Chapter 1

Going Beyond Beginning Algebra

In This Chapter

- Applying order of operations and algebraic properties
- Using FOIL and other products
- Solving linear and absolute value equations
- Dealing with inequalities

The nice thing about the rules in algebra is that they apply no matter what level of mathematics or what area of math you're studying. Everyone follows the same rules, so you find a nice consistency and orderliness. In this chapter, I discuss and use the basic rules to prepare you for the topics that show up in Algebra II.

Good Citizenship: Following the Order of Operations and Other Properties

The *order of operations* in mathematics deals with what comes first (much like the chicken and the egg). When faced with multiple operations, this *order* tells you the proper course of action.

ALGEBRA RULES

The *order of operations* states that you use the following sequence when simplifying algebraic expressions:

1. **Raise to powers or find roots.**
2. **Multiply or divide.**
3. **Add or subtract.**

Special groupings can override the normal order of operations. For instance, a^{b+c} asks you to add $b + c$ before raising a to the power, which is a sum. If groupings are a part of the expression, first perform whatever's in the grouping symbol. The most common grouping symbols are parentheses, ( ); brackets, [ ]; braces, { }; fraction bars, —; absolute value bars, | |; and radical signs, $\sqrt{}$.

If you find more than one operation from the same level, move from left to right performing those operations.

ALGEBRA RULES

The commutative, associative, and distributive properties allow you to rewrite expressions and not change their value. So, what do these properties say? Great question! And here are the answers:

- **Commutative property of addition and multiplication:** $a + b = b + a$, and $a \cdot b = b \cdot a$; the order doesn't matter.

  TIP

  Rewrite subtraction problems as addition problems so you can use the commutative (and associative) property. In other words, think of $a - b$ as $a + (-b)$.
- **Associative property of addition and multiplication:** $a + (b + c) = (a + b) + c$, and $a(b \cdot c) = (a \cdot b)c$; the order is the same, but the grouping changes.
- **Distributive property of multiplication over addition (or subtraction):** $a(b + c) = a \cdot b + a \cdot c$, and $a(b - c) = a \cdot b - a \cdot c$.

ALGEBRA RULES

The *multiplication property of zero* states that if the product of $a \cdot b = 0$, then either a or b (or both) must be equal to 0.

EXAMPLE

Q. Use the order of operations and other properties to simplify the expression $\dfrac{12\left(\left(\frac{5}{6}+\frac{3}{4}\right)-\frac{1}{6}\right)}{5\sqrt{2(3)^2+7}}$.

A. $\mathbf{\frac{17}{25}}$. The big fraction bar is a grouping symbol, so you deal with the numerator and denominator separately. Use the commutative and associative properties to rearrange the fractions in the numerator; square the 3 under the radical in the denominator. Next, in the numerator, combine the fractions that have a common denominator; below the fraction bar, multiply the two numbers under the radical. Reduce the first fraction in the numerator; add the numbers under the radical. Distribute the 12 over the two fractions; take the square root in the denominator. Simplify the numerator and denominator.

Here's what the process looks like:

$$\frac{12\left(\left(\frac{5}{6}-\frac{1}{6}\right)+\frac{3}{4}\right)}{5\sqrt{2(9)+7}}=\frac{12\left(\left(\frac{4}{6}\right)+\frac{3}{4}\right)}{5\sqrt{18+7}}=\frac{12\left(\frac{2}{3}+\frac{3}{4}\right)}{5\sqrt{25}}=\frac{8+9}{5(5)}=\frac{17}{25}$$

1. Simplify: $-3[4 - 2(3^2)]$

2. Simplify: $4^3(3^2 + 11)(7^4 - 7^4)(10^{10})$

3. Simplify: 153 + 187 – 153 + 270 – 471 – 270 + 471

4. Simplify: $(6^2+4)\sqrt{5^2-3^2}$

5. Simplify: $\dfrac{\sqrt[3]{4(3^2+7)}}{\sqrt{121}-\sqrt{49}}$

6. Simplify: $7+3\left|2(8)-5^2\right|$

(For info on absolute value, see the upcoming section, "Dealing with Linear Absolute Value Equations.")

Specializing in Products and FOIL

Multiplying algebraic expressions together can be dandy and nice or downright gruesome. Taking advantage of patterns and processes makes the multiplication quicker, easier, and more accurate.

When multiplying two binomials together, you have to multiply the two terms in the first binomial times the two terms in the second binomial — you're actually *distributing* the first terms over the second. The FOIL acronym describes a way of multiplying those terms in an organized fashion, saving space and time. *FOIL* refers to multiplying the two *F*irst terms together, then the two *O*uter terms, then the two *I*nner terms, and finally the two *L*ast terms. The *O*uter and *I*nner terms usually combine. Then you add the products together by combining like terms. So, if you have $(ax + b)(cx + d)$, you can do the multiplication of the terms, or FOIL, like so:

Terms	***Product***
First	$ax(cx)$
Outer	$ax(d)$
Inner	$b(cx)$
Last	$b(d)$

$$= acx^2 + adx + bcx + bd$$

$$= acx^2 + (ad + bc)x + bd$$

The following examples show some multiplication patterns to use when multiplying binomials (expressions with two terms).

Q. Find the square of the binomial: $(2x + 3)^2$

A. $\mathbf{4x^2 + 12x + 9}$. When squaring a binomial, you square both terms and put twice the product of the two original terms between the squares: $(a + b)^2 = a^2 + 2ab + b^2$. So, $(2x + 3)^2 = (2x)^2 + 2(2x)(3) + 3^2 = 4x^2 + 12x + 9$.

Q. Multiply the two binomials together using FOIL: $(3x - 5)(4x + 7)$

A. $\mathbf{12x^2 + x - 35}$. Find the products: *F*irst, $3x(4x) = 12x^2$, plus *O*uter, $3x(7) = 21x$, plus *I*nner, $(-5)(4x) = -20x$, plus *L*ast, $(-5)(7) = -35$. So, the product of $(3x - 5)(4x + 7)$ is $(3x)(4x) + (3x)(7) + (-5)(4x) + (-5)(7) = 12x^2 + 21x - 20x - 35 = 12x^2 + x - 35$.

Q. Find the product of the binomial and the trinomial: $(2x + 7)(3x^2 + x - 5)$

A. $\mathbf{6x^3 + 23x^2 - 3x - 35}$. Distribute the $2x$ over the terms in the trinomial, and then distribute the 7 over the same terms. Combine like terms to simplify. The product of $(2x + 7)(3x^2 + x - 5)$ is $2x(3x^2 + x - 5) + 7(3x^2 + x - 5) = 6x^3 + 2x^2 - 10x + 21x^2 + 7x - 35 = 6x^3 + 23x^2 - 3x - 35$.

7. Square the binomial: $(4x - 5)^2$

8. Multiply: $(5y - 6)(5y + 6)$

9. Multiply: $(8z - 3)(2z + 5)$

10. Multiply: $(2x - 5)(4x^2 + 10x + 25)$

Variables on the Side: Solving Linear Equations

A linear equation has the general format $ax + b = c$, where x is the variable and a, b, and c are constants. When you solve a linear equation, you're looking for the value that x takes on to make the linear equation a true statement. The general game plan for solving linear equations is to isolate the term with the variable on one side of the equation and then multiply or divide to find the solution.

EXAMPLE

Q. Solve for x in the equation $\frac{3(x+7)-5}{4} = 2x+9$.

A. $\boldsymbol{x = -4}$**.** First, multiply each side by 4 to get rid of the fraction. Then distribute the 3 over the terms in the parentheses. Combine the like terms on the left. Next, you want all variable terms on one side of the equation, so subtract $8x$ and 16 from each side. Finally, divide each side by –5.

$$\frac{\cancel{4}}{1} \cdot \frac{3(x+7)-5}{\cancel{4}} = 4(2x+9)$$
$$3(x+7)-5 = 8x+36$$
$$3x+21-5 = 8x+36$$
$$3x+16 = 8x+36$$
$$-5x = 20$$
$$x = -4$$

11. Solve for x: $4x - 5 = 3(3x + 10)$

12. Solve for x: $5[9 - x(x + 8)] = 5x(1 - x)$

13. Solve for x: $\frac{x}{4} + \frac{x-6}{5} = \frac{x+2}{3}$

14. Solve for x: $(x - 1) + 2(x - 2) + 3(x - 3) + 4(x - 4) + 5(x - 5) = 0$

Dealing with Linear Absolute Value Equations

The absolute value of a number is the number's distance from 0. The formal definition of *absolute value* is

$$|a| = \begin{cases} a, a \geq 0 \\ -a, a < 0 \end{cases}$$

In other words, the absolute value of a number is exactly that number unless the number is negative; when the number is negative, its absolute value is the opposite, or a positive. The absolute value of a number, then, is the number's value without a sign; it's never negative.

When solving linear absolute value equations, you have two possibilities: one that the quantity *inside* the absolute value bars is positive, and the other that it's negative. Because you have to consider both situations, you usually get two different answers when solving absolute value equations, one from each scenario. The two answers come from setting the quantity inside the absolute value bars first equal to a positive value and then equal to a negative value.

Before setting the quantity equal to the positive and negative values, first isolate the absolute value term on one side of the equation by adding or subtracting the other terms (if you have any) from each side of the equation.

TIP

If you find more than one absolute value expression in your problem, you have to get down and dirty — consider all the possibilities. A value inside absolute value bars can be either positive or negative, so look at all the different combinations: Both values within the bars are positive, or the first is positive and the second is negative, or the first is negative and the second positive, or both are negative. Whew!

EXAMPLE

Q. Solve for x in $|3x + 7| = 11$.

A. $\boldsymbol{x = \frac{4}{3}, -6}$**, –6.** First rewrite the absolute value equation as two separate linear equations. In the first equation, assume that the $3x + 7$ is positive and set it equal to 11. In the second equation, also equal to 11, assume that the $(3x + 7)$ is negative. For that one, negate (multiply by –1) the whole binomial, and then solve the equation.

$$3x + 7 = 11$$
$$3x = 4$$
$$x = \frac{4}{3}$$

$$-(3x + 7) = 11$$
$$-3x - 7 = 11$$
$$-3x = 18$$
$$x = -6$$

15. Solve for x in $|2x + 1| = 83$.

16. Solve for x in $5|4 - 3x| + 6 = 11$.

17. Solve for x in $|x+3| + |2x-1| = 6$.

18. Solve for x in $3|4x-5| + 10 = 7$.

Greater Math Skills: Equalizing Linear Inequalities

A linear inequality resembles a linear equation — except for the relationship between the terms. The basic form for a linear inequality is $ax + b > c$. When $\geq$ or $<$ or $\leq$ are in the statement, the methods used to solve the inequality stay the same. When the extra bar appears under the inequality symbol, it means "or equal to," so you read $\leq$ as "is less than or equal to."

The main time to watch out when solving inequalities is when you multiply or divide each side of the inequality by a negative number. When you do that — and yes, you're allowed — you have to reverse the sense or the relationship. The inequality $>$ becomes $<$, and vice versa.

ALGEBRA RULES

When solving *absolute value inequalities* (see the preceding section for more on absolute values), you first drop the absolute value bars. Then you apply one of two separate rules for absolute value inequalities, depending on which way the inequality symbol faces:

- Solve $|ax+b| > c$ using the two inequalities $ax + b > c$ and $ax + b < -c$.
- Solve $|ax+b| < c$ using the single compound inequality $-c < ax + b < c$.

Two ways of writing your answers are inequality notation and interval notation:

REMEMBER

- **Inequality notation:** This notation is just what it says it is: If your answer is all x's greater than or equal to 3, you write $x \geq 3$. To say that the answer is all x values between –5 and +5, including the –5 but not the positive 5, you write $-5 \leq x < 5$.
- **Interval notation:** Some mathematicians prefer interval notation because it's so short and sweet. You simply list the starting and stopping points of the numbers you want to use. When you see this notation, you just have to recognize that you're discussing intervals of numbers (and not, for instance, the coordinates of a point). The rule is that you use a bracket, [or], when you want to include the number, and use a parenthesis, (or), when you don't want to include the number. You always use a parenthesis with ∞ or $-\infty$. For example, to write $x \geq 3$ in interval notation, you use $[3, \infty)$. Writing $-5 \leq x < 5$ in interval notation, you have $[-5, 5)$.

Q. Solve the inequality $8x - 15 \geq 10x + 7$. Write the answer in both inequality and interval notation.

A. $\mathbf{x \leq -11}$ **and** $\mathbf{(-\infty, -11]}$**.** First subtract $10x$ from each side; then add 15 to each side. This step moves the variable terms to the left and the constants to the right: $-2x \geq 22$. Now divide each side by -2. Because you're dividing by a negative number, you need to reverse the inequality sign: $x \leq -11$. That's the answer in inequality notation. The solution is that x can be any number either equal to or smaller than -11. In interval notation, you write this as $(-\infty, -11]$.

Q. Solve the inequality $|5 - 6x| < 7$. Write the answer in both inequality and interval notation.

A. $\mathbf{-\frac{1}{3} < x < 2}$ **and** $\mathbf{\left(-\frac{1}{3}, 2\right)}$**.** Rewrite the absolute value inequality as the inequality $-7 < 5 - 6x < 7$. Subtract 5 from each of the three sections of the inequality to put the variable term by itself: $-12 < -6x < 2$. Now divide each section by -6, reversing the inequality symbols. Then, after you've gone to all the trouble of reversing the inequalities, rewrite the statement again with the smaller number on the left to correspond to numbers on the number line. This step requires reversing the inequalities again. Here are the details:

$$-12 < -6x < 2$$
$$2 > x > -\frac{1}{3}$$
$$-\frac{1}{3} < x < 2$$

In interval notation, the answer is $\left(-\frac{1}{3}, 2\right)$.

WARNING! This answer looks very much like the coordinates of a point. In instances like this, be very clear about what you're trying to convey with the interval notation.

19. Solve the inequality $5x + 7 \leq 22$.

20. Solve the inequality $8(y - 4) \geq 5(3y - 12)$.

21. Solve the inequality $|3x + 7| > 4$.

22. Solve the inequality $4|5 - 2x| + 3 < 7$.

Answers to Problems on Going Beyond Beginning Algebra

This section provides the answers (in bold) to the practice problems in this chapter.

1 Simplify: $-3[4 - 2(3^2)]$. The answer is **42.**

$-3[4 - 2(9)] = -3[4 - 18] = -3[-14] = 42$

2 Simplify: $4^3(3^2 + 11)(7^4 - 7^4)(10^{10})$. The answer is **0.**

The third factor is 0. This makes the whole product equal to 0. Remember, the *multiplication property of zero* says that if any factor in a product is equal to 0, then the entire product is equal to 0.

3 Simplify: $153 + 187 - 153 + 270 - 471 - 270 + 471$. The answer is **187.**

Use the associative and commutative properties to write the numbers and their opposites together:

$187 + 153 - 153 + 270 - 270 - 471 + 471 = 187 + 0 + 0 + 0 = 187$

4 Simplify: $(6^2 + 4)\sqrt{5^2 - 3^2}$. The answer is **160.**

$$(36+4)\sqrt{25-9} = (40)\sqrt{16} = (40)(4) = 160$$

5 Simplify: $\dfrac{\sqrt[3]{4(3^2+7)}}{\sqrt{121}-\sqrt{49}}$. The answer is **1.**

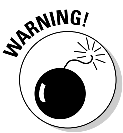
$$\frac{\sqrt[3]{4(9+7)}}{\sqrt{121}-\sqrt{49}} = \frac{\sqrt[3]{4(16)}}{11-7} = \frac{\sqrt[3]{64}}{4} = \frac{4}{4} = 1$$

6 Simplify: $7 + 3|2(8) - 5^2|$. The answer is **34.**

$7 + 3|16 - 25| = 7 + 3|-9| = 7 + 3(9) = 7 + 27 = 34$

7 Square the binomial: $(4x - 5)^2$. The answer is $\mathbf{16x^2 - 40x + 25}$**.**

When squaring a binomial, the two terms are each squared, and the term between them is twice the product of the original terms.



A common error in squaring binomials is to forget the middle term and just use the squares of the two terms in the binomial. If you tend to forget the middle term, you can avoid the error and get the correct answer through FOIL — $(4x - 5)(4x - 5) = 16x^2 - 20x - 20x + 25 = 16x^2 - 40x + 25$.

8 Multiply: $(5y - 6)(5y + 6)$. The answer is $\mathbf{25y^2 - 36}$**.**

The product of two binomials that contain the sum and difference of the same two terms results in a binomial that's the difference between the squares of the terms.

9 Multiply: $(8z - 3)(2z + 5)$. The answer is $\mathbf{16z^2 + 34z - 15}$**.**

Using FOIL, the first term in the answer is the product of the first two terms: $(8z)(2z)$. The middle term is the sum of the products of the *O*uter and *I*nner terms: $(8z)(5)$ and $(-3)(2z)$. The final term is the product of the two last terms: $(-3)(5)$.

10 Multiply: $(2x - 5)(4x^2 + 10x + 25)$. The answer is $\mathbf{8x^3 - 125}$.

Distribute the first term in the binomial ($2x$) over the terms in the trinomial, and then distribute the second term in the binomial (-5) over the terms in the trinomial. After that, combine like terms:

$$= 2x(4x^2 + 10x + 25) - 5(4x^2 + 10x + 25)$$
$$= 8x^3 + 20x^2 + 50x - 20x^2 - 50x - 125$$
$$= 8x^3 + 20x^2 - 20x^2 + 50x - 50x - 125$$
$$= 8x^3 - 125$$

11 Solve for x: $4x - 5 = 3(3x + 10)$. The answer is $\mathbf{x = -7}$.

Distribute the terms on the right to get $4x - 5 = 9x + 30$. Subtract $9x$ and add 5 to each side, which gives you $-5x = 35$. Then divide each side by -5: $x = -7$.

12 Solve for x: $5[9 - x(x + 8)] = 5x(1 - x)$. The answer is $\mathbf{x = 1}$.

First distribute the x in the bracket on the left and the $5x$ on the right. Then you can distribute the 5 outside the bracket on the left to see what the individual terms are. The squared terms disappear when you add $5x^2$ to each side of the equation. Solve for x:

$$\begin{aligned} 5\left[9 - x^2 - 8x\right] &= 5x - 5x^2 \\ 45 - 5x^2 - 40x &= 5x - 5x^2 \\ 45 - 40x &= 5x \\ 45 &= 45x \\ 1 &= x \end{aligned}$$

13 Solve for x: $\frac{x}{4} + \frac{x-6}{5} = \frac{x+2}{3}$. The answer is $\mathbf{x = 16}$.

First, multiply each fraction by 60, the least common multiple. Then distribute and simplify:

$$\begin{aligned} {}^{15}\cancel{60}\left(\frac{x}{\cancel{4}_1}\right) + {}^{12}\cancel{60}\left(\frac{x-6}{\cancel{5}_1}\right) &= {}^{20}\cancel{60}\left(\frac{x+2}{\cancel{3}_1}\right) \\ 15x + 12(x-6) &= 20(x+2) \\ 15x + 12x - 72 &= 20x + 40 \\ 7x &= 112 \\ x &= 16 \end{aligned}$$

14 Solve for x: $(x - 1) + 2(x - 2) + 3(x - 3) + 4(x - 4) + 5(x - 5) = 0$. The answer is $\mathbf{x = \frac{11}{3}}$.

Distribute over the terms. Combine like terms and solve for x.

$$\begin{aligned} x - 1 + 2x - 4 + 3x - 9 + 4x - 16 + 5x - 25 &= 0 \\ 15x - 55 &= 0 \\ 15x &= 55 \\ x &= \frac{11}{3} \end{aligned}$$

15 Solve for x in $|2x + 1| = 83$. The answer is $\mathbf{x = 41, -42}$.

First, let the value in the absolute value be positive, and solve $2x + 1 = 83$. You get $2x = 82$, or $x = 41$. Next, let the value in the absolute value be negative, and solve $-(2x + 1) = 83$. (If you multiply each side by -1, you don't have to distribute the negative sign over two terms. Instead, you have $2x + 1 = -83$.) Solving this, you get $2x = -84$, or $x = -42$.

16 Solve for x in $5|4-3x|+6=11$. The answer is $\boldsymbol{x = 1, \frac{5}{3}}$.

First subtract 6 from each side, and then divide by 5. (You can't apply the rule for changing an absolute value equation into linear equations unless the absolute value is isolated on one side of the equation.) Now you have $|4-3x|=1$. Letting the expression inside the absolute value be positive, you have $4-3x=1$, which means $-3x=-3$ and $x=1$. Now, if the expression inside the absolute value is negative, you make the expression positive by negating the whole thing: $-(4-3x)=1$ gives you $4-3x=-1$, which means $-3x=-5$ and $x=\frac{5}{3}$.

17 Solve for x in $|x+3|+|2x-1|=6$. The answer is $\boldsymbol{x = \frac{4}{3}, -2}$.

You need four different equations to solve this problem. Consider that both absolute values may be positive; then that the first is positive and the second, negative; then that the first is negative and the second, positive; and last, that both are negative.

REMEMBER

Unfortunately, not every equation gives you an answer that really works. Perhaps no value of x can make the first absolute value negative and the second positive. An extraneous solution to an equation or inequality is a false or incorrect solution. It occurs when you change the original format of the equation to a form that's more easily solved. The extraneous solution may be a solution to the new form, but it doesn't work in the original. After you change the format, simply solve each equation produced, and then check each answer in the original equation.

Here are the situations, one at a time:

- **Positive/Positive:** $(x+3)+(2x-1)=6$; $x+3+2x-1=6$; $3x+2=6$; $3x=4$; $x=\frac{4}{3}$. When you put this answer back into the original equation, it works — you get a true statement.
- **Positive/Negative:** $(x+3)+-(2x-1)=6$; $x+3-2x+1=6$; $-x+4=6$; $-x=2$; $x=-2$. Putting this value back into the original equation, you get a true statement, so this answer is valid, too.
- **Negative/Positive:** $-(x+3)+(2x-1)=6$; $-x-3+2x-1=6$; $x-4=6$; $x=10$. When you put this back into the original equation, you get $13+19=6$. That isn't true, so this solution is extraneous. It doesn't work.
- **Negative/Negative:** $-(x+3)+-(2x-1)=6$; $-x-3-2x+1=6$; $-3x-2=6$; $-3x=8$; $x=-\frac{8}{3}$.

  You get another extraneous solution. When you put this value for x back into the original equation, you get

  $$\left|-\frac{8}{3}+3\right|+\left|2\left(-\frac{8}{3}\right)-1\right|\stackrel{?}{=}6$$
  $$\left|\frac{1}{3}\right|+\left|-\frac{19}{3}\right|\stackrel{?}{=}6$$
  $$\frac{20}{3}\neq 6$$

  Not so! This solution doesn't work.

18 Solve for x in $3|4x-5|+10=7$. **No solution.**

This absolute value equation has no solution, because it asks you to find some number whose absolute value is negative. When you subtract 10 from each side and divide each side by 3, you get $|4x-5|=-1$. That's impossible. The absolute value of any number is either positive or 0; it's never negative.

19 Solve the inequality $5x + 7 \leq 22$. The answer is $\boldsymbol{x \leq 3}$.

First, subtract 7 from each side to get $5x \leq 15$. Then divide each side by 5. In interval notation, write the solution as $(-\infty, 3]$.

20 Solve the inequality $8(y - 4) \geq 5(3y - 12)$. The answer is $\boldsymbol{y \leq 4}$.

Distribute the 8 and 5 over their respective binomials to get $8y - 32 \geq 15y - 60$. Subtract $15y$ from each side and add 32 to each side: $-7y \geq -28$. When you divide each side by -7, you reverse the inequality to get $y \leq 4$. Write this answer as $(-\infty, 4]$ in interval notation.

21 Solve the inequality $|3x + 7| > 4$. The answer is $\boldsymbol{x > -1 \text{ or } x < -\frac{11}{3}}$.

Rewrite the absolute value inequality as two separate linear inequalities: $3x + 7 > 4$ and $3x + 7 < -4$. Solving the first, you get $3x > -3$, or $x > -1$. Solving the second, you get $3x < -11$, or $x < -\frac{11}{3}$. In interval notation, you write this solution as $(-1, \infty)$ or $\left(-\infty, -\frac{11}{3}\right)$.

22 Solve the inequality $4|5 - 2x| + 3 < 7$. The answer is $\boldsymbol{2 < x < 3}$.

First, subtract 3 from each side and then divide each side by 4 to get $|5 - 2x| < 1$. Then rewrite the absolute value inequality as $-1 < 5 - 2x < 1$. Subtract 5 from each interval: $-6 < -2x < -4$. Dividing each interval by -2 in the next step means reversing the inequality signs: $3 > x > 2$. The numbers should be written from smallest to largest, so switch the numbers and their inequality signs to get $2 < x < 3$. In interval notation, you write this answer as $(2, 3)$.