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Introduction with a Spring-Mass Frictionless Contact System

We start our introduction to contact mechanics from the simplest possible system: a mass point suspended on a spring, but free deformations of the system can be restricted by the additional plane. This chapter gives the general idea on how to handle contact by specifying contact constraints as well as illustrating numerical methods: Lagrange multipliers, Penalty and Augmented Lagrangian.

Contact from geometrical point of view can be observed as a restriction on certain motions. The most simple system to start, shown in Wriggers (2002), is a spring-mass system. Thus, a mass point m suspended on the spring with stiffness c gives us the simplest example of the contact problem, if we restrict the motion (deflection of the spring u) of the point by the rigid plane below, see Figure 1.1.

1.1 Structural Part – Deflection of Spring-Mass System

The equilibrium state of the given system, first without contact, can be described by the following three equivalent expressions:

1. Strong formulation or just an equation for equilibrium of forces

   $$F = mg$$

   with spring force represented by $F = cu$ gives us

   $$cu - mg = 0; \tag{1.1}$$

2. Weak or variational formulation is obtained if we consider the work of forces over small test displacement δu (variation).

   $$\delta u(cu - mg) = 0 \tag{1.2}$$

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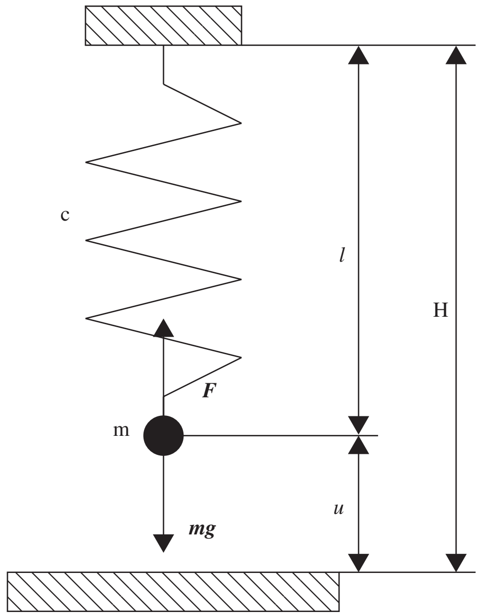

Figure 1.1 Spring-mass system in contact with a rigid plane

δu is also called virtual displacement or the variation of the displacement u. Treating the variation operation δ in a similar way to differentiation, from equation (1.2) we obtain by the integration the following expression

$$\delta\left(\frac{1}{2}cu^2 - mgu\right) = \delta\Pi = 0 \tag{1.3}$$

with Π representing the potential energy of the system. Equation (1.3) expresses in due course the necessary condition of the extremum and one can see from the positive polynomial of the second order for Π in equation (1.4) that the potential energy Π reaches its minimum. Thus we obtain;

3. Weak or variational formulation in extremal form

$$\Pi(u) = \frac{1}{2}cu^2 - mgu \rightarrow min \tag{1.4}$$

Namely, the last formulation in the extremal form will be employed to include contact conditions in that form of constraints using methods of optimization theory.

1.2 Contact Part – Non-Penetration into Rigid Plane

The non-penetration condition can be expressed by the geometrical inequality using the total height H of the constrained spring-mass system and the undeformed length of the spring l (undeformed). Thus, expression

$$l + u \leq H$$

follows. Equivalently, this can be expressed using a penetration function as $p(u)$

$$p(u) := l + u - H \leq 0. \tag{1.5}$$

Using methods of the optimization theory it is possible to solve the minimization problem $\Pi \to min$ subjected to such a restriction of p in equation (1.5). Summarizing all diversities of contact situations in the sense of optimization theory, so-called, the Karush–Kuhn–Tucker conditions (or KKT-conditions) are formed:

$$\begin{aligned} &\text{contact:} && p = 0 \ \text{ and } N > 0 \\ &\text{no contact:} && p < 0 \ \text{ and } N = 0 \\ &\text{complimentary condition:} && pN = 0 \end{aligned} \tag{1.6}$$

Formally, N is introduced as a Lagrange multiplier, however, with the mechanical interpretation as the normal contact force. The solution of the contact problem is formulated as minimizing the potential energy in equation (1.4) subject to the inequality conditions (KKT-conditions) in equation (1.6).

1.3 Contact Formulations

Various contact formulations can be obtained depending on the method to satisfy the inequality conditions. The most common methods are the Lagrange multiplier method, the Penalty method and the Augmented Lagrangian method, which are discussed in the following sections for the aforementioned spring-mass contact system.

1.3.1 Lagrange Multiplier Method

The Lagrange multiplier method is based on the construction of a Lagrange functional including constraints. The Lagrange functional L is constructed as a goal function – now potential energy Π – plus constraint equations times the Lagrange multiplier λ

$$L(u, \lambda) = \Pi + \lambda p(u). \tag{1.7}$$

The new functional $L(u, \lambda)$ is, hereby, not dependent on just u any more, but also on λ. Thus, the weak formulation in the form of the extremal problem in equation (1.4) is reformulated as

$$L(u, \lambda) = \Pi + \lambda p \to min.$$

The necessary condition for fulfilling this requirement is that the partial derivatives with respect to both variables u and λ have to be zero:

$$\begin{cases} \dfrac{\partial L}{\partial u} = 0 \Rightarrow \dfrac{\partial}{\partial u}\left(\dfrac{cu^2}{2} - mgu\right) + \lambda \dfrac{\partial}{\partial u}(l + u - H) = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \Rightarrow \qquad\qquad\qquad\qquad p = l + u - H = 0. \end{cases}$$

The following system is derived

$$\begin{cases} cu - mg + \lambda = 0 \\ l + u - H = 0. \end{cases} \tag{1.8}$$

The resulting two equations can be interpreted as that the Lagrange multiplier λ physically represents a force that equals the normal contact force ($\lambda = N$). Moreover, the non-penetration condition $p = 0$ is fulfilled exactly. The Karush–Kuhn–Tucker conditions are expressed as follows:

1. contact
   $p = 0$ and $\lambda > 0$;
2. no contact
   $p < 0$ and $\lambda = 0$;
3. complimentary condition
   $p\lambda = 0$
   either ($p = 0$ and $\lambda \neq 0$), or ($p \neq 0$ and $\lambda = 0$).

Remark 1.3.1 *The Lagrange multipliers method allows fulfillment the non-penetration condition exactly, however, the global system needs to be extended by an additional variable – the Lagrange multiplier λ. This can be regarded as an additional obstacle to solving the contact problem.*

1.3.2 Penalty Method

The Penalty method is based on the construction of a new functional without introducing any additional unknown, such as a Lagrange multiplier λ. The functional in equation (1.4) is extended by a penalty functional constructed with a penalty parameter ε and the penetration function $p(u)$

$$\Pi_p(u) = \Pi + \frac{1}{2}\varepsilon p^2(u). \tag{1.9}$$

The additional term $\frac{1}{2}\varepsilon p^2(u)$ is a penalty functional $W_\varepsilon(p)$ depending on both the penetration function p and the penalty parameter ε. We are now going to show that the contact problem will be solved if we increase the penalty parameter to infinity $p \to \infty$.

This is not the only possibility for formulating the penalty functional $W_\varepsilon(p)$ in equation (1.9). It has been proved in numerical methods for optimization that the penalty functional, in general, should fulfill the following conditions:

1. $W_\varepsilon(0) = 0$
2. $W_\varepsilon(p) > 0$ and strictly increases for both p and ε.
3. $\lim\limits_{p \to \infty} W_\varepsilon(p) = \infty$.

As an example of the penalty functional, fulfilling these mentioned conditions, it can be constructed in various forms as a positive Taylor series, for example

$$W_\varepsilon(p) = \frac{1}{2}\varepsilon p^2 + \varepsilon_4 p^4 + ... \tag{1.10}$$

These conditions are necessary for the fulfillment of constraints in limit case when $p \to 0$ if $\varepsilon(\varepsilon_4,...) \to \infty$.

We have to note that for this contact formulation the functional Π_p depends only on the primary variable u and the choice of the penalty parameter. Thus, the necessary condition for gaining a minimum for Π_p is only one equation as

$$\frac{\partial \Pi_p}{\partial u} = cu - mg + \frac{1}{2}\varepsilon\frac{\partial}{\partial u}(l + u - H)^2 = cu - mg + \varepsilon(l + u - H) = 0 \tag{1.11}$$

Here again, an interpretation for the normal contact force N can be gained as

$$N = \varepsilon(l + u - H) = \varepsilon p(u). \tag{1.12}$$

The last equation gives the mechanical interpretation of the penalty method, see Figure 1.2. In order to fulfill the non-penetration condition an additional spring with a stiffness ε is added at the point of penetration, thus leading to the pulling backwards force $N = \varepsilon p(u)$. It is obvious that the choice of a spring with the infinite

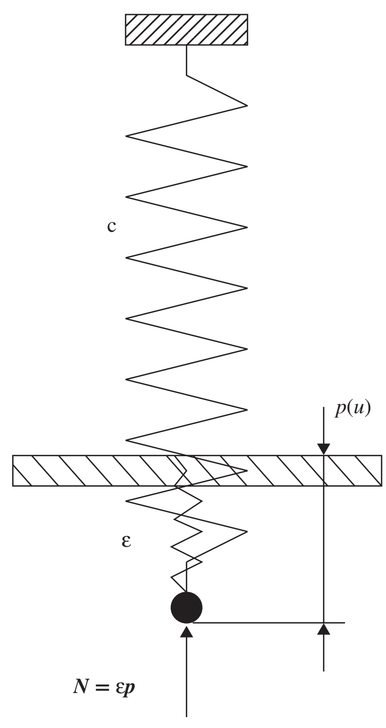

Figure 1.2 Mechanical interpretation of the Penalty method

stiffness leads to zero deformation of the additional spring, or to zero penetration. This backward pulling force is a contact force. This action is comparable to a spring with a stiffness ε and acting on the deflection $p = l + u - H$ attached to the rigid plane. There is no further restriction for the non-penetration condition $p = 0$ and thus penetration $p(u) > 0$ is allowed.

Let us study the deflection u at the limit case $\varepsilon \to \infty$

$$\lim_{\varepsilon\to\infty} u = \lim_{\varepsilon\to\infty} \frac{\varepsilon(H-l)+mg}{c+\varepsilon} = \lim_{\varepsilon\to\infty} \frac{H-l+\frac{mg}{\varepsilon}}{\frac{c}{\varepsilon}+1} = H-l. \tag{1.13}$$

The last means exactly that penetration in equation (1.5) vanishes if $\varepsilon \to \infty$

$$\lim_{\varepsilon\to\infty} p(u) = l + u - H = 0.$$

Remark 1.3.2 *The following remarks should be made:*

1. *The non-penetration condition is fulfilled exactly $p = 0$ if the penalty parameter ε goes to infinity.*
2. *In practical computations, "large values" of the penalty parameter ε are considered. This leads to "acceptable" small values for the penetration p. This will be studied during numerical tests in Part II.*
3. *The Penalty method fulfills the non-penetration condition only approximately, however, the global system is not extended and contains only the primary variable u.*

1.3.3 Augmented Lagrangian Method

The augmented Lagrangian method is introduced as an alternative approach to the Lagrange multipliers method and the Penalty method in order to find a way to fulfill the non-penetration condition exactly, however, without introducting an additional variable – Lagrange multiplier λ. The augmented Lagrange method is based on the following functional:

$$L(u) = \Pi + \lambda p(u) + \frac{1}{2}\varepsilon p^2(u). \tag{1.14}$$

Although the functional $L(u)$ contains terms introduced before in paragraphs 1.3.1 and 1.3.2 for both the Lagrange and the Penalty method with parts $\lambda p(u)$ and $\frac{1}{2}\varepsilon p^2(u)$, respectively, a special recursive scheme is constructed such that the functional depends only on the primary variable u. This fact results in a applied recursive scheme for computing the Lagrange multiplier λ in an iterative (repetitive) manner. This procedure is called augmentation of the Lagrange multiplier, hence the name of the method. The scheme consists of two main steps:

Step 1: Initialization of the Lagrange multiplier λ
$\lambda_0 = 0$
Definition of the initial displacement u_0 by using the system as for the Penalty method equation (1.11)
$cu_0 - mg + \varepsilon(l + u_0 - H) = 0$

Step 2: Solution of the minimization problem only via the primary variable u for the Lagrangian in equation (1.14)

$$\frac{\partial L}{\partial u} = \frac{\partial \Pi}{\partial u} + \lambda + \varepsilon p = 0 \tag{1.15}$$

using the following recursive scheme-augmentation

$$\begin{cases} \lambda_k = \lambda_{k-1} + \varepsilon(l + u_{k-1} - H) \\ cu_k - mg + \lambda_k + \varepsilon(l + u_k - H) = 0, \end{cases} \qquad k = 1, 2, 3, ... \tag{1.16}$$

where k is an augmentation counter.

Equation (1.16) will be justified in the following theorem.

Theorem 1.3.3 *The augmented Lagrangian scheme equation (1.16) converges where $\varepsilon > 0$ in the following sense*

$$\lim_{k\to\infty} p(u_k) = 0, \tag{1.17}$$

Proof. The statement $\lim\limits_{k\to\infty} p(u_k) = 0$ is proved using the analogy between the finite difference equations (FDE) and the ordinary differential equations (ODE). Using the recursive scheme equation (1.16) for iteration step k and $k+1$ of the primary variable u_{k-1} we can write the following sequence

$$\lambda_{k+1} = \lambda_k + \varepsilon(l + u_k - H) \tag{1.18}$$

$$cu_k - mg + \lambda_k + \varepsilon(l + u_k - H) = 0 \tag{1.19}$$

$$cu_{k+1} - mg + \lambda_{k+1} + \varepsilon(l + u_{k+1} - H) = 0 \tag{1.20}$$

Substituting λ_{k+1} and λ_k in equation (1.18) via equations (1.19) and (1.20) we obtain after transformations

$$(c + \varepsilon)(u_{k+1} - u_k) = -\varepsilon(l - H) - \varepsilon u_k. \tag{1.21}$$

The last is a finite difference equation. Considering the analogy between FDE and ODE by interpreting the iteration step k as the time t, equation (1.21) can be written as

$$(c + \varepsilon)(u_{t+1} - u_t) = -\varepsilon(l - H) - \varepsilon u_t. \tag{1.22}$$

By dividing the equation then with $\Delta t = 1$ and replacing the difference by $\Delta u = u_{t+1} - u_t$ the following equation is obtained

$$(c + \varepsilon)\frac{\Delta u}{\Delta t} + \varepsilon u_t = -\varepsilon(l - H)$$

where the assumption of the finite difference scheme for the first derivative holds ($\frac{\Delta u}{\Delta t} \approx \frac{du}{dt} = \dot{u}$). Thus, a proof for $k \to \infty$ for FDE can be reformulated in the sense ODE as $t \to \infty$ for the differential equation

$$(c+\varepsilon)\dot{u} + \varepsilon u = -\varepsilon(l - H) \quad \text{with the initial condition} \quad u|_{t=0} = u_0. \tag{1.23}$$

The solution of this non-homogenous ODE is given as

$$u = u_h + u_{nh} \tag{1.24}$$

The solution of the homogenous ODE u_h

$$(c+\varepsilon)\dot{u} + \varepsilon u = 0 \tag{1.25}$$

is obtained according to the characteristic polynomial

$$\lambda(c+\varepsilon) + \varepsilon = 0 \to \lambda = -\frac{\varepsilon}{c+\varepsilon} \tag{1.26}$$

This results in the exponential solution

$$u_h = Ae^{\lambda t} = Ae^{-\frac{\varepsilon}{c+\varepsilon}t}. \tag{1.27}$$

The particular solution of the non-homogeneous ODE u_{nh} is derived as

$$\varepsilon u_{nh} = -\varepsilon(l - H) \quad \Rightarrow \quad u_{nh} = H - l. \tag{1.28}$$

The full general solution is obtained as

$$u = Ae^{-\frac{\varepsilon}{c+\varepsilon}t} + H - l \tag{1.29}$$

which has to fulfill the initial condition $u|_{t=0} = u_0$ defining A as

$$u_0 = A + H - l.$$

Finally, the solution of our problem satisfying the initial condition is derived as

$$u = (u_0 - H + l)e^{-\frac{\varepsilon}{c+\varepsilon}t} + H - l. \tag{1.30}$$

Now we are coming back to the initial proof with the limit process if $t \to \infty$. In this case the counter of augmentations goes to infinity as $k \to \infty$. The limit of equation (1.30) is easily calculated

$$\lim_{t\to\infty} u = H - l \quad \text{for } c, \varepsilon > 0. \tag{1.31}$$

Thus, the penetration function in equation (1.5) is zero in this limit process

$$\lim_{t\to\infty} p(u) = 0 \quad \Rightarrow \quad \lim_{k\to\infty} p(u_k) = 0. \tag{1.32}$$

We prove that the augmented Lagrangian method is independent of the positive penalty parameter $\varepsilon > 0$ and converges to the exact non-penetration condition ($p(u) = 0$) if the number of augmentations goes to infinity $k \to \infty$.

Remark 1.3.4 *In practical computation the iteration is continued until the penetration is smaller than some prescribed tolerance variable* ϵ_{tol}

$$|p_k| = |l + u_k - H| < \epsilon_{tol} \tag{1.33}$$

Remark 1.3.5 *The augmented Lagrangian method fulfills the non-penetration condition within the prescribed tolerance* ϵ_{tol}*, which is the second parameter of the iterative scheme (first parameter to supply is the penalty parameter* ε*). The global system is not extended and contains only the primary variable* u*.*

Remark 1.3.6 *The contact force on each augmentation step* k *is computed as the augmented Lagrange multiplier in equation (1.18):* $N_k = \lambda_k$*.*