. It is true.

(3b) Denote “The sun rotates around the earth.” by p, “The earth rotates around the moon.” by q and “The sun rotates around the moon.” by r. Then the composite proposition in symbolic form is $p \vee q \to r$. It is true.

(3c) Denote “The moon rotates around the earth.” by p, “The sun rotates around the earth.” by q and “The earth rotates around the moon.” by r. Then the composite proposition in symbolic form is $\neg q \wedge \neg r \to \neg p$. It is false.

(3d) Denote “The earth rotates around itself.” by p, “The sun rotates around the earth.” by q and “The moon rotates around the earth.” by r. Then the composite proposition in symbolic form is $p \to q \vee \neg r$. It is false.

(3e) Denote “The earth rotates around itself.” by p, “The sun rotates around itself.” by q and “The moon rotates around itself.” by r. Then the composite proposition in symbolic form is $p \leftrightarrow (\neg q \vee \neg r)$. It is true.

(4a) Since B is true and $B \to A$ must be true, A must be true.

(4b) Since B is false and $A \to B$ must be true, A must be false.

(4c) B is false since $\neg B$ is true. Hence, since $A \vee B$ must be true, A is true.

(4d) $\neg C$ must be false, and since $\neg B \to \neg C$ is true, $\neg B$ must be false, i.e. B is true. Hence, since $B \to \neg A$ is true, $\neg A$ is true and so A is false.

(4e) Since $\neg C \wedge B$ is true, B and $\neg C$ are true. Hence, C is false. Since $\neg(A \vee C) \to C$ is true and C is false, $\neg(A \vee C)$ is false, i.e. $A \vee C$ is true. Therefore, since C is false, A must be true.

(5a) Not every number is different from 0. True in $\mathcal{R}$ and $\mathcal{N}$.

(5b) Every number is less than or equal to its cube. False in $\mathcal{R}$, take $x = -2$. True in $\mathcal{N}$.

(5c) Every number equal to its square is positive. False in $\mathcal{R}$ and $\mathcal{N}$, take $x = 0$.

(5d) There is a negative number equal to its square. False in both $\mathcal{R}$ and $\mathcal{N}$. The square of any number is always positive.

(5e) Any positive number is less than its square. False in both $\mathcal{R}$ and $\mathcal{N}$, take $x = 1$.

(5f) Every number is either zero or it is not equal to twice itself. True in $\mathcal{R}$ and $\mathcal{N}$.

(5g) For every pair of numbers x and y, one of them is less than the other. False in $\mathcal{R}$ and $\mathcal{N}$, take $x = y$.

(5h) Every number is greater than the square of some number. False in $\mathcal{R}$ and $\mathcal{N}$, as 0 is not greater than any square.

(5i) For every number x, there is a number y that is either positive or whose square is less than x. True in $\mathcal{R}$ and $\mathcal{N}$.

(5j) Every non-negative number is the square of a positive number. False in $\mathcal{R}$ and $\mathcal{N}$, as 0 is not equal to the square of any positive number.

(5k) For every number x, there is a number y such that if x is greater than y, then it is also greater than the square of y. True in $\mathcal{R}$ and $\mathcal{N}$. Given x, take $y = x$, which makes the antecedent false and hence the implication true.

(5l) For every number x, there is a number y such that, if it is different from x, then its square is less than x. True in $\mathcal{R}$ and $\mathcal{N}$. Given x, take $y = x$, which makes the antecedent false and hence the implication true.

(5m) There is number greater than all numbers. False in $\mathcal{R}$ and $\mathcal{N}$.

(5n) There is a number x such that adding any number to it again yields x. False in $\mathcal{R}$ and $\mathcal{N}$.

(5o) There is a number x that can be added to any number y to obtain y again. True in $\mathcal{R}$ and $\mathcal{N}$, take $x = 0$.

(5p) There is a number such that every number is either less than it or less than its additive inverse. False in $\mathcal{R}$. It is not a formula in the language $\mathcal{L}_{\mathcal{N}}$.

(5q) There is a number x that is greater than or not greater than any given number y. True in $\mathcal{R}$ and $\mathcal{N}$.

(5r) There is a number such that the square of every number greater than it is also greater than it. True in $\mathcal{R}$ and $\mathcal{N}$, take, for example, $x = 0$.

(5s) There is a number such that the square of every number less than it is also less than it. False in $\mathcal{R}$, while it is true in $\mathcal{N}$, take $x = 0$.

(5t) There exist two numbers whose sum is equal to their product. True in $\mathcal{R}$ and $\mathcal{N}$, take $x = y = 0$.

(5u) Between any two distinct numbers, there is another number. True in $\mathcal{R}$, while it is false in $\mathcal{N}$.

1.3. Mathematical induction

(1) For $n = 0$, $0^2 + 0 = 0$, which is clearly even. Our inductive hypothesis is that $k^2 + k$ is even. Now, for $n = k + 1$,

$$(k + 1)^2 + k + 1 = k^2 + 2k + 1 + k + 1 = k^2 + k + 2(k + 1).$$

However, we know that $k^2 + k$ is even from the inductive hypothesis and, furthermore, $2(k + 1)$ is also clearly even. Therefore, since the sum of two even numbers is even, $k^2 + k$ is even.

(2) For $n = 4$, $2^4 = 16$ and $4! = 24$, so, clearly, $2^4 < 4!$. Our inductive hypothesis is that $2^k < k!$ for some $k \geq 4$. Now, for $n = k + 1$,

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k! < 5 \cdot k! \leq (k + 1) \cdot k! = (k + 1)!$$

(3) First, the set $\{1\}$ has two subsets, namely $\{1\}$ and $\emptyset$, so, clearly, the power set of $\{1\}$ has 2^1 elements. Our inductive hypothesis is that the power set of $\{1, 2, 3, \dots, k\}$ has 2^k elements for some $k \geq 1$. Now, let $A = \{1, 2, \dots, k, k + 1\}$. Choose an element $a \in A$ and set $A' = A - \{a\}$. Note that $\mathcal{P}(A) = \{X \subseteq A \mid a \notin X\} \cup \{X \subseteq A \mid a \in X\}$. It is clear that these sets are disjoint, so to find the number of elements in $\mathcal{P}(A)$, we need only find the number of the elements in each of these sets and add them together. First, clearly, $\mathcal{P}(A') = \{X \subseteq A \mid a \notin X\}$, and so, since A' has k elements, $\{X \subseteq A \mid a \notin X\}$ has 2^k elements by the inductive hypothesis. Next, note that $X = Y \cup \{a\}$ for all $X \in \{X \subseteq A \mid a \in X\}$, where $Y \in \mathcal{P}(A')$. Since there are k elements in the set A', there are 2^k such Y by the inductive hypothesis. Hence, the set $\{X \subseteq A \mid a \in X\}$ has 2^k elements. Thus, in total, $\mathcal{P}(A)$ has $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ elements.