PART I

MEASURE THEORY

CHAPTER 1

SETS AND SEQUENCES

Sets are the most basic concepts in measure theory as well as in mathematics. In fact, set theory is a foundation of mathematics (Moschovakis, 2006). The algebra of sets develops the fundamental properties of set operations and relations. In this chapter, we shall introduce basic concepts about sets and some set operations such as union, intersection, and complementation. We will also introduce some set relations such as De Morgan's laws.

1.1 Basic Concepts and Facts

Definition 1.1 (Set, Subset, and Empty Set). A set is a collection of objects, which are called *elements*. A set B is said to be a subset of a set A, written as $B \subseteq A$, if the elements of B are also elements of A. A set A is called an *empty set*, denoted by $\emptyset$, if A contains no elements.

Definition 1.2 (Countable Set). A set A is said to be countable if either A contains a finite number of elements or every element of A appears in an infinite sequence $x_1, x_2, \ldots$. A set A is said to be uncountable if it is not countable.

Definition 1.3 (Equality of Sets). Two sets A and B are said to be equal, written as $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Definition 1.4 (Union, Intersection, and Complement of Sets). Let A and B be two subsets of a set S. The union of A and B is defined as

$$A \cup B = \{s \in S : s \in A \text{ or } s \in B\}.$$

The intersection of A and B is defined as

$$A \cap B = \{s \in S : s \in A \text{ and } s \in B\}.$$

The complement of A relative to S is defined as

$$A^c = \{s \in S : s \notin A\}.$$

Definition 1.5 (Difference and Symmetric Difference of Sets). Let A and B be two sets. The difference between A and B, denoted by $A - B$ or $A \backslash B$, is defined as

$$A \backslash B = A \cap B^c.$$

The symmetric difference between A and B is defined as

$$A \Delta B = (A \backslash B) \cup (B \backslash A).$$

Definition 1.6 (Increasing and Decreasing Sequence of Sets). Let $\{A_n\}_{n \geq 1}$ be a sequence of sets. We say that $\{A_n\}_{n \geq 1}$ is an increasing sequence of sets with limit A, written as $A_n \uparrow A$, if

(a) $A_n \subseteq A_{n+1}$ for all $n \geq 1$.

(b) $\bigcup_{n=1}^{\infty} A_n = A$.

We say that $\{A_n\}_{n \geq 1}$ is a decreasing sequence of sets with limit A, written as $A_n \downarrow A$, if

(a) $A_n \supseteq A_{n+1}$ for all $n \geq 1$.

(b) $\bigcap_{n=1}^{\infty} A_n = A$.

Definition 1.7 (Indicator Function). Let A be a set. Then the indicator function of A is defined as

$$I_A(s) = \begin{cases} 1, & \text{if } s \in A; \\ 0, & \text{if } s \notin A. \end{cases}$$

Definition 1.8 (Upper Limit and Lower Limit of Sequences of Sets). Let $\{E_n\}_{n\geq 1}$ be a sequence of subsets of S. Then $\limsup E_n$ and $\liminf E_n$ are defined as

$$\limsup E_n = \bigcap_{j\geq 1}\left(\bigcup_{i\geq j} E_i\right)$$

and

$$\liminf E_n = \bigcup_{j\geq 1}\left(\bigcap_{i\geq j} E_i\right),$$

respectively.

Definition 1.9 (Upper Limit and Lower Limit of Sequences of Real Numbers). Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\limsup x_n$ and $\liminf x_n$ are defined as

$$\limsup x_n = \inf_{m\geq 1}\left\{\sup_{n\geq m} x_n\right\} = \lim_{m\to\infty}\left\{\sup_{n\geq m} x_n\right\}$$

and

$$\liminf x_n = \sup_{m\geq 1}\left\{\inf_{n\geq m} x_n\right\} = \lim_{m\to\infty}\left\{\inf_{n\geq m} x_n\right\},$$

respectively.

Definition 1.10 (Convergence of Sequences). A sequence $\{x_n\}_{n\in\mathbf{N}}$ of real numbers is said to be convergent if and only if

$$\limsup x_n = \liminf x_n.$$

The sequence is said to be convergent to x, written as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, if and only if x_n is convergent and $\limsup x_n = \liminf x_n = x$.

Definition 1.11 (Convergence of Sets). A sequence $\{A_n\}_{n\in\mathbf{N}}$ of sets is said to converge to A, written as $A_n \to A$ or $\lim_{n\to\infty} A_n = A$, if

$$\lim_{n\to\infty} I_{A_n}(s) = I_A(s)$$

for all $s \in S$.

Definition 1.12 (Partial Ordering, Totally Ordered Sets, and Chains). A partial ordering $\leq$ on a set S is a relation that satisfies the following conditions, where a, b, and c are arbitrary elements of S:

(a) $a \leq a$ (reflexivity).

(b) If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetricity).

(c) If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

Let S be a set with a partial ordering $\leq$. A subset C of S is said to be a totally ordered subset of S if and only if for all $a, b \in C$, we have either $a \leq b$ or $b \leq a$. A chain in S is a totally ordered subset of S.

Theorem 1.1 (De Morgan's Laws). *Let $\{A_n\}_{n\geq 1}$ be a sequence of sets. Then*

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

and

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

Theorem 1.2 (Zorn's Lemma). *Let S be a nonempty set with a partial ordering "$\leq$". Assume that every nonempty chain C in S has an upper bound, that is, there exists an element $x \in S$ such that $a \leq x$ for all $a \in C$. Then S has a maximal element; in other words, there exists an element $m \in S$ such that $a \leq m$ for all $a \in S$.*

1.2 Problems

1.1. Let I be a countable set. For each $i \in I$, let A_i be a countable set. Show that the union

$$A = \bigcup_{i \in I} A_i$$

is again countable.

1.2. Let $\mathbf{Q}$ be the set of all rational numbers, which have the form of a/b, where a and $b (b \neq 0)$ are integers. Let $\mathbf{R}$ be the set of all real numbers. Show that

(a) $\mathbf{Q}$ is countable.

(b) $\mathbf{R}$ is uncountable.

1.3. Let $\{E_n\}_{n\geq 1}$ be a sequence of sets. Show that

$$\liminf E_n \subseteq \limsup E_n.$$

1.4. Let $\{E_n\}_{n\geq 1}$ be a sequence of subsets of S. Show that

$$(\limsup E_n)^c = \liminf E_n^c$$

and

$$(\liminf E_n)^c = \limsup E_n^c,$$

where $A^c = S \backslash A$ for set A.

1.5. Let $\{E_n\}_{n\geq 1}$ be a sequence of subsets of a set S. Show that

$$I_{\limsup E_n}(s) = \limsup I_{E_n}(s), \quad \forall s \in S \tag{1.1}$$

and

$$I_{\liminf E_n}(s) = \liminf I_{E_n}(s), \quad \forall s \in S, \tag{1.2}$$

where I is the indicator function.

1.6. Let $\{A_n\}_{n\geq 1}$ be a sequence of sets of real numbers defined as follows:

$$A_n = \begin{cases} (-\frac{1}{n}, 1], & \text{if } n \text{ is odd;} \\ (-1, \frac{1}{n}], & \text{if } n \text{ is even.} \end{cases}$$

Calculate $\liminf A_n$ and $\limsup A_n$.

1.7. Let $\{x_n\}_{n\geq 1}$ a sequence of real numbers. Show that x_n converges (i.e., the limit $\lim_{n\to\infty} x_n$ exists) in $[-\infty, \infty]$ if and only if

$$\limsup x_n = \liminf x_n.$$

1.8. Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequences of real numbers. Let c be a constant in $(-\infty, \infty)$. Show that

(a) $\limsup(-x_n) = -\liminf x_n$.

(b) $\limsup x_n \geq \liminf x_n$.

(c) $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$.

(d) $\limsup x_n + \limsup y_n \geq \limsup(x_n + y_n)$.

(e) $\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n)$.

(f) $\liminf(c + x_n) = c + \liminf x_n$.

(g) $\liminf(c - x_n) = c - \limsup x_n$.

1.9. Let $\{A_n\}_{n\geq 1}$ be a sequence of subsets of a set S. Show that $\lim_{n\to\infty} A_n$ exists if and only if $\limsup A_n = \liminf A_n$. In addition, if $A = \lim_{n\to\infty} A_n$ exists, then $A = \limsup A_n = \liminf A_n$.

1.10. Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequences of real numbers. Suppose that

$$\lim_{n\to\infty} x_n \quad \text{and} \quad \lim_{n\to\infty} y_n$$

exist. Show that $\lim_{n\to\infty}(x_n + y_n)$ exists and

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

1.3 Hints

1.1. Try to construct a sequence in which every element in A appears.

1.2. To prove part (a), show that all rational numbers can be written as a sequence. Part (b) can be proved by the method of contradiction, that is, by assuming that $\mathbf{R}$ is countable and can be written as a sequence $(x_n)_{n\geq 1}$. Then represent every x_n as a decimal of finite digits and find a new number, which is not in the sequence.

1.3. This problem can be proved by using Definition 1.8.

1.4. This problem can be proved by using Definition 1.8 and Theorem 1.1.

1.5. An indicator function has only two possible values: 0 and 1. Hence the first equality of the problem can be proved by considering two cases: $s \in \limsup E_n$ and $s \notin \limsup E_n$. The second equality of the problem can be proved using the result of Problem 1.4.

1.6. The lower and upper limits of the sequence can be calculated by using Definition 1.8.

1.7. Use the definition of limits. For example, $\lim_{n\to\infty} x_n = \infty$ if and only if, given any $\epsilon > 0$, there exists an integer N_ϵ such that $x_n > \epsilon$ for all $n \geq N_\epsilon$. Similarly, $\lim_{n\to\infty} x_n = L$ for some $L \in (-\infty, \infty)$ if and only if, given any $\epsilon > 0$, there exists an integer N_ϵ such that $L - \epsilon < x_n < L + \epsilon$ for all $n \geq N_\epsilon$.

1.8. Use Definition 1.9 and the fact that $\sup_{n\geq m}(-x_n) = -\inf_{n\geq m} x_n$ to prove part (a). To prove part (b), try to establish

$$\sup_{i\geq j} x_i \geq \inf_{n\geq m} x_n, \quad j, m \geq 1.$$

To prove part (c), try to establish the following inequality

$$\inf_{n\geq m} x_n + \inf_{i\geq j} y_i \leq \sup_{s\geq 1} \inf_{r\geq s} (x_r + y_r).$$

Use parts (a) and (c) to prove part (d). Use parts (a) and (d) to prove part (e). Use part (c) to prove part (f). Use parts (a) and (f) to prove part (g).

1.9. Use Definition 1.11 and the results of Problems 1.5 and 1.7.

1.10. Use the results of Problems 1.7 and 1.8.

1.4 Solutions

1.1. Since I is countable, then $\{A_i : i \in I\}$ is countable. Note that A_i is countable for each $i \in I$. There exists a sequence $(B_n)_{n\geq 1}$ of countable sets such that every

A_i appears in the sequence. For each integer $n \geq 1$, as B_n is countable, there exists a sequence $(x_{n,m})_{m\geq 1}$ such that every element of B_n appears in the sequence. Now let $(y_i)_{i\geq 1}$ be a sequence given by

$$x_{1,1}, x_{2,1}, x_{1,2}, x_{3,1}, x_{2,2}, x_{1,3}, \ldots.$$

Then every element in A appears in the sequence $(y_i)_{i\geq 1}$. Hence A is countable. This completes the proof.

1.2.

(a) For each integer $n \geq 0$, let

$$A_n = \{r \in \mathbf{Q} : n \leq r \leq n+1\}, \quad B_n = \{r \in \mathbf{Q} : -n-1 \leq r \leq -n\}.$$

Then

$$\mathbf{Q} = \bigcup_{n=0}^{\infty} (A_n \cup B_n).$$

By Problem 1.1, it is sufficient to show that A_n and B_n are countable for each integer $n \geq 0$. Note that $A_n = \{r+n : r \in A_0\}$ and $B_n = \{r-n-1 : r \in A_0\}$. We only need to show that A_0 is countable.

Let $(x_n)_{n\geq 1}$ be a sequence given by

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots.$$

Then every number in A_0 appears in the sequence. Hence $\mathbf{Q}$ is countable.

(b) Assume that $\mathbf{R}$ is countable. Then the subset $(0, 1]$ of $\mathbf{R}$ is also countable. Let the numbers in $(0, 1]$ be written as a sequence $(x_n)_{n\geq 1}$. Since $x_n \in (0, 1]$, we can represent x_n as a decimal

$$0.z_{n,1}z_{n,2}\cdots,$$

where $z_{n,j}$ is one of the 10 digits. The decimal representation is not unique for some numbers. For example, we have

$$0.1 = 0.0999\ldots = \sum_{j=1}^{\infty} \frac{9}{10^{j+1}}.$$

In such cases, we choose the representation with 9s at the end. Now let $y = w_1w_2\cdots$ be a decimal given by

$$w_j = \begin{cases} 7, & \text{if } z_{j,j} \leq 5; \\ 2, & \text{if } z_{j,j} > 5. \end{cases}$$

Since $w_j \neq 9$ and $w_j \neq z_{j,j}$ for all $j \geq 1$, we have $y \neq x_n$ for all $n \geq 1$. But the sequence $(x_n)_{n\geq 1}$ includes all numbers in $(0, 1]$. Hence $y = x_{n_0}$ for some n_0. This is a contradiction. Hence $\mathbf{R}$ is uncountable.

This completes the proof.

1.3. Let $s \in \liminf E_n$. Then by Definition 1.8, we have $s \in \bigcap_{i\geq j_0} E_i$ for some $j_0 \geq 1$. It follows that $s \in E_i$ for all $i \geq j_0$. Hence we have

$$s \in E_{\max\{j,j_0\}} \subseteq \bigcup_{i\geq j} E_i$$

for all $j \geq 1$. Consequently, $s \in \limsup E_n$. Therefore, $\liminf E_n \subseteq \limsup E_n$.

1.4. By Definition 1.8 and Theorem 1.1, we have

$$\begin{aligned}(\limsup E_n)^c &= \left(\bigcap_{j\geq 1}\left(\bigcup_{i\geq j} E_i\right)\right)^c = \bigcup_{j\geq 1}\left(\bigcup_{i\geq j} E_i\right)^c \\ &= \bigcup_{j\geq 1}\left(\bigcap_{i\geq j} E_i^c\right) = \liminf E_n^c.\end{aligned}$$

Similarly, we can show that $(\liminf E_n)^c = \limsup E_n^c$.

1.5. To prove (1.1), we consider two cases: $s \in \limsup E_n$ and $s \notin \limsup E_n$. If $s \in \limsup E_n$, then

$$s \in \bigcap_{m\geq 1}\bigcup_{n\geq m} E_n,$$

which implies

$$s \in \bigcup_{n\geq m} E_n, \quad \forall m \geq 1.$$

Thus $s \in E_n$ for infinitely many n. Hence we have

$$\sup_{n\geq m} I_{E_n}(s) = 1, \quad \forall m \geq 1,$$

which gives

$$\inf_{m\geq 1}\sup_{n\geq m} I_{E_n}(s) = 1.$$

If $s \notin \limsup E_n$, then $s \in E_n$ for only finitely many n. Thus we have

$$\sup_{n\geq m} I_{E_n}(s) = 0, \quad \forall m \geq M_0,$$

where M_0 is a sufficient large number. Hence we have

$$\inf_{m\geq 1}\sup_{n\geq m} I_{E_n}(s) = 0.$$

Therefore (1.1) is true. From (1.1) and noting that $I_E(s) = 1 - I_{E^c}(s)$, we have

$$I_{\liminf E_n}(s) = 1 - I_{\limsup E_n^c}(s) = 1 - \limsup I_{E_n^c}(s) = \liminf I_{E_n}(s).$$

Thus (1.2) holds.

1.6. To find $\liminf A_n$, we first need to calculate $\bigcap_{i\ge j} A_i$ for all $j \ge 1$. To do that, we let $B_n = A_{2n-1}$ and $C_n = A_{2n}$ for $n = 1, 2, \ldots$. Then B_n and C_n are decreasing, and we have

$$\bigcap_{n\ge k} B_n = [0, 1]$$

and

$$\bigcap_{n\ge k} C_n = (-1, 0]$$

for all $k \ge 1$. Thus $\bigcap_{i\ge j} A_i = \{0\}$ for all $j \ge 1$. Hence $\liminf A_n = \{0\}$. Similarly, we have $\limsup A_n = (-1, 1]$.

1.7. We first prove the "only if" part. Suppose that $\lim_{n\to\infty} x_n$ exists. Let $L = \lim_{n\to\infty} x_n$ and $\epsilon > 0$. If $L = -\infty$, then there exists an $N_\epsilon \ge 1$ such that $x_n < -\epsilon$ for all $n \ge N_\epsilon$. Hence

$$\inf_{m\ge1} \sup_{n\ge m} x_n \le \sup_{n\ge N_\epsilon} \le -\epsilon.$$

Letting $\epsilon \to \infty$ in the above equation gives $\limsup x_n = -\infty$, which implies $\limsup x_n = \liminf x_n = -\infty$. Similarly, we can show that

$$\limsup x_n = \liminf x_n$$

for $-\infty < L < \infty$ and $L = \infty$. Now we prove the "if" part. Suppose that $\limsup x_n = \liminf x_n = L$ and let $\epsilon > 0$. If $-\infty < L < \infty$, then $\sup_{n\ge m_1} x_n \le L + \epsilon$ for some $m_1 \ge 1$ and $\inf_{n\ge m_2} x_n \ge L - \epsilon$ for some $m_2 \ge 1$. Therefore, we have

$$L - \epsilon \le x_n \le L + \epsilon, \quad n \ge \max(m_1, m_2).$$

Since this is true for every $\epsilon > 0$, $\lim_{n\to\infty} x_n$ exists and is equal to L. Similarly, we can show that $\lim_{n\to\infty} x_n$ exists for $L = -\infty$ and $L = \infty$.

1.8.

(a) By Definition 1.9, we have

$$\begin{aligned}
\limsup(-x_n) &= \inf_{m\ge1} \sup_{n\ge m} (-x_n) \\
&= \inf_{m\ge1} (-\inf_{n\ge m} x_n) \\
&= -\sup_{m\ge1} \inf_{n\ge m} x_n \\
&= -\liminf x_n.
\end{aligned}$$

(b) For $j, m \geq 1$, we have

$$\sup_{i \geq j} x_i \geq x_r \geq \inf_{n \geq m} x_n,$$

where $r = \max(j, m)$. Since this is true for all $j, m \geq 1$, we have

$$\inf_{j \geq 1} \sup_{i \geq j} x_i \geq \inf_{n \geq m} x_n, \quad m \geq 1,$$

which implies

$$\inf_{j \geq 1} \sup_{i \geq j} x_i \geq \sup_{m \geq 1} \inf_{n \geq m} x_n.$$

(c) For fixed $m, j \geq 1$,

$$\inf_{n \geq m} x_n + \inf_{i \geq j} y_i \leq x_r + y_r, \quad r \geq \max(m, j),$$

we have

$$\inf_{n \geq m} x_n + \inf_{i \geq j} y_i \leq \inf_{r \geq \max(m,j)} (x_r + y_r) \leq \sup_{s \geq 1} \inf_{r \geq s} (x_r + y_r) \quad m, j \geq 1.$$

Since this is true for every $m, j \geq 1$, we have

$$\sup_{m \geq 1} \inf_{n \geq m} x_n + \sup_{j \geq 1} \inf_{i \geq j} y_i \leq \sup_{s \geq 1} \inf_{r \geq s} (x_r + y_r);$$

that is, $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$.

(d) From parts (a) and (c) of this proof, we have

$$\begin{aligned} \limsup x_n + \limsup y_n &= -\liminf(-x_n) - \liminf(-y_n) \\ &\geq -\liminf(-x_n - y_n) \\ &= \limsup(x_n + y_n). \end{aligned}$$

(e) From parts (a) and (d) of this proof, we have

$$\begin{aligned} \limsup x_n + \liminf y_n &= \limsup(x_n + y_n - y_n) + \liminf y_n \\ &\leq \limsup(x_n + y_n) + \limsup(-y_n) + \liminf y_n \\ &= \limsup(x_n + y_n). \end{aligned}$$

(f) Since $c = \liminf c$ and $-c = \liminf(-c)$, by part (c), we have

$$c + \liminf x_n = \liminf c + \liminf x_n \leq \liminf(c + x_n),$$

and

$$\liminf(c + x_n) - c = \liminf(c + x_n) + \liminf(-c) \leq \liminf x_n.$$

It follows that $c + \liminf x_n = \liminf(c + x_n)$.

(g) From parts (a) and (f) of this proof, we have

$$\liminf(c - x_n) = c + \liminf(-x_n) = c - \limsup x_n.$$

This finishes the proof.

1.9. First, we prove the "if" part. Suppose that $\limsup A_n = \liminf A_n$. Then by Problem 1.5, we have $\liminf I_{A_n}(s) = \limsup I_{A_n}(s)$ for all $s \in S$. It follows that $\lim I_{A_n}(s)$ exists for all $s \in S$. Hence $\lim_{n\to\infty} A_n$ exists.

Next, we prove the "only if" part. Suppose that $\lim_{n\to\infty} A_n$ exists. Then by definition, we have $\lim_{n\to\infty} I_{A_n}(s)$ exists for all $s \in S$. It follows that

$$\limsup I_{A_n}(s) = \liminf I_{A_n}(s)$$

for all $s \in S$. By Problem 1.5, we have $I_{\limsup A_n}(s) = I_{\liminf A_n}(s)$ for all $s \in S$. Therefore, $\limsup A_n = \liminf A_n$. By Problem 1.5, we have

$$\begin{aligned} I_A(s) &= \lim_{n\to\infty} I_{A_n}(s) = \limsup I_{A_n}(s) = \liminf I_{A_n}(s) \\ &= I_{\limsup A_n}(s) = I_{\liminf A_n}(s), \quad \forall s \in S. \end{aligned}$$

Hence $A = \limsup A_n = \liminf A_n$.

1.10. Since $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist, it follows from Problem 1.7 that

$$\lim_{n\to\infty} x_n = \liminf x_n = \limsup x_n$$

and

$$\lim_{n\to\infty} y_n = \liminf y_n = \limsup y_n.$$

Then by parts (c) and (d) of Problem 1.8, we have

$$\begin{aligned} \liminf(x_n + y_n) &\geq \liminf x_n + \liminf y_n \\ &= \limsup x_n + \limsup y_n \\ &\geq \limsup(x_n + y_n), \end{aligned}$$

which shows that

$$\limsup(x_n + y_n) = \liminf(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

This completes the proof.

1.5 Bibliographic Notes

In this chapter, we introduced some concepts in set theory as well as some set operations and relations. For further information about these concepts, readers are referred to Papoulis (1991), Williams (1991), Ash and Doleans-Dade (1999), Jacod and Protter (2004), and Reitano (2010).

We also introduced some concepts related to sequences of real numbers, which are connected to sequences of sets via indicator functions. The properties of sequences of real numbers and sets are frequently used in later chapters.

Zorn's lemma is an axiom of set theory and is equivalent to the axiom of choice. For a proof of the equivalence, readers are referred to Vaught (1995, p80), Dudley (2002, p20), and Moschovakis (2006, p114).