# 1

# MATHEMATICAL PRELIMINARIES



## 1.1 ARITHMETIC PROGRESSION

We may define an *arithmetic progression* as a set of numbers in which each one after the first is obtained from the preceding one by adding a fixed number called the *common difference*. Suppose we denote the common difference of an arithmetic progression by $d$, the first term by $a_1$, …, and the $n$th term by $a_n$. Then the terms up to and including the $n$th term can be written as

$$a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + (n-1)d (= a_n). \tag{1.1}$$

If $S_n$ denotes the sum of the first $n$ terms of an arithmetic progression, then

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d). \tag{1.2}$$

If the $n$ terms on the right-hand side of Equation 1.2 are written in reverse order, then $S_n$ can also be expressed as

$$S_n = (a_1 + (n-1)d) + (a_1 + (n-2)d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1. \tag{1.3}$$

Upon adding Equations 1.2 and 1.3, we obtain

$$2S_n = n(2a_1 + (n-1)d)$$

*Growth Curve Modeling: Theory and Applications*, First Edition. Michael J. Panik.

or

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) = \frac{n}{2}(a_1 + a_n). \tag{1.4}$$

**EXAMPLE 1.1** Given the arithmetic progression −3, 0, 3, …, determine the 50th term and the sum of the first 100 terms. For $a_1 = -3$, the second term (0) minus the first term is $0 - (-3) = 3 = d$, the common difference. Then, from Equation 1.1,

$$a_{50} = -3 + (50-1)3 = 144;$$

and, from Equation 1.4,

$$S_{100} = \frac{100}{2}(2 \cdot 3 + (100-1)3) = 15{,}150.$$

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## 1.2 GEOMETRIC PROGRESSION

A *geometric progression* is any set of numbers having a *common ratio*; that is, the quotient of any term (except the first) and the immediately preceding term is the same. Suppose we represent the common ratio of a geometric progression by $r$, the first term by $a_1(\neq 0), \ldots,$ and the $n$th term by $a_n$. Then the terms up to and including the $n$th term are

$$a_1, a_1 r, a_1 r^2, \ldots, a_1 r^{n-1} (= \mathrm{a}_n). \tag{1.5}$$

(Note that, as required,

$$\frac{\text{3rd term}}{\text{2nd term}} = \frac{a_1 r^2}{a_1 r} = r, \frac{\text{4th term}}{\text{3rd term}} = \frac{a_1 r^3}{a_1 r^2} = r, \text{ etc.)}$$

If the sum of the first $n$ terms of a geometric progression is denoted as $S_n$, then

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}. \tag{1.6}$$

Using Equation 1.6, let us form

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n \tag{1.7}$$

so that, upon subtracting Equation 1.7 from Equation 1.6, we obtain

$$S_n - rS_n = a_1 - a_1 r^n$$

or

$$S_n = \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1 - r a_n}{1-r}, \qquad r \neq 1. \tag{1.8}$$

**EXAMPLE 1.2** Given the geometric progression 1/2, 3/4, 9/8, …, determine the sixth term and the sum of the first nine terms. For $a_1 = 1/2$, the second term (3/4) divided by the first term (1/2) is $(3/4)/(1/2) = 3/2 = r$, the common ratio. Then, from Equation 1.5,

$$a_6 = \frac{1}{2}\left(\frac{3}{2}\right)^5 = \frac{243}{64};$$

and, from Equation 1.8,

$$S_9 = \frac{(1/2) - (1/2)(3/2)^9}{1 - (3/2)} = \frac{19{,}171}{512}.$$

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Suppose we have a geometric progression with infinitely many terms. The sum of the terms of this type of geometric progression, in which the value of $n$ can increase without bound, is called a *geometric series* and has the form

$$S = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots. \tag{1.9}$$

If we again designate the sum of the first $n$ terms in Equation 1.9 as $S_n$ (here $S_n$ is called a *finite partial sum* of the first $n$ terms) or Equation 1.6, then, via Equation 1.8,

$$S_n = \frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}. \tag{1.10}$$

If $|r| < 1$, then the second term in the difference on the right-hand side of Equation 1.10 decreases to zero as $n$ increases indefinitely ($r^n \to 0$ as $n \to \infty$). Hence,

$$S = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}, \quad |r| < 1. \tag{1.11}$$

Thus, the geometric series $S$ is said to *converge* to the value $a_1/(1-r)$. If $|r| > 1$, the finite partial sums $S_n$ do not approach any limiting value—the geometric series $S$ does not converge; it is said to *diverge* since $|r^n| \to \infty$ as $n \to \infty$.

**EXAMPLE 1.3** Given the geometric progression

$$1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^{n-1}}, \ldots,$$

does the geometric series

$$S = 1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}} + \cdots$$

converge? If so, to what value? Given $r = 1/3$, the $n$th finite partial sum is

$$S_n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}}$$

and, via Equation 1.10,

$$S_n = \frac{1}{1-(1/3)} - \frac{1 \cdot (1/3)^n}{1-(1/3)}.$$

Then

$$S = \lim_{n\to\infty} S_n = \frac{1}{1-(1/3)} = \frac{3}{2}.$$

■

## 1.3 THE BINOMIAL FORMULA

Suppose we are interested in finding $(a+b)^n$, where $n$ is a positive integer. According to the *binomial formula*,

$$\begin{aligned}(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 \\ + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots + b^n,\end{aligned} \tag{1.12}$$

with the coefficients of the terms on the right-hand side of Equation 1.12 termed *binomial coefficients* corresponding to the exponent $n$. For instance, from Equation 1.12,

$$\begin{aligned}(a+b)^5 = a^5 + 5a^4b + \frac{5\cdot 4}{2}a^3b^2 + \frac{5\cdot 4\cdot 3}{2\cdot 3}a^2b^3 \\ + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4}ab^4 + b^5.\end{aligned}$$

Note that, in general:

1. There are $n + 1$ terms in the binomial expansion of $(a + b)^n$.
2. The exponent of $a$ decreases by 1 from term to term, while the exponent of $b$ increases by 1 from term to term, and the sum of the exponents of $a$ and $b$ is $n$.
3. The coefficients of the terms equidistant from the ends of the binomial expansion are equal.

A glance back at Equation 1.12 reveals that the $(r+1)$st term in the binomial expansion of $(a+b)^n$ is

$$\binom{n}{r} a^{n-r} b^r = \frac{n!}{r!(n-r)!} a^{n-r} b^r, \quad r = 0,1,\ldots,n.^1 \tag{1.13}$$

That is, for

$$r = 0, \quad \frac{n!}{0!n!} a^n b^0 = a^n;$$

$$r = 1, \quad \frac{n!}{1!(n-1)!} a^{n-1} b;$$

$$r = 2, \quad \frac{n!}{2!(n-2)!} a^{n-2} b^2; \text{ etc.}$$

If, as in the preceding text, $n=5$, then the preceding three binomial expansion terms are

$$(r=0)\, a^5;$$

$$(r=1) \quad \frac{5!}{1!4!} a^4 b = \frac{5 \cdot 4!}{4!} a^4 b = 5a^4 b;$$

$$(r=2) \quad \frac{5!}{2!3!} a^3 b^2 = \frac{5 \cdot 4 \cdot 3!}{2!3!} = 10a^3 b^2; \text{ etc.}$$

Given Equation 1.13, we can now write the general binomial expansion formula as

$$\begin{aligned}(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} a^{n-r} b^r.\end{aligned} \tag{1.14}$$

## 1.4 THE CALCULUS OF FINITE DIFFERENCES

Suppose that the real-valued function $y=f(x)$ is defined on an interval containing $x$ and $\Delta x$ (i.e., $x$ has been increased by an amount $\Delta x$). Since the *difference interval* $\Delta x$ is generally a constant, we may simply denote this constant as $h$. Then the *difference operator* $\Delta$ applied to $f(x)$ is defined as

$$\Delta f(x) = f(x+h) - f(x). \tag{1.15}$$

Furthermore, while $h$ may be any constant value, it is usually the case that $h=1$. Hence, in what follows, the interval of differencing in $x$ is unity. Thus, Equation 1.15 becomes

[1] Here $r!$ is called *r factorial* and is calculated as $r! = r(r-1)(r-2)\cdots 3\cdot 2\cdot 1$ with $0! \equiv 1$.

$$\Delta f(x) = f(x+1) - f(x). \tag{1.15.1}$$

Given Equation 1.15.1, it is readily verified that:

1. $\Delta[cf(x)] = c\Delta f(x)$, $c$ an arbitrary constant. (1.16)

Clearly

$$\begin{aligned}\Delta[cf(x)] &= cf(x+1) - cf(x)\\ &= c[f(x+1) - f(x)] = c\Delta f(x).\end{aligned}$$

For real-valued functions $f(x)$ and $g(x)$ both defined over an interval containing $x$ and $x+1$,

2. $\Delta\left[f(x) \pm g(x)\right] = \Delta f(x) \pm \Delta g(x).$ (1.17)

Here

$$\begin{aligned}\Delta[f(x) \pm g(x)] &= [f(x+1) \pm g(x+1)] - [f(x) \pm g(x)]\\ &= [f(x+1) - f(x)] \pm [g(x+1) - g(x)]\\ &= \Delta f(x) \pm \Delta g(x).\end{aligned}$$

(Note that if $c_1$ and $c_2$ are arbitrary constants, then $\Delta[c_1 f(x) \pm c_2 g(x)] = c_1\Delta f(x) \pm c_2\Delta g(x)$.)

3. $\Delta\left[f(x)g(x)\right] = g(x)\Delta f(x) + f(x)\Delta g(x) + \Delta f(x)\Delta g(x).$ (1.18)

We first find

$$\Delta\left[f(x)g(x)\right] = f(x+1)g(x+1) - f(x)g(x).$$

If we now add and subtract $f(x)g(x+1)$ on the right-hand side of the previous expression, then we obtain

$$\begin{aligned}\Delta[f(x)g(x)] &= f(x+1)g(x+1) + f(x)g(x+1) - f(x)g(x+1) - f(x)g(x)\\ &= f(x)[g(x+1) - g(x)] + g(x+1)[f(x+1) - f(x)]\\ &= f(x)\Delta g(x) + g(x+1)\Delta f(x).\end{aligned}$$

Substituting $g(x+1) = g(x) + \Delta g(x)$ into the preceding expression yields Equation 1.18.

4. $\Delta\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+1)}, g(x)g(x+1) \neq 0.$ (1.19)

To see this, let

$$\Delta\left(\frac{f(x)}{g(x)}\right)=\frac{f(x+1)}{g(x+1)}-\frac{f(x)}{g(x)}=\frac{g(x)f(x+1)-f(x)g(x+1)}{g(x+1)g(x)}$$
$$=\frac{g(x)(\Delta f(x)+f(x))-f(x)(\Delta g(x)+g(x))}{g(x)g(x+1)}$$
$$=\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)g(x+1)}.$$

5. $\Delta a^x = a^x(a-1).$ (1.20)

Here

$$\Delta a^x = a^{x+1}-a^x = a^x(a-1).$$

6. $\Delta \log_a x = \log_a\left(1+\frac{1}{x}\right).$ (1.21)

We simply set

$$\Delta \log_a x = \log_a(x+1)-\log_a x = \log_a\left(\frac{x+1}{x}\right)$$
$$=\log_a\left(1+\frac{1}{x}\right).$$

7. $\Delta e^x = e^x(e-1).$ (1.22)

Set

$$\Delta e^x = e^{x+1}-e^x = e^x(e-1).$$

8. $\Delta x^n = nx^{n-1}+\binom{n}{2}x^{n-2}+\cdots+\binom{n}{n-1}x+1$, $n$ a positive integer. (1.23)

We first find

$$\Delta x^n = (x+1)^n - x^n.$$

Then from the binomial expansion formula (Eq. 1.14) applied to $(x+1)^n$, we have

$$\Delta x^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}-x^n = \sum_{r=1}^{n}\binom{n}{r}\ x^{n-r}$$

or Equation 1.23.

Given the real-valued function $f(x)$, we can, via the difference operator $\Delta$, define a new function $\Delta f(x)$. If we apply the operator $\Delta$ to this new function $\Delta f(x)$, then we obtain the *second difference of f(x)* as the difference of the first difference or

$$\begin{aligned} \Delta^2 f(x) &= \Delta(\Delta f(x)) \\ &= \Delta[f(x+1) - f(x)] \\ &= \Delta f(x+1) - \Delta f(x). \end{aligned} \tag{1.24}$$

Similarly, the *third difference of f(x)*, which is the difference of the second difference, is

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x)). \tag{1.25}$$

In general, the *nth difference of* $f(x)$, which is the difference of the $(n-1)$st difference of $f(x)$, is

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)), \quad n = 2, 3, 4, \ldots. \tag{1.26}$$

**EXAMPLE 1.4** Given the real-valued function $y = f(x) = x^3 + 2x^2$, find $\Delta^3 f(x)$. First,

$$\Delta f(x) = (x+1)^3 + 2(x+1)^2 - x^3 - 2x^2.$$

Then, via the binomial expansion formula,

$$\Delta f(x) = x^3 + 3x^2 + 3x + 1 + 2(x^2 + 2x + 1) - x^3 - 2x^2 = 3x^2 + 7x + 3.$$

Next,

$$\begin{aligned} \Delta^2 f(x) &= \Delta(\Delta f(x)) = \Delta(3x^2 + 7x + 3) \\ &= 3(x+1)^2 + 7(x+1) + 3 - 3x^2 - 7x - 3 \\ &= 6x + 10. \end{aligned}$$

Finally,

$$\begin{aligned} \Delta^3 f(x) &= \Delta(\Delta^2 f(x)) = \Delta(6x + 10) \\ &= 6(x+1) + 10 - 6x - 10 = 6. \end{aligned}$$

■

The preceding example problem serves as a nice lead-in to the following result:

9. Let $f(x)$ be a polynomial of degree $n$ in $x$ or $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where the $a_j$, $j = 0, 1, \ldots, n$, are arbitrary constants and $a_n \neq 0$. Then the $n$th difference of $f(x)$ is the constant function $\Delta^n f(x) = n!\, a_n$, and all succeeding differences vanish or $\Delta^p f(x) = 0$, $p > n$.

To see this we have, from property or result no. 2 earlier,

$$\Delta f(x) = a_0\Delta 1 + a_1\Delta x + a_2\Delta x^2 + \cdots + a_n\Delta x^n. \tag{1.27}$$

By property no. 8, $\Delta$ operating on $x^n$ renders a finite number of terms with $n-1$ as the highest power of $x$. Applying this observation to Equation 1.27 enables us to conclude that $\Delta$ operating on a polynomial of degree $n$ results in a polynomial of degree $n-1$. In a similar vein, $\Delta^2 f(x)$ will be a polynomial of degree $n-2$, and $\Delta^n f(x)$ will thus be a polynomial of degree 0 (i.e., a constant). Moreover, for $p>n$, $\Delta^p$ applied to a constant must be zero.

## 1.5 THE NUMBER *e*

Let us consider the *sequence* (an ordered countable set of numbers not necessarily all different) $x_1, x_2, \ldots, x_n, \ldots$, where

$$x_n = \left(1+\frac{1}{n}\right)^n. \tag{1.28}$$

If we expand the right-hand side of Equation 1.28 by the binomial formula (Eq. 1.14), then

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+\frac{n}{1!}\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}\\ &\quad+\cdots+\frac{n(n-1)\cdots 2\cdot 1}{n!}\frac{1}{n^n}\\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\\ &\quad+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right).\end{aligned} \tag{1.29}$$

Suppose we now replace $n$ by $n+1$ in Equation 1.28 so as to obtain

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}. \tag{1.30}$$

Again using the binomial expansion formula,

$$\begin{aligned}\left(1+\frac{1}{n+1}\right)^{n+1} &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\\ &\quad+\cdots+\left[\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(\frac{2}{n+1}\right)\left(\frac{1}{n+1}\right)\right].\end{aligned} \tag{1.31}$$

A term-by-term comparison of Equations 1.29 and 1.31 reveals that $x_{n+1}$ is always larger than $x_n$. In fact, Equation 1.31 has one more term than Equation 1.29. Hence, $x_{n+1}>x_n$; that is, the sequence of values specified by Equation 1.28 is strictly monotonically increasing.

Next, looking to the expansion of $x_n$ (Eq. 1.29), we see that

$$x_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = y_n.$$

Since $y_n$ is a geometric progression (the common ratio $r=1/2$), we have

$$x_n < 1+\frac{1-(1/2)^{n+1}}{1-(1/2)} < 1+\frac{1}{1/2} = 3.$$

Hence, Equation 1.28 is bounded from above. And since any monotone bounded sequence has a limit, we can denote the limit of Equation 1.28 as

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n &= 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!} = e.\end{aligned} \tag{1.28.1}$$

To five decimal places, $e=2.71828$.

## 1.6 THE NATURAL LOGARITHM

We may define the *natural logarithm* of $x$, for positive $x$, as

$$F(x) = \ln x = \int_1^x \frac{1}{r}\,dr, \quad x > 0 \tag{1.32}$$

(see Fig. 1.1). For $x=1$, obviously $\ln 1=0$; and for $x<1$,

$$\ln x = -\int_x^1 \frac{1}{r}\,dr.$$

Given $F(x)=\ln x$, it follows that $F'(x)=d\ln x/dx=1/x$.

Looking to the graph of the *logarithmic function* $y=\ln x$ (Fig. 1.2a), we see that $\ln x$ is continuous, single valued, and monotonically increasing with $dy/dx=1/x>0$, while $d^2y/dx^2=-1/x^2<0$. Since $\ln 1=0$, the curve passes through the point (1,0). Moreover, $\ln x\to+\infty$ as $x\to+\infty$; $\ln x=-\ln 1/x\to-\infty$ as $x\to 0+$.

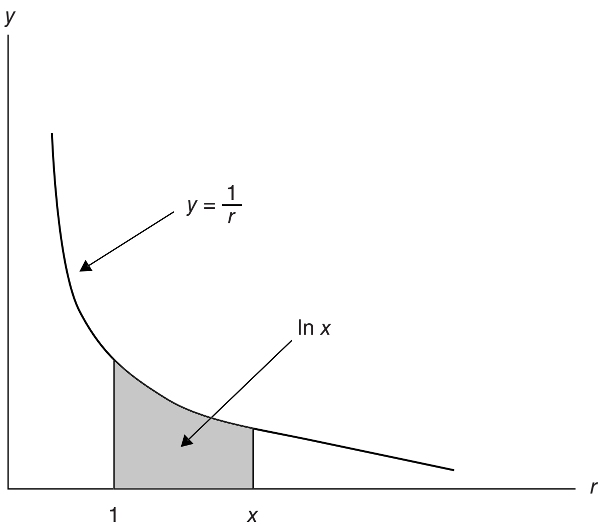


**FIGURE 1.1** The natural logarithm of $x$.

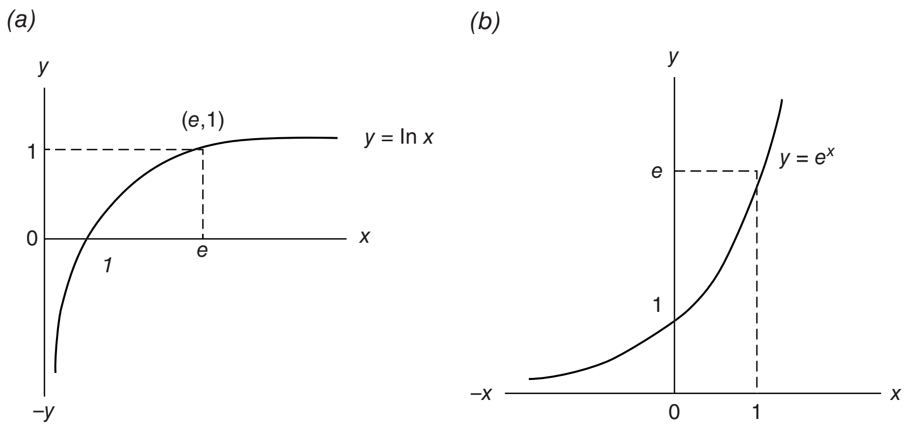


**FIGURE 1.2** (**a**) Logarithmic function and (**b**) Exponential function.

## 1.7 THE EXPONENTIAL FUNCTION

Given the logarithmic function $y = \ln x$, if $x = e$, then $\ln e = 1$ (Fig. 1.2a). Hence, the value of $x$ for which $\ln x = 1$ is $e$. Also, $\ln e^n = n \ln e = n$. Thus, the number whose natural logarithm is $n$ is $e^n$ so that the anti-natural logarithm of $n$ is $e^n$ or the *antilogarithm* is the inverse of the logarithm.

This said, given the logarithmic function $y = \ln x$, its inverse function is

$$y = \text{inverse natural logarithm of } x$$

or

$$x = \ln y, \tag{1.33}$$

where $x > 0$ for $y > 1$, $x = 0$ for $y = 1$, and $x < 0$ for $y < 1$.

Thus, there exists a one-to-one correspondence between the sets $Y = \{y | y > 0\}$ and $X = \{x | -\infty < x < +\infty\}$. In this regard, we can define $y$ ($> 0$) as a real-valued function of $x$ for $-\infty < x < +\infty$. Hence,

$$y = e^x \tag{1.34}$$

is equivalent to Equation 1.33 and is called the *exponential function*. So with the logarithmic function continuous, single valued, and monotonically increasing, it follows that the exponential function, its inverse, exists and has the same exact properties. In sum,

$$y = e^x \text{ if and only if } x = \ln y.$$

Hence, $e^x$, defined for all real $x$, is that positive number $y$ whose natural logarithm is $x$ (Fig. 1.2b).

Some useful relationships between the exponential function and the (natural) logarithmic function are:

$$\ln e^x = x \ln e = x \qquad e^{(\ln x)^2} = x^{\ln x}$$

$$\ln e^{-x} = \ln(1/e^x) = -x \qquad e^{(a \ln x)^2} = x^{a^2 \ln x}$$

$$e^{\ln x} = x \qquad e^{(a \ln x)(b \ln y)} = y^{ab \ln x} = x^{ab \ln y}$$

$$e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Moreover,

if $\ln y = \ln a + w(\ln r - \ln s)$, then

$$y = a\left(\frac{r}{s}\right)^w;$$

if $\ln y = a + w(r - s)$, then

$$y = e^a\left(\frac{e^r}{e^s}\right)^w;$$

and if $\ln y = a + w(\ln r - s)$, then

$$y = e^a\left(\frac{r}{e^s}\right)^w.$$

## 1.8 EXPONENTIAL AND LOGARITHMIC FUNCTIONS: ANOTHER LOOK

For the exponential function (Eq. 1.34), $e$ served as the (fixed) *base* of this expression. Moreover, the unique inverse of Equation 1.34 is the logarithmic function $y = \ln x$, where it is to be implicitly understood that "$y$ is the natural logarithm of $x$ to the base $e$." However, other bases can be used.

Specially, let us alternatively specify an *exponential function* of $x$ as

$$y = b^x, b > 1, \tag{1.35}$$

where $b$ is the (fixed) *base* of the function. The base $b$ will be taken to be a number greater than unity (since any positive number ($y$) can be expressed as a power ($x$) of a given number ($b$) greater than unity). Hence, Equation 1.35 is a continuous single-valued function, which is monotonically increasing for $-\infty < x < +\infty$ (Fig. 1.3a).

Since Equation 1.35 is continuous and single valued, it has a unique inverse called the *logarithmic function*

$$x = \log_b y, \quad b > 0 \text{ and } b \neq 1, \tag{1.36}$$

read "$x$ is the logarithm of $y$ to the base $b$" (Fig. 1.3b). In this regard, a number $x$ is said to be the *logarithm* of a positive real number $y$ to a given base $b$ if $x$ is the power to which $b$ must be raised in order to obtain $y$. Hence,

$$y = b^x \text{ if and only if } x = \log_b y.$$

(Thus, $\log_5 25 = 2$ since $5^2 = 25$; and $\log_2 8 = 3$ since $2^3 = 8$.)

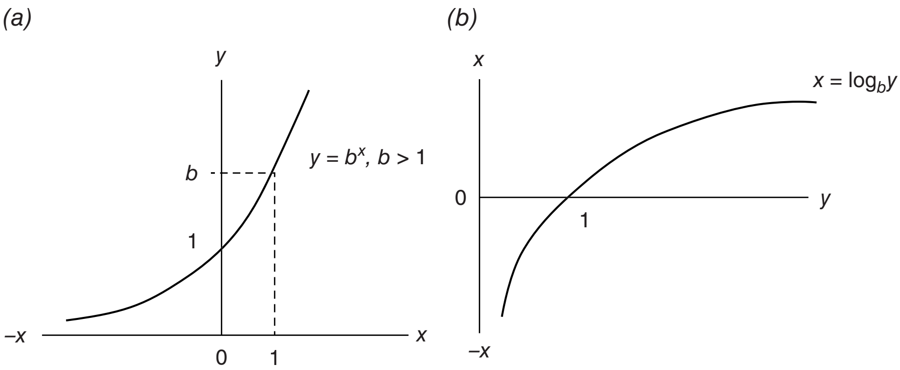


**FIGURE 1.3** (**a**) Exponential function (fixed base $b$) and (**b**) Logarithmic function (fixed base $b$).

Clearly a logarithm is simply a fancy way to write an exponent. Note that $\log_b y$ is not defined for negative values of $y$ or zero; that is, if $0<y<1$, then $\log_b y < 0$; if $y=1$, then $\log_b 1=0$; and if $y>1$, then $\log_b y>0$.

Some useful properties of logarithms are

| ***base b*** | ***base e*** |
|---|---|
| 1. $\log_b uv = \log_b u + \log_b v$ | 1. $\ln uv = \ln u + \ln v$ |
| 2. $\log_b \frac{u}{v} = \log_b u - \log_b v$ | 2. $\ln \frac{u}{v} = \ln u - \ln v$ |
| 3. $\log_b u^n = n \log_b u$ | 3. $\ln u^n = n \ln u$ |
| 4. $\log_b \frac{1}{p} = -\log_b p$ | 4. $\ln \frac{1}{p} = -\ln p$ |

Also, some useful differentiation formulas are

1. $\dfrac{da^x}{dx} = a^x \ln a$
2. $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$
3. $\dfrac{d\log_b x}{dx} = \dfrac{1}{x}\log_b e$
4. $\dfrac{de^x}{dx} = e^x$
5. $\dfrac{de^{cx}}{dx} = ce^{cx}$, $c$ a constant

## 1.9 CHANGE OF BASE OF A LOGARITHM

Our goal is to develop a method for transforming the logarithm of $x$ to the base $b$ to the logarithm of $x$ to the base $a$. To this end, we know from the preceding discussion of logarithms that $y=\log_a x$ is equivalent to $x=a^y$. Let us now take the logarithm of this latter expression with respect to the base $b$; that is,

$$\log_b x = y \log_b a$$

or

$$y = \frac{\log_b x}{\log_b a} = \log_a x. \tag{1.37}$$

Here Equation 1.37 will be termed our *Change of Base Rule*: the logarithm of $x$ to the base $a$ is the logarithm of $x$ to the base $b$ divided by logarithm of $a$ to the base $b$.

It is well known that a *common logarithm* is taken to the base 10 ($\log_{10} 100=2$ since $10^2=100$), while a *natural logarithm*, as defined earlier, is taken to the base $e=2.71828$ ($\ln 20.08554=3$ since $e^3=20.08554$). We may employ Equation 1.37 to facilitate the conversion between them, that is,

1. To convert from common to natural logarithms,

$$\ln x = \frac{\log_{10} x}{\log_{10} e} = 2.303 \log_{10} x.$$

2. To convert from natural to common logarithms,

$$\log_{10} x = \frac{\ln x}{\ln 10} = 0.4343 \ln x.$$

## 1.10 THE ARITHMETIC (NATURAL) SCALE VERSUS THE LOGARITHMIC SCALE

Suppose the observations $x_i$, $i = 1,2, \ldots$, on a variable $X$ are measured relative to some specific scale and appear as points on the $X$-axis, with distances along this axis taken from some specific base or reference point. Now:

1. If distance along the $X$-axis is taken to be equal (or proportional) to the actual value of the $X$ point plotted, then the observations on $X$ are measured on an *arithmetic scale* (for $X = x_r$, the distance between the base of 0 and the plotted point is $x_r$ units).
2. If an $X$ value is plotted at a distance along the $X$-axis, which is equal (or proportional) to its logarithm (to, say, the base 10), then the observations on $X$ are measured on a *logarithmic scale* (for $X = x_r$, the distance between the base value and the plotted point is $\log_{10} x_r$ units).

Looking to Figure 1.4, we see that on an arithmetic scale, either (i) the points appear at equal distances from each other (scale a) or (ii) the points appear at increasing distances from each other (scale b).

But as Figure 1.5 reveals, (i) taking the base 10 logarithms of the values on arithmetic scale a of Figure 1.4 produces a sequence of points exhibiting decreasing distances from each other (scale a′), or (ii) taking base 10 logarithms of the values on

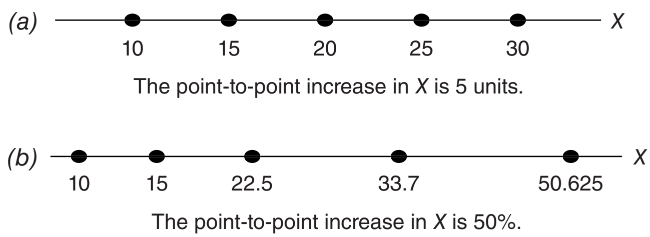


**FIGURE 1.4** Arithmetic scale.

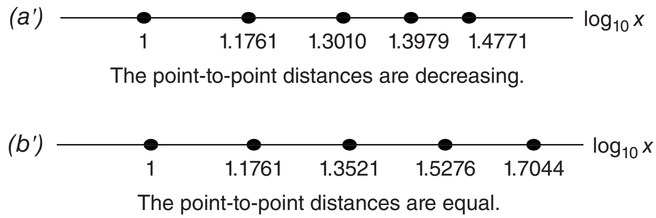


**FIGURE 1.5** Logarithmic scale.

arithmetic scale b of Figure 1.4 renders a sequence of points that are located at equal distances from each other (scale b′).

If on an arithmetic scale a sequence of $X$ values exhibits equal point-to-point *decreases* (e.g., consider 50, 40, 30, 20, 10), then the corresponding base 10 logarithmic scale displays values at increasing distances (to the left). And if the $X$ values decrease by a fixed percentage on an arithmetic scale (e.g., for a 20% decrease we get 50, 40, 32, 25.6, 20.48), then the corresponding base 10 logarithmic values display equal point-to-point distances. (The reader is asked to verify these assertions for the given sets of arithmetic values.)

On the basis of the preceding discussion, it is evident that equal point-to-point distances on an arithmetic scale indicate *equal absolute changes* in a variable $X$; but equal point-to-point distances on a (base 10) logarithmic scale reflect equal proportional or percentage changes in $X$. For instance, if $x_1$,$x_2$, and $x_3$ are values of $X$ plotted at equal distances on an arithmetic scale, then $X$ increases by equal absolute amounts since $x_3 - x_2 = x_2 - x_1$. However, if the (base 10) logarithms of these $X$ values are plotted at equal distances on a logarithmic scale, then $X$ increases by *equal proportional amounts* since $\log_{10}x_3 - \log_{10}x_2 = \log_{10}x_2 - \log_{10}x_1$ or $\log_{10}(x_3/x_2) - \log_{10}(x_2/x_1)$ or $x_3/x_2 = x_2/x_1$.

Suppose that instead of dealing with a single variable $X$, we introduce a second variable $Y$ and posit a functional relationship between them of the form $y = f(x)$, where $f$ is a rule or law of correspondence (i.e., a mapping), which associates with each admissible value $x$ of $X$ a unique admissible value $y$ of $Y$. Then in terms of our measurement scales, we note that:

1. If absolute changes in the variables are of interest, then we can model $y$ as a *linear* function of $x$ or $y = a + bx$, where $a$ is the vertical intercept and $b$ is the slope ($b = \Delta y/\Delta x$). Hence, the absolute change in $y$ is always the same constant proportion ($b$) of the absolute change in $x$.
2. If proportional or percentage increases in $y$ (or the rate of growth in $y$) are of interest as $x$ (measured on an arithmetic scale) increases in value, then we can model $y$ as an *exponential function* of $x$ or $y = ab^x$. Thus, $\log_{10}y = \log_{10}a + (\log_{10}b)x$, where $\log_{10}a$ is the vertical intercept and $\log_{10}b$ is the slope. Since only $y$ is measured on a logarithmic scale, this relationship is termed a *semilogarithmic function* of $x$ and, with $\log_{10}b$ constant, is linear in form or plots as a straight

line. In this circumstance, as $x$ varies over a given interval, $\log_{10}y$ increases by equal increments; that is, $y$ exhibits equal proportional or percentage increases in its value.

3. If proportional or percentage changes in both $x$ and $y$ are of interest, then we can model $y$ as a *power function* of $x$ or $y = ax^b$. Now $\log_{10}y = \log_{10}a + b\log_{10}x$, where $\log_{10}a$ is the vertical intercept, $b$ is the slope, and both $x$ and $y$ are measured on a logarithmic scale. With both variables measured on a logarithmic scale, this relationship is referred to as a *double-logarithmic function*. Here proportional or percentage changes in $y$ are explained by proportional or percentage changes in $x$, and if equal percentage changes in $x$ precipitate equal percentage changes in $y$, then, with $b$ constant, this function plots as a straight line. Here

$$b = \frac{\Delta \log_{10} y}{\Delta \log_{10} x}.$$

## 1.11 COMPOUND INTEREST ARITHMETIC

Suppose a principal amount of \$100.00 is invested and accumulates at a *compound interest rate* of 5% per year and interest is declared yearly. Then the following time profile of accumulation emerges:

Start year 1 $\quad$ \$100.00 invested

End year 1 $\quad \$100.00 + \underbrace{\$100.00(0.05)}_{\text{interest}} = \$100.00(1.05) = \$105.00$

End year 2 $\quad \$105.00 + \underbrace{\$105.00(0.05)}_{\text{interest}} = \$105.00(1.05)$

$= \$100.00(1.05)(1.05)$

$= \$100.00(1.05)^2 = \$110.25$

End year 3 $\quad \$110.25 + \underbrace{\$110.25(0.05)}_{\text{interest}} = \$110.25(1.05)$

$= \$100.00(1.05)^2(1.05)$

$= \$100.00(1.05)^3 = \$115.7625$

⋮

etc.

In general, after $t$ time periods or years, the accumulated amount at compound interest with annual compounding is

$$A_t = P(1+r)^t, \tag{1.38}$$

where $P$ is the principal invested, $100r\%$ is the yearly interest rate, and $t$ indexes time in years. (In the preceding example, $P=\$100.00$ and $r=0.05$.) So for, say, $t=10$, $A_{10} = 100(1+0.05)^{10} = 100(1.62889) = \$162.889$. As this example problem reveals, the nature of compound interest is that, over the entire investment period, the interest itself earns interest.

What if interest is added twice a year rather than just once at the end of each year? Since the yearly interest rate is 5%, it follows that the half-yearly rate must be 2.5% so that 2.5% is added in each first half year and 2.5% is added in each second half year. So if a principal of \$100.00 is invested and accumulates at a compound interest rate of 2.5% per half year and interest is declared at the end of each half year, then the revised time profile is:

Start year 1 $\quad \$100.00$ invested

End year 1 $\quad \underbrace{\$100.00(1.025)}_{\text{first half year}} \times \underbrace{(1.025)}_{\text{second half year}} = \$100.00(1.025)^2 = \$105.0625$

End year 2 $\quad \underbrace{\$105.0625(1.025)}_{\text{first half year}} \times \underbrace{(1.025)}_{\text{second half year}} = \$105.0625(1.025)^2$

$= \$100.00(1.025)^4$

$= \$110.38129$

End year 3 $\quad \underbrace{\$110.38129(1.025)}_{\text{first half year}} \times \underbrace{(1.025)}_{\text{second half year}} = \$110.38129(1.025)^2$

$= \$100.00(1.025)^6$

$= \$115.96934.$

⋮

etc.

To summarize, if interest is declared half-yearly, $P$ is the principal, and the yearly interest rate is $100r\%$, then after $t$ years the accumulated amount is

$$A_{2,t} = P\left(1+\frac{r}{2}\right)^{2t}.$$

Again taking $t=10$, $A_{2,10} = 100(1+0.025)^{20} = \$163.86144$. A comparison of $A_{10} = \$162.89$ with $A_{2,10} = \$163.86$ reveals that the more frequently interest is added, the larger is the accumulated amount at the end of a given period. In general, the *accumulated amount at compound interest with interest declared j times a year* is

$$A_{j,t} = P\left(1+\frac{r}{j}\right)^{jt}. \tag{1.39}$$

A moment's reflection concerning the structure of Equation 1.39 reveals that investment growth over time behaves as a geometric progression; that is, each amount is a fixed multiple $(1+(r/j))^j$ of the previous period's amount. That is, the sequence of terms of this geometric progression is:

$$P, P\left(1+\frac{r}{j}\right)^j, P\left(1+\frac{r}{j}\right)^{2j}, P\left(1+\frac{r}{j}\right)^{3j}, \ldots.$$

Hence, the growth process represented by Equation 1.39 can be expressed as the exponential function

$$A_{j,t} = Pb^{jt},\ b = 1+\frac{r}{j} \tag{1.40}$$

and is referred to as a *compound interest growth curve* (alternatively called a *geometric or exponential growth curve*). Transforming to logarithms gives

$$\log_{10} A_{j,t} = \log_{10} P + \log_{10}\left(1+\frac{r}{j}\right)(jt). \tag{1.41}$$

Clearly this semilogarithmic expression plots as a straight line with vertical intercept $\log_{10}P$ and (constant) slope $\log_{10}(1+(r/j))$. Obviously the magnitude of the slope depends upon $r$ and $j$. In this regard, $r/j$ is the proportionate rate of change in $A_{j,t}$ per unit period of time (i.e., per year if $j=1$, per half year if $j=2$, per quarter if $j=4$). Note that the independent variable on the right-hand side of Equation 1.41 is $jt$; it represents the total number of subperiods $j$ within a year times the number of years $t$.

For instance, if $j=4$, then $jt=4t$ represents the total number of quarters spanned by the entire accumulation period; that is, if $t=1$, $4t=4$ spans one year; if $t=2$, $4t=8$ spans a two-year time interval.

A special case of Equation 1.41 is, from Equation 1.38,

$$\log_{10} A_t = \log_{10} P + \log_{10}(1+r)t, \tag{1.41.1}$$

where $r$ is the proportionate rate of growth in $A_t$ per unit of time.

Given Equation 1.39, let us assume that $j$ increases without limit or, equivalently, that the compounding or conversion periods become shorter and shorter. In this instance the term $(1+(r/j))$ in Equation 1.39 is replaced by $e$, and we consequently have what is termed the case of *continuous compounding* or *continuous conversion*.

To see this, let us rewrite Equation 1.39 as

$$P\left[\left(1+\frac{r}{j}\right)^{j/r}\right]^{rt} = P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt},$$

where $n=j/r$. Now, as the number of compounding or conversion periods $j\to+\infty$, it follows that $n\to+\infty$. Hence, via Equation 1.28.1,

$$\lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt} = Pe^{rt} = A.\text{[2]} \qquad (1.42)$$

Here Equation 1.42 depicts a *natural exponential growth curve and represents the accumulated amount at the end of t years if the principal (P) grows at an exponential rate of 100r% per year or is compounded continuously at 100r% per year*. For instance, if we again take $P=\$100$, $r=0.05$, and $t=10$, then $A=100e^{0.05(10)}=\$164.872$.

An assortment of points concerning Equation 1.42 merits our attention. First, the variable $t$ in Equation 1.39 is discontinuous since interest is declared at specific (discrete) intervals over the investment period. However, as $j\to+\infty$ and interest is declared with increasing frequency, $t$ tends to become a continuous variable in Equation 1.42. Second, it is instructive to view the term $\left(1+\frac{1}{n}\right)^{n}$ in Equation 1.42 as the yearly accumulated amount of \$1.00 invested when interest is compounded at 100% per annum and declared $n$ times over the year. Then as $n$ increases without bound,

$$\lim_{n\to\infty} 1\cdot\left(1+\frac{1}{n}\right)^{n} = \$e.$$

So as compound interest is declared more and more frequently, the \$1.00 invested at 100% interest approaches \$$e$ at the end of a year. Third, transforming Equation 1.42 to logarithms yields

$$\ln A = \ln P + rt, \qquad (1.43)$$

which plots as a straight line with vertical intercept ln $P$ and (constant) slope $r$. Finally, if an amount is compounded yearly at $100r\%$ (e.g., see Eq. 1.38) and that same amount is compounded continuously at $100g\%$ and $100r\%$ and $100g\%$ are equivalent interest rates, then obviously $g=\ln(1+r)$.

[2] It is instructive to view the derivation of this expression in an alternative light. To this end, let us write Equation 1.39 as

$$P\left[\left(1+\frac{r}{j}\right)^{j/r}\right]^{rt} = P\left[(1+x)^{1/x}\right]^{rt},$$

where $x=r/j$. Then applying the binomial formula 1.12 to the term in square brackets yields

$$P\left[1^{1/x}+\frac{1}{x}\cdot 1^{\frac{1}{x}-1}x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}\cdot 1^{\frac{1}{x}-2}x^{2}+\cdots\right]^{rt} = P\left[1+1+\frac{1-x}{2!}+\frac{(1-x)(1-2x)}{3!}+\cdots\right]^{rt}.$$

As $j\to+\infty$, it follows that $x\to 0$ so that

$$\lim_{x\to\infty} P[.]^{rt} = P\left[1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right]^{rt} = Pe^{rt}$$

via Equation 1.28.1.