

PART

# One

## Mathematics

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## CHAPTER 1

# Elementary Mathematics

This chapter covers the measurement and presentation of economic and financial data. Data consists of numbers and graphics, which are essential for recording and understanding financial data. All financial transactions are represented by numbers. For instance, the price of a commodity  $A$  in terms of commodity  $B$  is a number; it is the number of units of commodity  $B$  that is paid to get one unit of commodity  $A$ . Usually, unknown amounts are expressed as variables, designated by symbols such as  $x$ ,  $y$ , or any other symbol, and the equations that contain these variables may be expressed in the form of monomials, binomials, or polynomials. The applications of equations, sequences, and series are important concepts to understand in finance.

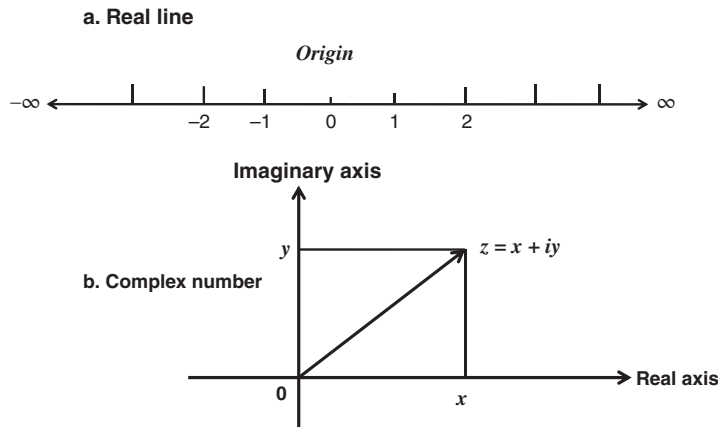
## **BASIC MATHEMATICAL OBJECTS**

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Numbers play a fundamental role in economics and finance. There are real numbers and complex or imaginary numbers. Real numbers are a subset of complex numbers. This section covers real numbers, complex numbers, the absolute value of a number, vectors and arrays, angles and directions, graphics, and the reporting of economic and financial data.

### **Real Numbers**

The set of real numbers, denoted by  $R$ , is represented by a real line  $(-\infty, \infty)$  where the symbols  $-\infty$  and  $\infty$  stand for minus infinity and plus infinity, respectively (Figure 1.1a). A real number is a value that represents a quantity along a continuous line. Real numbers include all the rational numbers, such as the integer  $-5$  and the fraction  $4/3$ , and all the irrational numbers such as  $\sqrt{2} = 1.41421356 \dots$  (an irrational algebraic number) and  $\pi = 3.14159265 \dots$  (a transcendental number). Real numbers can be thought of as points on an infinitely long line called the *number line* or *real line*, where the points corresponding to integers are equally spaced. A noninteger real number has a decimal representation such as that of  $8.632$ , where each consecutive digit is measured in units one tenth the size of the previous one. The real numbers are uncountable.



**FIGURE 1.1** Real and Complex Numbers

The set of real numbers,  $R$ , is a field, meaning that addition and multiplication are defined and have the usual properties. The field  $R$  is ordered, meaning that there is a total order  $\geq$  such that, for all real numbers  $x$ ,  $y$ , and  $z$ :

- If  $x \geq y$  then  $x + z \geq y + z$ .
- If  $x \geq 0$  and  $y \geq 0$  then  $xy \geq 0$ .

### Complex Numbers

A complex number is written in the form

$$z = x + iy \quad (1.1)$$

where  $x$  and  $y$  are real numbers and  $i$  is the imaginary unit, where

$$i^2 = -1 \quad (1.2)$$

In this expression,  $x$  is the real part of  $z$  denoted by  $Re(z)$ , and  $y$  is a real number called the *imaginary part* of  $z$  and is denoted by  $Im(z)$ . The set of all complex numbers is denoted by  $\mathbb{C}$ . Complex numbers extend the idea of the one-dimensional number line to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part. The complex number

$$z = x + iy$$

can be identified with the point  $(x, y)$  in the complex plane as shown in Figure 1.1b. A complex number whose real part is zero is said to be purely imaginary, whereas a complex number whose imaginary part is zero is a real number. In this way the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone. The real line can be thought of as a part of the complex plane, and, correspondingly, complex numbers include real numbers as a special case.

The set  $\mathbb{C}$  of complex numbers is a field. Briefly, this means that the following facts hold: first, any two complex numbers can be added and multiplied to yield another complex number. Second, for any complex number  $z$ , its additive inverse  $-z$  is also a complex number; and third, every nonzero complex number has a reciprocal complex number. Moreover, these operations satisfy a number of laws, for example, the law of commutativity of addition and multiplication for any two complex numbers  $z_1$  and  $z_2$ :

$$z_1 + z_2 = z_2 + z_1 \text{ and } z_1 z_2 = z_2 z_1$$

These two laws and the other requirements on a field can be proven using the fact that the real numbers themselves form a field.

In Islamic finance, we make use of logarithm numbers, exponential numbers, and trigonometric numbers. Trigonometric numbers are important in studying the slope of a curve. These numbers will be introduced later in the book. Nonetheless, we may provide some examples.

**Example:** Consider number 1. Its natural logarithm is zero. Its exponential is 2.718282. Consider the number  $\pi/2$ ; its cosine,  $\cos\pi/2$ , is zero; its sine,  $\sin\pi/2$  is 1.

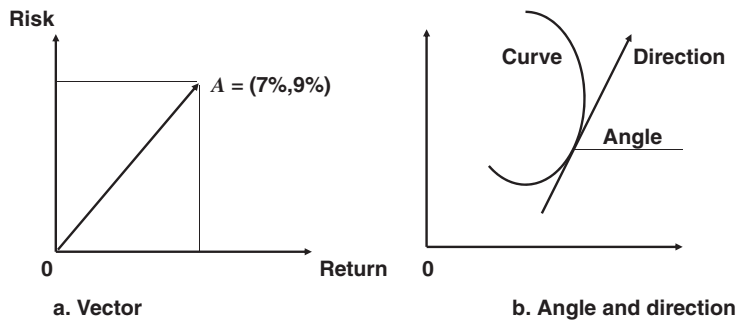
### Absolute Value of a Number

Numbers are also described by their absolute value or moduli. If a number is represented by a point on the real line, then the absolute value is a measure of the length of the distance between the number and point zero. The numbers 5 and  $-5$  have the same absolute value:  $|5| = |-5| = 5$ . In other words, when we see a number  $|2|$ , the corresponding number to this distance is either 2 or  $-2$ . The moduli of a complex number is the distance between the origin zero and the point represented by this number (Figure 1.1b). If  $z = -4 + 3i$ , then  $|z| = \sqrt{(-4)^2 + 3^2} = 5$ .

### Vectors and Arrays

Economic data may be represented by a point on the real line. For instance, real gross domestic product is \$14 billion in 2012. We show it by a point in the real line. However, economic information may have many dimensions. We need to go from  $R$ , the real line, to higher dimension Euclidian space such as  $R^2, R^3, \dots, R^n$ . For instance, an Islamic stock has an expected return and a risk. Traditionally, if a share has an expected return of 7 percent and a risk of 9 percent, this share is represented with a point whose coordinates are 7 percent and 9 percent (see Figure 1.2a). Further, we may be interested in the beta and alpha of the share. In this case, we have to go to  $R^4$ . If beta is 1.2 and alpha is 3.5 percent, then we describe the share by a vector in  $R^4$ :

$$V = \begin{bmatrix} 7\% \\ 9\% \\ 1.2 \\ 3.5\% \end{bmatrix} \quad (1.3)$$



**FIGURE 1.2** Vector, Angle, and Direction

Economic and financial information may need to be presented in the form of an array. A portfolio may have four Islamic shares. We present information about these shares in the form of a matrix:

$$A = \begin{bmatrix} & \text{Share1} & \text{Share2} & \text{Share4} & \text{Share5} \\ \text{Return} & 7\% & 12\% & 3\% & 7\% \\ \text{Risk} & 9\% & 16\% & 6\% & 8\% \\ \text{Beta} & 1.2 & 2 & 1.1 & 0.9 \\ \text{Alpha} & 3.5 & 5.4\% & 2\% & 1.1 \end{bmatrix} \quad (1.4)$$

A matrix is also used to describe the structure of international trade such as exports and imports, the structure of an economy, or the production processes of a farm or an industry.

### Angles and Directions

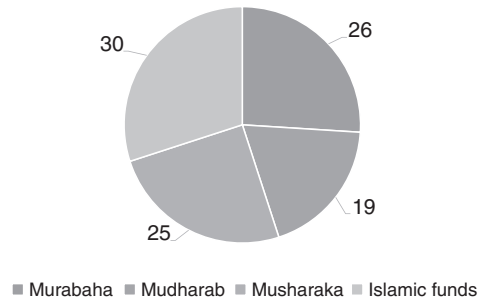
The notions of an angle and direction are important in economics and finance. We are interested in the slope of a curve as well as the direction of economic motion. An angle shows the slope of a tangent line to a curve. A vector shows the direction of a motion along the curve (see Figure 1.2b).

### Graphics

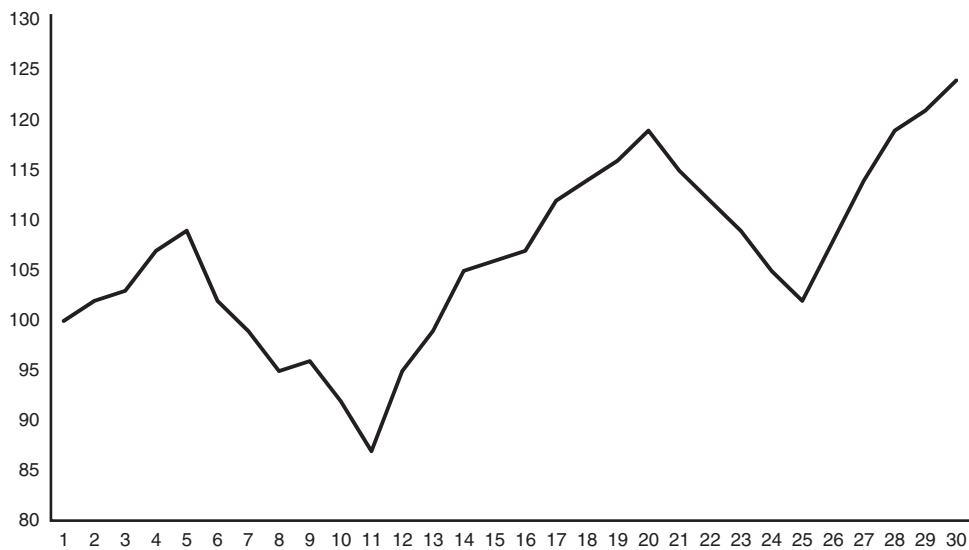
Graphics are essential tools in reporting economic and financial information and in teaching economics and finance. They facilitate economic and financial analysis. In fact, graphics are essential in all fields. For instance, Google maps show us directions in the form of a graph. A building or a house is designed in graphics before it is actually constructed. A contractor cannot build any house before he has the mapping of the house.

Consider an Islamic bank that has a portfolio composed of Murabaha (26 percent), Mudarabha (19 percent), Musharaka (25 percent), and Islamic funds (30 percent). This information is shown in Figure 1.3.

A portfolio manager uses graphics to track the market value of his portfolio. Figure 1.4 shows the value of the portfolio over a period of 30 weeks.



**FIGURE 1.3** Portfolio of an Islamic Bank



**FIGURE 1.4** Market Value of an Islamic Portfolio

### Reporting Economic and Financial Data

Besides graphics, economic and financial data is reported in special ways. If we say the real gross domestic product (GDP) of Malaysia rose by 2 billion (RM) and that of Burundi by 550 million (BF), this information is not easy to interpret. All we can say is that real GDP did not fall in either country. However, if we say real GDP rose by 7 percent in Malaysia and 2 percent in Burundi, this information is easier to interpret because it is placed in context of the existing GDP.

Economic and financial information is reported in the form of indicators; these are percent changes, ratios, indices, elasticities, and other specific indicators. For instance, the balance sheet of a company is described in terms of ratios such as liquidity ratio, solvency ratio, and equity ratio. Macroeconomic indicators use ratios such as external deficit ratio, debt ratio, and fiscal deficit ratio. Indices are important. An index normalizes data to a base of 100, then compares the evolution of data in relation to this base. For instance, the price index measures the price of a basket of commodities in reference to a base of 100, called *base year*, and computes the period

change in relation to this base. Elasticities are a way to describe economic and financial variables. We say that demand for bread is inelastic, implying that consumers are unable to change their demand for bread whether prices of bread go skyward or drop substantially. In contrast, the demand for apples is elastic, implying that when the price of apples increases, demand may decrease.

A percent change is defined as

$$\Delta x/x \quad (1.5)$$

where  $\Delta x$  denotes change in a variable  $x$ . A percent change can be positive, zero, or negative. A ratio involves two variables, one is the numerator and the other is the denominator:

$$x/y \quad (1.6)$$

For instance, per capita income is the ratio of real GDP in money terms divided by the size of the population, that is, the number of citizens of a country. An index is referred to by its abbreviation. For instance, *CPI* means consumer price index. *S&P 500* refers to Standard & Poor's stock price index. Elasticity is computed as the ratio of two percent changes:

$$(\Delta x/x)/(\Delta y/y) \quad (1.7)$$

It could be positive, zero, or negative. If it is close to zero, there is inelasticity of  $x$  in relation to  $y$ ; if it is  $\pm\infty$ , then there is high elasticity of  $x$  in relation to  $y$ .

## **VARIABLES, MONOMIALS, BINOMIALS, AND POLYNOMIALS**

This section covers monomials, binomials, polynomials, polynomial lags, identities, and factorization of a polynomial.

A variable is designated by the symbol  $x$ . We perform algebraic operations on the variable  $x$ . We may multiply  $x$  by any number  $a \in R$ , and we obtain  $ax$ . We may compute powers of  $x$  such as

$$x^{-2}, x^{-1}, x^0, x^1, x^2, \dots, x^n$$

We may perform operation such as

$$x + a, (x + a)^2, \dots, (x + a)^n$$

For instance, we perform the following multiplications:

$$(x + a)^2 = x^2 + 2ax + a^2 \quad (1.8)$$

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \quad (1.9)$$

$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4 \quad (1.10)$$

### Monomials, Binomials, and Polynomials

A monomial is the product of nonnegative integer powers of variables. Consequently, a monomial has no variable in its denominator. It has one term (*mono* implies one):

$$6, 2x, -18, x^2, 5y^2, -2xy, \text{ or } 97x^2y^3$$

We notice that there are no negative exponents and no fractional exponents. The number 6 is a monomial since it can be written as  $6x^0 = 6$ .

A binomial is the sum of two monomials. It has two unlike terms (*bi* implies two):

$$3x + 1, x^7 - 4x^2, 2x + y, \text{ or } y - y^2$$

A trinomial is the sum of three monomials; it has three unlike terms (*tri* implies three):

$$\begin{aligned} x^2 + 2x + 1 \\ 5x^4 - 3x^3 + 10 \\ 2x + y + 9 \end{aligned}$$

A polynomial is the sum of one or more terms (*poly* implies many):

$$\begin{aligned} x^2 + 2x \\ 2x^3 + x^2 + 5x + 2 \\ 4x - 6y + 5 \end{aligned}$$

The degree of a polynomial is the highest exponent of its monomials. Polynomials are in the simplest form when they contain no like terms. For instance, the polynomial

$$x^2 + 2x + 1 + 3x^2 - 4x$$

when simplified becomes

$$4x^2 - 2x + 1$$

### Polynomial Lags

In statistics we use a lag operator, denoted by  $L$  (lag). For instance, the price of tomatoes today is denoted as  $x_t$ , the price of tomatoes last month is  $x_{t-1}$ , the price two months past is  $x_{t-2}$ , three months past is  $x_{t-3}$ , . . . ,  $n$  months past is  $x_{t-n}$ . We present this information as  $x_{t-1} = Lx_t$ ,  $x_{t-2} = L^2x_t$ ,  $x_{t-3} = L^3x_t$ , . . . ,  $x_{t-n} = L^n x_t$ . Our notation of tomato prices  $\{x_t, x_{t-1}, x_{t-2}, x_{t-3}, \dots, x_{t-n}\}$  can be written in a polynomial lag as

$$\{1, L, L^2, L^3, \dots, L^n\}x_t \quad (1.11)$$

An example of a polynomial lag is

$$2 + 0.5L + 1.2L^2 - 1.8L^3 + 0.88L^4 - 2.6L^5$$

A polynomial lag is very useful in performing operations on a time series, such as tomato prices, or any other time series such as the daily values of the Dow Jones Islamic stock index. We may perform operations on polynomial lag in the same way as on any regular polynomial. For instance:  $1 - 2L + L^2$  may be written as  $(1 - L)(1 + L)$ .

### Identities

Often in Islamic finance, we need to use identities; we provide some useful identities:

$$(x + 1)^2 = x^2 + 2x + 1 \quad (1.12)$$

$$(x + 1)(x - 1) = x^2 - 1 \quad (1.13)$$

$$(x^2 + x + 1)(x - 1) = x^3 - 1 \quad (1.14)$$

$$(x^{n-1} + x^{n-2} + \dots + x + 1)(x - 1) = x^n - 1 \quad (1.15)$$

The binomial identity is an important one. It is stated as

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \dots + \frac{n(n-1)\dots(n-k+1)}{k!}x^{n-k}y^k + \dots + y^n \quad (1.16)$$

Here  $k!$  is called *factorial* of  $k$ ; it is written as  $k! = k \times (k - 1) \times (k - 2) \times \dots \times 2 \times 1$ .

### Factorization of a Polynomial

Let us consider the following product:

$$P(x) = (x - 1)(x + 5)(x - 2)(x + 1) = x^4 + 3x^3 - 11x^2 - 3x + 10 \quad (1.17)$$

We may reverse the path and start from the polynomial  $P(x) = x^4 + 3x^3 - 11x^2 - 3x + 10$  and try to factorize it into  $(x - 1)(x + 5)(x - 2)(x + 1)$ . The values  $x = 1$ ,  $x = -5$ ,  $x = 2$ , and  $x = -1$  are called the *roots* of the polynomials  $P(x)$ . If we replace  $x = 1$  into the polynomial we find

$$P(1) = 1 + 3 - 11 - 3 + 10 = 0$$

If we replace  $x = -5$  into the polynomial we find

$$P(-5) = 625 - 375 - 275 + 15 + 10 = 0$$

We observe that  $P(x)$  is different from 0 for any value of  $x$  different from 1, -5, 2, -1. For instance, for  $x = 0$  we have  $P(0) = 10$ , and for  $x = 0.4$  we  $P(0.4) = 7.2576$ .

## EQUATIONS

Equations are basic notions of finance. A large part of Islamic finance consists of solving equations such as computing internal rates of return, replicating portfolios, structuring products, pricing assets, and computing costs or break-even points. A simple equation is of the form

$$x - 2 = 1 \quad (1.18)$$

Obviously the solution is  $x = 3$ . We may have to deal with equations of the second degree, called *quadratic* equations, such as

$$ax^2 + bx + c = 0 \quad (1.19)$$

This equation has two roots. Completing the square can be used to derive a general formula for solving quadratic equations. Dividing the quadratic equation by  $a$  (which is allowed because  $a$  is nonzero), gives

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad (1.20)$$

$$\text{Or } x^2 + \frac{b}{a}x = -\frac{c}{a} \quad (1.21)$$

The quadratic equation is now in a form to which the method of completing the square can be applied. To “complete the square” is to add a constant to both sides of the equation such that the left hand side becomes a complete square

$$x^2 + \frac{b}{a}x + \left(\frac{1b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{1b}{2a}\right)^2 \quad (1.22)$$

that produces

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{1b}{2a}\right)^2 \quad (1.23)$$

The right side can be written as a single fraction with common denominator  $4a^2$ . This gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad (1.24)$$

Taking the square root of both sides yields

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad (1.25)$$

Isolating  $x$  gives

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1.26)$$

The plus-minus symbol “ $\pm$ ” indicates that

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (1.27)$$

In the above formula, the expression underneath the square root sign is called the *discriminant* of the quadratic equation, and is often represented using an uppercase Greek delta,  $\Delta$ :

$$\Delta = b^2 - 4ac \quad (1.28)$$

A quadratic equation with real coefficients can have either one or two distinct real roots, or two distinct complex roots. In this case the discriminant determines the number and nature of the roots. There are three cases:

1. If the discriminant is positive, then there are two distinct roots

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{\Delta}}{2a} \quad (1.29)$$

both of which are real numbers.

2. If the discriminant is zero, then there is exactly one real root, sometimes called a *double* root:

$$x = -\frac{b}{2a} \quad (1.30)$$

3. If the discriminant is negative, then there are no real roots. Rather, there are two distinct (nonreal) complex roots, which are complex conjugates of each other. In these expressions  $i$  is the imaginary unit:

$$x_1 = \frac{-b + i\sqrt{-\Delta}}{2a} \text{ and } x_2 = \frac{-b - i\sqrt{-\Delta}}{2a} \quad (1.31)$$

Thus the roots are distinct if and only if the discriminant is nonzero, and the roots are real if and only if the discriminant is nonnegative.

**Example:** Complete the square for  $3x^2 - 6x - 11 = 0$ .

We apply the formula  $(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$ ; we obtain  $(x - 1)^2 = \frac{14}{3}$ .

**Example:** Solve the following quadratic function:

$$x^2 + x - 6 = 0$$

We compute  $\Delta = b^2 - 4ac = 1 + 24 = 25$ ,  $\sqrt{\Delta} = \sqrt{25} = 5$ ,

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-1 + 5}{2} = 2; \quad x_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-1 - 5}{2} = -3$$

**Example:** Solve the following quadratic equation:\*

$$x^2 + x + 6 = 0$$

We compute  $\Delta = b^2 - 4ac = 1 - 24 = -23$ ;  $\sqrt{-\Delta} = \sqrt{23}$ ,

We have two complex roots:  $x_1 = -\frac{b}{2a} + i\frac{\sqrt{-\Delta}}{2a} = -\frac{1}{2} + i\frac{\sqrt{23}}{2}$  and  $x_2 = -\frac{b}{2a} - i\frac{\sqrt{-\Delta}}{2a} = -\frac{1}{2} - i\frac{\sqrt{23}}{2}$ .

**Example:** You want to invest in two Islamic mutual funds A and B, in proportions  $x_1$  and  $x_2$ , respectively, with  $x_1 + x_2 = 1$ , and  $x_1 \geq 0$ ,  $x_2 \geq 0$ . The risk of mutual funds A and B are ( $\sigma_1 = 9$  percent) and ( $\sigma_2 = 14$  percent), respectively; the correlation coefficient between expected returns is  $\rho = 0.6$ . Find the composition that will achieve a portfolio risk of 11.2 percent.

The portfolio variance is

$$V(x_1, x_2) = \sigma_1^2 x_1^2 + 2\rho\sigma_1\sigma_2 x_1 x_2 + \sigma_2^2 x_2^2 = 81x_1^2 + 151.2x_1 x_2 + 196x_2^2$$

We equate the portfolio variance to  $11.2^2 = 125.44$ . We obtain

$$81x_1^2 + 151.2x_1 x_2 + 196x_2^2 = 125.44$$

\*Many online calculators are available for solving quadratic equations; for example, [www.math.com/students/calculators/source/quadratic.htm](http://www.math.com/students/calculators/source/quadratic.htm).

We note that  $x_2 = 1 - x_1$ ; hence,  $x_2^2 = (1 - x_1)^2 = 1 - 2x_1 + x_1^2$ . We replace  $x_2$  into the equation, we obtain

$$\begin{aligned} 81x_1^2 + 151.2x_1x_2 + 196x_2^2 \\ &= 81x_1^2 + 151.2x_1(1 - x_1) + 196(1 - 2x_1 + x_1^2) \\ &= 125.8x_1^2 - 240.8x_1 + 196 = 125.44 \end{aligned}$$

Or

$$125.8x_1^2 - 240.8x_1 + 70.56 = 0$$

This is an equation of the form  $ax^2 + bx + c = 0$ ; its roots are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}; \text{ we find } x_1 = 0.36 \text{ and } x_1 = 1.56.$$

The latter root means 156 percent of savings are invested in mutual fund A; since we cannot invest more than 100 percent in any fund, we discard this root. The root  $x_1 = 0.36$  is acceptable. It implies  $x_2 = 1 - 0.36 = 0.64$ . Hence 36 percent of the savings are invested in fund A and 64 percent are invested in fund B.

## EQUATIONS OF HIGHER ORDER

Entrepreneurs and planners undertake investment projects with a life span of many years. They are interested in determining the profitability of the project in which they will invest considerable money. By deciding to invest in a specific project such as a textile plant, they have to renounce many other alternative or competing projects. They tend to choose the project that has the highest returns. The investment selection problem necessitates solving an equation of the order  $n \geq 2$ , where  $n$  is an integer equal to the life span of the project or the investment horizon. Assuming a discount rate  $r$ , we may summarize the cash flow (CF) of the project in table form, as shown here.

	0	1	2	3	4	5	...	$n$
Net cash flow	$CF_0$	$CF_1$	$CF_2$	$CF_3$	$CF_4$	$CF_5$	...	$CF_n$
Discounted cash flow	$\frac{CF_0}{(1+r)^0}$	$\frac{CF_1}{(1+r)^1}$	$\frac{CF_2}{(1+r)^2}$	$\frac{CF_3}{(1+r)^3}$	$\frac{CF_4}{(1+r)^4}$	$\frac{CF_5}{(1+r)^5}$	...	$\frac{CF_n}{(1+r)^n}$

The entrepreneur is interested in solving an equation of the form

$$\frac{CF_0}{(1+r)^0} + \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \frac{CF_3}{(1+r)^3} + \frac{CF_4}{(1+r)^4} + \frac{CF_5}{(1+r)^5} + \dots + \frac{CF_n}{(1+r)^n} = 0 \quad (1.32)$$

The rate of return  $r$  that solves this equation is called the *internal rate of return* (IRR). The equation can be rewritten in polynomial form as

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots + a_nx^n = 0 \quad (1.33)$$

where  $x = \frac{1}{1+r}$ .

The equation admits  $n$  roots (real and complex). Unfortunately, there are no formulas that can be readily employed for finding the roots of the equation. We have to proceed by iteration. We may draw a chart for the equation and see the points where it intersects the horizontal line. We may also start from an initial value for  $x$  such as  $x_0$  and iterate until the polynomial becomes very close to zero. Since there are  $n$  roots, we may have to discard negative roots and consider only positive roots. Nonetheless, there is software such as Microsoft Excel that can provide the solutions for an equation of order  $n$ .

**Example:** Malay Palm Oil Corporation contemplates an investment project, which has the cash flow shown in the following table. We want to find the internal rate of return (IRR). We apply Excel formula *IRR*, we find  $r = 11$  percent.

	0	1	2	3	4	5	6	7	8	9	10
Net cash flow	-6,796	-500	-750	200	1,350	1,570	2,037	2,500	2,560	3,200	3,500

## SEQUENCES

A sequence is an ordered list of objects (or events). Like a set, it contains members (also called *elements* or *terms*). The number of ordered elements (possibly infinite) is called the *length of the sequence*. Unlike in a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence. Most precisely, a sequence can be defined as a function whose domain is a countable, totally ordered set, such as the natural numbers.

An example of a sequence is (1, 2, 3, 5, 8). Sequences can be finite, as in this example, or infinite, such as the sequence of all even positive integers (2, 4, 6, . . .). Finite sequences are sometimes known as *strings* or *words* and infinite sequences as *streams*. The empty sequence ( $\emptyset$ ) is included in most notions of sequence, but may be excluded depending on the context.

There are a number of ways to denote a sequence, some of which are more useful for specific types of sequences. One way to specify a sequence is to list the elements. For example, the first four odd numbers form the sequence (1, 3, 5, 7). This notation can be used for infinite sequences as well. For instance, the infinite sequence of positive odd integers can be written (1, 3, 5, 7, . . .). Listing is most useful for infinite sequences with a pattern that can be easily discerned from the first few elements.

There are many important integer sequences. The prime numbers are numbers that have no divisors but 1 and themselves. Taking these in their natural

order gives the sequence  $(2, 3, 5, 7, 11, 13, 17, 19, \dots)$ . The study of prime numbers has important applications for mathematics and specifically number theory.

The Fibonacci numbers are the integer sequence whose elements are the sum of the previous two elements. The first two elements are either 0 and 1 or 1 and 1 so that the sequence is  $(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ .

The terms of a sequence are commonly denoted by a single variable, say  $x_n$ , where the index  $n$  indicates the  $n$ th element of the sequence. The sequence  $x_n$  is written as

$$x_1, x_2, x_3, \dots, x_{n-1}, x_n, x_{n+1}, \dots \quad (1.34)$$

Indexing notation is used to refer to a sequence in the abstract. It is also a natural notation for sequences whose elements are related to the index  $n$  (the element's position) in a simple way. For instance, the sequence of the first 10 square numbers could be written as

$$x_k = k^2, k = 1, 2, \dots, 10 \quad (1.35)$$

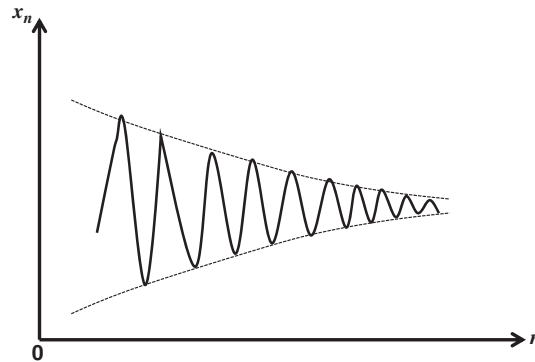
This represents the sequence  $(1, 4, 9, \dots, 100)$ . The length of a sequence is defined as the number of terms in the sequence. A sequence of a finite length  $n$  is also called an  $n$ -tuple. Finite sequences include the empty sequence  $(\emptyset)$  that has no elements.

Normally, the term *infinite sequence* refers to a sequence that is infinite in one direction, and finite in the other—the sequence has a first element, but no final element (a singly infinite sequence). A sequence that is infinite in both directions—it has neither a first nor a final element—is called a *bi-infinite sequence*, two-way infinite sequence, or doubly infinite sequence. For instance, a function from all integers into a set, such as the sequence of all even integers  $(\dots, -4, -2, 0, 2, 4, 6, 8, \dots)$ , is bi-infinite. This sequence could be denoted  $(2n)_{n=-\infty}^{n=\infty}$ .

A sequence is said to be *monotonically increasing* if each term is greater than or equal to the one before it. For a sequence  $(x_n)_{n=1}^{\infty}$  this can be written as  $x_n \leq x_{n+1}$  for all  $n \in \mathbb{N}$ , where  $\mathbb{N}$  is the set of all integers. If each consecutive term is strictly greater ( $>$ ) than the previous term then the sequence is called *strictly monotonically increasing*. A sequence is *monotonically decreasing* if each consecutive term is less than or equal to the previous one, and strictly monotonically decreasing if each is strictly less than the previous. If a sequence is either increasing or decreasing it is called a *monotone sequence*. This is a special case of the more general notion of a monotonic function.

The terms *nondecreasing* and *nonincreasing* are often used in place of increasing and decreasing in order to avoid any possible confusion with strictly increasing and strictly decreasing, respectively.

**Example:** Generate the first 10 terms of the sequence:  $x_n = x_{n-1} + x_{n-2}$  with  $x_0 = 0$  and  $x_1 = 1$ . We obtain  $(x_n)_{n=0}^{10} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34\}$ .



**FIGURE 1.5** Example of a Cauchy Sequence

A Cauchy sequence is a sequence whose terms become arbitrarily close together as  $n$  gets very large. A sequence  $x_1, x_2, x_3, \dots$ , of real numbers is called a *Cauchy* sequence, if for every positive real number  $\varepsilon$ , there is a positive integer  $N$  such that for all natural numbers  $m, n > N$

$$|x_m - x_n| < \varepsilon \quad (1.36)$$

where the vertical bars denote the absolute value. In a similar way one can define Cauchy sequences of rational or complex numbers. Cauchy formulated such a condition by requiring  $|x_m - x_n|$  to be infinitesimal for every pair of infinite  $m, n > N$ . Figure 1.5 depicts an example of a Cauchy sequence.

One of the most important properties of a sequence is convergence. Informally, a sequence converges if it has a limit. Informally, a (singly-infinite) sequence has a limit if it approaches some value  $\ell$ , called the *limit*, as  $n$  becomes very large. That is, for an abstract sequence  $(x_n)$  (with  $n$  running from 1 to infinity) the value of  $x_n$  approaches  $\ell$  as  $n \rightarrow \infty$ , denoted  $\lim_{n \rightarrow \infty} x_n = \ell$ .

More precisely, the sequence converges if there exists a limit  $\ell$  such that the remaining  $x_n$ 's are arbitrarily close to  $\ell$  for some  $n$  large enough. If a sequence converges to some limit, then it is convergent; a sequence is convergent if and only if it is a Cauchy sequence. If  $x_n$  gets arbitrarily large as  $n \rightarrow \infty$  we write  $\lim_{n \rightarrow \infty} x_n = \infty$  and say that the sequence diverges to infinity.

Sequences arise in finance. For instance, if we consider any budget such as the budget of a state, corporation, or household, we obtain sequences over time for receipts, expenditure, and the balance, as shown in the following table:

Time	1	2	3	4	5	6	7	8	9
Receipts	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$x_9$
Expenditure	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$
Balance	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$	$b_9$

Income from an asset such as a stock or a sukuk describes a sequence over periods of time during which the income accrues. Likewise, the cash flow of a company may be described by the following sequence:

Time	1	2	3	...	$n$	...
Cash flow	$CF_1$	$CF_2$	$CF_3$	...	$CF_n$	...

The present value of \$1 to be received in future periods forms the following sequence of discounted values:

Time	1	2	3	...	$n$	...
Present value of a future \$1	$\frac{\$1}{(1+r)^1}$	$\frac{\$1}{(1+r)^2}$	$\frac{\$1}{(1+r)^3}$	...	$\frac{\$1}{(1+r)^n}$	...

We observe that the sequence  $\frac{\$1}{(1+r)^n}$  is a convergent sequence for  $r > 0$ .

## SERIES

The sum of the first  $n$  terms of a (one-sided) sequence forms the  $n$ th term in a sequence called a *series*. That is, the series of the sequence  $(x_n)$  is the sequence  $(S_n)$  given by

$$\begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 + x_2 \\ S_3 &= x_1 + x_2 + x_3 \\ S_n &= x_1 + x_2 + x_3 + \cdots + x_n \end{aligned}$$

We can also write the  $n$ th term of the series as

$$S_N = \sum_{n=1}^N x_n \quad (1.37)$$

Then the concepts used to talk about sequences, such as convergence, carry over to series (the sequence of partial sums) and the properties can be characterized as properties of the underlying sequences (such as  $x_n$ ). The limit, if it exists, of an infinite series (the series created from an infinite sequence) is written as

$$\lim_{n \rightarrow \infty} S_N = \sum_{n=1}^{\infty} x_n \quad (1.38)$$

To sum up, given an infinite sequence of numbers  $\{x_n\}$ , a series is informally the result of adding all those terms together:  $x_1 + x_2 + x_3 + \cdots$ . These can be written more compactly using the summation symbol  $\sum$ .

**Example:**  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

The terms of the series are often produced according to a certain rule, such as by a formula. As there are an infinite number of terms, this series is often called an *infinite series*. Unlike finite summations, infinite series need tools from mathematical analysis, and specifically the notion of limits, to be fully understood and manipulated. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, and finance.

### Convergence of a Series

If a series  $S_n = \sum_{n=0}^{\infty} x_n$  is convergent as  $n \rightarrow \infty$ , then it has a limit  $S = \sum_{n=0}^{\infty} x_n$ , and  $\lim_{n \rightarrow \infty} x_n = 0$ .

We consider the geometric series

$$S_n = a + ax + ax^2 + ax^3 + \dots + ax^{n-1} + \dots \quad (1.39)$$

This series converges and has the sum  $S = \frac{a}{1-x}$  if  $|x| < 1$ ; it diverges if  $|x| \geq 1$ .

**Example:** Prove that the following series converges, and find its sum:

$$S_n = 0.6 + 0.06 + 0.006 + \dots + \frac{6}{10^n} + \dots$$

This is a geometric series;  $a = 0.6$  and  $x = 0.1$ . Since  $|x| < 1$ , we conclude that the series converges and has the sum  $S = \frac{a}{1-x} = \frac{0.6}{1-0.1} = \frac{2}{3} = 0.666666$ .

**Example:** Prove that the following series converges, and find its sum:

$$S_n = 2 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots + \frac{2}{3^{n-1}} + \dots$$

This is a geometric series;  $a = 2$  and  $x = 1/3$ . Since  $|x| < 1$ , we conclude that the series converges and has the sum  $S = \frac{a}{1-x} = \frac{2}{1-\frac{1}{3}} = 3$ .

**Power Series** A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n \quad (1.40)$$

The Taylor series at a point  $c$  of a function is a power series that, in many cases, converges to the function in a neighborhood of  $c$ . For example, the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is the Taylor series of  $e^x$  at the origin 0 and converges to it for every  $x$ .

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (1.41)$$

where  $e$  denotes the exponential number.

**Laurent Series** Laurent series generalize power series by admitting terms into the series with negative as well as positive exponents. A Laurent series is thus any series of the form

$$\sum_{n=-\infty}^{\infty} a_n x^n \quad (1.42)$$

### **APPLICATIONS OF SERIES TO PRESENT VALUE OF ASSETS**

In Islamic finance sequences arise in relation to investment, production, budgeting, and many other economic and financial operations.

#### **Applications of Series to Present Value Computation**

The value  $V$  today ( $t = 0$ ) of an asset maturing at date  $n$  is the present value (PV) of the future cash flows expressed as

$$V = \frac{CF_1}{(1 + R)} + \frac{CF_2}{(1 + R)^2} + \cdots + \frac{CF_n}{(1 + R)^n} \quad (1.43)$$

Using the summation symbol  $\sum$  we can rewrite  $V$  as

$$V = \sum_{t=1}^n \frac{CF_t}{(1 + R)^t} \quad (1.44)$$

If the cash flows are constant and equal to  $CF$ , the value of the asset becomes

$$V = \sum_{t=1}^n \frac{CF}{(1 + R)^t} \quad (1.45)$$

It can be rewritten as

$$V = CF \times \frac{\left[ 1 - \left[ \frac{1}{((1 + R)^n)} \right] \right]}{R} \quad (1.46)$$

If the number of future periods  $n \rightarrow \infty$ , the asset is called a *perpetuity* and its value today becomes

$$V = \frac{CF}{R} \quad (1.47)$$

**Example:** Computing the price of an asset.

Using the Microsoft Excel present value function, compute the price of an asset today that provides a constant cash flow of \$100 per month for 20 years, assuming a discount rate of 5.77 percent. Compute the price of a perpetuity that pays a constant monthly cash flow of \$100, assuming a discount rate of 5.77 percent.

Here  $CF = \$100$ ,  $R = 5.77 \text{ percent} \times 1/12 = 0.4808 \text{ percent}$ , and  $n = 12 \times 20 = 240$ . Applying directly Equation (1.46) or using Excel's present value function  $PV(0.004808, 240, 100)$ , we find  $V = \$14,220.18$ . The price of a perpetuity is  $V = 100/0.004808 = \$20,797.23$ .

**Example:** Constant payment.

Compute the monthly payment  $CF$  on a 30-year mortgage loan of \$300,000 and a mortgage rate equal to 7 percent per year. Here  $V = \$300,000$ ;  $n = 12 \times 30 = 360$ ; and  $R = 7 \text{ percent} \times 1/12 = 0.00583$ . Inserting these values into the PV formula (Equation 1.46), we find  $CF = \$1,999.9$ , or  $CF = \sim \$2,000$ .

**Application of series to stock valuation:** Expected dividends serve as the basis for stock valuation. Like all financial assets, equilibrium stock prices are the present value of a stream of cash flows. The value of stock,  $V_0$ , is the present value of expected future dividends; it is expressed as

$$V_0 = \frac{D_1}{(1+r_s)^1} + \frac{D_2}{(1+r_s)^2} + \frac{D_3}{(1+r_s)^3} + \cdots + \frac{D_\infty}{(1+r_s)^\infty} = \sum_{t=1}^{\infty} \frac{D_t}{(1+r_s)^t} \quad (1.48)$$

Here  $D_t$  is the dividend the stockholder expects to receive at the end of the year  $t$ ;  $r_s$  is the minimum acceptable rate of return on the stock. A simplified approach to stock valuation is the constant growth stock model. In practice, it is difficult to get an accurate forecast of the future dividends. However, in many cases, the stream of dividends is expected to grow at a constant rate  $g$ . If this is the case, the above equation may be written as

$$\begin{aligned} V_0 &= \frac{D_0(1+g)^1}{(1+r_s)^1} + \frac{D_0(1+g)^2}{(1+r_s)^2} + \cdots + \frac{D_0(1+g)^\infty}{(1+r_s)^\infty} \\ &= D_0 \sum_{t=1}^{\infty} \frac{(1+g)^t}{(1+r_s)^t} = \frac{D_0(1+g)}{r_s - g} = \frac{D_1}{r_s - g} \end{aligned} \quad (1.49)$$

Here  $D_0$  is the most recent dividend, which has already been paid. As illustration of the constant growth stock model, assume that Corporation XYZ just paid a dividend of \$1.15, so  $D_0 = \$1.15$ . Its stock has a required rate of return  $r_s = 13.4$  percent and investors expect the dividend to grow at a constant 8 percent in the future. The stock's intrinsic value is \$23, computed as  $V_0 = \frac{\$1.15(1.08)}{0.134 - 0.08} = \$23$ .

## SUMMARY

This chapter deals with elementary mathematics applied in Islamic finance. It introduces basic mathematical objects such as real numbers, complex numbers, absolute value of a number, vectors, arrays, graphics, and the reporting of economic and

financial data using ratios and elasticities. It introduces the notions of variables, monomials, binomials, polynomials, polynomial lags, identities, and factorization of a polynomial. The chapter covers equations and their methods of solutions, and the notions of sequences and series and their convergence to a finite limit. The chapter illustrates application of equations and series to Islamic finance.

By becoming familiar with the content of this chapter, the reader will be able to analyze, in terms of ratios and graphically, Islamic financial data; compute the returns to investment projects; and compute price of stocks as well as mortgage payments. The chapter provides notions that are basic for Islamic finance.

## QUESTIONS

1. An Islamic bank has its assets distributed as following: farming (28 percent), manufacturing (15 percent), mining (24 percent), commerce (25 percent), and construction (8 percent). Using Microsoft Excel, show a graph of the assets.
2. The price of bread increased by 25 percent, and the demand for bread fell by 0.1 percent. Compute the elasticity of bread. The price of an exotic fruit rose by 5 percent, and the demand for this fruit fell by 25 percent. Compute the price elasticity. The price of gold rose by 400 percent, and the supply of gold rose by 0.5 percent. Compute the supply price elasticity of gold.
3. The consumer price index in a country increased from 145.2 to 153.7. Compute the rate of inflation.
4. The population of the United States rose by 10 million, and that of Burundi by 300,000. The population is growing faster in the United States compared to Burundi. Do you agree?
5. Complete the square for  $3x^2 - 6x - 14 = 0$ ,  $7x^2 + 2x + 14 = 0$ , and  $\frac{x^2}{3} + \frac{x}{7} - 9 = 0$ .
6. Solve  $x^2 + x + 4 = 0$ ,  $-x^2 + x + 4 = 0$ , and  $x^2 + x - 210 = 0$ .
7. You want to invest your savings in two Islamic mutual funds,  $A$  and  $B$ , in proportions  $x_1$  and  $x_2$ , respectively, with  $x_1 + x_2 = 1$ , and  $x_1 \geq 0$ ,  $x_2 \geq 0$ . The risk of mutual funds  $A$  and  $B$  are ( $\sigma_1 = 9$  percent) and ( $\sigma_2 = 14$  percent), respectively; the correlation coefficient between expected returns is  $\rho = 0.35$ . Find the composition that will achieve a portfolio risk of 10.8 percent. Note that the portfolio variance is

$$V(x_1, x_2) = \sigma_1^2 x_1^2 + 2\rho\sigma_1\sigma_2 x_1 x_2 + \sigma_2^2 x_2^2$$

8. Malay Palm Oil Corporation contemplates an investment project that has the cash flow shown in the following table. Using Microsoft Excel, find the project's internal rate of return.

	0	1	2	3	4	5	6	7	8	9	10	11	12
Net cash flow	-7,796	-1,500	-750	1,200	1,350	1,570	2,037	2,500	2,560	2,800	2,600	3,200	3,500

9. A farm is expected to provide a net income of \$330,000 per year; what is its worth today if the market rate of return is 7 percent? What is its worth if the rate of return is 2 percent? What is its worth if the rate of return is 0 percent? Compare and explain the difference in the farm's value.
10. An asset promises to pay a net cash flow as shown below; using Microsoft Excel, compute its value today if the market rate of return is 6.23 percent per year.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Net cash flow	796	1,500	750	1,200	1,350	1,570	2,037	2,500	2,560	2,800	2,600	3,200	1,211	1,768	2,030

11. Compute the monthly payment  $CF$  on a 30-year mortgage loan of \$300,000 and a mortgage rate equal to 4.7 percent per year.
12. A stock promises to pay a net dividend as shown below; using Microsoft Excel, compute its value today if the minimum acceptable rate of return is 7.23 percent per year.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Net cash flow	796	856	904	1,020	1,350	1,570	1,590	1,500	1,560	1,800	1,700	1,720	1,211	1,768	1,600

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Net cash flow															

13. A stock paid today has a dividend of \$53. The minimum acceptable rate of return is 8.5 percent. The dividend is expected to grow at 7 percent per year. How much is the stock worth today?
14. Consider the sequence  $a_n = \frac{n}{3n+2}$ ; write the terms of this sequence for  $n = 1, 2, 3, 4, 5$ . Compute the limit of  $a_n$  when  $n \rightarrow \infty$ .
15. Consider the sequence  $a_n = \frac{2}{\sqrt{n^2+9}}$ ; write the terms of this sequence for  $n = 1, 2, 3, 4, 5$ . Compute the limit of  $a_n$  when  $n \rightarrow \infty$ .
16. Consider the sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$ ; write the terms of this sequence for  $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ ; do you expect  $a_n$  to converge to a limit as  $n \rightarrow \infty$ ?
17. Find the values for which the series converges, and find the sum of series:
- $S_n = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^{n+1}x^n$
  - $S_n = 1 + x^2 + x^4 + \dots + x^{2n}$
  - $S_n = \frac{1}{2} + \frac{(x-3)}{4} + \frac{(x-3)^2}{8} + \dots + \frac{(x-3)^n}{2^{n+1}} + \dots$

