PART I

THE BASICS OF ENUMERATIVE COMBINATORICS

1

INITIAL EnCOUNTers WITH COMBINATORIAL REASONING

PROBLEM SET 1.2

1.2.1. A bag contains 7 blue, 4 red, and 9 green marbles. How many marbles must be drawn from the bag without looking to be sure that we have drawn

(a) a pair of red marbles?

(b) a pair of marbles of the same color?

(c) a pair of marbles with different colors?

(d) three marbles of the same color?

(e) a red, blue, and green marble?

Answer

(a) 18 **(b)** 4 **(c)** 10 **(d)** 7 **(e)** 17

1.2.3. There are 10 people at a dinner party. Show that at least two people have the same number of acquaintances at the party.

Answer

Each person can know any where from 0 (no one) to 9 (everyone) people. But if someone knows no one, there cannot be someone who knows

Solutions Manual to Accompany Combinatorial Reasoning: An Introduction to the Art of Counting, First Edition. Duane DeTemple and William Webb.

everyone, and vice versa. Thus, place the 10 people into the 9 boxes that are labeled 1, 2, … , 8, and 0|9. By the pigeonhole principle, some box has at least 2 members. That is, there are at least two people at the party with the same number of acquaintances.

1.2.5. Given any five points in the plane, with no three on the same line, show that there exists a subset of four of the points that form a convex quadrilateral.

[*Hint*: Consider the *convex hull* of the points; that is, consider the convex polygon with vertices at some or all of the given points that encloses all five points. This scenario can be imagined as the figure obtained by bundling the points within a taut rubber band that has been snapped around all five points. There are then three cases to consider, depending on whether the convex hull is a pentagon, a quadrilateral containing the fifth point, or a triangle containing the other two given points.]

Answer
If the convex hull is a pentagon, each set of 4 points are the vertices of a convex quadrilateral. If the convex hull is a quadrilateral, the convex hull itself is the sought quadrilateral. If the convex hull is a triangle, the line formed by the two points within the triangle separates the vertices of the triangle into opposite half planes. By the pigeonhole principle, there are two points of the triangle in the same half plane. These two points, together with the two points within the triangle, can be combined to form the desired convex quadrilateral.

1.2.7. Given five points on a sphere, show that some four of the points lie in a closed hemisphere.

[*Note*: A closed hemisphere includes the points on the bounding great circle.]

Answer
Pick any two of the five points and draw a great circle through them. At least two of the remaining three points belong to the same closed hemisphere determined by the great circle. These two points, and the two starting points, are four points in the same closed hemisphere.

1.2.9. Suppose that 51 numbers are chosen randomly from $[100] = \{1, 2, \ldots, 100\}$. Show that two of the numbers have the sum 101.

Answer
Each of the 51 numbers belongs to one of the 50 sets $\{1, 100\}, \{2, 99\}, \ldots, \{50, 51\}$. Some set contains two of the chosen numbers, and these sum to 101.

1.2.11. Choose any 51 numbers from $[100] = \{1, 2, \dots, 100\}$. Show that there are two of the chosen numbers that are relatively prime (i.e., have no common divisor other than 1).

Answer
Place each of the 51 numbers into one of the 50 sets $\{1, 2\}, \{3, 4\}, \dots, \{99, 100\}$. One of the sets contains a pair of consecutive integers that are relatively prime.

1.2.13. Choose any 51 numbers from $[100] = \{1, 2, \dots, 100\}$. Show that there are two of the chosen numbers for which one divides the other.

Answer
Any natural number has the form $m = 2^{d_m} k_m$, where $d_m \geq 0$ and k_m is odd. Call k_m the *odd factor* of m. For example, the odd factor of $100 = 2^2 \cdot 25$ is $k_{100} = 25$. Thus, the odd factors of the 51 chosen numbers are in the set $\{1, 3, 5, \dots, 99\}$. Since this is a set with 50 members, two of the 51 chosen numbers have the same odd factor. The smaller is then a divisor of the larger, with a quotient that is a power of two.

1.2.15. Consider a string of $3n$ consecutive natural numbers. Show that any subset of $n + 1$ of the numbers has two members that differ by at most 2.

Answer
Suppose the $3n$ consecutive numbers are $a, a + 1, \dots, b$. Each of the $n + 1$ numbers in the given subset belongs to one of the sets $\{a, a + 1, a + 2\}$, $\{a + 3, a + 4, a + 5\}. \dots, \{b - 2, b - 1, b\}$. By the pigeonhole principle, one of these sets has two members of the subset and these differ by at most 2.

1.2.17. Suppose that the numbering of the squares along the spiral path shown in Example 1.9 is continued. What number k is assigned to the square S whose lower left corner is at the point (9, 5)?

Answer
We want to find a solution to the equations $k = 11i + 9$ and $k = 16j + 5$ for some integers i and j. This gives us $11i + 4 = 16j$. Both 4 and 16 are divisible by 4, so we see that i is divisible by 4. If we let $i = 4$, then $j = 3$ and we obtain the solution $k = 53$. The next multiple of 4 giving a solution is $i = 20$, but then $k = 229$ and we see that the spiral is overlapping itself with repeated squares covered a second time.

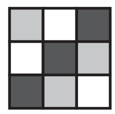
1.2.19. Generalize the results of Problem 1.2.18.

(a) How many spiral paths exist on the torus if $m = n$?

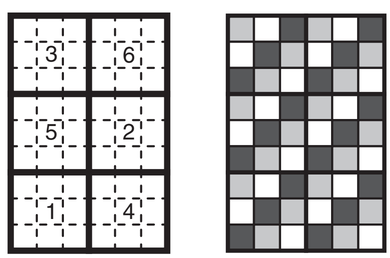
(b) Suppose $d \geq 2$ is the largest common divisor of m and n. How many distinct spiral paths exist on the torus?

Answer

(a) Any path returns to its starting position in m steps, so there are m spirals each covering m squares. For example, there are three paths when $m = n = 3$, as shown here.



(b) Since m/d and n/d are relatively prime, there is a unique spiral with mn/d^2 steps that covers a d by d square at each step. For example, if $m = 6$ and $n = 9$, then $d = 3$, and there is a unique spiral of length $\frac{mn}{d^2} = \frac{6 \cdot 9}{3^2} = 6$ of 3×3 squares that covers the torus. This is seen at the left in the figure below. By part (a) we see there are d nonintersecting spirals on the torus, each of length $\frac{mn}{d}$. The case $m = 6$, $n = 9$, $d = 3$, is shown at the right below, with the $d = 3$ paths each of length $\frac{mn}{d} = \frac{6 \cdot 9}{3} = 18$ shown in black, white, and gray.



PROBLEM SET 1.3

1.3.1. Consider an $m \times n$ chessboard, where m is even and n is odd. Prove that if two opposite corners of the board are removed, the trimmed board can be tiled with dominoes.

Answer

The left and right hand columns of height $n - 1$ of the trimmed board can each be tiled with vertical dominoes. The remaining board is has all of its rows of even length $m - 2$, so it can be tiled with horizontal dominoes.

1.3.3. Suppose that the lower left $j \times k$ rectangle is removed from an $m \times n$ chessboard, leaving an angle-shaped chessboard. Prove that that angular board can be tiled with dominoes if it contains an even number of squares.

Answer

Since $mn - jk = (m - k)\,n + (n - j)\,k$ is even, $(m - k)\,n$ and $(n - j)\,k$ have the same parity. If both are even, we can tile the resulting $(m - k) \times n$ and $(n - j) \times k$ rectangles. If both are odd, then n and k are odd thus m and j must be even. We can then tile the $m \times (n - j)\,n$ and $(m - k) \times j$ rectangles.

Alternate answer

View the angular region as the union of rectangles A, B, and C, where the corner rectangle B shares an edge with each of A and C. If all three rectangles have even area, the angle can be tiled since A, B, and C can each be tiled individually. If A and B, or B and C, each have odd area, then combining the odd rectangles shows that the angle is a union of two even area rectangles and therefore can be tiled. If A and C are odd, their edges are all of odd length and therefore rectangle B is also odd; the angular board therefore is not of even area.

1.3.5. Consider a rectangular solid of size $l \times m \times n$, where l, m, and n are all odd positive integers. Imagine that the unit cubes forming the solid are alternately colored gray and black, with a black cube at the corner in the first column, first row, and first layer.

(a) What is the color of each of the remaining corner cubes of the solid?

(b) How can the color of the cube in column j, row k, and layer h of the solid be determined?

(c) Prove that removing any black cube leaves a trimmed solid that can be filled with solid $1 \times 1 \times 2$ dominoes.

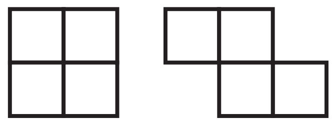
Answer

(a) Since the colors alternate, all eight corners of the solid are black.

(b) The cube is black if and only if the sum $j + k + h$ is odd. For example, the cube in column 1, row 1, and layer 1 is black since $1 + 1 + 1 = 3$, an odd number. That is, j, k, and h must all be odd, or one must be odd and the other two even.

(c) If the cube that is removed is black, and is in column j, row k, and layer h, then j, k, and h are all odd or two are even and one is odd. With no loss in generality assume that $j + k$ is even and h is odd. Theorem 1.21 tells us that layer h can be tiled with dominoes confined to that layer. When layer h is removed it leaves two (possibly one if $h = 1$ or n) rectangular solids with an even dimension and so it can be tiled with solid dominoes.

1.3.7. A *tetromino* is formed with four squares joined along common edges. For example, the *O* and the *Z* tetromino are shown here.



(a) Find the three other tetrominoes, called the *I, J,* and *T* tetrominoes.

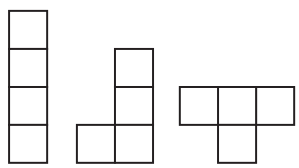
(b) The set of five tetrominoes has a total area of 20 square units. Explain why it is not possible to tile a 4×5 rectangle with a set of tetrominoes.

(c) Show that a 4×10 rectangle can be tiled with two sets of tetrominoes.

(d) Show that a 5×8 rectangle can be tiled with two sets of tetrominoes.

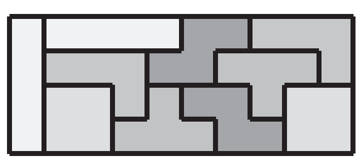
Answer

(a)

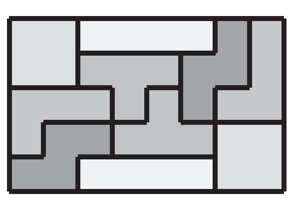


(b) A 4×5 chessboard has 10 unit squares of each color. The *O, Z, I,* and *J* tetrominoes each cover 2 unit squares of each color, but the *T* tetromino covers 3 squares of one color and one of the other color. Therefore the 4×5 square cannot be tiled with a set of tetrominoes.

(c)

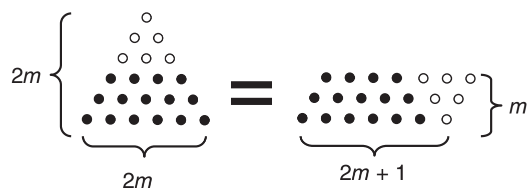


(d)



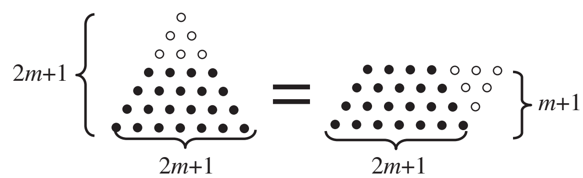
PROBLEM SET 1.4

1.4.1. The following diagram illustrates that $t_{2m} = m(2m + 1)$



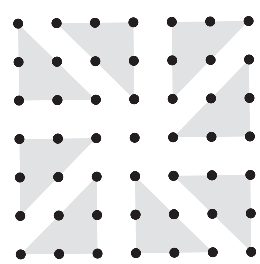
Create a similar diagram that illustrates the formula $t_{2m+1} = (2m + 1)(m + 1)$.

Answer



1.4.3. Use both algebra and dot patterns to show that the square of an odd integer is congruent to 1 modulo 8. That is, show that $s_{2n+1} = 8u_n + 1$ for some integer u_n. Be sure to identify the integer u_n by its well-known name.

Answer
The answer is $s_{2n+1} = 8t_n + 1$, since $(2n + 1)^2 = 4n^2 + 4n + 1 = 8\dfrac{n(n + 1)}{2} + 1 = 8t_n + 1$. See the following diagram:



1.4.5. The *centered square numbers* are obtained much like the centered triangle numbers of Problem 1.4.4, except that squares with an increasing number of dots per side surround a center dot.

(a) Create a diagram that shows the sequence of centered square numbers beginning with 1, 5, 13, 25, and 41.

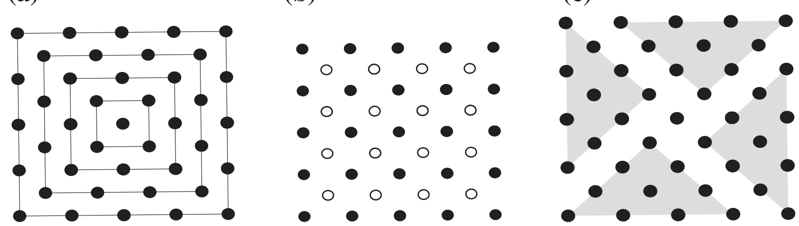
(b) Color the dots in the diagram from part (a) to show that the n^{th} centered square number is given by $(n + 1)^2 + n^2$.

(c) Shade your diagram from part (a) to shows that every centered square number is congruent to 1 modulo 4.

(d) Verify part (c) with algebra.

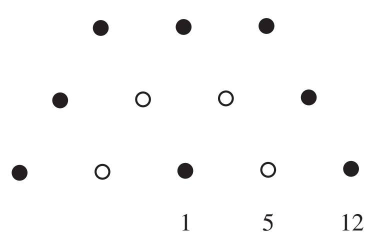
Answer

(a) **(b)** **(c)**



(d) $(n + 1)^2 + n^2 = 2n^2 + 2n + 1 = 1 + 2n(n + 1) = 1 + 4\frac{n(n + 1)}{2} = 1 + 4t_n$

1.4.7. The first three *trapezoidal numbers* are 1, 5, and 12, as shown by the dot pattern here.



(a) Continue the trapezoidal pattern to find the next three trapezoidal numbers.

(b) Draw some lines on your diagram from part (a) to explain why the trapezoidal numbers are simply an alternative pattern for the pentagonal numbers $p_1 = 1, p_2 = 5, p_3 = 12, \ldots$.

(c) Use the trapezoidal diagram to show why each pentagonal number is the sum of a triangular number and a square number. Give an explicit formula for p_n in terms of the triangular and square numbers.

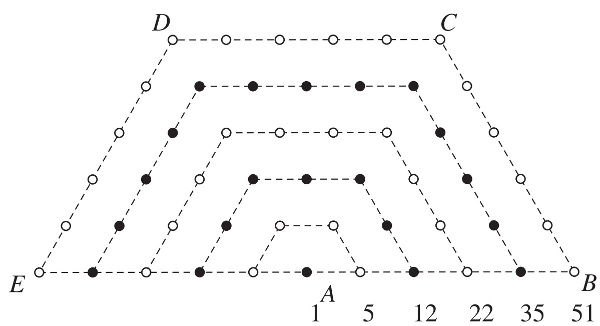
(d) The trapezoidal diagram shows that each pentagonal number is the difference of two triangular numbers. Determine the two triangular numbers corresponding to p_n and express this result in a formula.

(e) Construct a diagram showing that each pentagonal number is one-third of a triangular number. Give an explicit formula of this property.

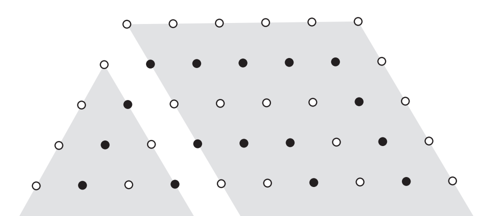
Answer

(a) The next three trapezoidal numbers are 22, 35, and 51.

(b) View a trapezoidal number as a distorted pentagon *ABCDE,* with sides *EA* and *AB* along the same line.

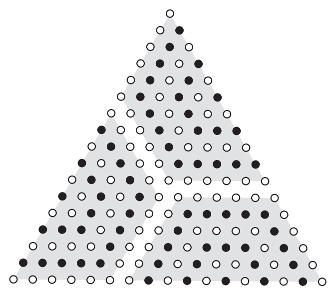


(c) $p_n = t_n + n^2$. For example, $p_6 = 51 = t_{6-1} + 6^2 = \frac{5 \cdot 6}{2} + 36 = 15 + 36$



(d) Extend the trapezoid to a triangle of side $2n - 1$. We then see that $p_n = t_{2n-1} - t_{n-1}$. For example, $p_4 = 22 = t_7 - t_3 = \frac{7 \cdot 8}{2} - \frac{3 \cdot 4}{2} = 28 - 6$.

(e) The shading in the diagram below shows that $p_n = \frac{1}{3}t_{3n-1}$.



For example, $p_6 = 51 = \frac{1}{3}t_{3\cdot 6-1} = \frac{1}{3}t_{17} = \frac{1}{3}\left(\frac{17 \cdot 18}{2}\right) = \frac{17 \cdot 18}{6} = 17.3.$

1.4.9. Dominoes, as described in Problem 1.4.8 also come in double-9, double-12, double-15, and even double-18 sets. Consider, more generally, a double-n set, so each half-domino is imprinted with 0 to n pips.

(a) Derive a formula for the number of dominoes in a double-n set. Use the formula to determine the number of dominoes in a double-n set for $n = 6, 9, 12, 15$, and 18.

(b) Derive a formula for the total number of pips in a double-n set. Use the formula to determine the total number of pips in a double-n set for $n = 6, 9, 12, 15$, and 18.

Answer

(a) Consider the array of dots in the x-,y-coordinate plane with a dot at (p, q) that represents the $p - q$ domino, with $0 \le q \le p \le n$. This array is a triangle with $n + 1$ dots per side, so there are $t_{n+1} = \frac{1}{2}(n + 1)(n + 2)$ dominoes in a double-n set.

For $n = 3, 6, 9, 12, 15$, and 18, the number of dominoes are the triangular numbers 10, 28, 55, 91, 136, and 190.

(b) Imagine that you have two double-n sets, so that each $p - q$ domino from one set can be paired with the $(n - p) - (n - q)$ complementary domino from the second set. For example, in a double-15 set, pair the $11 - 6$ domino from one set with the complementary $4 - 9$ domino from the second set. Each pair of complementary dominoes has a total of $2n$ pips, so by part (a) there are $(2n)t_{n+1} = (2n)\frac{1}{2}(n+1)(n+2) = n(n+1)(n+2)$ pips in the two double-n sets. Therefore, a single double-n set has a total of $\frac{1}{2}n(n+1)(n+2)$ pips.

Alternate answer
Each half-domino with k pips, $0 \le k \le n$, occurs $n+2$ times in a double-n set, so the total number of pips is given by $(n+2)(0+1+2+\cdots+n) = (n+2)t_n = (n+2)\dfrac{n(n+1)}{2} = \dfrac{1}{2}n(n+1)(n+2)$.

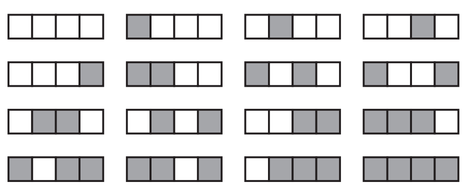
PROBLEM SET 1.5

1.5.1. **(a)** Extend Figure 1.9 to depict the set of 16 tilings of a board of length 4, where each tile is either gray or white.

(b) Explain how it is easy to use the 8 tilings of boards of length 3 to draw all of the tiled boards of length 4.

Answer

(a)



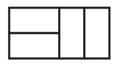
(b) Add a white tile at the right end of each of the 8 tiled boards of length 3, and then add a gray tile at the right end of each of the 8 tiled boards of length 3. Altogether, this forms all 16 of the tilings of boards of length 4.

1.5.3. Use formulas (1.20) and (1.21) to prove Pascal's identity (1.24).

Answer

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$
$$= \frac{(n-1)!}{k!\,(n-k)!}\,[k+n-k] = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

1.5.5. **(a)** Find all of the ways that a 2×4 rectangular board can be tiled with 1×2 dominoes. Here is one way to tile the board.

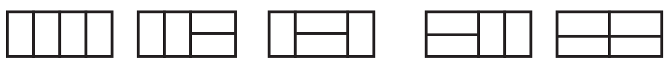


(b) Draw all of the ways to tile a 2×4 board with dominoes.

(c) How many ways can a $2 \times n$ board be tiled with dominoes?

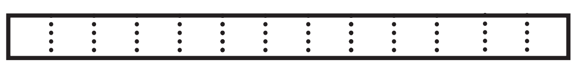
Answer

(a) There are five tilings:



(b) Draw a horizontal midline through each tiling by dominoes. The lower $1 \times n$ row board is equivalent to a tiling by squares and dominoes, showing there are f_n tilings of a $2 \times n$ board with dominoes.

1.5.7. The following train (see Problem 1.5.6 for the definition of a train) has just one car of length 13.



(a) How many ways can a train of length 13 be formed with 2 cars?

(b) Why are there $\binom{12}{4}$ trains of length 13 that can be formed with 5 cars?

(c) Generalize your answer to part (b) to give a binomial coefficient that expresses the number of trains of length n with r cars.

Answer

(a) Any one of the 12 vertical dashed lines can be chosen to end a car and start a new one, so there are $\binom{12}{1} = 12$ ways to form a train of length 13 with 2 cars.

(b) Any choice of 4 of the 12 vertical dashed lines forms a train with 5 cars. Thus, there are $\binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495$ trains of length 13 with 5 cars.

(c) There are $\binom{n-1}{r-1}$ trains of length n with r cars.

1.5.9. There are 4 ways to express 5 as a sum of two ordered summands, namely $4 + 1$, $3 + 2$, $2 + 3$, and $1 + 4$.

(a) How many ways can 5 be expressed as a sum of three ordered summands? (see Problem 1.5.8)

(b) How many ways can a positive integer n be expressed as a sum of k summands?

Answer

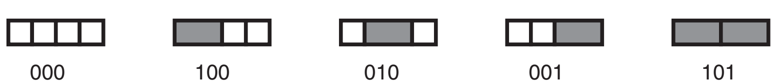
(a) Six ways

(b) $\binom{n-1}{k-1}$ since the summands can be viewed as the lengths of k cars that form a train of length n.

1.5.11. How many binary sequences of length n have no two consecutive ones? (A binary sequence in an ordered list of ones and zeroes, such as 100101001.) For example, there are 5 binary sequences of length 3 with no two consecutive ones, namely 000, 100, 010, 001, and 101.

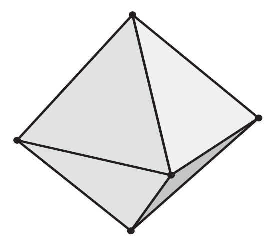
Answer

There are 2 sequences of length 1 (0 and 1), 3 of length 2 (00, 10, and 01), 5 of length 3, and 8 of length 4 (0000, 1000, 0100, 0010, 0001, 1010, 0101, and 1001). This suggests that the number of binary sequences of length n with no consecutive ones is given by the Fibonacci number F_{n+2}, or, equivalently, by the combinatorial Fibonacci number f_{n+1}. But this is the number of tilings of a $1 \times (n + 1)$ board with squares and dominoes. To see the connection, consider a board of length 4. Each of the three vertical dashed segments can be labeled with a 0 to represent a break between tiles or labeled with a 1 to represent the midline of a domino. Since the distance between midlines of any two dominos is at least two, no two 1s can be consecutive. Here is the correspondence between tilings and binary sequences with no consecutive ones in the case of a sequence of length 3 and a board of length 4:



PROBLEM SET 1.6

1.6.1. The 2006 World Cup used a soccer ball called the "+teamgeist" ("team spirit," with the + sign to allow the German word to be copyrighted) Mathematically, the ball is a spherical analog of the truncated octahedron, obtained by starting with the octahedron of 8 triangles as shown, then replacing ("truncating") each corner with a square, and finally rounding the faces to become spherical. What counting principle can be used to determine the number of panels on the +teamgeist soccer ball?



Answer
When truncated, the 6 corners of the octahedron are replaced with squares, and the 8 triangles become hexagons. The sets of squares and hexagons partition the set of all of the ball's panels, so there are $8 + 6 = 14$ panels in all by the addition principle.

1.6.3. There are two types of seams of the traditional soccer ball shown in Example 1.30, those that separate two hexagonal panels and those that separate a pentagonal panel from a hexagonal panel. Find the total number of seams on the ball, and the number that separate one hexagonal panel from another.

Answer
Let S denote the set of all of the seams on the ball, A the set of seams between panels of different types, and B the set of seams between abutting hexagonal panels. Since every seam in set A borders exactly one of the 12 pentagons, we see that $|A| = 5 \cdot 12 = 60$. Similarly, each of the 20 hexagonal panels is bordered by 6 seams, but the product $6 \cdot 20 = 120$ counts the seams between abutting hexagons twice and the seams along pentagons once. This means that $120 + 60 = 180$ counts every seam of the ball twice, so we conclude there are $|S| = 180/2 = 90$ seams on the ball. By the subtraction principle, $90 - 60 = 30$ seams separate the hexagonal panels from one another.

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1.6.5. Maria likes to order double-scoop ice cream cones, with chocolate or strawberry on the bottom, and chocolate, vanilla, or mint on the top. Describe the ways Maria can order her ice cream cones with a Cartesian product, and count the number of types of cones she likes.

Answer
Let $A = \{C, S\}$ and $B = \{C, V, M\}$ be the sets of choices of flavors. Then Maria will have $|A \times B| = |A||B| = 2 \cdot 3 = 6$ types of cones to her liking.

1.6.7. Generalize the results of Example 1.45 and Problem 1.6.6. That is, provide formulas for the number of ways to split up m people into singles and doubles matches. For doubles, create as many matches as possible and then set up a singles match if enough people remain not already playing doubles.

Answer
If $m = 2n$, we can form $(2n - 1)!!$ singles matches, and if $m = 2n + 1$ is odd we can form n matches in $(2n + 1)!!$ ways, with someone sitting out. If $m = 4n$, we can form $2n$ doubles partners in $(4n - 1)!!$ ways, and then form doubles matches in $(2n - 1)!!$ ways. That is, there are $(4n - 1)!!(2n - 1)!!$ ways to split up for n games of doubles. If $m = 4n + 2$, we can form $2n + 1$ pairings in $(4n + 1)!!$ ways, and then form doubles matches and one singles match in $(2n + 1)!!$ ways. That is, there are $(4n + 1)!!(2n + 1)!!$ ways to split up for n doubles and one singles match. If $m = 4n + 1$, we can choose a player to sit out in $4n + 1$ ways, so there are $(4n + 1)(4n - 1)!!(2n - 1)!! = (4n + 1)!!(2n - 1)!!$ ways to form the n doubles matches and choose who sits out. Similarly, for If $m = 4n + 3$, we can choose a player to sit out in $4n + 3$ ways, so there are $(4n + 3)(4n + 1)!!(2n + 1)!! = (4n + 3)!!(2n + 1)!!$ ways to form the n doubles matches, one singles match, and choose who sits out.

1.6.9. Construct Pascal's triangle that shows rows 0 through 6.

(a) Draw a loop around the terms in the hockey stick identity (1.32) for the case $n = 3$ and $r = 2$, showing that a hockey stick shape is formed.

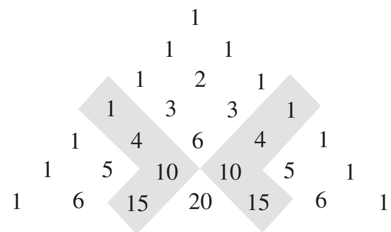
(b) Repeat part (a) but for the hockey stick identity of Problem 1.6.8 in the case that $n = 5$ and $r = 3$.

(c) How do the terms in the handle of the hockey stick relate to the term in the blade of the hockey stick?

Answer

(a) and **(b)**: See the diagram below.

(c) The sum of the terms in the handle is the term in the blade.



1.6.11. There are three roads from Sylvan to Tacoma, four roads from Tacoma to Umpqua, and two roads from Sylvan directly to Umpqua. How many routes, with no backtracking, can be taken from Sylvan to Umpqua?

Answer

$3 \cdot 4 + 2 = 14$

SECTION 1.7

1.7.1. **(a)** Let A be any set of 51 numbers chosen from [100]. Show that two members of A differ by 50.

(b) State and prove a generalization of the result of part (a).

Answer

(a) Consider the 50 pigeonholes given by the sets $\{1, 51\}$, $\{2, 52\}$, … , $\{50, 100\}$. By the pigeonhole principle, two of the 51 numbers belong to the same set and their difference is 50.

(b) Theorem. Any subset of $[2n]$ with $n + 1$ members contains two members whose difference in n.

Proof. Consider the n pigeonholes $\{1, n + 1\}$,$\{2, n + 2\}$, … , $\{n, 2n\}$. At least two members of any set with $n + 1$ members has 2 members in the same pigeonhole, and their difference is n.

1.7.3. Show that any set of 10 natural numbers, each between 1 and 100, contains two disjoint subsets with the same sum of its members.

Answer

There are $2^{10} = 1024$ subsets of a set with 10 members. The sum of the numbers in any subset less than $10 \cdot 100 = 1000$. By the pigeonhole principle, there are at least two distinct subsets with the same sum. If there are any numbers common to both sets, these can be deleted from both sets to leave two disjoint sets still with the same sum.

1.7.5. A traditional dart board divides the circular board into 20 sectors that are numbered clockwise from the top with the sequence 20 - 1 - 18 - 4 - 13 - 6 - 10 - 15 - 2 - 17 - 3 - 19 - 7 - 16 - 8 - 11 - 14 - 9 - 12 - 5.

There is considerable variation in the sum of three successive numbers, from 23 = 1 + 18 + 4 to 42 = 19 + 7 + 16. Can the numbers 1 through 20 be rearranged so that the sum of each group of three successive numbers is smaller than 32?

Answer

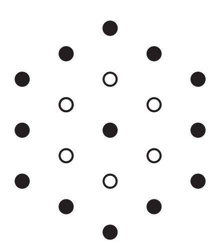
Consider an arrangement $a_1, a_2, \ldots, a_{20}$ and the 20 sums $s_1 = a_1 + a_2 + a_3, s_2 = a_2 + a_3 + a_4, \ldots, s_{20} = a_{20} + a_1 + a_2$. Then $s_1 + s_2 + \cdots + s_{20} = 3\left(a_1 + a_2 + \cdots + a_{20}\right) = 3\,(1 + 2 + \cdots + 20) = 3\frac{20 \cdot 21}{2} = 630$.

If each sum were no larger than 31, then $s_1 + s_2 + \cdots + s_{20} \leq 20 \cdot 31 = 620$. This is a contradiction, so at least one of the sums s_j must be 32 or larger.

Alternate answer

The average of the sums $s_1, s_2, \ldots, s_{20}$ is $\frac{s_1 + s_2 + \cdots + s_{20}}{20} = \frac{3\left(a_1 + a_2 + \cdots + a_{20}\right)}{20} = \frac{3\,(1 + 2 + \cdots + 20)}{20} = 3\frac{20 \cdot 21}{2 \cdot 20} = 31.5$. If each sum were no larger than 31, then their average would also be no larger than 31, a contradiction.

1.7.7. The centered hexagon numbers (or *hex* numbers), H_n, are obtained by starting with a single dot and then surrounding it by hexagons with 6, 12, 18, … dots on its sides. The diagram below shows that $H_0 = 1$, $H_1 = 7$, and $H_2 = 19$.

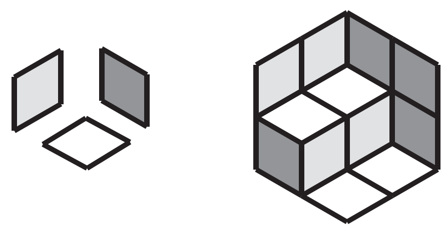


(a) Extend the diagram with two more surrounding hexagons to determine H_3 and H_4.

(b) Derive a formula that gives H_n in terms of the triangular numbers.

(c) Obtain an expression for H_n as a function of n.

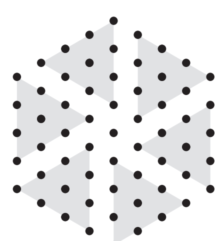
(d) Suppose that *tridominoes* are formed with a pair of equilateral triangles joined along a common edge, and are colored gray, white, or black according to their orientation. The figure here shows the three types of tridominoes and one way they can be used to tile the hex pattern for the H_2 array of dots.



Show that every hex pattern of H_n dots can be tiled with tridominoes, and give the number of tridominoes of each color that are used in the tiling. [*Suggestion*: The hex numbers might also be called the *corner* numbers!]

Answer

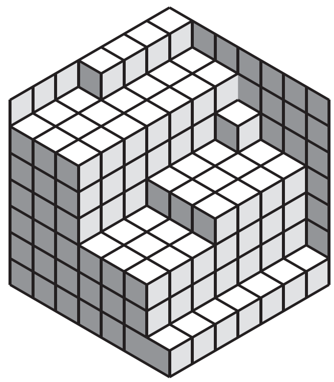
(a) $H_3 = 37, H_4 = 61$



(b) The shading of the diagram shows that $H_n = 1 + 6t_n$.

(c) $H_n = 1 + 6t_n = 1 + 6\left(\frac{1}{2}n(n+1)\right) = 3n^2 + 3n + 1$

(d) The tiling with colored tridominoes makes it appear that cubes have been stacked into a cubical $n \times n \times n$ corner. When viewed from above, only the n^2 white tridominoes show, and similarly n^2 black and n^2 gray tridominoes are seen from the right and left, respectively. Altogether, $3n^2$ tridominoes are needed for the tiling. For example, if $n = 7$, there are 49 tridominoes of each color, as seen in the tiling that follows.



1.7.9. Use the result of Problem 1.7.8 to show that hockey stick identity 1 (1.32) can be rewritten to become $\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$ (hockey stick identity 2)

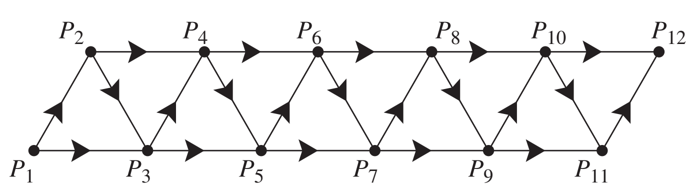
Answer

Hockey stick identity 1 (1.32), with n replaced by s and r replaced by $m - s$, becomes $\binom{m+1}{m-s} = \binom{s}{0} + \binom{s+1}{1} + \binom{s+2}{2} + \cdots + \binom{m}{m-s}$.

Using the result $\binom{n}{r} = \binom{n}{n-r}$ from Problem 1.7.8, this identity becomes $\binom{m+1}{s+1} = \binom{s}{s} + \binom{s+1}{s} + \binom{s+2}{s} + \cdots + \binom{m}{s}$.

This is hockey stick identity 2 once s is replaced with r and m is replaced n.

1.7.11. **(a)** How many paths extend from point P_1 to each of the points $P_2, P_3, \ldots, P_{12}$ in the following directed graph? Each step along any path must be in the direction indicated by the arrow. For example, there are 2 paths from P_1 to P_3.



What famous number sequence gives the number of paths to the points $P_1, P_2, \dots, P_n$? Provide a justification for your answer.

Answer

(a) There are 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, and 144 paths, respectively, to the points $P_2, P_3, \dots, P_{12}$.

(b) The set of paths that reach point P_n can be partitioned into two disjoints subsets: the set of paths that end crossing the arc $P_{n-1}P_n$ and the set of paths that end by crossing the arc $P_{n-2}P_n$. This partition shows that the number of paths to vertex P_n is the sum of the number of paths to vertices P_{n-1} and P_{n-2}. This is the Fibonacci recursion relation, and there is one path to P_2 and two paths to P_3. Therefore, the number of paths to points P_n is the Fibonacci number F_n.

1.7.13. There are six positive divisors of $12 = 2^2 \times 3^1$, namely $1 = 2^0 \times 3^0$, $2 = 2^1 \times 3^0$, $4 = 2^2 \times 3^0$, $3 = 2^0 \times 3^1$, $6 = 2^1 \times 3^1$, and $12 = 2^2 \times 3^1$. What is the number of positive divisors of these integers?

(a) 660 **(b)** $2^5 \times 3^7 \times 11^2 \times 23^4$ **(c)** 10^{100}

Answer

(a) $660 = 2^2 \times 3^1 \times 5^1 \times 11^1$, so there are $3 \times 2 \times 2 \times 2 = 24$ positive divisors, since we can choose the powers of 2, 3, 5, and 11 in 3, 2, 2, and 2 ways, respectively in each prime divisor.

(b) $6 \times 8 \times 3 \times 5 = 720$ positive divisors

(c) $10^{100} = 2^{100} \times 5^{100}$ has $101^2 = 10201$ divisors