CHAPTER 1

THE KINEMATICS AND DYNAMICS OF AIRCRAFT MOTION

1.1 INTRODUCTION

In this chapter the end point will be the equations of motion of a rigid vehicle moving over the oblate, rotating Earth. The flat-Earth equations, describing motion over a small area of a nonrotating Earth, with constant gravity, are sufficient for many aircraft simulation needs and will be derived first. To reach this end point we will use the vector analysis of classical mechanics to set up the equations of motion, matrix algebra to describe operations with coordinate systems, and concepts from geodesy (a branch of mathematics dealing with the shape of the Earth), gravitation (the mass attraction effect of the Earth), and navigation, to introduce the effects of Earth's shape and mass attraction.

The moments and forces acting on the vehicle, other than the mass attraction of the Earth, will be abstract until Chapter 2 is reached. At this stage the equations can be used to describe the motion of any type of aerospace vehicle, including an Earth satellite, provided that suitable force and moment models are available. The term *rigid* means that structural flexibility is not allowed for, and all points in the vehicle are assumed to maintain the same relative position at all times. This assumption is good enough for flight simulation in most cases as well as for flight control system design provided that we are not trying to design a system to control structural modes or to alleviate aerodynamic loads on the aircraft structure.

The vector analysis needed for the treatment of the equations of motion often causes difficulties for the student, particularly the concept of the angular velocity vector. Therefore, a review of the relevant topics is provided. In some cases we have gone beyond the traditional approach to flight mechanics. The introduction of topics from geodesy, gravitation, and distance and position calculations allows us to accurately simulate the trajectories of aircraft that can fly autonomously at very high altitudes and over long distances, including "point-to-point suborbital flight" (e.g., White Knight 2 and SpaceShipTwo). Some topics have been reserved for an "optional" advanced section (e.g., quaternions), Section 1.8.

The equations of motion will be organized as a set of simultaneous first-order differential equations, explicitly solved for the derivatives. For n independent variables, X_i (such as components of position, velocity, etc.), and m control inputs, U_i (such as throttle, control surface deflection, etc.), the general form will be

$$\begin{aligned} \dot{X}_1 &= f_1(X_1, X_2, \ldots, X_n, U_1, U_2, \ldots, U_m) \\ \dot{X}_2 &= f_2(X_1, X_2, \ldots, X_n, U_1, U_2, \ldots, U_m) \\ &\vdots \\ \dot{X}_n &= f_n(X_1, X_2, \ldots, X_n, U_1, U_2, \ldots, U_m), \end{aligned} \tag{1.1-1}$$

where the functions f_i are the nonlinear functions that can arise from modeling real systems. If the variables X_i constitute the smallest set of variables that, together with given inputs U_i, completely describe the behavior of the system, then the X_i are a set of *state variables* for the system, and Equations (1.1-1) are a *state-space* description of the system. The functions f_i are required to be single-valued continuous functions. Equations (1.1-1) are often written symbolically as

$$\dot{X} = f(X, U), \tag{1.1-2}$$

where the *state vector* X is an $(n \times 1)$ column array of the n state variables, the *control vector* U is an $(m \times 1)$ column array of the control variables, and f is an array of nonlinear functions. When U is held constant, the nonlinear state equations (1.1-1), or a subset of them, usually have one or more *equilibrium points* in the multidimensional state and control space, where the derivatives vanish. The equations are usually approximately linear for small perturbations from equilibrium and can be written in matrix form as the *linear state equation*:

$$\dot{x} = Ax + Bu \tag{1.1-3}$$

Here, the lowercase notation for the state and control vectors indicates that they are perturbations from equilibrium, although the derivative vector contains the actual values (i.e., perturbations from zero). The "*A-matrix*" is square and the "*B-matrix*" has dimensions determined by the number of states and controls.

The state-space formulation will be described in more detail in Chapters 2 and 3. At this point we will simply note that a major advantage of this formulation is that the nonlinear state equations can be solved numerically. The simplest numerical solution method is *Euler integration*, described by

$$X_{k+1} = X_k + f(X_k, U_k)\, \delta t, \tag{1.1-4}$$

in which X_k is the kth value of the state vector computed at discrete times $k\,\delta t$, $k = 0, 1, 2, \ldots$, starting from an *initial condition* X_0. The *integration time step*, δt, must be made small enough that, for every δt interval, U can be approximated by a constant value, and $\dot{X}\,\delta t$ provides a good approximation to the increment in the state vector. This numerical integration allows the state vector to be stepped forward, in time increments of δt, to obtain a *time-history simulation*.

1.2 VECTOR OPERATIONS

Definitions and Notation

Kinematics can be defined as the study of the motion of objects without regard to the mechanisms that cause the motion. The motion of physical objects can be described by means of vectors in three dimensions, and in performing kinematic analysis with vectors we will make use of the following definitions:

Frame of Reference: A rigid body or set of rigidly related points that can be used to establish distances and directions (denoted by F_i, F_e, etc.). In general, a subscript used to indicate a frame will be lowercase, while a subscript used to indicate a point will be uppercase.

Inertial Frame: A frame of reference in which Newton's laws apply. Our best inertial approximation is probably a "helio-astronomic" frame in which the center of mass (cm) of the sun is a fixed point, and fixed directions are established by the normal to the plane of the ecliptic and the projection on that plane of certain stars that appear to be fixed in position.

Vector: A vector is an abstract geometrical object that has both magnitude and direction. It exists independently of any coordinate system. The vectors used here are Euclidean vectors that exist only in three-dimensional space and come in two main types:

Bound Vector: A vector from a fixed point in a frame (e.g., a position vector).

Free Vector: Can be translated parallel to itself (e.g., velocity, torque).

Coordinate System: A measurement system for locating points in a frame of reference. We may have multiple coordinate systems (with no relative motion) within one frame of reference, and we sometimes loosely refer to them also as "frames."

In choosing a notation the following facts must be taken into account. For position vectors, the notation should specify the two points whose relative position the vector describes. Velocity and acceleration vectors are relative to a frame of reference, and the notation should specify the frame of reference as well as the moving point. The derivative of a vector depends on the observer's frame of reference, and this frame must be specified in the notation. A derivative may be taken in a different frame from that in which a vector is defined, so the notation may require two frame designators with one vector. We will use the following notation:

Vectors will be in boldface type fonts.

Right subscripts will be used to designate two points for a position vector, and a point and a frame for a velocity or acceleration vector. A "/" in a subscript will mean "with respect to."

A *left superscript* will specify the frame in which a derivative is taken, and the dot notation will indicate a derivative.

A *right superscript* on a vector will specify a coordinate system. It will therefore denote an array of the components of that vector in the specified system.

Vector length will be denoted by single bars, for example, $|\mathbf{p}|$.

Examples of the notation are:

$$\mathbf{p}_{A/B} \equiv \text{Position vector of point } A \text{ with respect to point } B$$

$$\mathbf{v}_{A/i} \equiv \text{Velocity vector of point } A \text{ in frame } F_i$$

$$^{b}\dot{\mathbf{v}}_{A/i} \equiv \text{Vector derivative of } \mathbf{v}_{A/i} \text{ taken in frame } F_b$$

$$\mathbf{v}^{c}_{A/i} \equiv (\mathbf{v}_{A/i})^{c} \equiv \text{Array of components of } \mathbf{v}_{A/i} \text{ in coordinate system } c$$

$$^{b}\dot{\mathbf{v}}^{c}_{A/i} \equiv \text{Components in system } c \text{ of the derivative taken in } F_b$$

The individual components of a vector will have subscripts that indicate the coordinate system or be denoted by the vector symbol with subscripts x, y, and z to indicate the coordinates. All component arrays will be column arrays unless otherwise indicated by the transpose symbol, a right superscript T. For example, arrays of components in a coordinate system b could be shown as

$$\mathbf{p}^{b}_{A/B} = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \quad \text{or} \quad \mathbf{v}^{b}_{A/i} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = [v_x \; v_y \; v_z]^{T}$$

Vector Properties

Vectors are independent of any Cartesian coordinate system. Addition and subtraction of vectors can be defined independently of coordinate systems by means of geometrical constructions (the "parallelogram law"). Thus, we can draw vectors on charts to determine the *track* of a vehicle through the air or on or under the sea. Some vector operations yield *pseudovectors* that are not independent of a "handedness" convention. For example, the result of the *vector cross-product* operation is a vector whose direction depends on whether a right-handed or left-handed convention is being used. We will always use the right-hand rule in connection with vector direction.

It is usually most convenient to manipulate vectors algebraically by decomposing them into a sum of appropriately scaled unit-length vectors usually written as

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ (i.e., $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$). These *unit vectors* are normally chosen to form a right-handed orthogonal set, that is, the right-hand rule applied to **i** and **j** gives the direction of **k** (i.e., $\mathbf{i} \times \mathbf{j} = \mathbf{k}$). The use of orthogonal unit vectors leads naturally to using Cartesian coordinate systems for their scaling factors and thence to manipulating the coordinates with matrix algebra (next section).

The direction of a vector **p** relative to a coordinate system is commonly described in two different ways: first by rotations in two orthogonal planes, for example, an *azimuth rotation* to point in the right direction and then an *elevation rotation* above the azimuth plane (used with El-over-Az mechanical gimbals), and second by three *direction angles* α, β, γ to the coordinate axes (used with some radar antennas). The *direction cosines* of **p**—$\cos\ \alpha, \cos\ \beta, \cos\ \gamma$—give the projections of **p** on the coordinate axes, and two applications of the theorem of Pythagoras yield

$$|\mathbf{p}|^2\cos^2\alpha + |\mathbf{p}|^2\cos^2\beta + |\mathbf{p}|^2\cos^2\gamma = |\mathbf{p}|^2$$

$$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1.2-1}$$

The *dot product* of two vectors, say **u** and **v**, is a scalar defined by

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta, \tag{1.2-2}$$

where θ is the included angle between the vectors (it may be necessary to translate the vectors so that they intersect). The dot product is commutative and distributive; thus,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$

The principal uses of the dot product are to find the projection of a vector, to establish orthogonality, and to find length. For example, if (1.2-2) is divided by $|\mathbf{v}|$, we have the projection of **u** on **v**,

$$(\mathbf{u} \cdot \mathbf{v}) / |\mathbf{v}| = |\mathbf{u}| \cos\theta$$

If $\cos\theta = 0$, $\mathbf{u} \cdot \mathbf{v} = 0$, and the vectors are said to be *orthogonal*. If a vector is dotted with itself, then $\cos\theta = 1$, and we obtain the square of its length. Orthogonal unit vectors satisfy the dot product relationships

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

Using these relationships, the dot product of two vectors can be evaluated in terms of components in any convenient orthogonal coordinate system (say a, with components x, y, z),

$$(\mathbf{u} \cdot \mathbf{v})^a = u_x v_x + u_y v_y + u_z v_z \tag{1.2-3}$$

The *cross-product* of **u** and **v**, denoted by $\mathbf{u} \times \mathbf{v}$, is a vector **w** that is normal to the plane of **u** and **v** and is in a direction such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ (in that order) form a right-handed system (again, it may be necessary to translate the vectors so that they intersect). The length of $\mathbf{w}$ is defined to be $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where θ is the included angle between $\mathbf{u}$ and $\mathbf{v}$. It has the following properties:

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{(anticommutative)}$$

$$a(\mathbf{u} \times \mathbf{v}) = (a\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (a\mathbf{v}) \quad (\textit{associative; ``a'' scalar})$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \quad \text{(distributive)}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \quad \text{(scalar triple product)}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{w} \cdot \mathbf{u}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \quad \text{(vector triple product)} \qquad (1.2\text{-}4)$$

As an aid for remembering the form of the triple products, note the *cyclic permutation* of the vectors involved. Alternatively, the vector triple product can be remembered phonetically using "ABC = BAC-CAB."

The cross-products of the unit vectors describing a right-handed orthogonal coordinate system satisfy the equations

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

and, using cyclic permutation,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Also remember that $\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}$, and so on.

An example of the use of the cross-product is finding the vector moment $\mathbf{r} \times \mathbf{F}$ of a force $\mathbf{F}$ acting at a point whose position vector is $\mathbf{r}$.

Rotation of a Vector

It is intuitively obvious that a vector can be made to point in an arbitrary direction by means of a single rotation around an appropriate axis. Here we follow Goldstein (1980) to derive a formula for vector rotation.

Consider Figure 1.2-1, in which a vector $\mathbf{u}$ has been rotated to form a new vector $\mathbf{v}$ by defining a rotation axis along a unit vector $\mathbf{n}$ and performing a left-handed rotation through μ around $\mathbf{n}$. The two vectors that must be added to $\mathbf{u}$ to obtain $\mathbf{v}$ are shown in the figure and provide a good student exercise in using the vector cross-product (Problem 1.2-4). By doing this addition, we get

$$\mathbf{v} = \mathbf{u} + (1 - \cos\mu)(\mathbf{n} \times (\mathbf{n} \times \mathbf{u})) - (\mathbf{n} \times \mathbf{u}) \sin\mu \qquad (1.2\text{-}5a)$$

or

$$\mathbf{v} = (1 - \cos\mu)\,\mathbf{n}(\mathbf{n} \cdot \mathbf{u}) + \mathbf{u}\cos\mu - (\mathbf{n} \times \mathbf{u}) \sin\mu \qquad (1.2\text{-}5b)$$

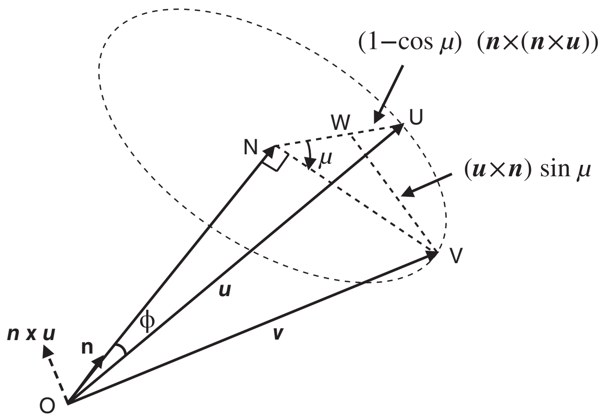

Figure 1.2-1 Rotation of a vector.

Equations (1.2-5) are sometimes called the *rotation formula*; they show that, after choosing **n** and μ, we can operate on **u** with dot and cross-product operations to get the desired rotation; no coordinate system is involved, and the rotation angle can be arbitrarily large.

1.3 MATRIX OPERATIONS ON VECTOR COORDINATES

As noted earlier, the coordinate system components of a vector will be written as a (3 × 1) column array. Here, we shall show how those components are manipulated in correspondence with operations performed with vectors.

The Scalar Product

If $\mathbf{u}^a$ and $\mathbf{v}^a$ are column arrays of the same dimension, their scalar product is $(\mathbf{u}^a)^T\mathbf{v}^a$, and, for example, in three dimensions,

$$(\mathbf{u}^a)^T\mathbf{v}^a = [u_x\ u_y\ \ u_z]\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = u_x v_x + u_y v_y + u_z v_z \tag{1.3-1a}$$

This result is identical to Equation (1.2-3) obtained from the vector dot product. The scalar product allows us to find the *2-norm* of a column matrix:

$$|\mathbf{v}^a| = \left[(\mathbf{v}^a)^T\mathbf{v}^a\right]^{\frac{1}{2}} \tag{1.3-1b}$$

In Euclidean space this is the length of the vector.

The Cross-Product Matrix

From the unit-vector cross-products, given earlier, we can derive a formula for the components of the cross-product of two vectors by writing them in terms of a sum of unit vectors. A convenient mnemonic for remembering the formula is to write it so that it resembles the expansion of a determinant, as follows:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \tag{1.3-2}$$

where subscripts $x, y, z,$ indicate components in a coordinate system whose axes are aligned respectively with the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. We often wish to directly translate a vector equation into a matrix equation of vector components. From the above mnemonic it is easy to see that

$$(\mathbf{u} \times \mathbf{v})^a = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \equiv \widetilde{\mathbf{u}}^a \, \mathbf{v}^a \tag{1.3-3}$$

A skew-symmetric matrix of the above form will be denoted by the tilde overbar and referred to as the *tilde matrix* or *cross-product matrix*. An example of the use of the cross-product matrix involves the centripetal acceleration at a point described by a position vector $\mathbf{r}$ rotating with an angular velocity vector ω (see also Equation 1):

$$\text{centripetal acceleration} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

In the case of a vector triple product, the vector operation in parentheses must be performed first, but the corresponding matrix operations may be performed collectively in any order:

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))^a = (\widetilde{\mathbf{u}}^a \, \widetilde{\mathbf{v}}^a)\mathbf{w}^a = \widetilde{\mathbf{u}}^a(\widetilde{\mathbf{v}}^a \, \mathbf{w}^a)$$

Here, the third term requires only postmultiplication by a column array and hence fewer operations to evaluate than the second term.

Coordinate Rotation, the DCM

When the rotation formula (1.2-5*b*) is resolved in a coordinate system *a,* the result is

$$\mathbf{v}^a = \left[(1 - \cos\mu)\, \mathbf{n}^a(\mathbf{n}^a)^T + (\cos\mu)\mathbf{I} - (\sin\mu)\widetilde{\mathbf{n}}^a\right] \mathbf{u}^a, \tag{1.3-4}$$

where $\mathbf{n}^a(\mathbf{n}^a)^T$ is a square matrix, $\mathbf{I}$ is the identity matrix, and $\widetilde{\mathbf{n}}^a$ is a cross-product matrix. This formula was developed as an "active" vector operation in that a vector was being rotated to a new position by means of a left-handed rotation about the specified unit vector. In component form, the new array can be interpreted as the components of a new vector in the same coordinate system, or as the components of the original vector in a new coordinate system, obtained by a right-handed coordinate

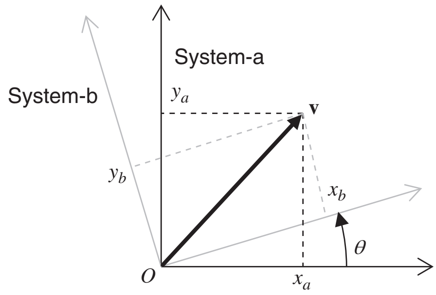

Figure 1.3-1 A plane rotation of coordinates.

rotation around the specified axis. This can be visualized in Figure 1.3-1, which shows the new components of a vector **v** after a right-handed coordinate system rotation, θ, around the z-axis. Instead, if the vector is given a left-handed rotation of the same amount, then (x_b, y_b) will become the components of the vector in the original system. Taking the coordinate system rotation viewpoint and combining the matrices in (1.3-4) into a single coefficient matrix, this linear transformation can be written as

$$\mathbf{u}^b = C_{b/a}\mathbf{u}^a \tag{1.3-5}$$

Here $C_{b/a}$ is a matrix that transforms the coordinates of the vector **u** from system a to system b and is called a *direction cosine matrix* (DCM), or simply a *rotation matrix*.

In Figure 1.3-1 a new coordinate system is formed by a *right-handed rotation around the z-axis* of the original orthogonal coordinate system; the DCM can easily be found by applying Equation (1.3-4) using

$$\mathbf{n}^a = \mathbf{n}^b = [0 \;\; 0 \;\; 1\,]^T, \qquad \tilde{\mathbf{n}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The DCM and the *components of* **u** *in system b* are then found to be

$$\mathbf{u}^b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \tag{1.3-6}$$

The direction cosine matrix is so called because its elements are direction cosines between corresponding axes of the new and old coordinate systems. Let **i**, **j**, **k**, with appropriate subscripts, be unit vectors defining the axes of our orthogonal coordinate systems a and b. The x_b-component of an arbitrary vector **r** can be written as

$$x_b = (\mathbf{r} \cdot \mathbf{i}_b)^b = (\mathbf{r} \cdot \mathbf{i}_b)^a = x_a(\mathbf{i}_a \cdot \mathbf{i}_b) + y_a(\mathbf{j}_a \cdot \mathbf{i}_b) + z_a(\mathbf{k}_a \cdot \mathbf{i}_b)$$

This equation defines the first row of the DCM; the other b-system components can be found in the same way and consist of dot products of unit vectors, which are equivalent to direction cosines.

The above two methods of constructing a DCM are not very convenient for a general three-dimensional rotation; Euler Rotations (following) provide a more convenient way.

Direction Cosine Matrix Properties

We will look briefly at some of the properties of the rotation matrix and then at how it may be determined in applications. A coordinate rotation must leave the length of a vector unchanged. The change of length under the rotation above is

$$|\mathbf{u}|^2 = (\mathbf{u}^b)^T \mathbf{u}^b = (C_{b/a}\mathbf{u}^a)^T C_{b/a}\mathbf{u}^a = (\mathbf{u}^a)^T C_{b/a}^T C_{b/a}\mathbf{u}^a$$

and the length is preserved if

$$C_{b/a}^T C_{b/a} = I = C_{b/a} C_{b/a}^T \tag{1.3-7}$$

This is the definition of an *orthogonal matrix*, and it makes the inverse matrix particularly easy to determine ($C^{-1} = C^T$). It also implies that the columns (and also the rows) of the rotation matrix form an orthonormal set:

$$C_{b/a} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \rightarrow c_i^T c_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Also, since

$$c_1 \equiv C[1 \;\; 0 \;\; 0]^T$$

columns of the rotation matrix give us the components in the new system of a unit vector in the old system.

If a vector is expressed in a new coordinate system by a sequence of rotations as

$$\mathbf{u}^d = C_{d/c}\, C_{c/b}\, C_{b/a}\, \mathbf{u}^a \tag{1.3-8}$$

then the inverse operation is given by

$$\begin{aligned} \mathbf{u}^a &= (C_{d/c}C_{c/b}C_{b/a})^{-1}\mathbf{u}^d = C_{b/a}^{-1}C_{c/b}^{-1}C_{d/c}^{-1}\mathbf{u}^d = C_{b/a}^T C_{c/b}^T C_{d/c}^T \mathbf{u}^d \\ &= (C_{d/c}C_{c/b}C_{b/a})^T \mathbf{u}^d = C_{d/a}^T \mathbf{u}^d \end{aligned} \tag{1.3-9}$$

Summary of DCM (Rotation Matrix) Properties

(a) Successive rotations are described by the product of the individual DCMs; cf. (1.3-8).

(b) Rotation matrices are not commutative, for example, $C_{c/b}C_{b/a} \neq C_{b/a}C_{c/b}$.

(c) Rotation matrices are orthogonal matrices.

(d) The determinant of a DCM is unity.

(e) A nontrivial DCM has one, and only one, eigenvalue equal to unity [see Euler's Rotation Theorem].

Euler Rotations

Here we will determine the rotation matrix in a way that is better suited to visualizing vehicle orientation.

The orientation of one Cartesian coordinate system with respect to another can always be described by three successive rotations around the orthogonal coordinate axes, and the angles of rotation are called the *Euler angles* (or Eulerian angles). The axes and the order of the rotations are chosen in various ways in different fields of science. When we rule out two successive Euler rotations about the same axis, there are twelve possibilities, six without repetition of an axis (counting both forward and reverse) and six with repetition.

In the aerospace field Euler rotations are performed, in an x, y, z or z, y, x order. Each rotation has a form similar to Equation (1.3-6); the zeros and the "1" are placed so that the appropriate coordinate is unchanged (the z-coordinate in (1.3-6)). The remaining terms are placed with cosines on the main diagonal and sines in the remaining off-diagonal positions, so that the matrix reverts to the identity matrix when the rotation angle is zero. The negative sine term is placed on the row above the "1" term when a positive angle corresponds to a right-handed rotation around the current axis. Henceforth the plane rotation matrix will be written immediately by inspection, and three-dimensional coordinate rotations will be built up as a sequence of plane rotations. The fact that the individual rotations are not commutative can be checked by performing sequences of rotations with any convenient solid object. Therefore, although the order of the sequence can be defined arbitrarily, the same order must be maintained ever after.

The sequence of three Euler rotations leading to a given DCM is not unique, and for a particular DCM we could, in general, find a different set of Euler rotations that would lead to the same final attitude. The Euler angles would then differ from the prescribed angles, and they may be impossible to perform because of physical constraints, for example, aircraft aerodynamic constraints, or mechanical gimbal constraints (think of a simple elevation-over-azimuth sensor-pointing system, where there is a mechanical constraint of zero roll angle). Knowing the Euler rotation convention that was used with the DCM allows the correct Euler angles to be extracted from the DCM, as shown earlier.

Note that Euler angles do not form the components of a vector (though infinitesimal rotations can be treated as such), as will be further elaborated in Section 1.4.

Rotations Describing Aircraft Attitude

Standard aircraft practice is to describe aircraft orientation by the z, y, x (also called 3, 2, 1) right-handed Euler rotation sequence that is required to get from a reference system on the surface of Earth into alignment with an aircraft body-fixed coordinate system. The usual choice for the reference system, on Earth, is a North-East-down (*ned*) system, with the x-axis pointing true North, the z-axis pointing down, and the y-axis completing the right-handed set. The exact meaning of "down" will be explained in Section 1.6. The aircraft axes are normally aligned (x, y, z), forward, right, and down (*frd*), with "forward" aligned with the *longitudinal reference line* of the aircraft, and the forward and down axes in the aircraft plane of symmetry. Therefore, starting from the reference system, the sequence of rotations is:

1. Right-handed rotation about the z-axis, or positive ψ (compass heading)
2. Right-handed rotation about the new y-axis, or positive θ (pitch)
3. Right-handed rotation about the new x-axis, or positive ϕ (roll)

The rotations are often described as a yaw-pitch-roll sequence, starting from the reference system.

The plane rotation matrices can be written down immediately with the help of the rules established in the preceding subsection. Thus, abbreviating cosine and sine to c and s, we have,

$$C_{frd/ned} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{frd/ned} = \begin{bmatrix} c\theta\, c\psi & c\theta\, s\psi & -s\theta \\ (-c\phi\, s\psi + s\phi\, s\theta\, c\psi) & (c\phi\, c\psi + s\phi\, s\theta\, s\psi) & s\phi\, c\theta \\ (s\phi\, s\psi + c\phi\, s\theta\, c\psi) & (-s\phi\, c\psi + c\phi\, s\theta\, s\psi) & c\phi\, c\theta \end{bmatrix} \qquad (1.3\text{-}10)$$

This matrix represents a standard transformation and will be used throughout the text.

The defined ranges for the rotation angles are

$$-\pi < \phi \le \pi$$
$$-\pi/2 \le \theta \le \pi/2$$
$$-\pi < \psi \le \pi$$

If the pitch angle, θ, had been allowed to have a $\pm 180°$ range then the airplane could be inverted and heading South with the roll and heading angles reading zero, which is obviously undesirable from a human factors viewpoint! The restriction on theta can be enforced naturally, simply by interpretation of the DCM, as we see in the next subsection

Euler Angles from the DCM

In a control system it is often necessary to extract the Euler angles, from a continuously computed DCM, for display to a human operator. For the z-y-x sequence used in Equation (1.3-10), taking account of the chosen angular ranges, the Euler angles are easily seen to be

$$\begin{aligned} \phi &= \operatorname{atan2}(c_{23}, c_{33}), && -\pi < \phi \le \pi \\ \theta &= -\operatorname{asin}(c_{13}), && -\pi/2 \le \theta \le \pi/2 \\ \psi &= \operatorname{atan2}(c_{12}, c_{11}), && -\pi < \psi \le \pi, \end{aligned} \qquad (1.3\text{-}11)$$

where atan2(*) is the four-quadrant inverse tangent function, available in most programming languages. These equations also work for only two Euler rotations (when the order and positive reference directions are the same), for example, the elevation-over-azimuth gimbal system with zero roll angle.

Finite precision computer arithmetic occasionally causes the DCM element C_{13} to very slightly exceed unit magnitude; in computer code we simply detect this condition and set the pitch attitude to 90°. Since θ is usually a low-precision "output" variable, not a state variable, this does not cause any accuracy problems. A more significant problem is the ambiguity introduced into the DCM (1.3-10) at vertical pitch. When $\theta = \pm\pi/2$ the condition $C_{11} = C_{12} = C_{23} = C_{33} = 0$ occurs, and the remaining elements can be written as sine and cosine of $(\phi - \psi)$, or $(\phi + \psi)$ when $\theta = -\pi/2$. Heading is undefined at vertical pitch, and so roll cannot be computed. For aerobatic aircraft, missiles, and spacecraft, the problem can be avoided by using the quaternion representation of attitude. For most aircraft simulations, the condition $\theta = 90.000\ldots$ degrees has a very low probability of occurrence and an aircraft simulation can usually fly through vertical pitch without numerical problems.

Linear Transformations

Linear transformations occur both in the state equation (1.1-3), via the A-matrix, and in a coordinate rotation. A little knowledge of linear transformations is required in order to use some of the properties of eigenvalues and eigenvectors, described in the next subsection.

Consider the matrix equation

$$v = A\,u, \tag{1.3-12}$$

where v and u are $n \times 1$ matrices (e.g., vector component arrays) and A is an $n \times n$ constant matrix, not necessarily nonsingular. Each element of v is a linear combination of the elements of u, and so this equation is a *linear transformation* of the matrix u. In Euclidean space the geometrical interpretation of the transformation is that a vector is being changed in length and/or direction.

Next, suppose that in an analysis we change to a new set of variables through a reversible linear transformation. If L is the matrix of this transformation, then L^{-1} must exist (i.e., L is nonsingular) for the transformation to be reversible, and the new variables corresponding to u and v are

$$u_1 = Lu, \qquad v_1 = Lv$$

Therefore, the relationship between the new variables must be

$$v_1 = LA\,u = LA\,L^{-1}\,u_1 \tag{1.3-13a}$$

The transformation LAL^{-1} is a *similarity transformation* of the original coefficient matrix A. A special case of this transformation occurs when the inverse of the matrix L is given by its transpose (i.e., L is an orthogonal matrix) and the similarity transformation becomes a *congruence transformation*, LAL^{T}. An important example of a similarity transformation is a change of state variables, $z = Lx$, in the linear state equation (1.1-3), leading to the new state equation

$$\dot{z} = (LAL^{-1})z + (LB)u \tag{1.3-13b}$$

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are introduced here because of the insight they provide into coordinate rotations; they will also be used extensively in Chapter 3 to provide information on the dynamic behavior of systems described by a linear, time-invariant state equation.

A square-matrix linear transformation with an arbitrary matrix $A(n, n)$ has the property that vectors exist whose components are only scaled by the transformation. If **v** is such an "invariant" vector, its component array, v, must satisfy the equation

$$Av = \lambda v, \qquad v(n \times 1) \tag{1.3-14}$$

where λ is a (scalar) constant of proportionality. A rearrangement of (1.3-14) gives the set of homogeneous linear equations

$$(A - \lambda I)v = 0 \tag{1.3-15}$$

which has a nonnull solution for v if and only if the determinant of the coefficient matrix is zero (Strang, 1980); that is,

$$|A - \lambda I| = 0 \tag{1.3-16}$$

This determinant is an nth-order polynomial in λ, called the *characteristic polynomial* of A, so there may be up to n distinct solutions for λ. Each solution, λ_i, is known as an *eigenvalue* or *characteristic value* of the matrix A. The associated invariant vector defined by (1.3-14) is known as a *right eigenvector* of A (the left eigenvectors of A are the right eigenvectors of its transpose A^T).

In the mathematical model of a physical system, a reversible change of model state variables does not change the behavior of the model if observed at the same outputs. An example of this is the invariance of the eigenvalues of a linear system, described by the state equation (1.1-3), under the similarity transformation (1.3-13). After the similarity transformation, the eigenvalues are given by

$$|\lambda I - LAL^{-1}| = 0,$$

which can be rewritten as

$$|\lambda LL^{-1} - LAL^{-1}| = 0$$

The determinant of a product of square matrices is equal to the product of the individual determinants; therefore,

$$|L|\,|\lambda I - A|\,|L^{-1}| = 0 \tag{1.3-17}$$

This equation is satisfied by the eigenvalues of the matrix A, so the eigenvalues are unchanged by the transformation.

Now consider a special similarity transformation that will reduce the linear equations with square coefficient matrix A to a canonical (standard) form. First, consider the case when all of the n eigenvalues of A are distinct. Then the n eigenvectors v_i can be shown to form a linearly independent set; therefore, their components can be used to form the columns of a nonsingular transformation matrix. This matrix is called the *modal matrix, M*, and

$$M \equiv [v_1\, v_2 \ldots v_n]$$

Then, according to the eigenvector/eigenvalue defining equation (1.3-14),

$$AM = MJ, \qquad \text{where } J = \text{diag}(\lambda_1 \ldots .\ \lambda_n)$$

or

$$M^{-1}AM = J \tag{1.3-18}$$

When some of the eigenvalues of A are repeated (i.e., multiple), it may not be possible to find a set of n linearly independent eigenvectors. Also, in the case of repeated eigenvalues, the result of the similarity transformation (1.3-18) is in general a *Jordan form matrix* (Wilkinson and Golub, 1976). In this case the matrix J may have some unit entries on the superdiagonal. These entries are associated with blocks of repeated eigenvalues on the main diagonal.

As an example, the linear state equation (1.1-3), with $x = Mz$, becomes

$$\dot{z} = Jz + M^{-1}Bu \tag{1.3-19}$$

This corresponds to a set of state equations with minimal coupling between them. For example, if the eigenvalue λ_i is of multiplicity 2 and the associated Jordan block has a superdiagonal 1, we can write the corresponding equations as

$$\begin{aligned} \dot{z}_i &= \lambda_i z_i + z_{i+1} + b_i' u \\ \dot{z}_{i+1} &= \lambda_i z_i + b_{i+1}' u \end{aligned} \tag{1.3-20}$$

The variables z_i are called the modal coordinates. In the above case these two equations are coupled; when the eigenvalues are all distinct, the modal coordinates yield a set of uncoupled first-order differential equations. Their homogeneous solutions (i.e., response to initial conditions, with $u = 0$) are the exponential functions $e^{\lambda_i t}$, and these are the natural modes of (behavior of) the dynamic system. A disadvantage of the modal coordinates is that the state variables usually lose their original physical significance.

Euler's Rotation Theorem

A better understanding of coordinate rotations can be obtained by examining the eigenvalues of the DCM. Any nontrivial (3×3) rotation matrix has one, and only one, eigenvalue equal to +1 (see, for example, Goldstein, 1980). The other two eigenvalues are a complex conjugate pair with unit magnitude and can be written as $(\cos\phi \pm j\sin\phi)$ apart from a special case of two "−1" eigenvalues (see below).

Because the eigenvalues are distinct, the +1 eigenvalue has an associated unique, real eigenvector and, for this eigenvector of an arbitrary rotation matrix C, Equation (1.3-14) can be written as

$$Cv = v$$

Now, let v (suitably normalized) be the direction cosine array of an axis passing through the coordinate origin. The only way in which it is possible for the direction cosines to remain unchanged by an arbitrary C is for C to be equivalent to a single rotation around the axis given by the eigenvector of eigenvalue +1. Therefore, any compound rotation, made up of rotations about various axes, is equivalent to a single rotation around an axis corresponding to the +1 eigenvector of the compound rotation matrix. (The special case of two "−1" eigenvalues occurs when this rotation is 180°.) This is a modern version of a *fixed-point theorem* proven by Leonhard Euler in 1775.

Euler showed that *if a sphere is rigidly rotated about its center, then there is a diameter that remains fixed*. The principle is fundamentally important and forms the basis of the quaternion representation of rotation that we describe in Section 1.8.

1.4 ROTATIONAL KINEMATICS

In this section we will develop kinematic equations for a time-varying orientation, specifically, the relationship between the derivative of a translational vector and angular velocity expressed as a vector. We will follow this with the relationship of the Euler angle derivatives to the angular velocity vector, expressed in state-space form. These relationships will be required when we derive the equations for the six-degrees-of-freedom (6-DoF) motion of a rigid body in Section 1.7. We know from simple mechanics that rotation of a body around an axis induces translational velocities at points away from the axis. We now need to formalize this relationship by expressing the translational velocity as a vector and combining the direction of the axis of rotation with the rate of rotation as a single *angular velocity vector*.

The Derivative of a Vector

Here we will define the derivative of a vector, show how it depends on the observer's frame of reference, and relate the derivatives of a vector, taken in two different frames, through the relative angular velocity vector of the frames.

In general terms, the derivative of a vector is defined in the same way as the derivative of a scalar:

$$\frac{d}{dt}\mathbf{p}_{A/B} = \lim_{\delta t \to 0}\left[\frac{\mathbf{p}_{A/B}(t+\delta t) - \mathbf{p}_{A/B}(t)}{\delta t}\right]$$

This is a new vector created by the changes in length and direction of $\mathbf{p}_{A/B}$. If $\mathbf{p}$ is a free vector (e.g., velocity), then we expect its derivative to be independent of translation, and the changes in length and direction come from the motion of the tip of **p** relative to its tail. If **p** is a bound vector (e.g., a position vector) in some frame, its derivative in that frame is a free vector, corresponding to motion of the tip of **p**.

Angular Velocity as a Vector

Using Figure 1.2-1, make a small right-handed rotation, $\delta\mu \ll 1$ rad, and define $\mathbf{v} = \mathbf{u} + \boldsymbol{\delta}\mathbf{u}$; then Equation (1.2-5*a*) gives

$$\delta\mathbf{u} \approx -\sin(-\delta\mu)\mathbf{n} \times \mathbf{u} \approx (\mathbf{n} \times \mathbf{u})\,\delta\mu$$

Now divide by δt, take the limit as $\delta t \to 0$, and define the vector $\boldsymbol{\omega} \equiv \dot{\mu}\,\mathbf{n}$, giving

$$\dot{\mathbf{u}} = \boldsymbol{\omega} \times \mathbf{u} \qquad (1.4\text{-}1)$$

This equation relates the translational velocity of the tip of the constant-length bound vector **u** to the vector ω. The vector ω is made up of a unit vector defining the axis of rotation, scaled by the rotation rate; it is the *angular velocity vector* of this rotation. It is a free vector (can be translated parallel to itself) and an axial or pseudovector (it would change direction if we had chosen a left-handed rotation convention).

Because ω is a free vector, we associate it with the rigid body (i.e., frame), not just a bound vector in the frame, and give it subscripts to indicate that it is the angular velocity of that body relative to some other body. The orientation of a rotating rigid body is described by a time-varying DCM and it follows from Euler's theorem that the body has a unique *instantaneous axis of rotation*; the angular velocity vector is parallel to this axis and is unique to the body.

Vector Derivatives and Rotation

To understand the derivative of a vector, observed from another frame, in relative motion, we can proceed as follows.

Figure 1.4-1 shows a frame F_b in arbitrary motion with respect to another frame F_a and with angular velocity $\boldsymbol{\omega}_{b/a}$. Fixed point Q has translational velocity $\mathbf{v}_{Q/a}$ with respect to F_a, and vector **p** from Q is the vector of interest. An observer in F_b watching the tip of this vector would see the new vector $\mathbf{p}_1$ corresponding to a nonzero derivative ${}^b\dot{\mathbf{p}}$. An observer in F_a would see, in addition, the effect of the angular velocity of F_b with respect to F_a, giving the vector $\mathbf{p}_2$. The F_a observer would also see $\mathbf{p}_2$ translated parallel to itself because of the translational velocity $\mathbf{v}_{Q/a}$. However, the derivative in F_a is a free vector and this translation does not entail a change in length or direction of $\mathbf{p}_2$. The derivative ${}^a\dot{\mathbf{p}}$ is obtained by comparing $\mathbf{p}_2$ with **p** as $\delta t \to 0$.

In time δt, $\mathbf{p}_2 - \mathbf{p}$ is given by

$$\mathbf{p}_2 - \mathbf{p} = {}^b\dot{\mathbf{p}}\;\delta t + (\boldsymbol{\omega}_{b/a} \times \mathbf{p})\delta t$$

Dividing by δt and taking the limit as $\delta t \to 0$ give

$${}^a\dot{\mathbf{p}} = {}^b\dot{\mathbf{p}} + \boldsymbol{\omega}_{b/a} \times \mathbf{p} \qquad (1.4\text{-}2)$$

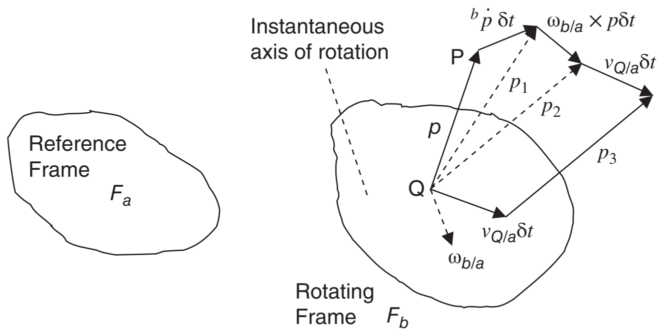

Figure 1.4-1 A vector derivative in a rotating frame.

Equation (1.4-2) is sometimes called the *equation of Coriolis* (Blakelock, 1965) and will be an essential tool in developing equations of motion from Newton's laws. It is much more general than is indicated above and applies to any physical quantity that has a vector representation. The derivatives need not even be taken with respect to time. Angular velocity can be defined as the vector that relates the derivatives of any arbitrary vector in two different frames, according to (1.4-2). In the interests of having a vector diagram and intuitive feel, we have derived the equation in a rather restricted fashion. A more rigorous derivation (with no diagram) has been given by McGill and King (1995) and a longish derivation with a different kind of diagram by Pestel and Thompson (1968).

Some formal properties of the angular velocity vector are:

(a) It is a unique vector that relates the derivatives of a vector taken in two different frames.
(b) It satisfies the relative motion condition $\boldsymbol{\omega}_{b/a} = -\boldsymbol{\omega}_{a/b}$.
(c) It is additive over multiple frames, e.g., $\boldsymbol{\omega}_{c/a} = \boldsymbol{\omega}_{c/b} + \boldsymbol{\omega}_{b/a}$ (not true of angular acceleration).
(d) Its derivative is the same in either frame, ${}^a\dot{\boldsymbol{\omega}}_{b/a} = {}^b\dot{\boldsymbol{\omega}}_{b/a}$. [Use (1.4-2) to find the derivative.]

A common problem is the determination of an angular velocity vector after the frames have been defined in a practical application. This can be achieved by finding one or more intermediate frames in which an axis of rotation and an angular rate are physically evident. Then the additive property can be invoked to combine the intermediate angular velocities. An example of this is given later, with the "rotating-Earth" equations of motion of an aerospace vehicle.

The derivative of a vector in some frame can be found from the derivatives of its components in a coordinate system fixed in that frame, that is, if

$$\mathbf{v}^{af} = [v_x \;\; v_y \;\; v_z]^T$$

where system *af* is fixed in frame *a*, then

$$^{a}\dot{\mathbf{v}}^{af} = [\dot{v}_x \;\; \dot{v}_y \;\; \dot{v}_z]^T$$

If the vector is from a fixed point in that frame, it is a velocity, acceleration, etc., with respect to that frame. If the vector is from a fixed point in a different frame, then it is a relative velocity, acceleration, etc., taken in the derivative frame.

Example 1.4-1: Centripetal Acceleration on Earth's Surface If $\mathbf{p}$ is a position vector from Earth's cm to a fixed point P on the surface rotating with Earth's (constant) inertial angular velocity $\boldsymbol{\omega}_{e/i}$, then the inertial acceleration vector $\mathbf{a}$ of P can be found from

$$\mathbf{v}_{P/i} = {}^{i}\dot{\mathbf{p}} = {}^{e}\dot{\mathbf{p}} + \boldsymbol{\omega}_{e/i} \times \mathbf{p}$$

$$\mathbf{a} = {}^{i}\dot{\mathbf{v}}_{P/i} = {}^{i}\dot{\boldsymbol{\omega}}_{e/i} \times \mathbf{p} + \boldsymbol{\omega}_{e/i} \times {}^{i}\dot{\mathbf{p}} = \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}) \tag{1}$$

It is easy to confirm that this *centripetal acceleration* is orthogonal to the angular velocity vector and to show that this equation leads to the well-known scalar formulae (v^2/r and $r\omega^2$) for centripetal acceleration in a plane perpendicular to the angular velocity vector. ■

Euler Angle Kinematics

With the idea in mind of relating the Euler angle rates, describing the changing attitude of a body, to its angular velocity, we proceed as follows. We define a reference frame F_r and a body frame F_b with a relative angular velocity vector $\omega_{b/r}$ and a sequence of Euler angles that define the attitude of the body as the orientation of a coordinate system fixed in the body relative to a coordinate system fixed in the reference frame. Each Euler angle rate provides the magnitude and direction information for an individual angular velocity vector (i.e., along a particular coordinate axis). These three vectors can be added to find the resultant angular velocity vector of the vehicle whose Euler angle rates are being considered. Equivalently, we can find the components of the resultant angular velocity vector.

To make this process more concrete we take the common case of motion over Earth, with a *frd* coordinate system in the body, a *ned* system in the reference frame, and a yaw-pitch-roll Euler angle sequence from *ned* to *frd*. In the case of the flat-Earth equations (Section 1.7) the *ned* system is fixed in the Earth as the reference frame, and the relative angular velocity is that of the body with respect to Earth. In the case of the more general 6-DoF equations the *ned* system moves over the Earth, underneath the body, and we must define an abstract reference frame which has its own angular velocity with respect to the Earth frame (determined by latitude and longitude rates).

The coordinate transformations are

$$\boldsymbol{\omega}^{frd}_{b/r} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + C_\phi \left(\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + C_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \right)$$

where C_ϕ and C_θ are the right-handed plane rotations through the particular Euler angles, as given in Equations (1.3-10). After multiplying out the matrices, the final result is

$$\boldsymbol{\omega}_{b/r}^{frd} \equiv \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{1.4-3}$$

where P, Q, R, are standard symbols for, respectively, the roll, pitch, and yaw rate components of the aircraft angular velocity vector in *frd* coordinates. The inverse transformation is

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{1.4-4}$$

For brevity, we will define $\Phi \equiv [\phi\ \theta\ \psi]^T$ and write (1.4-4) as

$$\dot{\Phi} = H(\Phi)\, \boldsymbol{\omega}_{b/r}^{frd} \tag{1.4-5}$$

Equations (1.4-3) and (1.4-4) will be referred to as the *Euler kinematical equations*. Note that the coefficient matrices are not orthogonal matrices representing ordinary coordinate rotations. Note also that Equations (1.4-4) have a singularity when $\theta = \pm\pi/2$. In addition, if these equations are used in a simulation, the Euler angle rates may integrate up to values outside the Euler angle range. Therefore, logic to deal with this problem must be included in the computer code. Despite these disadvantages the Euler kinematical equations are commonly used in aircraft simulation.

1.5 TRANSLATIONAL KINEMATICS

In this section we introduce the equations for relative velocity and relative acceleration between rigid bodies in motion and, in particular, introduce *centripetal* and *Coriolis acceleration*. The equations are then applied to motion over Earth, and the results are used in the 6-DoF motion in Section 1.7.

Velocity and Acceleration in Moving Frames

Figure 1.5-1 shows a point P with position vector $\mathbf{p}$ moving with respect to two frames F_a and F_b, in relative motion, and with fixed points O and Q, respectively. Suppose that we wish to relate the velocities in the two frames and also the accelerations. We must first relate the position vectors shown in the figure and then take derivatives in F_a to introduce velocity (we are arbitrarily choosing F_a to be the "reference" frame):

$$\mathbf{p}_{P/O} = \mathbf{p}_{Q/O} + \mathbf{p}_{P/Q} \tag{1.5-1}$$

$${}^{a}\dot{\mathbf{p}}_{P/O} = {}^{a}\dot{\mathbf{p}}_{Q/O} + {}^{a}\dot{\mathbf{p}}_{P/Q} \tag{1.5-2}$$

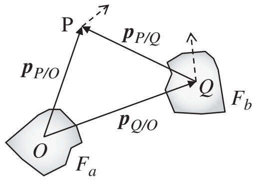

Figure 1.5-1 Velocity and acceleration in moving frames.

Starting from the left-hand side of Equation (1.5-2), the first two terms are velocities in F_a but the last term involves the position of P relative to a fixed point in F_b, with the derivative taken in F_a. Let $\mathbf{v}$ with an appropriate subscript represent a velocity vector. Then, by applying the equation of Coriolis, Equation (1.5-2) gives

$$\mathbf{v}_{P/a} = \mathbf{v}_{Q/a} + (\mathbf{v}_{P/b} + \boldsymbol{\omega}_{b/a} \times \mathbf{p}_{P/Q}) \tag{1.5-3}$$

Note that (1.5-3) can be written as

$$\mathbf{v}_{P/a} = \mathbf{v}_{P/b} + (\mathbf{v}_{Q/a} + \boldsymbol{\omega}_{b/a} \times \mathbf{p}_{P/Q}),$$

where the term in parentheses is the velocity in F_a of a fixed point in F_b that is instantaneously coincident with P and is called the *transport velocity of P in F_a*.

As an application of Equation (1.5-3), let F_a be an inertial reference frame and F_b a body moving with respect to the reference frame. Assume that a navigator on the moving body determines, from an onboard inertial navigation system, his velocity in the inertial reference frame $\mathbf{v}_{Q/a}$ and his inertial angular velocity vector $\boldsymbol{\omega}_{b/a}$. Also, using a radar set, he measures the velocity $\mathbf{v}_{P/b}$, of P in F_b and the position $\mathbf{p}_{P/Q}$ of P with respect to Q. He can then use Equation (1.5-3), with appropriately chosen coordinate systems, to calculate the velocity of the object in the inertial reference frame and, knowing the equation of motion in the inertial frame, predict its trajectory.

We next find the acceleration of P by taking derivatives of (1.5-3) in F_a. Starting from the left, the first two terms are velocities in F_a and these become accelerations in F_a. The third term is a velocity in F_b and must be differentiated by the equation of Coriolis. The last term involving a cross-product can be differentiated by the "product rule," and the derivative of angular velocity is an angular acceleration vector, denoted by $\boldsymbol{\alpha}$. Therefore, denoting translational acceleration by $\mathbf{a}$, (1.5-3) yields

$$\mathbf{a}_{P/a} = \mathbf{a}_{Q/a} + (\mathbf{a}_{P/b} + \boldsymbol{\omega}_{b/a} \times \mathbf{v}_{P/b}) + \boldsymbol{\alpha}_{b/a} \times \mathbf{p}_{P/Q} + \boldsymbol{\omega}_{b/a} \times (\mathbf{v}_{P/b} + \boldsymbol{\omega}_{b/a} \times \mathbf{p}_{P/Q})$$

Regrouping terms, we get

$$\underset{\text{Total Accel.}}{\mathbf{a}_{P/a}} = \underset{\text{Relative Accel.}}{\mathbf{a}_{P/b}} + \underbrace{\mathbf{a}_{Q/a} + \boldsymbol{\alpha}_{b/a} \times \mathbf{p}_{P/Q} + \underset{\text{Centripetal Acceleration}}{\boldsymbol{\omega}_{b/a} \times (\boldsymbol{\omega}_{b/a} \times \mathbf{p}_{P/Q})}}_{\text{Transport Acceleration of P in Frame-a}}$$

$$+ \underset{\text{Coriolis Acceleration}}{2\boldsymbol{\omega}_{b/a} \times \mathbf{v}_{P/b}} \tag{1.5-4}$$

Here, the names *total acceleration* and *relative acceleration* apply to the reference and secondary frames, respectively. If P were fixed in F_b the first and last right-hand-side terms would vanish, leaving only the *transport acceleration*; this is defined as the acceleration in F_a of a fixed point in F_b that is instantaneously coincident with P. As could be anticipated, the transport acceleration term contains the effects of the motion of F_b in terms of the acceleration of the reference point Q and the angular acceleration and angular velocity of the frame [see (1) for more detail of the centripetal term]. A comparison of the acceleration equation with the velocity equation shows that a new type of term has appeared, namely the *Coriolis acceleration*. The significance of Coriolis acceleration is examined in the following subsection.

Example 1.5-1: Sensor Fixed in A Moving Body A sensor (e.g., accelerometer, radar, etc.) fixed to a rigid vehicle has no velocity or acceleration in that frame, so according to Equation (1.5-3) or (1.5-4) only the transport term in these equations is nonzero. Sensor motion must often be related analytically to the motion of the vehicle cm (or perhaps some other fixed point). For example, with the same notation as Equation (1.5-4), an accelerometer at position P, with position vector $\mathbf{p}_{P/Q}$ relative to the point Q, has an acceleration given by

$$\mathbf{a}_{P/a} = \mathbf{a}_{Q/a} + \boldsymbol{\alpha}_{b/a} \times \mathbf{p}_{P/Q} + \boldsymbol{\omega}_{b/a} \times (\boldsymbol{\omega}_{b/a} \times \mathbf{p}_{P/Q}),$$

where a and b denote, respectively, the reference and vehicle frames. ■

Acceleration Relative to Earth

This book is concerned with the motion of aerospace vehicles over the Earth, and acceleration relative to Earth is the starting point for equations of motion. Using the results of the previous subsection, let F_a become an inertial frame F_i and F_b become the rigid Earth frame F_e. Let the points Q and O coincide, at Earth's cm (Earth is assumed to have no translational acceleration) so that the acceleration $\mathbf{a}_{Q/a}$ vanishes and $\mathbf{p}_{P/Q}$ is a geocentric position vector. Earth's angular velocity is closely constant and so the derivative of $\boldsymbol{\omega}_{b/a}$ vanishes. This leaves only the relative acceleration, centripetal acceleration, and Coriolis acceleration terms and gives the fundamental equation, relating true (inertial) acceleration to relative acceleration, that we will use in Section 1.7 to apply Newton's laws to motion of a point P over Earth:

$$\mathbf{a}_{P/i} = \mathbf{a}_{P/e} + \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{P/O}) + 2\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{P/e} \qquad (1.5\text{-}5)$$

For a particle of mass m at P, the relative acceleration $\mathbf{a}_{P/e}$ corresponds to an "apparent force" on the particle and produces the trajectory observed by a stationary observer on Earth. The true acceleration $\mathbf{a}_{P/i}$ corresponds to "true" forces (e.g., mass attraction, drag); therefore, writing (1.5-5) in terms of force,

$$\text{Apparent force} = \text{true force} - m[\boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{P/O})] - m(2\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{P/e})$$

The second term on the right is the *centrifugal force*, directed normal to the angular velocity vector. The third term is usually referred to as the *Coriolis force* and will cause a ballistic trajectory over Earth to curve to the left or right.

The true force is the sum of the *contact forces*, say F, and the mass attraction of *Earth's gravitational field*, $m\mathbf{G}$ (see next section). The Earth gravity vector is $\mathbf{g} = \mathbf{G}$ – centripetal acceleration (see next section), so Equation (1.5-5) is often written (for a body of mass m) as

$$\mathbf{a}_{P/e} = \frac{F}{m} + g - 2\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{P/e} \tag{1.5-6}$$

An often-quoted example of the Coriolis force is the circulation of winds around a low-pressure area (a cyclone) on Earth. The true force is radially inward along the pressure gradient. In the Northern Hemisphere, for example, Earth's angular velocity vector points outward from Earth's surface and, whichever way the velocity vector $\mathbf{v}_{P/e}$ is directed, the Coriolis force is directed to the right of $\mathbf{v}_{P/e}$. Therefore, in the Northern Hemisphere the winds spiral inward in a counterclockwise direction around a cyclone.

The Coriolis acceleration is also significant in high-speed flight; it is zero for an aircraft flying due North or South at the equator and reaches its maximum value at the poles or for flight due East or West at any latitude. Its significance can be estimated by equating its value to the centripetal acceleration, in low, constant-altitude flight, at 45° latitude, and solving for the speed over Earth:

$$2\ |\boldsymbol{\omega}_{e/i}|\,|\mathbf{v}_{cm/e}|\ \sin(45^{\circ}) = |\mathbf{v}_{cm/e}|^2 / r_E$$

$$|\mathbf{v}_{cm/e}| = \sqrt{2}\, r_E\, |\boldsymbol{\omega}_{e/i}| \approx 657\ \text{m/s}\ (2156\ \text{ft/s})$$

At this speed the Coriolis acceleration is equal to the centripetal acceleration and is very small compared to $\boldsymbol{g}$ but causes a position error that grows quadratically with time.

1.6 GEODESY, COORDINATE SYSTEMS, GRAVITY

Introduction

Geodesy is a branch of mathematics that deals with the shape and area of the Earth. Some ideas and facts from geodesy are needed to simulate the motion of an aerospace vehicle around Earth. In addition, some knowledge of Earth's gravitation is required. Useful references are *Encyclopaedia Britannica* (1987), Heiskanen and Moritz (1967), Kuebler and Sommers (1981), NIMA (1997), and Vanicek and Krakiwsky (1982).

The Shape of the Earth, WGS-84

Simulation of high-speed flight over large areas of Earth's surface, with accurate equations of motion and precise calculation of position, requires an accurate model of Earth's shape, rotation, and gravity. Meridional cross sections of Earth are approximately elliptical and the polar radius of Earth is about 21 km less than the equatorial radius, so the solid figure generated by rotating an appropriately scaled ellipse about its minor axis will provide a model of Earth's shape. Organizations from many countries participate in making accurate measurements of the parameters of these *spheroidal* (i.e., ellipsoids of revolution) models. In the United States the current model is the Department of Defense World Geodetic System 1984, or WGS-84, and the agency responsible for supporting this model is the National Imagery and Mapping Agency (NIMA, 1997). The Global Positioning System (GPS) relies on WGS-84 for the ephemerides of its satellites.

The equipotential surface of Earth's gravity field that coincides with the undisturbed mean sea level extended continuously underneath the continents is called the *geoid*. Earth's irregular mass distribution causes the geoid to be an undulating surface, and this is illustrated in Figure 1.6-1. Note that the *local vertical* is defined by the direction in which a plumb-bob hangs and is accurately normal to the geoid. The angle that it makes with the spheroid normal is called the *deflection of the vertical* and is usually less than 10 arc-s (the largest deflections over the entire Earth are about 1 arc-min).

Figure 1.6-2 shows the *Earth spheroid*, with the oblateness greatly exaggerated. The coordinate system shown has its origin at Earth's center of mass (indirectly determined from satellite orbits), z up the spin axis, and its x and y axes in the equatorial plane. Based on this coordinate system, the equation of the spheroidal model is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \qquad (1.6\text{-}1)$$

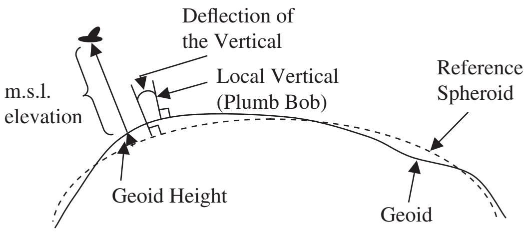

Figure 1.6-1 The geoid and definitions of height.

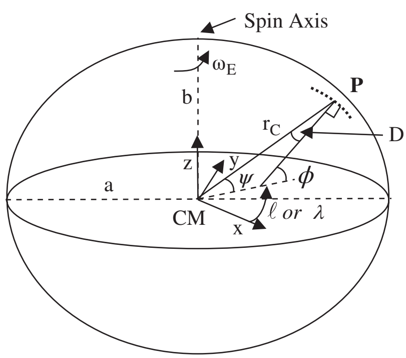

Figure 1.6-2 The oblate spheroidal model of the Earth.

In the figure, a and b are respectively the semimajor and semiminor axes of the generating ellipse. Two other parameters of the ellipse (not shown) are its *flattening*, f, and its *eccentricity*, e.

The WGS-84 spheroid was originally (1976–1979 data) a least-squares best fit to the geoid. More recent estimates have slightly changed the "best fit" parameters, but the current WGS-84 spheroid now uses the original parameters as its defining values. Based on a 1° × 1° (latitude, longitude) worldwide grid, the root-mean-square (rms) deviation of the geoid from the spheroid is only about 30 m! The WGS-84 defined and derived values are:

$$a \equiv 6\,378\,137.0\,\text{m} \qquad \text{(defined)}$$

$$f = \frac{a-b}{a} \equiv 1/298.257\,223\,563 \qquad \text{(defined)}$$

$$b = 6\,356\,752\,\text{m} \qquad \text{(derived)}$$

$$e = \frac{(a^2-b^2)^{\frac{1}{2}}}{a} = .0818\,191\,908\,426 \qquad \text{(derived)} \qquad (1.6\text{-}2a)$$

Two additional parameters are used to define the complete WGS-84 reference frame; these are the fixed (scalar) Earth rotation rate, ω_E, and the Earth's gravitational constant (GM) with the mass of the atmosphere included. In WGS-84 they are defined to be

$$\omega_E \equiv 7.292\,1150 \times 10^{-5}\,\text{rad/s}$$

$$GM \equiv 398\,6004.418 \times 10^8\,\text{m}^3/\text{s}^2 \qquad (1.6\text{-}2b)$$

The ω_E value is called the *sidereal rate of rotation* (rate relative to the "fixed" stars); it actually corresponds to a component of Earth's angular velocity in the heliocentric frame ($\omega_E = (2\pi/(24 \times 3600)) \times (1 + 1/365.25)$), neglecting the inclination of Earth's axis).

Frames, Earth-Centered Coordinates, Latitude and Longitude

The *reference frames* used here are the Earth, considered to be a rigid body, and an *inertial frame* (Kaplan, 1981) containing Earth's cm as a fixed point (this neglects the small centripetal acceleration of Earth's orbit and any acceleration of the Sun with respect to the Galaxy). An inertial frame must also be nonrotating; so the small, low-amplitude wobble of Earth's axis will be neglected, and a line from the cm, in the plane of the ecliptic, parallel to a line from the Sun's cm to a very distant "fixed" star will be taken to be a fixed line. Several polar and Cartesian coordinate systems are defined in these frames; they use Earth's spin axis and equatorial plane (defined as orthogonal to the spin axis and containing Earth's cm) for reference.

The Earth-centered–Earth-fixed (ECEF) system is fixed in the Earth frame, has its origin at the cm, its z-axis up the spin axis, and its x and y axes in the equatorial plane (as in Figure 1.6-2) with the x-axis passing through the Greenwich Meridian (actually a few arc-seconds off the Meridian (see NIMA, 1997)). The Earth-centered inertial (ECI) system is fixed in the inertial frame, and defined in the same way as ECEF, except that its x-axis is always parallel to a line from the Sun's cm to Earth's position in orbit at the vernal equinox.

Terrestrial longitude, ℓ, and *celestial longitude*, λ, are shown in the figure, measured easterly from the appropriate x-axis to the projection of the position vector on the equatorial plane. In a given time interval, an increment in celestial longitude is equal to the increment in terrestrial longitude plus the increment in Earth's rotation angle. This can be written as

$$\lambda - \lambda_0 = \ell - \ell_0 + \omega_E t, \tag{1.6-3}$$

where λ_0 and ℓ_0 are the values at $t = 0$. Absolute celestial longitude is often unimportant, and $\lambda_0 \equiv 0$ can be used. *Latitude angles* are angles subtended by the position vector, above the equatorial plane, and are positive in the Northern Hemisphere.

Geocentric Coordinates of a Point The *geocentric coordinates* of a point P, on the spheroid, are shown in Figure 1.6-2 and, in cross section, for a point above the spheroid in Figure 1.6-3. They are referenced to the common origin of the ECI and ECEF systems and the equatorial plane. They are:

The geocentric latitude of P: angle ψ
The geocentric radius of P: distance r
(The geocentric radius of the spheroid is r_c.)

Geodetic Coordinates of a Point The *geodetic coordinates* of point P, in Figures 1.6-2 and 1.6-3, are used for maps and navigation and are referenced to the normal to the spheroid from point P. They are:

Geodetic latitude, ϕ: the angle of the normal with the equatorial plane.
Geodetic height, h: the height above the spheroid, along the normal.

Geodetic height can be determined from a database of tabulated geoid height versus latitude and longitude plus the elevation above mean sea level (msl). The elevation above msl is in turn obtained from a barometric altimeter or from the land elevation (in a *hypsographic database*) plus the altitude above land (e.g., radar altimeter).

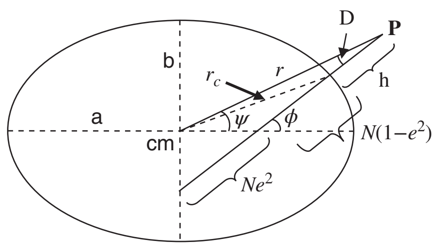

Figure 1.6-3 The geometry of a point above the spheroid.

In Figure 1.6-3, the triangle that defines latitude shows that

$$\phi = \psi + D \tag{1.6-4}$$

and the very small angle D is called *the deviation of the normal* and has a maximum value of 11.5 arc-min when P is on the spheroid and the latitude is 45°.

Local Coordinate Systems

Local coordinate systems have their origins on the spheroid. The *local geocentric system* (c-system) has its down axis aligned with the geocentric position vector and its “horizontal” axes aligned geographically (usually true North and East). The *local geographic systems* have their down axis aligned with the spheroid normal and are oriented North-East-down (*ned*) or East-North-up (*enu*). These systems move with the vehicle (i.e., origin vertically below the vehicle cm) and the latitude and longitude of the vehicle determine their orientation relative to the Earth-centered systems [see Earth-Related Coordinate Transformations]. If required we could define an imaginary frame in which these systems would be fixed (e.g., a “vehicle-carried” frame) with an angular velocity determined by the vehicle latitude and longitude rates (found from radii of curvature, following). A *tangent-plane coordinate system* is aligned as a geographic system but has its origin fixed at a point of interest on the spheroid; this coordinate system is used with the flat-Earth equations of motion (Section 1.7).

Radii of Curvature

A *radius of curvature* is a radial length that relates incremental distance along a geometrical arc to an increment in the angle subtended by the arc on a coordinate axis. Discussions of curvature and formulae for radii of curvature can be found in calculus textbooks; the simplest example is a circular arc, where the radius of curvature is the radius of the circle. For the spheroidal model of the Earth, the radii of curvature relating North-South distance along a meridian to an increment in latitude and East-West distance to an increment in longitude are required for estimating distances and speeds over the real Earth.

The *meridian radius of curvature*, M, of the spheroid is the radius of curvature in a meridian plane that relates North-South distance to increments in geodetic latitude; it is determined by the gradient of the generating ellipse. Applying a general formula for radius of curvature to the generating ellipse, it is easy to show that M is given by

$$M = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}, \qquad \frac{b^2}{a} \le M \le \frac{a^2}{b} \tag{1.6-5}$$

A radius of curvature, integrated with respect to angle, gives arc length. In this case the integral cannot be found in closed form, and it is much easier to compute distance over the Earth approximately using spherical triangles. The usefulness of this radius of curvature lies in calculating components of velocity. Thus, at geodetic height h, the geographic system North component of velocity, v_N, over Earth is related to latitude rate by

$$v_N = (M + h)\,\dot{\phi} \tag{1.6-6}$$

The *prime vertical radius of curvature*, N, is the radius of curvature in a plane perpendicular to the meridian plane and containing the *prime vertical* (the normal to the spheroid at the pertinent latitude). By rotational symmetry, the *center of curvature* (origin for the radius of curvature) is on the minor axis of the generating ellipse, as shown in Figure 1.6-3, and N is the distance to the ellipse (two parts of N are shown in the figure). Note that N occurs in almost all of the geodesy calculations that we will use. The formula for N is more easily (and instructively) found from the following simple argument than from an algebraically messy application of the standard formula for the radius of curvature. From the figure, we find the radius r of a *small circle* (of constant latitude) where N meets the ellipse and, from the rectangular coordinates on the spheroid, the meridian gradient and the gradient of the normal:

$$r = N\cos\phi$$

$$\text{Meridian gradient} = \frac{dz}{dr} = -\frac{b^2 r}{a^2 z}$$

$$\text{Gradient of normal} = \tan\phi = -\frac{1}{dz/dr} = \frac{a^2 z}{b^2 r}$$

From these equations, we find that the z-component on the spheroid at geodetic latitude, ϕ, is

$$z = \frac{b^2}{a^2} N\sin\phi = N(1 - e^2)\sin\phi \tag{1.6-7}$$

Equation (1.6-7) shows the very useful property that N can be divided into two parts, above and below the equatorial plane, as shown in the figure,

$$N = \underset{(\text{below } x\text{–}y)}{Ne^2} + \underset{(\text{above } x\text{–}y)}{N(1 - e^2)} \tag{1.6-8}$$

Next, we write the spheroid equation in terms of r and z, substitute the above expressions for r and z, and solve for N:

$$N = \frac{a}{(1 - e^2\sin^2\phi)^{1/2}}, \qquad a \le N \le \frac{a^2}{b}, \tag{1.6-9}$$

where N is needed for coordinate calculations and is useful for calculating velocity components. Using the constant-latitude circle we find that the geographic system East component of velocity over Earth, v_E, is related to longitude rate by

$$v_E = (N + h)\,\cos(\phi)\,\dot{\ell} \tag{1.6-10}$$

Trigonometric Relationships for the Spheroid

Some useful relationships can be derived from the spheroid equation using simple trigonometry. The geocentric radius at any point on the spheroid is given by

$$r_c{}^2 = \frac{a^2}{1 + \frac{e^2}{1-e^2}\sin^2\psi} = \frac{b^2}{1 - e^2\cos^2\psi} = \frac{a^2[1 - e^2(2 - e^2)\sin^2\phi]}{1 - e^2\sin^2\phi} \tag{1.6-11}$$

The deviation of the normal can be found from

$$\tan D = \frac{n\ \sin\phi\ \cos\phi}{1 - n\ \sin^2\phi} = \frac{n\ \sin\psi\ \cos\psi}{1 - n\ \cos^2\psi}, \quad \text{where } n = \frac{e^2N}{N+h} \approx e^2 \tag{1.6-12}$$

and the relationships between geodetic and geocentric latitude are

$$\sin\psi = \frac{(1-n)\sin\phi}{[1 - n(2-n)\sin^2\phi]^{1/2}}, \qquad \cos\psi = \frac{\cos\phi}{[1 - n(2-n)\sin^2\phi]^{1/2}} \tag{1.6-13}$$

so,

$$\tan\psi = (1-n)\tan\varphi \tag{1.6-14}$$

The geocentric radius to a point P at geodetic height h is

$$r = (N+h)[1 - n(2-n)\sin^2\phi]^{1/2} \tag{1.6-15}$$

but, because the deviation of the normal, D, is so small, the geocentric radius at P is closely equal to the sum of the geocentric radius of the spheroid and the geodetic height:

$$r \approx r_c + h \tag{1.6-16}$$

The error in this approximation is insensitive to altitude and greatest at 45° latitude, where it is still less than 6×10^{-4} %. The use of the approximation is described in the next subsection.

Cartesian/Polar Coordinate Conversions

Cartesian position coordinates (ECI or ECEF) can be readily calculated from polar coordinates using the prime vertical radius of curvature. The projection of N on the x-y plane gives the x- and y-components; the z-component was given in Equation (1.6-7). Therefore, the ECEF position can be calculated from either geocentric or geodetic coordinates by

$$\mathbf{p}^{ecef} = \begin{bmatrix} r\ \cos\psi & \cos\ell \\ r\ \cos\psi & \sin\ell \\ r\ \sin\psi & \end{bmatrix} = \begin{bmatrix} (N+h)\ \cos\phi\ \cos\ell \\ (N+h)\ \cos\phi\ \sin\ell \\ [N(1-e^2)+h]\ \sin\phi \end{bmatrix} \tag{1.6-17}$$

The position in ECI coordinates is of the same form as (1.6-17), but with celestial longitude λ replacing terrestrial longitude ℓ.

Geocentric coordinates are easily found from the Cartesian coordinates, but geodetic coordinates are more difficult to find. An exact formula exists but requires the solution of a quartic equation in tan ϕ (Vanicek and Krakiwsky, 1982). Therefore, an iterative algorithm is often used. Referring to Figure 1.6-3, we see that

$$\sin\phi = \frac{z}{N(1-e^2)+h} \tag{1.6-18}$$

Using the large triangle with hypotenuse $N+h$ and sides $\sqrt{x^2+y^2}$, $[z+Ne^2\sin\phi]$, we can write

$$\tan\phi = \frac{[z+Ne^2\sin\phi]}{\sqrt{x^2+y^2}} \tag{1.6-19}$$

If (1.6-18) is substituted for $\sin(\phi)$ in (1.6-19) and simplified, we obtain

$$\tan\phi = \frac{z}{(x^2+y^2)^{\frac{1}{2}}\,[1-Ne^2/(N+h)]}$$

Because N is a function of ϕ, this formula is implicit in ϕ, but it can be used in the following iterative algorithm for the geodetic coordinates:

$$\begin{aligned}
&\ell = \operatorname{atan2}(y,x)\\
&h = 0;\quad N = a\\
\rightarrow\ &\phi = \tan^{-1}\left[\frac{z}{\left(x^2+y^2\right)^{1/2}}\left(1-\frac{e^2N}{N+h}\right)^{-1}\right]\\
\uparrow\ &N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}}\\
\uparrow\ &(h+N) = \frac{(x^2+y^2)^{1/2}}{\cos\phi}\\
\leftarrow\ &h = (N+h)-N
\end{aligned} \tag{1.6-20}$$

Latitudes of $\pm 90^\circ$ must be dealt with as a special case, but elsewhere the iterations converge very rapidly, and accuracy of 10 to 12 decimal digits is easily obtainable. If the algorithm is modified to eliminate the inverse tangent function, convergence is badly affected.

In most practical applications the algorithm can be replaced by the approximation (1.6-16) to find h, with the geocentric radius of the spheroid found from geocentric latitude ψ and ψ found directly from the position vector. Single-precision arithmetic (seven decimal digits) is inadequate when the height above Earth is calculated from the small difference of large quantities (e.g., N or a). The geodetic latitude can be found from $\phi = \psi + D$, where the deviation of the normal, D, is found from ψ

using (1.6-12) with the approximation $h \ll N$. These approximations can also be used to initialize the iterative algorithm above.

Earth-Related Coordinate Transformations

The rotation from the ECI to the ECEF system is a plane rotation around the z-axis, and the rotation angle increases steadily as Earth rotates. The conventions chosen for the directions of the two systems (ECEF x-axis through Greenwich and ECI x-axis aligned with the line from the Sun to Earth's position in orbit at the vernal equinox) allow the rotation angle to be tabulated as a daily function of *Greenwich Mean Time* (*GMT*). The angle is known as the *Greenwich Hour Angle* (*GHA,* positive East) and tabulated in nautical almanacs published annually for use by navigators. Since the vernal equinox position originally aligned with the First Point of Aries, the angle is given the astronomical symbol for Aries, $\wp$. Therefore, the rotation from ECI to ECEF can be written as

$$C_{ecef/eci} = \begin{bmatrix} c\left(\mathrm{GHA}_{\wp}\right) & s(\mathrm{GHA}_{\wp}) & 0 \\ -s(\mathrm{GHA}_{\wp}) & c(\mathrm{GHA}_{\wp}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6-21}$$

When going from the ECEF to a local system, the convention is to perform the longitude rotation first. For example, consider the coordinate rotation from ECEF to NED. After rotating around the ECEF z-axis to the correct longitude, a left-handed rotation through 90°, around the y-axis, is needed to get the x-axis pointing north and the z-axis down. It is then only necessary to move to the correct latitude and fall into alignment with the NED system by means of an additional left-handed rotation around the y-axis, through the latitude angle. Therefore, the transformation is

$$\begin{aligned} C_{ned/ecef} &= \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\ell & s\ell & 0 \\ -s\ell & c\ell & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -s\phi\, c\ell & -s\phi\, s\ell & c\phi \\ -s\ell & c\ell & 0 \\ -c\phi\, c\ell & -c\phi\, s\ell & -s\phi \end{bmatrix} \end{aligned} \tag{1.6-22}$$

The rotation to a geocentric system is found similarly, except that geocentric latitude is used in place of geodetic. For example, a straight replacement of variables in Equation (1.6-22) gives

$$C_{c/eci} = \begin{bmatrix} -s\psi\, c\lambda & -s\psi\, s\lambda & c\psi \\ -s\lambda & c\lambda & 0 \\ -c\psi\, c\lambda & -c\psi\, s\lambda & -s\psi \end{bmatrix} \tag{1.6-23}$$

Gravitation and Gravity

The term *gravitation* denotes a mass attraction effect, as distinct from *gravity*, meaning the combination of mass attraction and centrifugal force experienced by a body constrained to move with Earth's surface.

The WGS-84 datum includes an amazingly detailed model of Earth's gravitation. This model is in the form of a (scalar) potential function, V, such that components of specific mass attraction force along each of three axes can be found from the respective gradients of the potential function. The current potential function, for use with WGS-84, is *Earth Gravitational Model* 1996 (EGM96). This has 130,676 coefficients and is intended for very precise satellite and missile calculations. The largest coefficient is two orders of magnitude bigger than the next coefficient and, if we retain only the largest coefficient, the result is still a very accurate model. Neglecting the other coefficients removes the dependence on terrestrial longitude, leaving the following potential function at point $P(r,\psi)$:

$$V(r,\psi) = \frac{GM}{r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 J_2(3\sin^2\psi - 1)\right], \tag{1.6-24}$$

where r is the length of the geocentric position vector and ψ is the geocentric latitude. The Earth'*s gravitational constant*, *GM,* is the product of Earth's mass and the *universal gravitational constant* of the inverse square law. Its EGM96 value, with the mass of the atmosphere included, was given in Equation (1.6-2*b*). The constant J_2 is given by

$$J_2 = -\sqrt{5}\,\overline{C}_{2,0} = 1.082\,626\,684 \times 10^{-3}, \tag{1.6-25}$$

where $\overline{C}_{2,0}$ is the actual EGM96 coefficient.

The gradients of the potential function are easily evaluated in geocentric coordinates. When this is done and the results are transformed into the ECEF system, we obtain the following gravitation model for the *gravitational acceleration*, **G**:

$$G^{ecef} = -\frac{GM}{r^2}\begin{bmatrix}\left[1 + \frac{3}{2}\left(\frac{a}{r}\right)^2 J_2(1 - 5\sin^2\psi)\right] p_x/r \\ \left[1 + \frac{3}{2}\left(\frac{a}{r}\right)^2 J_2(1 - 5\sin^2\psi)\right] p_y/r \\ \left[1 + \frac{3}{2}\left(\frac{a}{r}\right)^2 J_2(3 - 5\sin^2\psi)\right] p_z/r\end{bmatrix}, \tag{1.6-26}$$

where the geocentric position vector is **p**, with ECEF components p_x, p_y, p_z and length r, and the geocentric latitude is given by $\sin\psi = p_z/r$. This model is accurate to about 30×10^{-3} to 35×10^{-3} cm/s^2 rms, but local deviations can be quite large. Note that the x- and y-components are identical because there is no longitude dependence. The model can also be converted to geodetic coordinates using the relationships given earlier.

The weight of an object on Earth is determined by the gravitational attraction ($m\mathbf{G}$) minus the mass times the centripetal acceleration needed to produce the circular motion in inertial space at geocentric position vector **p**. Dividing the weight of the object by its mass gives the *gravity vector* **g**. Therefore, the vector equation for **g** is

$$\mathbf{g} = \mathbf{G} - \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}) \tag{1.6-27}$$

As noted earlier, at Earth's surface **g** is accurately normal to the geoid, points downward, and defines the local vertical. When Equation (1.6-26) is substituted for **G** in (1.6-27) and the equation is resolved in the NED system, we find that **g** is almost entirely along the down axis with a variable *north component* of only a few micro-gs. This is a modeling error, since deflection of the vertical is not explicitly included in the model. The *down component* of **g** given by the model, at Earth's surface, varies sinusoidally from 9.780 m/s^2 at the equator to 9.806 m/s^2 at 45° geodetic latitude and 9.832 m/s^2 at the poles. Our simplified "flat-Earth" equations of motion will use a **g** vector that has only a down component, $\mathbf{g}_D$, and is measured at Earth's surface. When a constant value of $\mathbf{g}_D$ is to be used (e.g., in a simulation), the value at 45° latitude is taken as the standard value of gravity (actually defined to be 9.80665 m/s^2).

Gravitation and Accelerometers

The basic principle used in nearly all accelerometers is measurement, indirectly, of the force, **F**, that must be applied (mechanically or by means of a magnetic or electrostatic field) to prevent a known "proof" mass from accelerating with respect to its instrument case when the case is being accelerated. Thus, apart from a small transient and/or steady-state error determined by the dynamics of the proof mass "rebalancing" servo and the type of acceleration signal, the acceleration of the proof mass is the same as the acceleration of the case. An accelerometer is usually a single-axis device, but here we will write vector equations for the acceleration experienced by the accelerometer. The gravitational field always acts on the proof mass, m, and so the acceleration, **a**, of the proof mass is given by the vector sum:

$$\mathbf{a} = \frac{\mathbf{F}}{m} + \mathbf{G} \equiv \mathbf{f} + \mathbf{G},$$

where $\mathbf{F}/m$ is the force per unit mass applied to the proof mass, called the *specific force*, **f**. The accelerometer calibration procedure determines the *scale factor*, s, relating the output quantity to the specific force, and so the accelerometer equation is

$$\text{Accelerometer output} = s\mathbf{f} = s(\mathbf{a} - \mathbf{G}) \tag{1.6-28}$$

Equation (1.6-28) shows that an accelerometer is basically a specific force measuring device. When acceleration must be measured precisely (as in inertial navigation), an accurate model of **G** (as a function of position) is essential. In other, lower precision applications the accelerometer "bias" of **G** is not a limitation.

When an accelerometer's sensitive axis is horizontal, the bias is zero. When an accelerometer, with geocentric position vector **p**, is stationary with respect to Earth, its acceleration and specific force reading are given by

$$\mathbf{a} = \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p})$$
$$\mathbf{f} = \mathbf{a} - \mathbf{G} = -\mathbf{g} \tag{1.6-29}$$

This specific force equation shows that the true force acting on the accelerometer has a magnitude equal to the accelerometer weight and is in the negative direction of the **g** vector (i.e., the reaction of the surface on which the accelerometer is sitting). Single-axis accelerometers intended to be used with their sensitive axis vertical can be calibrated in "g-units," using the standard gravity or the local gravity, so that the specific force reading is

$$\mathbf{f}_D^{ned} = -1\,\text{g-unit}$$

At a different position from the calibration location, the accelerometer could be corrected to the local gravity, but if acceleration is to be calculated accurately, a correction would have to be applied for the different (in general) centripetal acceleration. For accelerometers in motion over Earth, we must evaluate a transport acceleration to relate accelerometer acceleration to vehicle acceleration, as shown in Chapter 4 for the normal-acceleration control augmentation system.

1.7 RIGID-BODY DYNAMICS

In this section we finally put together the ideas and equations from the previous sections to obtain a set of state equations that describe the 6-DoF motion of a rigid aerospace vehicle (defined to be frame F_b). We shall deal first with the angular motion of the vehicle in response to torques generated by aerodynamic, thrust, or any other forces, whose lines of action do not pass through the vehicle cm. By using the vehicle cm as a reference point, the rotational dynamics of the aircraft can be separated from the translational dynamics (Wells, 1967); we therefore use a body-fixed coordinate system, *bf*, with its origin at the cm to compute moments about the origin. A (nonzero) torque vector produces a rate of change of angular momentum vector, but then, to relate angular momentum to the mass distribution of a specific body, we must use the coordinate system *bf* and switch to matrix equations to obtain the components of the angular acceleration vector. Angular acceleration components integrate to angular velocity components, but then the three degrees of freedom in angular displacement are obtained from nonlinear equations such as the Euler equations (1.4-4). For an aircraft, the coordinate system *bf* is usually forward-right-down, *frd*, as described in Section 1.3.

The translational equations are more straightforward, the acceleration of the vehicle cm is obtained from the vector sum of the various forces, and their lines of action do not have to pass through the cm because the effect of any offset is incorporated into the moment equations. The equations are expressed in terms of motion relative to Earth and introduce the usual Coriolis and centripetal terms. Aerodynamic and thrust effects depend on motion relative to the surrounding atmosphere and so, when the atmosphere is moving relative to Earth, we must introduce an auxiliary equation to compute the *relative wind*.

Angular Motion

Here, we develop the equations for the rotational dynamics, which will be the same for both the flat-Earth and oblate-rotating-Earth equations of motion. The following definitions will be needed:

$$F_i = \text{an inertial reference frame}$$

$$F_b = \text{the body of the rigid vehicle}$$

$$\mathbf{v}_{cm/i} = \text{velocity of vehicle cm in } F_i$$

$$\boldsymbol{\omega}_{b/i} = \text{angular velocity of } F_b \text{ with respect to } F_i$$

$$\mathbf{M} = \text{vector sum of all moments about the cm}$$

The moment, $\mathbf{M}$, may be generated by aerodynamic effects, any propulsion-force components not acting through the cm, and attitude control devices.

Let the angular momentum vector of a rigid body in the inertial frame and taken about the cm be denoted by $\mathbf{h}$. It is shown in textbooks on classical mechanics (Goldstein, 1980) that the derivative of $\mathbf{h}$ taken in the inertial frame is equal to the vector moment $\mathbf{M}$ applied about the cm. Therefore, analogously to Newton's law for translational momentum, we write

$$\mathbf{M} = {}^{i}\dot{\mathbf{h}}_{cm/i} \tag{1.7-1}$$

The angular momentum vector can be found by considering an element of mass δm with position vector $\mathbf{r}$ relative to the cm. Its translational momentum is given by

$$(\mathbf{v}_{cm/i} + \boldsymbol{\omega}_{b/i} \times \mathbf{r})\delta m$$

The angular momentum of this particle about the cm is the moment of the translational momentum about the cm, or

$$\boldsymbol{\delta}\mathbf{h} = \mathbf{r} \times (\mathbf{v}_{cm/i} + \boldsymbol{\omega}_{b/i} \times \mathbf{r})\delta m$$

and for the whole body,

$$\mathbf{h}_{cm/i} = \iiint \mathbf{r} \times (\mathbf{v}_{cm/i} + \boldsymbol{\omega}_{b/i} \times \mathbf{r})\, dm$$

In order to integrate this equation over the whole body, we must choose a coordinate system and, to avoid a time-varying integrand, the coordinate system must be fixed in the body. Let the position coordinates in body-fixed axes, *bf*, be

$$\mathbf{r}^{bf} = [x \;\; y \;\; z]^T$$

The corresponding matrix equation for $\mathbf{h}$ is obtained by replacing the cross-products by $\widetilde{\mathbf{r}}$ and noting that $\mathbf{v}$ and $\boldsymbol{\omega}$ are constants for the purposes of integration. The first term contains only integrals whose integrands have a position coordinate integrated dm and, by definition of the cm, they all integrate to zero. The second term is

$$\mathbf{h}^{bf}_{cm/i} = -\iiint \tilde{\mathbf{r}}(\tilde{\mathbf{r}}\boldsymbol{\omega}^{bf}_{b/i})\,dm = -\iiint \tilde{\mathbf{r}}^2 dm\,\boldsymbol{\omega}^{bf}_{b/i}$$

giving

$$\mathbf{h}^{bf}_{cm/i} = \iiint \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm\,\boldsymbol{\omega}^{bf}_{b/i} \tag{1.7-2}$$

The result of this integration is a 3×3 constant matrix that is defined to be the *inertia matrix* J^{bf} for the rigid body; it contains scalar *moments* and *cross-products of inertia*, for example:

$$\text{Moment of Inertia about } x\text{-axis} = J_{xx} = \int (y^2 + z^2)\,dm$$

$$\text{Cross-Product of Inertia } J_{xy} \equiv J_{yx} = \int xy\,dm$$

and so,

$$\mathbf{h}^{bf}_{cm/i} = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{yy} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{zz} \end{bmatrix} \boldsymbol{\omega}^{bf}_{b/i} \equiv J^{bf}\,\boldsymbol{\omega}^{bf}_{b/i} \tag{1.7-3}$$

It was necessary to choose a coordinate system to obtain this matrix and, consequently, it is not possible to obtain a vector equation of motion that is completely coordinate free. In more advanced treatments this paradox is avoided by the use of tensors. Note also that J is a real symmetric matrix and therefore has special properties that are discussed below. Various formulae and theorems are available for calculating J^{bf} for a composite body like an aircraft, and it can be estimated experimentally with the aircraft mounted on a turntable.

With the angular momentum expressed in terms of the inertia matrix and angular velocity vector of the complete rigid body, Equation (1.7-1) can be evaluated. Since the inertia matrix is known, and constant in the body frame, it will be convenient to replace the derivative in (1.7-1) by a derivative taken in the body frame:

$$\mathbf{M} = {}^{i}\dot{\mathbf{h}}_{cm/i} = {}^{b}\dot{\mathbf{h}}_{cm/i} + \boldsymbol{\omega}_{b/i} \times \mathbf{h}_{cm/i} \tag{1.7-4}$$

Now, differentiating (1.7-3) in F_b, with J constant, and taking body-fixed components, we obtain

$$\mathbf{M}^{bf} = J^{bf}\,{}^{b}\dot{\boldsymbol{\omega}}^{bf}_{b/i} + \tilde{\boldsymbol{\omega}}^{bf}_{b/i} J^{bf}\,\boldsymbol{\omega}^{bf}_{b/i}$$

A rearrangement of this equation gives the *state equation for angular velocity*:

$${}^{b}\dot{\boldsymbol{\omega}}^{bf}_{b/i} = (J^{bf})^{-1}\left[\mathbf{M}^{bf} - \tilde{\boldsymbol{\omega}}^{bf}_{b/i} J^{bf}\,\boldsymbol{\omega}^{bf}_{b/i}\right] \tag{1.7-5}$$

This state equation is widely used in simulation and analysis of rigid-body motion from satellites to ships. It can be solved numerically for the angular velocity given the inertia matrix and the torque vector, and its features will now be described.

The assumption that the inertia matrix is constant is not always completely true. For example, with aircraft the inertias will change slowly as fuel is transferred and burned. Also, the inertias will change abruptly if an aircraft is engaged in dropping stores. These effects can usually be adequately accounted for in a simulation by simply changing the inertias in (1.7-5) without accounting for their rates of change. As far as aircraft control system design is concerned, point designs are done for particular flight conditions, and interpolation between point designs can be used when the aircraft mass properties change. This is more likely to be done to deal with movement of the vehicle cm and the resultant effect on static stability (Chapter 2).

The inverse of the inertia matrix occurs in (1.7-5), and because of symmetry this has a relatively simple form:

$$J^{-1} = \frac{1}{\Delta}\begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_4 & k_5 \\ k_3 & k_5 & k_6 \end{bmatrix} \tag{1.7-6}$$

where

$$\begin{aligned} k_1 &= (J_{yy}J_{zz} - J_{yz}^2)/\Delta, \quad & k_2 &= (J_{yz}J_{zx} + J_{xy}J_{zz})/\Delta \\ k_3 &= (J_{xy}J_{yz} + J_{zx}J_{yy})/\Delta, \quad & k_4 &= (J_{zz}J_{xx} - J_{zx}^2)/\Delta \\ k_5 &= (J_{xy}J_{zx} + J_{yz}J_{xx})/\Delta, \quad & k_6 &= (J_{xx}J_{yy} - J_{xy}^2)/\Delta \end{aligned}$$

$$\Delta = J_{xx}J_{yy}J_{zz} - 2J_{xy}J_{yz}J_{zx} - J_{xx}J_{yz}^2 - J_{yy}J_{zx}^2 - J_{zz}J_{xy}^2$$

A real symmetric matrix has real eigenvalues and, furthermore, a repeated eigenvalue of order p still has associated with it p linearly independent eigenvectors. Therefore, a similarity transformation can be found that reduces the matrix to a real diagonal form. In the case of the inertia matrix this means that we can find a set of coordinate axes in which the inertia matrix is diagonal. These axes are called the *principal axes*. The inverse of a diagonal matrix is also diagonal and the angular velocity state equation takes its simplest form, known as *Euler's equations of motion*.

At this point it is convenient to be more specific and choose the body-fixed axes to be *frd*, so that we can use standard aircraft yaw, pitch, and roll symbols:

$$M^{frd} = [\ell \ m \ n]^T, \qquad \omega_{b/i}^{frd} = [P \ Q \ R]^T \tag{1.7-7}$$

Then Euler's equations of motion are

$$\begin{aligned} \dot{P} &= \frac{(J_y - J_z)QR}{J_x} + \frac{\ell}{J_x} \\ \dot{Q} &= \frac{(J_z - J_x)RP}{J_y} + \frac{m}{J_y} \\ \dot{R} &= \frac{(J_x - J_y)PQ}{J_z} + \frac{n}{J_z} \end{aligned} \tag{1.7-8}$$

and the double-subscript notation on the moments of inertia has been dropped. The equations involve cyclic permutation of the rate and inertia components; they are inherently coupled because angular rates about any two axes produce an angular acceleration about the third. This *inertia coupling* has important consequences for aircraft maneuvering rapidly at high angles of attack; we examine its effects in Chapter 4. The stability properties of the Euler equations are interesting and will be studied in Problem 1.7-3.

The angular velocity state equation is again simplified when applied to aircraft since for most aircraft the *frd* x-z plane is a plane of symmetry. Under this condition, for every product $y_i\, z_j$ or $y_i\, x_j$ in an inertia computation there is a product that is identical in magnitude but opposite in sign. Therefore, only the J_{xz} cross-product of inertia is nonzero. A notable exception is an oblique-wing aircraft (Travassos et al., 1980), which does not have a plane of symmetry. Under the plane-of-symmetry assumption the inertia matrix and its inverse reduce to

$$J^{frd} = \begin{bmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{bmatrix}, \quad (J^{frd})^{-1} = \frac{1}{\Gamma} \begin{bmatrix} J_z & 0 & J_{xz} \\ 0 & \left(\frac{\Gamma}{J_y}\right) & 0 \\ J_{xz} & 0 & J_x \end{bmatrix} \tag{1.7-9}$$

$$\Gamma = J_x J_z - J_{xz}^2$$

If the angular velocity state equation (1.7-5) is expanded using the torque vector in (1.7-7) and the simple inertia matrix given by (1.7-9), the coupled, nonlinear angular acceleration equations are

$$\begin{aligned} \Gamma \dot{P} &= J_{xz}(J_x - J_y + J_z)PQ - \left[J_z\left(J_z - J_y\right) + J_{xz}^2\right] QR + J_z \ell + J_{xz} n \\ J_y \dot{Q} &= (J_z - J_x)RP - J_{xz}(P^2 - R^2) + m \\ \Gamma \dot{R} &= -J_{xz}(J_x - J_y + J_z)QR + \left[J_x\left(J_x - J_y\right) + J_{xz}^2\right] PQ + J_{xz} \ell + J_x n \end{aligned} \tag{1.7-10}$$

In the analysis of angular motion we have so far neglected the angular momentum of any spinning rotors. Technically this violates the rigid-body assumption, but the resulting equations are valid. Note that, strictly, we require axial symmetry of the spinning rotors; otherwise the position of the vehicle cm will vary. This is not a restrictive requirement because it is also a requirement for dynamically balancing the rotors. The effects of the additional angular momentum may be quite significant. For example, a number of World War I aircraft had a single "rotary" engine that had a fixed crankshaft and rotating cylinders. The gyroscopic effects caused by the large angular momentum of the engine gave these aircraft tricky handling characteristics. In the case of a small jet with a single turbofan engine on the longitudinal axis, the effects are smaller. To represent the effect, a constant vector can be added to the angular momentum vector in (1.7-3); therefore, let

$$\mathbf{h}^{bf}_{cm/i} = J^{bf}\boldsymbol{\omega}^{bf}_{b/i} + \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}_{\text{engines}} \qquad (1.7\text{-}11a)$$

If this analysis is carried through, the effect is to add the following terms, respectively, to the right-hand sides of the three equations (1.7-10):

$$\begin{aligned} &J_z(Rh_y - Qh_z) + J_{xz}(Qh_x - Ph_y) \\ &-Rh_x + Ph_z \\ &J_{xz}(Rh_y - Qh_z) + J_x(Qh_x - Ph_y) \end{aligned} \qquad (1.7\text{-}11b)$$

To complete the set of equations for angular motion, an attitude state equation is required. Here, with the flat-Earth equations in mind, we will assume that this will be the Euler kinematical equations (1.4-4). A direction cosine matrix can be computed from the Euler angles and will be needed in the translational equations. Thus, the translational equations will be coupled to the rotational equations. We now have all of the state equations for the angular motion dynamics, and we will turn our attention to the translational motion of the cm.

Translational Motion of the Center of Mass

Vector State Equations We begin by applying Newton's second law to the motion of a constant-mass rigid body near the surface of Earth to find the inertial derivative of velocity under the influence of aerodynamic, propulsion, and mass attraction forces. We shall find state equations, in vector form at first, for position and velocity.

Some considerations for the choice of state variables are that the velocity vector can be chosen for convenience in either navigation over Earth or aerodynamic force and moment calculations on the body. The position vector can be taken from an arbitrary fixed point in the rigid-Earth frame, F_e. If the variation in gravity is significant over the trajectory, then the position vector should be taken from Earth's cm but will be over six million meters long. For short-range navigation it can more conveniently be taken from an initial point on the surface of Earth, but then latitude and longitude cannot easily be calculated. We will first derive equations for convenience in navigation using velocity of the vehicle cm in F_e and taking the position vector from Earth's cm.

Earth's cm is a fixed point common to both the inertial frame, F_i, and the Earth frame, F_e, so the derivative of a position vector from the cm will give either inertial velocity or Earth velocity, according to the frame in which the derivative is taken. Derivatives in F_i and F_e are related through Earth's angular velocity vector, $\omega_{e/i}$, according to the Coriolis equation. In addition to the above frames, the rigid-body frame F_b will be required. We also define the following scalars and vectors:

$$
\begin{aligned}
\mathrm{m} &\equiv \text{Vehicle (constant) mass} \\
O &\equiv \text{Earth cm} \\
\mathbf{p}_{cm/O} &\equiv \text{Vehicle cm position relative to } O \\
\mathbf{v}_{cm/i} &\equiv {}^{i}\dot{\mathbf{p}}_{cm/O} = \text{Velocity of the cm in } F_i \\
\mathbf{v}_{cm/e} &\equiv {}^{e}\dot{\mathbf{p}}_{cm/O} = \text{Velocity of the cm in } F_e \\
\boldsymbol{\omega}_{x/y} &\equiv \text{Angular velocity of frame } x \text{ with respect to frame } y \\
\mathbf{F} &\equiv \text{Vector sum of forces at cm} \\
\mathbf{G} &\equiv \text{Earth's gravitation vector} \\
\mathbf{g} &\equiv \text{Earth's gravity vector, } \mathbf{g} = \mathbf{G} - \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{cm/O})
\end{aligned}
$$

From the above explanation and definitions, the position and velocity state equations, in vector form, are

$$
\begin{aligned}
{}^{e}\dot{\mathbf{p}}_{cm/O} &= \mathbf{v}_{cm/e} \\
{}^{e}\dot{\mathbf{v}}_{cm/e} &= \mathbf{a}_{cm/e}
\end{aligned}
\tag{1.7-12}
$$

To apply Newton's laws we use Equation (1.5-5), substitute ($\frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{G}$) for the inertial acceleration, and solve for the relative acceleration (i.e., the derivative of the velocity state in F_e):

$$
{}^{e}\dot{\mathbf{v}}_{cm/e} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{G} - \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{cm/O}) - 2\,\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{cm/e} \tag{1.7-13}
$$

This velocity state equation together with the position state equation is suitable for accurate simulation of flight around the oblate, rotating Earth. Latitude and longitude and **G** can be calculated from the geocentric position vector. The Coriolis term was examined in Section 1.5; a rule of thumb is to consider the Coriolis effect significant for speeds over about 2000 ft/s.

Finally, true airspeed is needed for calculating aerodynamic effects and propulsion system performance; therefore we define a relative velocity vector, $\mathbf{v}_{rel}$, by

$$
\mathbf{v}_{rel} = \mathbf{v}_{cm/e} - \mathbf{v}_{W/e}, \tag{1.7-14}
$$

where $\mathbf{v}_{w/e}$ is the wind velocity taken in F_e. This is an auxiliary equation that will be needed with the state equations.

The full set of 6-DoF oblate, rotating-Earth matrix equations will be illustrated in Section 1.8, and we will now simplify the vector equations (1.7-12) and (1.7-13) to obtain the much more commonly used *flat-Earth equations*. If the **g** vector can be considered to be independent of latitude and taken to be approximately constant or dependent only on height above Earth's surface, the position vector can be taken from a point of interest, Q, on Earth's surface. The position vector then need no longer be six million meters long, but latitude and longitude cannot be calculated from it, and it will give only approximate distances over Earth's surface. If Q is the fixed point on Earth's surface, then the position state equation is

$$^{e}\dot{\mathbf{p}}_{cm/Q} = \mathbf{v}_{cm/e} \qquad (1.7\text{-}15)$$

As noted at the beginning of this subsection, the velocity state equation can alternatively be expressed in terms of a vector derivative taken in the vehicle body frame, and then when components are taken in a body-fixed coordinate system, we have a set of component derivatives that can be integrated to provide velocity components that determine aerodynamic effects. Changing derivatives in Equation (1.7-13) and substituting for $\boldsymbol{g}$, we have

$$^{b}\dot{\mathbf{v}}_{cm/e} + \boldsymbol{\omega}_{b/e} \times \mathbf{v}_{cm/e} = {}^{e}\dot{\mathbf{v}}_{cm/e} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{g} - 2\,\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{cm/e}$$

$$^{b}\dot{\mathbf{v}}_{cm/e} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{g} - (\boldsymbol{\omega}_{b/e} + 2\,\boldsymbol{\omega}_{e/i}) \times \mathbf{v}_{cm/e} \qquad (1.7\text{-}16a)$$

Alternatively, using the additive property of angular velocity vectors,

$$^{b}\dot{\mathbf{v}}_{cm/e} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{g} - (\boldsymbol{\omega}_{b/i} + \boldsymbol{\omega}_{e/i}) \times \mathbf{v}_{cm/e} \qquad (1.7\text{-}16b)$$

A further assumption is that Earth is an inertial reference frame; Earth's angular velocity can then be neglected, and $\boldsymbol{\omega}_{b/i} \equiv \boldsymbol{\omega}_{b/e}$, and Equations (1.7-16*a*) and (1.7-16*b*) both reduce to

$$^{b}\dot{\mathbf{v}}_{cm/e} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{g} - \boldsymbol{\omega}_{b/e} \times \mathbf{v}_{cm/e} \qquad (1.7\text{-}16c)$$

These approximations are the basis of the flat-Earth equations of motion, described in the next section.

The Flat-Earth Equations, Matrix Form As explained above, the flat-Earth equations are not suitable for precise determination of position over Earth, but they are widely used in simulations to study aircraft performance and dynamic behavior and are used to derive linear state-space models for analytical studies and flight control system design. The assumptions for the flat-Earth equations will now be formally listed:

(i) The Earth frame is an inertial reference frame.
(ii) Position is measured in a tangent-plane coordinate system, *tp*.
(iii) The gravity vector is normal to the tangent plane and constant in magnitude.

Some consequent assumptions are:

(iv) Height above the tangent plane is a good approximation to true height above Earth's surface, and the horizontal projection of the position vector gives a good approximation to distance traveled over Earth's surface (this will be reasonable up to a few hundred miles from the tangent-plane origin).

(v) The attitude of the vehicle in the tangent-plane coordinate system is a good approximation to true geographic attitude at the position of the vehicle.

Equation (1.7-16*c*) already incorporates the first flat-Earth assumption, and the position state variable is already referred to the tangent-plane origin; we must now make choices of coordinate systems for the variables in the state equations and make provisions to calculate a rotation matrix to convert from one system to another where necessary.

A *frd* coordinate system fixed in F_b is very convenient for the velocity vector derivative in F_b, for the aerodynamic and propulsion forces, and for the vehicle angular velocity vector (which uses body-axes components in the angular motion equations). This choice is less convenient for the $\boldsymbol{g}$ vector and Earth angular velocity vector; these are known in Earth-fixed coordinate systems and must be rotated into the body axes using a vehicle-attitude DCM obtained from the attitude state equations. In the flat-Earth equations, the changing attitude of the vehicle is almost invariably adequately modeled with the simple Euler angle kinematical equations, (1.4-4). These relate a *frd* body-fixed system to a *ned* system, here the *ned* system is the *tangent-plane system*, *tp*, fixed in the Earth. The Euler angles can then be used to construct the $C_{frd/tp}$ DCM, which must be done before the position and velocity state equations can be evaluated.

The set of 6-DoF state equations will be completed by the addition of the angular velocity state equation (1.7-5), with $\boldsymbol{\omega}_{b/i} \equiv \boldsymbol{\omega}_{b/e}$ as the state variable. The state vector is now

$$X^T = \left[\left(\mathbf{p}_{cm/Q}^{tp}\right)^T \ \boldsymbol{\Phi}^T \ \left(\mathbf{v}_{cm/e}^{frd}\right)^T \ \left(\boldsymbol{\omega}_{b/e}^{frd}\right)^T \right] \tag{1.7-17}$$

Using the current values of the state variables we evaluate the state derivatives as follows. The rotation matrix is calculated before the position and velocity state equations as noted above. Aerodynamic angle derivatives can be calculated from the translational velocity derivatives, and therefore the translational velocity state equation is placed ahead of the angular velocity state equation, where those derivatives are more significant (this is explained in detail in Chapter 2) and we have the following set of equations:

$$\begin{aligned}
C_{frd/tp} &= fn(\Phi) \\
\dot{\boldsymbol{\Phi}} &= H(\Phi)\, \boldsymbol{\omega}_{b/e}^{frd} \\
{}^{e}\dot{\mathbf{p}}_{cm/Q}^{tp} &= C_{tp/frd}\, \mathbf{v}_{cm/e}^{frd} \\
{}^{b}\dot{\mathbf{v}}_{cm/e}^{frd} &= \frac{1}{\mathrm{m}} \mathbf{F}^{frd} + C_{frd/tp}\mathbf{g}^{tp} - \widetilde{\boldsymbol{\omega}}_{b/e}^{frd}\, \mathbf{v}_{cm/e}^{frd} \\
{}^{b}\dot{\boldsymbol{\omega}}_{b/e}^{frd} &= \left(J^{frd}\right)^{-1} \left[\mathbf{M}^{frd} - \widetilde{\boldsymbol{\omega}}_{b/e}^{frd} J^{frd} \boldsymbol{\omega}_{b/e}^{frd}\right]
\end{aligned} \tag{1.7-18}$$

The 6-DoF flat-Earth equations contained in (1.7-18) are twelve (scalar) coupled, nonlinear, first-order differential equations and an auxiliary equation. Coupling exists because angular acceleration integrates to angular velocity, which determines the Euler angle rates, which in turn determine the direction cosine matrix. The direction cosine matrix is involved in the state equations for position and velocity; position (the altitude component) and velocity determine aerodynamic effects which determine angular acceleration. Coupling is also present through the translational velocity. These interrelationships will become more apparent in Chapter 2.

To complete the flat-Earth assumptions, **g** in tangent-plane coordinates will be

$$\mathbf{g}^{tp} = [0 \;\; 0 \;\; g_D]^T$$

with the down component, g_D, equal to the standard gravity (9.80665 m/s^2), or the local value. Aerodynamic calculations will require the equation for the velocity vector relative to the surrounding air [from Equation (1.7-14)]:

$$\mathbf{v}_{rel}^{frd} = \mathbf{v}_{cm/e}^{frd} - \mathbf{C}_{frd/tp}\,\mathbf{v}_{W/e}^{tp} \tag{1.7-19}$$

Some additional auxiliary equations will be needed to compute all of the aerodynamic effects, but these will be introduced in Chapter 2.

An interesting alternative to the translational velocity state equation in (1.7-18) can be derived by using relative velocity as the state variable. The vector form of the relative velocity equation is (1.7-14). If this equation is differentiated in the body-fixed frame and used to eliminate $\mathbf{v}_{cm/e}$ and its derivative from the vector equation for the translational acceleration, we obtain

$$^{b}\dot{\mathbf{v}}_{rel} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{g} - \boldsymbol{\omega}_{b/e} \times \mathbf{v}_{rel} - \left(\boldsymbol{\omega}_{b/e} \times \mathbf{v}_{W/e} + {}^{b}\dot{\mathbf{v}}_{W/e}\right)$$

The term in parentheses is the derivative of the wind velocity, taken in F_e, so we can write

$$^{b}\dot{\mathbf{v}}_{rel} = \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{g} - \boldsymbol{\omega}_{b/e} \times \mathbf{v}_{rel} - {}^{e}\dot{\mathbf{v}}_{W/e} \tag{1.7-20}$$

The last term on the right can be used as a way of introducing gust inputs into the model or can be set to zero for steady winds. Taking the latter course and introducing components in the body-fixed system give

$$^{b}\dot{\mathbf{v}}_{rel}^{frd} = \frac{1}{\mathrm{m}}\mathbf{F}^{frd} + C_{frd/tp}\mathbf{g}^{tp} - \widetilde{\boldsymbol{\omega}}_{b/e}^{frd}\,\mathbf{v}_{rel}^{frd} \tag{1.7-21}$$

This equation is an alternative to the velocity state equation in (1.7-18), and the position state equation therein must then be modified to use the sum of the relative and wind velocities.

The negative of $\mathbf{v}_{rel}$ is the *relative wind*, which determines the aerodynamic forces and moments on the aerodynamic vehicle and hence its dynamic behavior. In Chapter 2 we will use Equation (1.7-21) to make a model that is suitable for studying the dynamic behavior. Chapter 2 shows how the flat-Earth equations can be "solved" analytically. Chapter 3 shows how they can be solved simultaneously by numerical integration for the purposes of flight simulation.

1.8 ADVANCED TOPICS

In this section we have derived two additional sets of kinematical equations for the attitude of a rotating body. The resulting attitude state equations have better numerical properties than the Euler angle state equations [Equations (1.4-4)]. The first set of kinematical equations relates the derivatives of the elements of a direction cosine matrix to the components of the associated angular velocity vector; we will refer to them as the *Poisson kinematical equations* (PKEs). They involve more mathematical operations than the Euler kinematical equations but are free from the singularity at 90° pitch attitude.

The second set of kinematical equations are based on quaternions, a complex number algebra invented by Sir William Rowan Hamilton in 1843 in an attempt to generalize ordinary complex numbers to three dimensions. We have derived many properties of quaternions, applied them to coordinate rotations, and related them to the direction cosine matrix and to Euler angles. Next, we have derived a set of quaternion state equations for the attitude of a rotating body. These quaternion state equations have additional numerical advantages over the PKE.

Finally, we have returned to the 6-DoF equations of motion for a rigid body moving around the oblate, rotating Earth, examined their properties, and explained how they are used. For this motion it is more appropriate to use the PKEs or the quaternion state equations to track the attitude of the rigid body.

Poisson's Kinematical Equations

Consider, once again, the coordinate transformation, $C_{bf/rf}$, between systems fixed in a reference frame, $\mathbf{F}_r$, and in a rigid body, $\mathbf{F}_b$, when the body has an angular velocity vector $\boldsymbol{\omega}_{b/r}$ with respect to the reference frame. Applying the transformation to the components of an arbitrary vector, $\mathbf{u}$, we have

$$\mathbf{u}^{bf} = C_{bf/rf}(t)\,\mathbf{u}^{rf}$$

A fixed unit vector in $\mathbf{F}_r$ corresponds to a unit-length vector with time-varying components in $\mathbf{F}_b$. Let this be the vector $\boldsymbol{c}_i$, with components in $\mathbf{F}_b$ given by the ith column of $C_{bf/rf}$. Now, applying the equation of Coriolis to the derivative of this vector in the two frames, we have

$$0 = {}^r\dot{\mathbf{c}}_i = {}^b\dot{\mathbf{c}}_i + \boldsymbol{\omega}_{b/r} \times \mathbf{c}_i, \quad i = 1, 2, 3$$

Take body-fixed coordinates:

$$0 = {}^b\dot{\mathbf{c}}_i^{bf} + \tilde{\boldsymbol{\omega}}_{b/r}^{bf}\,\mathbf{c}_i^{bf}, \qquad i = 1, 2, 3$$

The term ${}^b\dot{\mathbf{c}}_i^{bf}$ is the derivative of the ith column of $C_{bf/rf}$. If we combine the three equations into one matrix equation, the result is

$$\dot{C}_{bf/rf} = -\tilde{\boldsymbol{\omega}}_{b/r}^{bf}\,C_{bf/rf} \tag{1.8-1}$$

These equations are known as *Poisson's kinematical equations,* PKEs, or in inertial navigation as the s*trapdown equation*. Whereas Equations (1.4-4) deal with the Euler angles, this equation deals directly with the elements of the rotation matrix. Compared to the Euler kinematical equations, the PKEs have the advantage of being singularity free and the disadvantage of a large amount of redundancy (nine scalar equations). When they are used in a simulation, the Euler angles are not directly available and must be calculated from the direction cosine matrix as in Equations (1.3-11).

The Equation of Coriolis

In Section 1.4 the equation of Coriolis was derived using vectors; a matrix form can be derived with the use of the PKEs. Starting from a time-varying coordinate transformation of the components of a general vector **u**,

$$\mathbf{u}^{bf} = C_{bf/af}\,\mathbf{u}^{af},$$

with coordinate systems *af* and *bf* fixed in $\mathbf{F}_a$ and $\mathbf{F}_b$, respectively, differentiate the arrays on both sides of the equation. Differentiating the arrays is equivalent to taking derivatives in their respective frames, with components taken in the systems fixed in the frames; therefore,

$${}^{b}\dot{u}^{bf} = C_{bf/af}\,{}^{a}\dot{\mathbf{u}}^{af} + \dot{C}_{bf/af}\,\mathbf{u}^{af}$$

or

$${}^{b}\dot{\mathbf{u}}^{bf} = {}^{a}\dot{\mathbf{u}}^{bf} + \dot{C}_{bf/af}\,\mathbf{u}^{af}$$

Now use the Poisson equations to replace $\dot{C}_{bf/af}$ (note that we used the equation of Coriolis to derive the Poisson equations, but they could have been derived in other ways),

$${}^{b}\dot{\mathbf{u}}^{bf} = {}^{a}\dot{\mathbf{u}}^{bf} - \tilde{\boldsymbol{\omega}}^{bf}_{b/a}\,C_{bf/af}\,\mathbf{u}^{af}$$

or

$${}^{b}\dot{\mathbf{u}}^{bf} = {}^{a}\dot{\mathbf{u}}^{bf} + \tilde{\boldsymbol{\omega}}^{bf}_{a/b}\,\mathbf{u}^{bf} \tag{1.8-2}$$

Equation (1.8-2) is the equation of Coriolis resolved in coordinate system *bf*.

Quaternions

Quaternions are introduced here because of their "all-attitude" capability and numerical advantages in simulation and control. They are now widely used in simulation, robotics, guidance and navigation calculations, attitude control, and graphics animation. We will review enough of their properties to use them for coordinate rotation in the following subsections. W. R. Hamilton (1805–1865) introduced the quaternion form:

$$x_0 + x_1 i + x_2 j + x_3 k \tag{1.8-3a}$$

with the *imaginary operators* given by

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad ki = j = -ik, \text{ etc.}$$

in an attempt to generalize complex numbers in a plane to three dimensions.

Quaternions obey the normal laws of algebra, except that multiplication is not commutative. Multiplication, indicated by "$*$", is defined by the associative law. For example, if,

$$r = p * q = (p_0 + p_1 i + p_2 j + p_3 k) * (q_0 + q_1 i + q_2 j + q_3 k)$$

then,

$$r = p_0 q_0 + p_0 q_1 i + p_0 q_2 j + p_0 q_3 k + p_1 q_0 i + p_1 q_1 i^2 + \cdots$$

By using the rules for i, j, k, products, and collecting terms, the answer can be written in various forms, for example,

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Alternatively, by interpreting i, j, k as unit vectors, the quaternion (1.8-3*a*) can be treated as $q_0 + \mathbf{q}$, where $\mathbf{q}$ is the quaternion vector part, with components q_1, q_2, q_3 along, **i**, **j**, **k**, respectively. We will write the quaternion as an array, formed from q_0 and the vector components, thus:

$$p = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \Rightarrow \begin{bmatrix} p_0 \\ \mathbf{p}^r \end{bmatrix}, \qquad q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \Rightarrow \begin{bmatrix} q_0 \\ \mathbf{q}^r \end{bmatrix}, \tag{1.8-3b}$$

in which the components of the vectors are taken in a reference system r, to be chosen when the quaternion is applied. The above multiplication can be written as

$$p * q = \begin{bmatrix} p_0 q_0 - (\mathbf{p} \cdot \mathbf{q})^r \\ (p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})^r \end{bmatrix} \tag{1.8-4}$$

We will use (1.8-3*b*) and (1.8-4) as the definitions of quaternions and quaternion multiplication. Quaternion properties can now be derived using ordinary vector operations.

Quaternion Properties

(i) ***Quaternion Noncommutativity*:** Consider the following identity:

$$p * q - q * p = \begin{bmatrix} 0 \\ (\mathbf{p} \times \mathbf{q} - \mathbf{q} \times \mathbf{p})^r \end{bmatrix} = \begin{bmatrix} 0 \\ 2(\mathbf{p} \times \mathbf{q})^r \end{bmatrix}$$

It is apparent that, in general,

$$p * q \neq q * p$$

(ii) ***The Quaternion Norm*:** The *norm* of a quaternion is defined to be the sum of the squares of its elements:

$$\text{norm}(q) = \sum_{i=0}^{i=3} q_i^2$$

(iii) ***Norm of a Product*:** Using the definition of the norm and vector operations, it is straightforward to show (Problem 1.8-1) that the norm of a product is equal to the product of the individual norms:

$$\text{norm}\,(p * q) = \text{norm}\,(p)\;\text{norm}\,(q)$$

(iv) ***Associative Property over Multiplication:*** The associative property, $(p * q) * r = p * (q * r)$, is proven in a straightforward manner.

(v) ***The Quaternion Inverse*:** Consider the following product:

$$\begin{bmatrix} q_0 \\ \mathbf{q}^r \end{bmatrix} * \begin{bmatrix} q_0 \\ -\mathbf{q}^r \end{bmatrix} = \begin{bmatrix} q_0^2 + \mathbf{q} \cdot \mathbf{q} \\ (q_0\mathbf{q} - q_0\mathbf{q} - \mathbf{q} \times \mathbf{q})^r \end{bmatrix} = \begin{bmatrix} \sum q_i^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that multiplying a quaternion by another quaternion, which differs only by a change in sign of the vector part, produces a quaternion with a scalar part only. A quaternion of the latter form will have very simple properties in multiplication (i.e., multiplication by a constant) and, when divided by the quaternion norm, will serve as the "identity quaternion." Therefore, the inverse of a quaternion is defined by

$$q^{-1} = \begin{bmatrix} q_0 \\ \mathbf{q}^r \end{bmatrix}^{-1} = \frac{1}{\text{norm}(q)} \begin{bmatrix} q_0 \\ -\mathbf{q}^r \end{bmatrix} \tag{1.8-5}$$

However, we will work entirely with unit-norm quaternions, thus simplifying many expressions.

(vi) ***Inverse of a Product*:** The inverse of a quaternion product is given by the product of the individual inverses in the reverse order. This can be seen as follows:

$$(p * q)^{-1} = \frac{1}{\text{norm}(p * q)} \begin{bmatrix} p_0 q_0 - (\mathbf{p} \cdot \mathbf{q})^r \\ -(p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})^r \end{bmatrix}$$

$$= \frac{1}{\text{norm}(q)} \begin{bmatrix} q_0 \\ -\mathbf{q}^r \end{bmatrix} * \begin{bmatrix} p_0 \\ -\mathbf{p}^r \end{bmatrix} \frac{1}{\text{norm}(p)}$$

Therefore,

$$(p * q)^{-1} = q^{-1} * p^{-1} \tag{1.8-6}$$

Vector Rotation by Quaternions A quaternion can be used to rotate a Euclidean vector in the same manner as the rotation formula, and the quaternion rotation is much simpler in form. The vector part of the quaternion is used to define the rotation axis and the scalar part to define the angle of rotation. The rotation axis is specified by its direction cosines in the reference coordinate system, and it is convenient to impose a unity norm constraint on the quaternion. Therefore, if the direction angles of the axis are α, β, γ and a measure of the rotation angle is δ, the rotation quaternion is written as

$$q = \begin{bmatrix} \cos\delta \\ \cos\alpha\ \sin\delta \\ \cos\beta\ \sin\delta \\ \cos\gamma\ \sin\delta \end{bmatrix} = \begin{bmatrix} \cos\delta \\ \sin\delta\ \mathbf{n}^r \end{bmatrix} \tag{1.8-7}$$

where $\mathbf{n}$ is a unit vector along the rotation axis,

$$\mathbf{n}^r = [\cos\alpha\ \ \cos\beta\ \ \cos\gamma]^T$$

and

$$\text{norm}(q) = \cos^2\delta + \sin^2\delta\,(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 1$$

This formulation also guarantees that there is a unique quaternion for every value of δ in the range $\pm 180°$, thus encompassing all possible rotations.

Now consider the form of the transformation, which must involve multiplication. For compatibility of multiplication between vectors and quaternions, a Euclidean vector is written as a quaternion with a scalar part of zero; thus

$$u = \begin{bmatrix} 0 \\ \mathbf{u}^r \end{bmatrix}$$

The result of the rotation must also be a quaternion with a scalar part of zero, the transformation must be reversible by means of the quaternion inverse, and Euclidean length must be preserved. The transformation $v = q * u$ obviously does not satisfy the first of these requirements. Therefore, we consider the transformations

$$v = q * u * q^{-1} \quad \text{or} \quad v = q^{-1} * u * q,$$

which are reversible by performing the inverse operations on v. The second of these transformations leads to the convention most commonly used:

$$v = q^{-1} * u * q = \begin{bmatrix} q_0(\mathbf{q}\cdot\mathbf{u}) - (q_0\mathbf{u} - \mathbf{q}\times\mathbf{u})\cdot\mathbf{q} \\ ((\mathbf{q}\cdot\mathbf{u})\mathbf{q} + q_0(q_0\mathbf{u} - \mathbf{q}\times\mathbf{u}) + (q_0\mathbf{u} - \mathbf{q}\times\mathbf{u})\times\mathbf{q})^r \end{bmatrix},$$

which reduces to

$$v = q^{-1} * u * q = \begin{bmatrix} 0 \\ \left(2\mathbf{q}(\mathbf{q}\cdot\mathbf{u}) + (q_0^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{u} - 2q_0(\mathbf{q}\times\mathbf{u})\right)^r \end{bmatrix} \tag{1.8-8}$$

Therefore, this transformation meets the requirement of zero scalar part. Also, because of the properties of quaternion norms, the Euclidean length is preserved. For a match with the rotation formula, we require agreement between:

Rotation Formula	Quaternion Rotation
$(1 - \cos\mu)\mathbf{n}(\mathbf{n}\cdot\mathbf{u})$	$2\sin^2\delta\,\mathbf{n}(\mathbf{n}\cdot\mathbf{u})$
$\cos\mu\,\mathbf{u}$	$(\cos^2\delta - \sin^2\delta)\mathbf{u}$
$-\sin\mu\,(\mathbf{n}\times\mathbf{u})$	$-2\cos\delta\,\sin\delta\,(\mathbf{n}\times\mathbf{u})$

The corresponding terms agree if $\delta = \mu/2$ and half-angle trigonometric identities are applied. Therefore, the quaternion

$$q = \begin{bmatrix} \cos(\mu/2) \\ \sin(\mu/2)\,\mathbf{n}^r \end{bmatrix} \tag{1.8-9a}$$

and transformation

$$u = q^{-1} * u * q \tag{1.8-9b}$$

give a left-handed rotation of a vector $\mathbf{u}$ through an angle μ around $\mathbf{n}$ when μ is positive.

Quaternion Coordinate Rotation Refer to the quaternion rotation formulae (1.8-9) and take the viewpoint that positive μ is a right-handed coordinate rotation rather than a left-handed rotation of a vector. We will define the quaternion that performs the coordinate rotation to system b from system a to be $q_{b/a}$; therefore,

$$q_{b/a} = \begin{bmatrix} \cos(\mu/2) \\ \sin(\mu/2)\,\mathbf{n}^r \end{bmatrix} \tag{1.8-10a}$$

and the coordinate transformation is

$$u^b = q_{b/a}^{-1} * u^a * q_{b/a} \tag{1.8-10b}$$

Equation (1.8-10b) can take the place of the direction cosine matrix transformation (1.3-5), and the coordinate transformation is thus achieved by a single rotation around an axis aligned with the quaternion vector part, $\mathbf{n}\ \sin(\mu/2)$. Euler's theorem shows that the same coordinate rotation can be achieved by a plane rotation around the unique axis corresponding to an eigenvector of the rotation matrix. Therefore, the vector $\mathbf{n}$ must be parallel to this eigenvector, and so

$$\mathbf{n}^b = C_{b/a}\mathbf{n}^a = \mathbf{n}^a,$$

which shows that the quaternion vector part has the same components in system a or system b. In (1.8-10a) the reference coordinate system r may be either a or b. We will postpone, for the moment, the problem of finding the rotation quaternion without finding the direction cosine matrix and its eigenstructure and instead examine the properties of the quaternion transformation.

Performing the inverse transformation to (1.8-10b) shows that

$$q_{b/a}^{-1} = q_{a/b} \tag{1.8-11}$$

Also, for multiple transformations,

$$u^c = q_{c/b}^{-1} * q_{b/a}^{-1} * u^a * q_{b/a} * q_{c/b} \tag{1.8-12}$$

which, because of the associative property, means that we can also perform this transformation with the single quaternion given by

$$q_{c/a}^{-1} = q_{c/b}^{-1} * q_{b/a}^{-1} \tag{1.8-13a}$$

or

$$q_{c/a} = q_{b/a} * q_{c/b} \tag{1.8-13b}$$

The quaternion coordinate transformation (1.8-10b) actually involves more arithmetical operations than premultiplication of $\mathbf{u}^a$ by the direction cosine matrix. However, when the coordinate transformation is evolving with time, the time update of the quaternion involves differential equations (following shortly) that are numerically preferable to the Euler kinematical equations and more efficient than the Poisson kinematical equations. In addition, the quaternion formulation avoids the singularity of the Euler equations and is easily renormalized (to reduce error accumulation).

The Quaternion Kinematical Equations With the goal of finding an expression for the derivative of a time-varying quaternion, and hence obtaining a state equation for vehicle attitude, we will derive an expression for an incremental increase $q(t + \delta t)$ from an existing state $q(t)$ in response to a nonzero angular velocity vector. Following the order of Equation (1.8-13b) for multiplication of two "forward" quaternions, we have

$$q(t + \delta t) = q(t) * \delta q(\delta t)$$

using

$$q(t) = \begin{bmatrix} \cos\frac{\mu}{2} \\ \mathbf{n}^r \sin\frac{\mu}{2} \end{bmatrix}, \quad \delta q(\delta t) \cong \begin{bmatrix} 1 \\ \mathbf{n}^r\,\frac{\delta\mu}{2} \end{bmatrix},$$

where $\mu(t)$ is the rotation angle, $\mathbf{n}$ the Euler axis, and r the reference coordinate system. The definition of a derivative gives

$$\dot{q} = \lim_{\delta t \to 0} \frac{q(t+\delta t) - q(t)}{\delta t} = \lim_{\delta t \to 0} \frac{q(t) * (\delta q - I_{(q)})}{\delta t} = \lim_{\delta t \to 0} q * \begin{bmatrix} 0 \\ \frac{1}{2}\mathbf{n}^r \frac{\delta \mu}{\delta t} \end{bmatrix},$$

where $I_{(q)}$ is the identity quaternion. Now, take the indicated limit and recognize the angular velocity vector $\boldsymbol{\omega}$ (as in Section 1.4) associated with the evolving quaternion,

$$\dot{q} = \frac{1}{2} q * \begin{bmatrix} 0 \\ \mathbf{n}^r \frac{d\mu}{dt} \end{bmatrix} = \frac{1}{2} q * \begin{bmatrix} 0 \\ \boldsymbol{\omega}^r \end{bmatrix}$$

Let this equation be associated with a coordinate rotation from *system a* to *system b*. Then, in terms of our notation, it is written as

$$\dot{q}_{b/a} = \frac{1}{2} q_{b/a} * \boldsymbol{\omega}^b_{b/a} \qquad (1.8\text{-}14)$$

The above quaternion can also be written as the matrix equation

$$\dot{q}_{b/a} = \frac{1}{2} \begin{bmatrix} 0 & -\left(\boldsymbol{\omega}^b_{b/a}\right)^T \\ \boldsymbol{\omega}^b_{b/a} & -\tilde{\boldsymbol{\omega}}^b_{b/a} \end{bmatrix} \begin{bmatrix} q_0 \\ \mathbf{q}^b \end{bmatrix}$$

Writing this out in full using the body system components of $\boldsymbol{\omega}_{b/a}$ gives

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -P & -Q & -R \\ P & 0 & R & -Q \\ Q & -R & 0 & P \\ R & Q & -P & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \equiv \frac{1}{2} \Omega q \qquad (1.8\text{-}15)$$

These quaternion state equations (1.8-14) and (1.8-15) are widely used in simulation of rigid-body angular motion, and in discrete form they are used in digital attitude control systems (e.g., for satellites) and for inertial navigation digital processing. We will illustrate their use in 6-DoF simulation for tracking the attitude of a body in motion around the oblate, rotating Earth in the next section.

Initializing a Quaternion In simulation and control, we often choose to keep track of orientation with a quaternion and construct the direction cosine matrix and/or Euler angles from the quaternion as needed. It is easy to initialize the quaternion for a simple plane rotation since the Euler axis is evident. For a compound rotation (e.g., yaw, pitch, and roll combined) an eigenvector analysis of the DCM would be needed to formally determine the Euler axis and construct a quaternion. Fortunately, this is not necessary, for a specific rotation the Euler axis is unique and so the quaternion is unique. Therefore, if we construct the quaternion in some other manner, the rotation axis will be implicitly correct. We shall now give two examples of constructing a quaternion for a particular set of rotations.

Example 1.8-1: Quaternion for a Yaw, Pitch, Roll Sequence For the yaw, pitch, roll sequence described by (1.3-10) the quaternion formulation is

$$v^{frd} = q_\phi^{-1} * q_\theta^{-1} * q_\psi^{-1} * v^{ned} * q_\psi * q_\theta * q_\phi$$

The rotation axes for the individual quaternions are immediately evident:

$$q_\phi = \begin{bmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \\ 0 \\ 0 \end{bmatrix}, \qquad q_\theta = \begin{bmatrix} \cos\frac{\theta}{2} \\ 0 \\ \sin\frac{\theta}{2} \\ 0 \end{bmatrix}, \qquad q_\psi = \begin{bmatrix} \cos\frac{\psi}{2} \\ 0 \\ 0 \\ \sin\frac{\psi}{2} \end{bmatrix}$$

These transformations can be multiplied out, using quaternion multiplication, with only a minor amount of pain. The result is

$$q_{frd/ned} = q_\psi * q_\theta * q_\phi = \begin{bmatrix} \pm\left(\cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\right) \\ \pm\left(\sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\right) \\ \pm\left(\cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\right) \\ \pm\left(\cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\right) \end{bmatrix} \qquad (1)$$

A plus or minus sign has been added to these equations because neither (1.8-10*b*) nor (1.8-15) is affected by the choice of sign. The same choice of sign must be used in all of Equations (1). ■

Example 1.8-2: Quaternion for an ECEF-to-NED Rotation The sequence of rotations required to arrive at an NED orientation, starting from the ECEF system is:

(i) A right-handed rotation about the ECEF z-axis to a positive longitude, ℓ.
(ii) A left-handed rotation of $(90 + \phi)$ degrees, around the new y-axis, to a positive geodetic latitude of ϕ. (This is easily seen by letting ℓ be zero.)

The quaternion description is

$$q_{ned/ecf} = q_\ell * q_{(-90-\phi)} = \begin{bmatrix} \cos\frac{\ell}{2} \\ 0 \\ 0 \\ \sin\frac{\ell}{2} \end{bmatrix} * \begin{bmatrix} \cos\left(\frac{\phi}{2} + 45^\circ\right) \\ 0 \\ -\sin\left(\frac{\phi}{2} + 45^\circ\right) \\ 0 \end{bmatrix}$$

Following the rules of quaternion multiplication, with a cross-product matrix used on the vector part, gives

$$q_{ned/ecf} = \begin{bmatrix} \cos\frac{\ell}{2}\cos\left(\frac{\phi}{2} + 45^\circ\right) \\ \sin\frac{\ell}{2}\sin\left(\frac{\phi}{2} + 45^\circ\right) \\ -\cos\frac{\ell}{2}\sin\left(\frac{\phi}{2} + 45^\circ\right) \\ \sin\frac{\ell}{2}\cos\left(\frac{\phi}{2} + 45^\circ\right) \end{bmatrix} \qquad (1)$$

■

Direction Cosine Matrix from Quaternion If we write the quaternion rotation formula (1.8-8) in terms of array operations, using the vector part of the quaternion, we get

$$\mathbf{u}^b = \left[2\mathbf{q}^a(\mathbf{q}^a)^T + (q_0^2 - (\mathbf{q}^a)^T\mathbf{q}^a)\,\mathbf{I} - 2q_0\widetilde{\mathbf{q}}^a\right]\,\mathbf{u}^a \equiv C_{b/a}\mathbf{u}^a \tag{1.8-16}$$

The cross-product matrix $\widetilde{\mathbf{q}}^a$ is given by

$$\widetilde{\mathbf{q}}^a = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{1.8-17}$$

Now, evaluating the complete transformation matrix in (1.8-16), we find that

$$C_{b/a} = \begin{bmatrix} \left(q_0^2 + q_1^2 - q_2^2 - q_3^2\right) & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & (q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix} \tag{1.8-18}$$

This expression for the rotation matrix, in terms of quaternion parameters, corresponds to Equations (1.8-10) and the single right-handed rotation around **n** through the angle μ. Equation (1.8-18) is independent of any choice of Euler angles. Depending on the coordinate rotation that it represents, we can determine a set of Euler angles as in Equation (1.3-11).

Quaternion from Direction Cosine Matrix The quaternion parameters can also be calculated from the elements $\{c_{i,j}\}$ of the general direction cosine matrix. If terms on the main diagonal of (1.8-18) are combined, the following relationships are obtained:

$$\begin{aligned} 4q_0^2 &= 1 + c_{11} + c_{22} + c_{33} \\ 4q_1^2 &= 1 + c_{11} - c_{22} - c_{33} \\ 4q_2^2 &= 1 - c_{11} + c_{22} - c_{33} \\ 4q_3^2 &= 1 - c_{11} - c_{22} + c_{33} \end{aligned} \tag{1.8-19a}$$

These relationships give the magnitudes of the quaternion elements but not the signs. The off-diagonal terms in (1.8-18) yield the additional relationships

$$\begin{aligned} 4q_0q_1 &= c_{23} - c_{32}, &\qquad 4q_1q_2 &= c_{12} + c_{21} \\ 4q_0q_2 &= c_{31} - c_{13}, &\qquad 4q_2q_3 &= c_{23} + c_{32} \\ 4q_0q_3 &= c_{12} - c_{21}, &\qquad 4q_1q_3 &= c_{31} + c_{13} \end{aligned} \tag{1.8-19b}$$

From the first set of equations, (1.8-19*a*), the quaternion element with the largest magnitude (at least one of the four must be nonzero) can be selected. The sign associated with the square root can be chosen arbitrarily, and then this variable can be used as a divisor with (1.8-19*b*) to find the remaining quaternion elements. An interesting quirk of this algorithm is that the quaternion may change sign if the algorithm is restarted with a new set of initial conditions. This will have no effect on the rotation matrix given in (1.8-18). Algorithms like this are discussed by Shoemake (1985) and Shepperd (1978).

The Oblate Rotating-Earth 6-DoF Equations

The starting point here will be the position and velocity state equations (1.7-12) and (1.7-13) and the angular velocity state equation (1.7-4):

$$\begin{aligned}
{}^{\mathbf{e}}\dot{\mathbf{p}}_{cm/O} &= \mathbf{v}_{cm/e} \\
{}^{e}\dot{\mathbf{v}}_{cm/e} &= \frac{1}{\mathrm{m}}\mathbf{F} + \mathbf{G} - \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{cm/O}) - 2\,\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{cm/e} \\
\mathbf{M} = {}^{i}\dot{\mathbf{h}}_{cm/i} &= {}^{b}\dot{\mathbf{h}}_{cm/i} + \boldsymbol{\omega}_{b/i} \times \mathbf{h}_{cm/i}
\end{aligned}$$

This time, we will resolve the position and velocity equations on the coordinate axes of the ECEF system (abbreviated in the equations to *ecf*), instead of the tangent-plane system. The reference point, O, for the position vector will then be at Earth's cm, and latitude and longitude will then be easily calculated. The angular velocity equation must, as usual, be resolved in a body-fixed coordinate system in order to avoid a time-varying inertia matrix, and we will use the forward-right-down system, *frd*. A coordinate transformation will therefore be needed, and this time it will be obtained from a quaternion, $q_{frd/ecf}$. With these choices of coordinate systems the state vector, for the set of 6-DoF equations, will be

$$X = \left[q_{frd/ecf},\ \mathbf{p}_{cm/O}^{ecf},\ \mathbf{v}_{cm/e}^{ecf},\ \boldsymbol{\omega}_{b/i}^{frd}\right]^T$$

Note that, here, the transpose designation is only meant to indicate that the element column arrays inside the brackets should be stacked into a single column.

The matrix state equations now follow from this choice of state vector as

$$\begin{aligned}
\dot{q}_{frd/ecf} &= {}^{1}\!/_{2}\, q_{frd/ecf} * \left(\omega_{b/i}^{frd} - \omega_{e/i}^{frd}\right) \\
{}^{e}\dot{\mathbf{p}}_{cm/o}^{ecf} &= v_{cm/e}^{ecf} \\
{}^{e}\dot{v}_{cm/e}^{ecf} &= \frac{F^{ecf}}{\mathrm{m}} - 2\widetilde{\omega}_{e/i}^{ecf} v_{cm/e}^{ecf} + g^{ecf} \\
{}^{b}\dot{\omega}_{b/i}^{frd} &= \left(J^{frd}\right)^{-1}\left(M^{frd} - \widetilde{\omega}_{b/i}^{frd} J^{frd} \omega_{b/i}^{frd}\right)
\end{aligned} \tag{1.8-20}$$

The following auxiliary equations must be executed first to compute Earth's angular velocity in body-fixed coordinates for the quaternion equation and the aerodynamic forces in Earth-fixed coordinates for the velocity state equation:

$$\begin{bmatrix} 0 \\ \omega_{e/i}^{frd} \end{bmatrix} = q_{frd/ecf}^{-1} * \begin{bmatrix} 0 \\ \omega_{e/i}^{ecf} \end{bmatrix} * q_{frd/ecf} \qquad \begin{bmatrix} 0 \\ \frac{F^{ecf}}{\mathrm{m}} \end{bmatrix} = q_{frd/ecf} * \begin{bmatrix} 0 \\ \frac{F^{frd}}{\mathrm{m}} \end{bmatrix} * q_{ecf/frd} \tag{1.8-21}$$

The gravity term will be calculated from the centripetal acceleration and **G** as a function of the geocentric position vector.

A short digression will be used here to bring out useful information contained in the velocity equation. We shall set the applied force, **F**, to zero and look for a steady-state Earth orbit around the equator, i.e., $p_z \equiv 0$ (this is the only Great Circle possibility). Also, let the y-component of position, p_y, be zero, so that the vehicle is crossing the *ecf* x-axis (zero longitude), and set the y and z acceleration components to zero. The *ecf* x-acceleration component will be set to the centripetal acceleration for a circular orbit at geodetic height h above the WGS-84 spheroid, $\dot{v}_x = -v_y^2/(a+h)$. Therefore, the x-axis equation of motion will be

$$-v_y^2/(a+h) = 2\omega_z v_y - G_D + \omega_z^2 (a+h),$$

where ω_z is the z-component of $\omega_{e/i}$. When this quadratic equation is solved for the velocity, we obtain the *circular orbit condition*,

$$v_y = \sqrt{G_D(a+h)} - \omega_z(a+h) \tag{1.8-22}$$

The first term on the right-hand side is the inertial velocity component, and the second is the easterly component of Earth's velocity at the equator. The inertial term simply boils down to the centripetal acceleration condition $v^2/r = G$. Some idea of the numbers involved can be obtained by using the value of a given in Section 1.6 and the **G** model given there and choosing a geodetic height. At 422 km above the spheroid the inertial component is 7.662 km/s. The International Space Station is stated to be in a nearly circular orbit, at an average height of 422 km above msl, and inclined at about 55° to the equatorial plane, and its orbital speed is stated to be 7.661 km/s (17,100 mph). The orbital velocity is quite insensitive to the orbit inclination and height, and most objects in low-Earth orbit (LEO) have about this velocity.

Returning to the 6-DoF equations, the relative wind, defined in Equation (1.7-14), could be computed for use in finding the aerodynamic forces and moments as

$$\begin{bmatrix} 0 \\ \mathbf{v}_{rel}^{frd} \end{bmatrix} = q_{frd/ecf}^{-1} * \begin{bmatrix} 0 \\ \mathbf{v}_{cm/e}^{ecf} - \mathbf{v}_{W/e}^{ecf} \end{bmatrix} * q_{frd/ecf} \tag{1.8-23}$$

The components of $\mathbf{v}_{rel}^{frd}$ determine the aerodynamic angles and these, together with the magnitude of this velocity vector, determine the aerodynamic forces and moments on the vehicle. There would be practical difficulties in providing the wind information for a simulation, unless it could be neglected for high-speed, high-altitude flight or taken as piecewise constant over different segments of a flight. Onboard a real vehicle the situation would be reversed, in that the velocity over Earth would be known from the INS (Inertial Navigation System), and the major component of the relative wind would be known from the aircraft pitot-static air-data system, so that some estimate of atmospheric wind could be calculated.

Output equations that are likely to be needed with the 6-DoF equations are a calculation of vehicle attitude in a geographic coordinate system and calculation of geodetic position coordinates. Referring to Section 1.6, longitude is easily calculated from the ECEF coordinates of the geocentric position vector in Equations (1.8-20),

$$\ell = \text{atan2}(p_y, p_x), \tag{1.8-24}$$

while geodetic latitude and height can be calculated from the approximations described in Section 1.6. The usual attitude reference for the vehicle is a geographic coordinate system (i.e., NED or ENU, and moving over the Earth with the vehicle) and the attitude is specified by the Euler angles of the vehicle body axes relative to the geographic system. We will choose the NED system, and the roll, pitch, and heading angles of the vehicle can be calculated as follows. First, calculate $q_{ned/ecf}$ from latitude and longitude, as in Example 1.8-2. Then, using the quaternion state variable, we can calculate the quaternion $q_{frd/ned}$. This quaternion determines the DCM, $C_{frd/ned}$, and from this we can find the Euler angles, as in (1.3-11). The equations are

$$\begin{aligned} q^{-1}_{frd/ned} &= q^{-1}_{frd/ecf} * q^{-1}_{ecf/ned} \\ C_{frd/ned} &= fn\,(q_{frd/ned}) \\ \phi &= \text{atan2}(c_{23}, c_{33}) \\ \theta &= -\text{asin}(c_{13}) \\ \psi &= \text{atan2}(c_{12}, c_{11}) \end{aligned} \tag{1.8-25}$$

This completes the discussion of the oblate, rotating-Earth 6-DoF simulation equations, and the following simulation example will illustrate their use.

Example 1.8-3: Simulation of Motion around Earth Equations (1.8-20), (1.8-21), (1.8-24), and (1.8-25) were programmed as a subroutine, with the state and derivative vectors as its arguments. The programming is almost trivially easy in a language that handles matrix operations (e.g., Fortran-95; MATLAB™). It is only necessary to write two additional routines, for quaternion multiplication and for the tilde matrix from vector elements. The "vehicle" was simply a "brick" with dimensions $2 \times 5 \times 8$ units, and the coordinate origin was at the center of mass with the x-axis parallel to the eight-unit side and y parallel to the five-unit side. For this simulation no aerodynamic effects were modeled, and the applied torque and applied specific force components were set to zero. A simple driver program was written to use the fourth-order Runge-Kutta routine in Chapter 2 to integrate the equations and handle input and output operations. The initial-condition inputs were geodetic position (latitude, longitude, and altitude), Euler angles, velocity over Earth in *frd* coordinates, simulation run time, and integration time step. Note that, because the equations are in terms of velocity over Earth, no input of Earth's inertial velocity components is required; inertial effects are accounted for with the Coriolis term in the state equation.

Much can be learned from running this simulation; the brick can be fired vertically to observe Coriolis effects, spun around its intermediate-inertia axis to observe its instability to small additional angular rate disturbances, or put into Earth orbit to study steady-state conditions, escape velocity, etc. Here, we have simulated an orbit starting from zero latitude and longitude and an altitude of 10^5 m. This low-altitude orbit

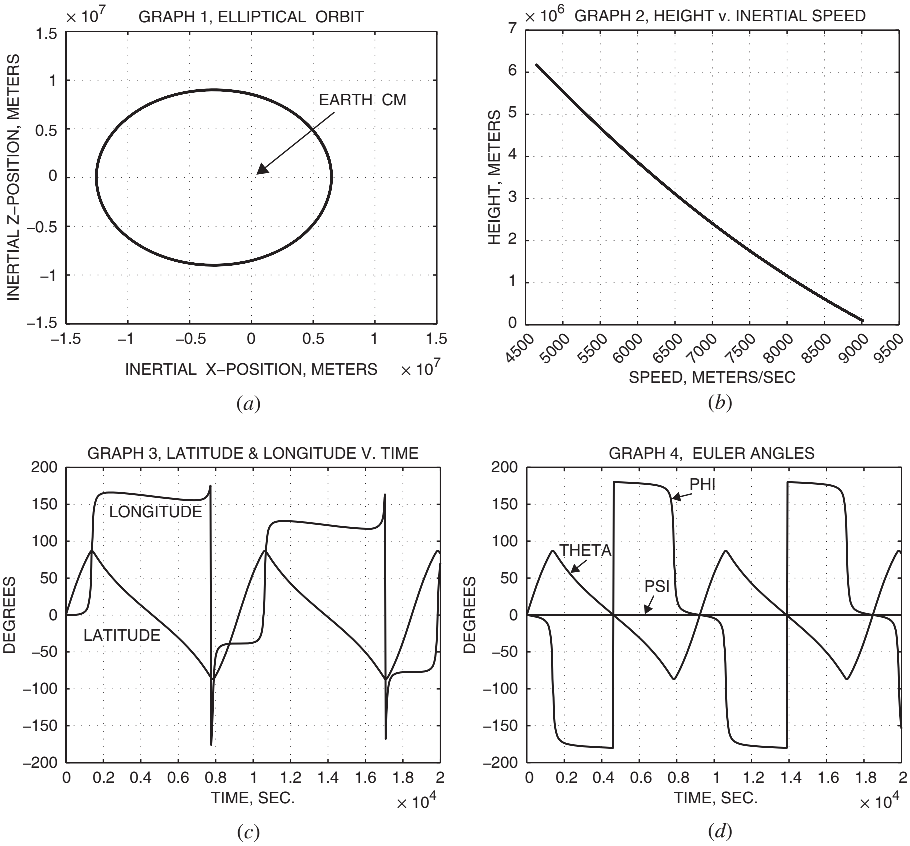

Figure 1.8-1

(below the usual “LEO” range) would show a rapid decay if atmospheric drag were modeled. The other initial conditions are a heading angle of zero degrees (aiming for the North Pole); zero pitch and roll, a forward component of velocity of 9.0 km/s (enough to give a moderately elliptical orbit), and zero initial angular rates. Integration step size is not critical; 0.01, 0.10, and 1.0 s step sizes give identical graphs (small step size would be needed for the spinning brick stability experiment).

Graph 1 shows a plot of the orbit in an inertial coordinate system initialized from the *ecf* system at $t = 0$. Earth’s cm is at the origin (a focus of the ellipse). Orbits that pass through high latitudes are significantly affected by the variation of Earth’s gravity with latitude, especially very low orbits such as this one. Thus, if the simulation is run for two or more orbits an inertial precession of the orbital plane will be observed.

Graph 2 shows height above the Spheroid versus the inertial speed, and this reaches minimum speed and maximum height at the Apogee (zero latitude, and 180° longitude).

Graph 3 shows latitude and longitude. The orbit will pass to the right of the North Pole because added to the initial Earth-velocity of 9 km/s North it has an initial inertial velocity of 465.1 m/s to the East, imparted at the Equator by the spin of the Earth. The orbital plane must contain the Earth's cm and so the orbit is tilted away from the poles, and the latitude never reaches $\pm 90°$. Longitude will decrease steadily as the Earth rotates under the inertially fixed orbit. The rate of change of longitude is determined by the eastward component of the relative inertial velocity of the orbit and points on the Earth below, and the convergence of the Meridians near the Poles. Therefore, longitude changes very slowly at first, and then changes rapidly near 90° latitude. At the maximum negative latitude the longitude changes by 180°, in the same way as near the North Pole, but the change is disguised by the 180° ambiguity.

The Euler angle graphs show the attitude of the brick relative to a local NED system. The brick maintains a fixed inertial orientation as it circles Earth (initial rates were zero, and no torques were applied), and so the Euler angle variations are caused by changing orientation of the NED system as it follows the trajectory. The local NED system never reaches zero tilt with respect to the equatorial plane since the trajectory does not pass over the poles. Consequently, the pitch attitude angle of the brick never reaches 90° as the brick approaches the poles. The roll angle of the brick shows the expected 180° transitions, and the shape of these closely matches the shape of changes in the longitude graph. The heading angle remains unchanged, at zero. ■

In concluding this chapter, we note that practically all of the concepts introduced in the chapter are used in Example 1.8-3, and a number of significantly different orbits and initial condition combinations can be simulated, leading to graphical results that are quite demanding in their interpretation. Lack of space prevented the use of the simulation to illustrate properties of spinning bodies, which is also very instructive.

REFERENCES

Blakelock, J. H. *Automatic Control of Aircraft and Missiles*. New York: Wiley, 1965.

Encyclopaedia Britannica. 15th ed., vol. 17, *Macropaedia*. Chicago: Encyclopaedia Britannica, 1987, pp. 530–539.

Goldstein, H. *Classical Mechanics*. 2d ed. Reading, Mass.: Addison-Wesley, 1980.

Heiskanen, W. A., and H. Moritz. *Physical Geodesy*. San Francisco and London: W. H. Freeman, 1967.

Kane, T. R., P. W. Likins, and D. A. Levinson. *Spacecraft Dynamics*. New York: McGraw-Hill, 1983.

Kaplan, G. H. The IAU Resolutions on Astronomical Constants, Time Scales, and the Fundamental Reference Frame. Circular no. 163. United States Naval Observatory, Washington, D.C., December 10, 1981.

Kuebler, W., and S. Sommers. "The Role of the Earth's Shape in Navigation: An Example." *Journal of the Institute of Navigation* 28, no. 1 (spring 1981): 29–33.

McGill, D. J., and W. W. King. *An Introduction to Dynamics*. 3d ed. Boston: PWS Engineering, 1995.

NIMA. "Department of Defense World Geodetic System 1984, Its Definition and Relationships with Local Geodetic Systems." Technical Report 8350.2. 3d ed. National Imagery and Mapping Agency, July 4, 1997 (Freely available by online download).

Pestel, E. C., and W. T. Thompson. *Dynamics*. New York: McGraw-Hill, 1968

Shepperd, S. W. "Quaternion from Rotation Matrix." *AIAA Journal of Guidance and Control* 1, no. 3 (May–June 1978): 223–224.

Shoemake, K. "Animating Rotation with Quaternion Curves." *Computer Graphics* 19, no. 3 (1985): 245–254.

Strang, G. *Linear Algebra and Its Applications*. New York: Academic, 1980.

Travassos, R. H., N. K. Gupta, K. W. Iliffe, and R. Maine. "Determination of an Oblique Wing Aircraft's Aerodynamic Characteristics." Paper no. 80-1630. *AIAA Atmospheric Flight Mechanics Conference*, 1980, p. 608.

Vanicek, P., and E. J. Krakiwsky. *Geodesy: The Concepts*. Amsterdam: North-Holland, 1982.

Wells, D. *Theory and Problems of Lagrangian Dynamics*. Schaum's Outline Series. New York: McGraw-Hill, 1967.

Wilkinson, J. H., and G. Golub. "Ill Conditioned Eigensystems and the Computation of the Jordan Canonical Form." *SIAM Review* 18 (October 1976): 578–619.

PROBLEMS

Section 1.2

1.2-1 If vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, from a common point, define the adjacent edges of a parallelepiped, show that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ represents the signed volume of the parallelepiped.

1.2-2 Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0$.

1.2-3 Two particles moving with constant velocity are described by the position vectors

$$\mathbf{p} = \mathbf{p_0} + \mathbf{v}t, \qquad \mathbf{s} = \mathbf{s_0} + \mathbf{w}t$$

(a) Show that the shortest distance between their trajectories is given by

$$d = |(\mathbf{s_0} - \mathbf{p_0}) \cdot (\mathbf{w} \times \mathbf{v})| \;/\; |\mathbf{w} \times \mathbf{v}|$$

(b) Find the shortest distance between the particles themselves.

1.2-4 Derive the vector expressions shown in Figure 1.2-1.

Section 1.3

1.3-1 Derive the cross-product matrix used in Equation (1.3-3).

1.3-2 Start with an airplane heading north in level flight and draw two sequences of pictures to illustrate the difference between a yaw, pitch, roll sequence and a roll, yaw, pitch sequence. Let the rotations (Euler angles) be yaw $\psi = -90^\circ$, pitch $\theta = -45^\circ$, and roll $\phi = 45^\circ$. State the final orientation.

1.3-3 Find the rotation matrix corresponding to (1.3-10) but for a heading, roll, pitch sequence. Find the formulae for the Euler angles and specify their ranges.

1.3-4 For the rotation in Equation (1.3-10), with heading, pitch, and roll angles all equal to $-90°$, find, by hand:

(a) The eigenvalues

(b) The eigenvector for the $+1$ eigenvalue

(c) The direction of the Euler axis in terms of an azimuth and an elevation angle

(d) The equivalent rotation around the Euler axis (by physical experiment)

1.3-5 Show that the rotation matrix between two coordinate systems can be calculated from a knowledge of the position vectors of two different objects if the position vectors are known in each system.

(a) Specify the rotation matrix in terms of the solution of a matrix equation.

(b) Show how the matrix equation can be solved for the rotation matrix.

Section 1.4

1.4-1 Prove that the derivative of the angular velocity vector of a frame F_b relative to frame F_a is the same when taken in either F_a or F_b.

1.4-2 Prove that the centripetal acceleration vector is always orthogonal to the angular velocity vector.

1.4-3 Find the Euler angle rates as in Equation (1.4-4) but for the rotation sequence heading, roll, pitch.

Section 1.5

1.5-1 Start from the vector equation (1.5-6).

(a) Obtain the matrix equation for the NED coordinates of the vectors. Assume that **g** has a *down component* only.

(b) Neglecting North motion, and the *y-dot* contribution to vertical acceleration, integrate the equations to obtain the y and z displacement equations (include initial condition terms with the indefinite integrals).

(c) Compare the Coriolis deflections of a mass reaching the ground for the following two cases: **(i)** thrown vertically upward with initial velocity u; **(ii)** dropped, with zero initial velocity, from the maximum height reached in (i).

Section 1.6

1.6-1 Starting from a calculus textbook definition of radius of curvature and the equation of an ellipse, derive the formula (1.6-5) for the meridian radius of curvature.

1.6-2 Derive the formulae (1.6-13) and hence the formula (1.6-14) for geocentric latitude in terms of geodetic latitude by using the geometry of the generating ellipse.

1.6-3 Derive the formula (1.6-26) for **G** starting from the potential function V in Equation (1.6-24). Use a geocentric coordinate system as mentioned in the text.

1.6-4 Starting from (1.6-26), write and test a program to evaluate $|\mathbf{g}|$ and $|\mathbf{G}|$ as functions of geodetic latitude and altitude. Plot them both on the same axes against latitude ($0 \rightarrow 90°$). Do this for $h = 0$ and 30,000 m.

1.6-5 Derive the conditions for a body to remain in a geostationary orbit of Earth. Use the gravity model and geodetic data to determine the geostationary altitude. What are the constraints on the latitude and inclination of the orbit?

Section 1.7

1.7-1 An aircraft is to be mounted on a platform with a torsional suspension so that its moment of inertia, I_{zz}, can be determined. Treat the wings as one piece equal to one-third of the aircraft weight and placed on the fuselage one-third back from the nose.

(a) Find the distance of the aircraft cm from the nose as a fraction of the fuselage length.

(b) The aircraft weight is 80,000 lb, the wing planform is a rectangle 40 ft by 16 ft, and the planview of the fuselage is a rectangle 50 ft by 12 ft. Assuming uniform density, calculate the aircraft moment of inertia (in slug-ft^2).

(c) Calculate the period of oscillation (in seconds) of the platform if the torsional spring constant is 10,000 lb-ft/rad.

1.7-2 Use Euler's equations of motion (1.7-8) and the Euler kinematical equations (1.4-4) to simulate the angular motion of a brick tossed in the air and spinning. Write a MATLAB program using Euler integration (1.1-4) to integrate these equations over a 300-s interval using an integration step of 10 ms. Add logic to the program to restrict the Euler angles to the ranges described in Section 1.3. Let the brick have dimensions $8 \times 5 \times 2$ units, corresponding to x, y, z axes at the center of mass. The moments ℓ, m, n are all zero, and the initial conditions are:

(a) $\phi = \theta = \psi = 0, \quad P = 0.1, \quad Q = 0, \quad R = 0.001$ rad/s

(b) $\phi = \theta = \psi = 0, \quad P = 0.001, \quad Q = 0, \quad R = 0.1$ rad/s

(c) $\phi = \theta = \psi = 0, \quad P = 0.0, \quad Q = 0.1, \quad R = 0.001$ rad/s

Plot the three angular rates (deg/s) on one graph, and the three Euler angles (in deg) on another. Which motion is stable and why?

1.7-3 Derive a set of linear state equations from Equations (1.7-8) by considering perturbations from a steady-state condition with angular rates P_e, Q_e, and R_e.

Find expressions for the eigenvalues of the coefficient matrix when only one angular rate is nonzero and show that there is an unstable eigenvalue if the moment of inertia about this axis is either the largest or the smallest of the three inertias. Deduce any practical consequences of this result.

1.7-4 The propeller and crankshaft of a single-engine aircraft have a combined moment of inertia of 45 slug-ft^2 about the axis of rotation and are rotating at 1500 rpm clockwise when viewed from in front. The moments of inertia of the aircraft are roll: 3000 slug-ft^2, pitch: 6700 slug-ft^2, yaw: 9000 slug-ft^2. If the aircraft rolls at 100 deg/s, while pitching at 20 deg/s, determine the angular acceleration in yaw. All inertias and angular rates are body-axes components.

1.7-5 Analyze the height and distance errors of the flat-Earth equations.

Section 1.8

1.8-1 Show that, for a quaternion product, the norm of the product is equal to the product of the individual norms.

1.8-2 Compare the operation count ($+,\ -,\ \times,\ \div$) of the vector rotation formula (1.2-5*b*) with that of the quaternion formula (1.8-9*b*).

1.8-3 If a coordinate system *b* is rotating at a constant rate with respect to a system *a* and only the components of the angular velocity vector in system *b* are given, find an expression for the quaternion that transforms coordinates from *b* to *a*.

1.8-4 **(a)** Write a subroutine or an M-file for the Round the Earth 6-DoF equations of motion as described in Example 1.8-3.

(b) Write a driver program to use these 6-DoF equations and reproduce the results of Example 1.8-3.