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Introduction

In this chapter, analytical methods for approximate solutions of periodic motions to chaos in nonlinear dynamical systems will be presented briefly. The Lagrange stand form, perturbation method, method of averaging, harmonic balance, generalized harmonic balance will be discussed. A brief literature survey will be completed to present a main development skeleton of analytical methods for periodic motions in nonlinear dynamical systems. The weakness of current approximate, analytical methods will also be discussed in this chapter, and the significance of analytical methods in nonlinear engineering will be presented.

1.1 Analytical Methods

Since the appearance of Newton's mechanics, one has been interested in periodic motion. From the Fourier series theory, any periodic function can be expressed by a Fourier series expansion with different harmonics. The periodic motion in dynamical systems is a closed curve in state space in the prescribed period. In addition to simple oscillations in mechanical systems, one has been interested in motions of moon, earth, and sun in the three-body problem. The earliest approximation method is the method of averaging, and the idea of averaging originates from Lagrange (1788).

1.1.1 Lagrange Standard Form

Consider an initial value problem for $\mathbf{x} \in D \subset \mathbf{R}^n$ and $t \geq 0$,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \varepsilon\mathbf{f}(\mathbf{x}, t), \mathbf{x}(0) = \mathbf{x}_0 \tag{1.1}$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix and continuous with time t. $\mathbf{f}(\mathbf{x}, t)$ is a C^r – continuous vector function of t and $\mathbf{x}$. The unperturbed system is linear ($\varepsilon = 0$) and such a linear system has n independent basic solution to form a fundamental matrix $\mathbf{\Phi}(t)$. That is,

$$\dot{\mathbf{x}}^{(0)} = \mathbf{A}(t)\mathbf{x}^{(0)} \Rightarrow \mathbf{x}^{(0)} = \mathbf{\Phi}(t)\mathbf{c} \tag{1.2}$$

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where **c** is constant, determined by initial conditions. As in Luo (2012a,b), a linear transformation is introduced as

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{y}. \tag{1.3}$$

Substitution of Equation (1.3) into Equation (1.1) gives

$$\dot{\mathbf{\Phi}}(t)\mathbf{y} + \mathbf{\Phi}(t)\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{\Phi}(t)\mathbf{y} + \varepsilon\mathbf{f}(\mathbf{\Phi}(t)\mathbf{y}, t). \tag{1.4}$$

With $\dot{\mathbf{\Phi}}(t) = \mathbf{A}(t)\mathbf{\Phi}(t)$, we obtain

$$\begin{aligned} &\dot{\mathbf{y}} = \varepsilon\mathbf{\Phi}^{-1}(t)\mathbf{f}(\mathbf{\Phi}(t)\mathbf{y}, t) \equiv \mathbf{g}(\mathbf{y}, t), \\ &\mathbf{y}_0 = \mathbf{\Phi}^{-1}(0)\mathbf{x}_0 \end{aligned} \tag{1.5}$$

The foregoing form is called the Lagrange standard form.

Consider a vibration problem as

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, t). \tag{1.6}$$

From the basic solution of the unperturbed system, we have a transformation as

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}. \tag{1.7}$$

Using this transformation, Equation (1.6) becomes

$$\begin{aligned} \dot{y}_1 &= \varepsilon g_1(y_1, y_2, t), \\ \dot{y}_2 &= \varepsilon g_2(y_1, y_2, t). \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} g_1(y_1, y_2, t) &= -\frac{1}{\omega} f(x, \dot{x}, t)\sin\omega t, \\ g_2(y_1, y_2, t) &= f(x, \dot{x}, t)\cos\omega t. \end{aligned} \tag{1.9}$$

If the function $g_1(y_1, y_2, t)$ and $g_2(y_1, y_2, t)$ is T-periodic with $T = 2\pi/\omega$,

$$\begin{aligned} \dot{y}_1 &= \varepsilon g_1^0(y_1, y_2), \\ \dot{y}_2 &= \varepsilon g_1^0(y_1, y_2). \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} g_1^0(y_1, y_2) &= \frac{1}{T}\int_0^T g_1(y_1, y_2, t)dt = -\frac{1}{T}\int_0^T \frac{1}{\omega} f(x, \dot{x}, t)\sin(\omega t)dt, \\ g_2^0(y_1, y_2) &= \frac{1}{T}\int_0^T g_2(y_1, y_2, t)dt = \frac{1}{T}\int_0^T f(x, \dot{x}, t)\cos(\omega t)dt. \end{aligned} \tag{1.11}$$

1.1.2 Perturbation Methods

In the end of the nineteenth century, Poincare (1890) provided the qualitative analysis of dynamical systems to determine periodic solutions and stability, and developed the perturbation theory for periodic solution. In addition, Poincare (1899) discovered that the motion of a nonlinear coupled oscillator is sensitive to the initial condition, and qualitatively stated that the motion in the vicinity of unstable fixed points of nonlinear oscillation systems may be stochastic under regular applied forces. In the twentieth century, one followed Poincare's ideas to develop and apply the qualitative theory to investigate the complexity of motions in dynamical systems. With Poincare's influence, Birkhoff (1913) continued Poincare's work, and a proof of Poincare's geometric theorem was given. Birkhoff (1927) showed that both stable and unstable fixed points of nonlinear oscillation systems with two degrees of freedom must exist whenever their frequency ratio (also called resonance) is rational. The sub-resonances in periodic motions of such systems change the topological structures of phase trajectories, and the island chains are obtained when the dynamical systems are renormalized with fine scales. In such qualitative and quantitative analysis, the Taylor series expansion and the perturbation analysis play an important role. However, the Taylor series expansion analysis is valid in the small finite domain under certain convergent conditions, and the perturbation analysis based on the small parameters, as an approximate estimate, is only acceptable for a very small domain with a short time period. From Verhulst (1991), the perturbation solution of dynamical system can be stated as follows.

Theorem 1.1 *Consider a dynamical system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \varepsilon), \mathbf{x}(t_0) = \mathbf{a} \tag{1.12}$$

with $\mathbf{x} \in D \subset \mathscr{R}^n,\ |t - t_0| < h$, *and* $0 \le \varepsilon \le \varepsilon_0 < 1$. $\mathbf{f}(\mathbf{x}, t, \varepsilon)$ *is a* C^r *– continuous vector function of* $t,\ \mathbf{x}$, *and* ε. *Assume* $\mathbf{f}(\mathbf{x}, t, \varepsilon)$ *can be expanded in a Taylor series with respect to* ε *as*

$$\mathbf{f}(\mathbf{x}, t, \varepsilon) = \sum_{k=0}^{m} \varepsilon^k \mathbf{f}_k(\mathbf{x}, t) + \varepsilon^{m+1}\mathbf{R}(\mathbf{x}, t, \varepsilon) \tag{1.13}$$

with $|t - t_0| \le h$ *and* $0 \le \varepsilon \le \varepsilon_0$. $\mathbf{f}_k(\mathbf{x}, t)$ $(k = 0, 1, 2, \dots, m)$ *is continuous in* t *and* $\mathbf{x}$ *with* $(m + 1 - k)$ *times continuously differentiable with* $\mathbf{x}$, *and* $\mathbf{R}(\mathbf{x}, t, \varepsilon)$ *is continuous in* $t,\ \mathbf{x}$, *and* ε, *and satisfies Lipschitz – continuous in* $\mathbf{x}$. *Suppose there is a* ε*-series of* $\mathbf{x}$ *as*

$$\mathbf{x}(t) = \sum_{k=0}^{m} \varepsilon^k \mathbf{x}_k(t). \tag{1.14}$$

Application of Equation (1.14) to Equation (1.12), using the Taylor series expansion of $\mathbf{f}_k(\mathbf{x}, t)$ *with respect to power of* ε, *and equating coefficients with the initial condition*

$$\mathbf{x}_0(t_0) = \mathbf{a} \text{ and } \mathbf{x}_k(t_0) = \mathbf{0}\ (k = 1, 2, \dots, m) \tag{1.15}$$

generates an approximate solution of $\mathbf{x}(t)$ *with*

$$\left\| \mathbf{x}\,(t) - \sum_{k=0}^{m} \varepsilon^k \mathbf{x}_k(t) \right\| = O(\varepsilon^{m+1}) \tag{1.16}$$

on the time-scale 1.

Proof. The proof can be referred to Verhulst (1991). ■

Assume that $\mathbf{f}(\mathbf{x}, t, \varepsilon)$ in Equation (1.12) can be expanded in a convergent Taylor series with respect to ε and $\mathbf{x}$ in a finite domain. Consider an unperturbed system in Equation (1.12) as

$$\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, t, 0), \mathbf{x}(t_0) = \mathbf{b} \tag{1.17}$$

Using a transform

$$\mathbf{x}(t) = y(t) + \mathbf{x}_0(t) \tag{1.18}$$

Equation (1.12) becomes

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t, \varepsilon), \mathbf{x}(t_0) = \mathbf{c} \tag{1.19}$$

where $\mathbf{c} = \mathbf{a} - \mathbf{b}$ and

$$\mathbf{F}(\mathbf{y}, t, \varepsilon) = \mathbf{f}(\mathbf{y} + \mathbf{x}_0, t, \varepsilon) - \mathbf{f}(\mathbf{x}_0, t, 0) \tag{1.20}$$

Thus, the Poincare perturbation theory for nonlinear dynamical systems can also be stated as follows:

Theorem 1.2 *(Poincare) Consider a dynamical system*

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t, \varepsilon), \mathbf{y}(t_0) = \mathbf{c} \tag{1.21}$$

with $\mathbf{y} \in D \subset \mathscr{R}^n$, $|t - t_0| < h$, *and* $0 \le \varepsilon < 1$. $\mathbf{F}(\mathbf{y}, t, \varepsilon)$ *is a* C^r *– continuous vector function of* t, $\mathbf{y}$, *and* ε. *If such a vector function can be expanded in a convergent power series with respect to* $\mathbf{y}$ *and* ε *for* $\|\mathbf{y}\| < \rho$ *and* $0 \le \varepsilon < 1$, *then* $\mathbf{y}(t)$ *can be expanded in a convergent power series with respect to* $\mathbf{c}$ *and* ε *in the vicinity of* $\mathbf{c} = \mathbf{0}$ *and* $\varepsilon = 0$ *on time scale 1.*

Proof. The proof can be referred to Verhulst (1991). ■

In the perturbation theory, the Poincare-Lindstedt method is discussed herein. Consider a vibration problem as

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, \varepsilon),\ (x, \dot{x}) \in D \subset \mathscr{R}^2. \tag{1.22}$$

For $\varepsilon = 0$, with initial condition $(x, \dot{x})|_{t=0} = (a, 0)$

$$x = a\cos t. \tag{1.23}$$

For variation of a foregoing solution with ε, the following transformation is introduced as

$$\omega t = \theta,\ \omega^{-2} = 1 - \varepsilon\eta(\varepsilon). \tag{1.24}$$

Application of Equation (1.24) to Equation (1.22) gives

$$\begin{aligned} x'' + x &= \varepsilon[\eta x + (1 - \varepsilon\pi)] f(x, (1 - \varepsilon\eta)^{-\frac{1}{2}} x', \varepsilon) \\ &\equiv \varepsilon g(x, x', \varepsilon, \eta) \end{aligned} \tag{1.25}$$

with initial conditions

$$x(0) = a(\varepsilon), x'(0) = 0. \tag{1.26}$$

From the solution of $\varepsilon = 0$, by the variation of constant, Equation (1.25) gives

$$x(\theta) = a\cos\theta + \varepsilon\int_0^{\theta} \sin(\theta - \tau) g(x(\tau), x'(\tau), \varepsilon, \eta) d\tau \tag{1.27}$$

From the periodicity, $x(\theta) = x(\theta + 2\pi)$ in the foregoing equation yields

$$\int_{\theta}^{\theta+2\pi} \sin(\theta - \tau) g(x(\tau), x'(\tau), \varepsilon, \eta) d\tau = 0. \tag{1.28}$$

Thus,

$$\begin{aligned} &\int_{0}^{2\pi} \sin\tau g(x(\tau), x'(\tau), \varepsilon, \eta) d\tau = 0, \\ &\int_{0}^{2\pi} \cos\tau g(x(\tau), x'(\tau), \varepsilon, \eta) d\tau = 0; \end{aligned} \tag{1.29}$$

from which we obtain

$$F_1(a, \eta) = 0 \text{ and } F_2(a, \eta) = 0. \tag{1.30}$$

If the following equation exists

$$\left| \frac{\partial \left(F_1, F_2\right)}{\partial(a, \eta)} \right| \neq 0, \tag{1.31}$$

then

$$a(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k a_k, \text{ and } \eta(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \eta_k \tag{1.32}$$

and the solution of Equation (1.25) is

$$x(\theta) = a(0) \cos\theta + \sum_{k=0}^{\infty} \varepsilon^k \gamma_k(\theta) \tag{1.33}$$

In the foregoing procedure, the nonlinear solution is based on the variation of linear solution, which may not be adequate. This method is the foundation of multiple-scale method. Introduce

$$\omega = 1 + \sum_{k=0}^{\infty} \varepsilon^k \omega_k. \tag{1.34}$$

The following quantities are assumed as

$$\theta_N = \left(1 + \sum_{k=1}^{N} \varepsilon^k \omega_k\right) t \text{ and } x_M = a_0 \cos\theta + \sum_{k=1}^{M} \varepsilon^k \gamma_k(\theta). \tag{1.35}$$

Such a procedure makes the problem more complicated.

1.1.3 Method of Averaging

Based on the Lagrange standard form, one developed the method of averaging. van der Pol (1920) used the averaging method to determine the periodic motions of self-excited systems in circuits, and the presence of natural entrainment frequencies in such a system was observed in van der Pol and van der Mark (1927). Cartwright and Littlewood (1945) discussed the periodic motions of the van der Pol equation and proved the existence of periodic motions. Levinson (1948) used a piecewise linear model to describe the van der Pol equation and determined the existence of periodic motions. Levinson (1949) further developed the structures of periodic solutions in such a second order differential equation through the piecewise linear model, and discovered that infinite periodic solutions exist in such a piecewise linear model.

Since the nonlinear phenomena was observed in engineering, Duffing (1918) used the hardening spring model to investigate the vibration of electro-magnetized vibrating beam, and after that, the Duffing oscillator has been extensively used in structural dynamics. In addition to determining the existence of periodic motions in nonlinear different equations of the second order in mathematics, one has applied the Poincare perturbation methods for periodic motions in nonlinear dynamical systems. Fatou (1928) provided the first proof of asymptotic validity of the method of averaging through the existence of solutions of differential equations. Krylov and Bogolyubov (1935) developed systematically the method of averaging and the detailed discussion can be found in Bogoliubov and Mitropolsky (1961). The method of averaging is presented as follows:

Theorem 1.3 *Consider a dynamical system*

$$\dot{\mathbf{x}} = \varepsilon \mathbf{f}(\mathbf{x}, t) + \varepsilon^2 \mathbf{g}(\mathbf{x}, t, \varepsilon), \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.36}$$

If the following conditions are satisfied, that is,

i. *the vector functions* $\mathbf{f}, \mathbf{g}$ *and* $\partial \mathbf{f}/\partial \mathbf{x}$ *are defined, continuous and bounded;*
ii. $\mathbf{f}(\mathbf{x}, t)$ *is T-periodic with*

$$\mathbf{f}^0(\mathbf{x}, t) = \frac{1}{T}\int_0^T \mathbf{f}(\mathbf{x}, t) dt \tag{1.37}$$

where T is constant independent of ε, *and*

$$\dot{\mathbf{y}} = \varepsilon \mathbf{f}^0(\mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0 \tag{1.38}$$

with $\mathbf{x}, \mathbf{y} \in D \subset \mathscr{R}^n$, $|t - t_0| < h$, *and* $0 \le \varepsilon \le \varepsilon_0 < 1$;
iii. $\mathbf{g}$ *is Lipschitz-continuous in* $\mathbf{x}$ *for* $\mathbf{x} \in D$, *and* $\mathbf{y}(t)$ *is in the subset of* D; *then*

$$\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon) \tag{1.39}$$

on the time scale $1/\varepsilon$.

Proof. The proof can be referred to Verhulst (1991). ■

The classic perturbation methods for nonlinear oscillators were presented (e.g., Stoker, 1950; Minorsky, 1962; Hayashi, 1964). Hayashi (1964) used the method of averaging and the harmonic balance method (HBM) to discuss the approximate periodic solutions of nonlinear systems and the corresponding stability. Nayfeh (1973) employed the multiple-scale perturbation method to develop approximate solutions of periodic motions in the Duffing oscillators. Holmes and Rand (1976) discussed the stability and bifurcation of periodic motions in the Duffing oscillator. Nayfeh and Mook (1979) applied the perturbation analysis to nonlinear structural vibrations via the Duffing oscillators, and Holmes (1979) demonstrated chaotic motions in nonlinear oscillators through the Duffing oscillator with a twin-well potential. Ueda (1980) numerically simulated chaos via period-doubling of periodic motions of Duffing oscillators. Thus, one continues using the perturbation analysis to determine the approximate analytical solution of periodic motions. Coppola and Rand (1990) determined limit cycles of nonlinear oscillators through elliptic functions in the averaging method. Wang *et al*. (1992) used the harmonic balance method and the Floquet theory to investigate the nonlinear behaviors of the Duffing oscillator with a bounded potential well (also see, Kao, Wang, and Yang, 1992). Luo and Han (1997) determined the stability and bifurcation conditions of periodic motions of the Duffing oscillator. However, only symmetric periodic motions of the Duffing oscillators were investigated. Luo and Han (1999) investigated the analytical prediction of chaos in nonlinear rods through the Duffing oscillator. Peng *et al*. (2008) presented the approximate symmetric solution of period-1 motions in the Duffing oscillator by the harmonic balance method with three harmonic terms. In addition, Buonomo (1998a,b) showed the procedure for periodic solutions of van der Pol oscillator in power series. Kovacic and Mickens (2012) applied the generalized Krylov-Bogoliubov method to the van der Pol oscillator with small nonlinearity for limit cycles.

For parametric oscillators, the Mathieu equation should be mentioned herein. Mathieu (1868) investigated the linear Mathieu equation (also see, Mathieu, 1873; McLachlan, 1947). Whittaker (1913) presented a method to find the unstable solutions for very weak excitation (also see, Whittaker and Watson, 1935). In engineering, Sevin (1961) used the Mathieu equation to investigate the vibration-absorber with parametric excitation. Hsu (1963) discussed the first approximation analysis and stability criteria for a multiple-degree of freedom dynamical system (also see, Hsu, 1965). Tso and Caughey (1965) discussed the stability of parametric, nonlinear systems. Mond *et al*. (1993) presented the stability analysis of nonlinear Mathieu equation. Zounes and Rand (2000) discussed the transient response for the quasi-periodic Mathieu equation. Luo (2004) discussed chaotic motions in the resonant separatrix bands of the Mathieu-Duffing oscillator with a twin-well potential. Shen *et al*. (2008) used the incremental harmonic balance method to investigate the bifurcation and route to chaos in the Mathieu-Duffing oscillator.

The rotor dynamics is about the vibration of rotating shaft with disks. The shaft is supported by bearings with seals. In industrial application, flexible rotors are extensively used, which is relatively long. In 1883, Gustav Delaval manufactured a gas turbine which can operate over the first critical rotation speed. The high performance machines always operate over the first critical speed. Jeffcott (1919) first developed equations of motion for linear rotor dynamics. For such a linear rotor system, it can be easily analyzed. However, the results may not be adequate for flexible rotors with high operation speed. Thus one considered the bearing clearance, squeezing film dampers, seals, and fluid dynamics effects in the flexible rotor systems. Begg (1974) investigated the stability of a friction-induced rotor whirl motion. Childs (1982) applied a perturbation method to investigate subharmonic responses of a rotor with a small nonlinearity. Choi and Noah (1987) used the harmonic balance method and fast Fourier transformation (FFT) to study the subharmonic and superharmonic responses in a rotor with a bearing clearance. Day (1987) used multiple-scale method to show the aperiodic motion. Ehrich (1988) numerically investigated higher order subharmonic responses in such a rotor system under a high operation speed. Kim and Noah (1990) used the harmonic balance method to discuss the bifurcation of periodic motions in a modified Jeffcott rotor with bearing clearings. Choi and Noah (1994) still used the harmonic balance method to investigate mode-locking motion and chaos in such a Jeffcott rotor. The quasi-periodic motions and stability for such a modified Jeffcott rotor was also presented through the harmonic balance method in Kim and Noah (1996). Chu and Zhang (1998) used the harmonic balance method to determine periodic motions and numerically show the bifurcation scenarios. In fact, the modified Jeffcott rotor is discontinuous. Thus, the harmonic balance method may not be an adequate method for periodic motions in such a modified rotor with discontinuity, which can be as a rough prediction. Jiang and Ulbrich (2001) investigated stability of sliding whirl in a nonlinear Jeffcott rotor.

1.1.4 Generalized Harmonic Balance

As mentioned in previous sections, those analytical methods are based on solutions of linear systems to solve the nonlinear dynamical systems. Variation of constants in the solution of a so-called related linear system enforces the original nonlinear system to satisfy under the perturbation expansion with a small parameter. One always thinks the periodic motion as like a circle with a harmonic term. Such a complicated procedure cannot give satisfactory solutions. In such a mathematical treatment, the original vector fields are changed through the perturbation expansion. Thus, the approximate solutions cannot represent the original dynamical systems for a long time period.

To determine periodic solutions in nonlinear systems, we should find a basis of periodic functions to represent the periodic solution in nonlinear dynamical systems instead of perturbation expansion. Luo (2012a) developed a generalized harmonic balance method to get the approximate analytical solutions of periodic motions and chaos in nonlinear dynamical systems. This method used the finite term Fourier series to express periodic motions and the coefficients are time-varying. With the principle of virtual work, a dynamical system of coefficients are obtained from which the steady-state solution are achieved and the corresponding stability and bifurcation are completed. Two theorems will be presented herein, which will be used in other chapters. The detailed description of such a theory can be referred to Luo (2012a, 2013, 2014). Without excitation, the corresponding theorem of a nonlinear vibration system is stated as follows.

Theorem 1.4 *Consider a nonlinear vibration system as*

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) \in \mathscr{R}^n \tag{1.40}$$

where $\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{p})$ *is a* C^r *– continuous nonlinear function vector* $(r \geq 1)$. *If such a dynamical system has a period-*m *motion* $\mathbf{x}^{(m)}(t)$ *with finite norm* $\|\mathbf{x}^{(m)}\|$ *and period* $T = 2\pi/\Omega$, *there is a generalized coordinate transformation with* $\theta = \Omega t$ *for the periodic motion of Equation (1.40) in the form of*

$$\mathbf{x}^{(m)}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{\infty} \mathbf{b}_{k/m}(t) \cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t) \sin\left(\frac{k}{m}\theta\right) \tag{1.41}$$

with

$$\begin{aligned}
\mathbf{a}_0^{(m)} &= (a_{01}^{(m)}, a_{02}^{(m)}, \ldots, a_{0n}^{(m)})^{\mathrm{T}},\\
\mathbf{b}_{k/m} &= (b_{k/m1}, b_{k/m2}, \ldots, b_{k/mn})^{\mathrm{T}},\\
\mathbf{c}_{k/m} &= (c_{k/m1}, c_{k/m2}, \ldots, c_{k/mn})^{\mathrm{T}}
\end{aligned} \tag{1.42}$$

and

$$\begin{aligned}
&\|\mathbf{x}^{(m)}\| = \|\mathbf{a}_0^{(m)}\| + \sum_{k=1}^{\infty} \|\mathbf{A}_{k/m}\|, \text{ and } \lim_{k\to\infty} \|\mathbf{A}_{k/m}\| = 0 \text{ but not uniform}\\
&\text{with } \mathbf{A}_{k/m} = (A_{k/m1}, A_{k/m2} \ldots, A_{k/mn})^{\mathrm{T}}\\
&\text{and } A_{k/mj} = \sqrt{b_{k/mj}^2 + c_{k/mj}^2}\ (j = 1, 2, \ldots, n).
\end{aligned} \tag{1.43}$$

For $\|\mathbf{x}^{(m)}(t)-\mathbf{x}^{(m)*}(t)\|<\varepsilon$ *with a prescribed small* $\varepsilon>0$, *the infinite term transformation* $\mathbf{x}^{(m)}(t)$ *of period-*m *motion of Equation (1.40), given by Equation (1.41), can be approximated by a finite term transformation* $\mathbf{x}^{(m)*}(t)$ *as*

$$\mathbf{x}^{(m)*}(t)=\mathbf{a}_0^{(m)}(t)+\sum_{k=1}^{N}\mathbf{b}_{k/m}(t)\cos\left(\frac{k}{m}\theta\right)+\mathbf{c}_{k/m}(t)\sin\left(\frac{k}{m}\theta\right) \tag{1.44}$$

and the generalized coordinates are determined by

$$\begin{aligned}
\ddot{\mathbf{a}}_0^{(m)}&=\mathbf{F}_0^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}^{(m)},\dot{\mathbf{c}}^{(m)}),\\
\ddot{\mathbf{b}}^{(m)}&=-2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{c}}^{(m)}+\frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{b}^{(m)}+\mathbf{F}_1^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}^{(m)},\dot{\mathbf{c}}^{(m)}),\\
\ddot{\mathbf{c}}^{(m)}&=2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{b}}^{(m)}+\frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{c}^{(m)}+\mathbf{F}_2^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}^{(m)},\dot{\mathbf{c}}^{(m)})
\end{aligned} \tag{1.45}$$

where

$$\begin{aligned}
&\mathbf{k}_1=diag(\mathbf{I}_{n\times n},2\mathbf{I}_{n\times n},\ldots,N\mathbf{I}_{n\times n}),\\
&\mathbf{k}_2=diag(\mathbf{I}_{n\times n},2^2\mathbf{I}_{n\times n},\ldots,N^2\mathbf{I}_{n\times n}),\\
&\mathbf{b}^{(m)}=(\mathbf{b}_{1/m},\mathbf{b}_{2/m},\ldots,\mathbf{b}_{N/m})^{\mathrm{T}},\\
&\mathbf{c}^{(m)}=(\mathbf{c}_{1/m},\mathbf{c}_{2/m},\ldots,\mathbf{c}_{N/m})^{\mathrm{T}},\\
&\mathbf{F}_1^{(m)}=(\mathbf{F}_{11}^{(m)},\mathbf{F}_{12}^{(m)},\ldots,\mathbf{F}_{1N}^{(m)})^{\mathrm{T}},\\
&\mathbf{F}_2^{(m)}=(\mathbf{F}_{21}^{(m)},\mathbf{F}_{22}^{(m)},\ldots,\mathbf{F}_{2N}^{(m)})^{\mathrm{T}}\\
&\text{for } N=1,2,\ldots,\infty
\end{aligned} \tag{1.46}$$

and for $k=1,2,\ldots,N$

$$\begin{aligned}
&\mathbf{F}_0^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}^{(m)},\dot{\mathbf{c}}^{(m)})\\
&\quad=\frac{1}{2m\pi}\int_0^{2m\pi}\mathbf{f}(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},\mathbf{p})d\theta;\\
&\mathbf{F}_{1k}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}^{(m)},\dot{\mathbf{c}}^{(m)})\\
&\quad=\frac{1}{m\pi}\int_0^{2m\pi}\mathbf{f}(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},\mathbf{p})\cos\left(\frac{k}{m}\theta\right)d\theta,\\
&\mathbf{F}_{2k}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}^{(m)},\dot{\mathbf{c}}^{(m)})\\
&\quad=\frac{1}{m\pi}\int_0^{2m\pi}\mathbf{f}(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},\mathbf{p})\sin\left(\frac{k}{m}\theta\right)d\theta.
\end{aligned} \tag{1.47}$$

The state-space form of Equation (1.45) is

$$\dot{\mathbf{z}}^{(m)}=\mathbf{z}_1^{(m)}\text{ and }\dot{\mathbf{z}}_1^{(m)}=\mathbf{g}^{(m)}(\mathbf{z}^{(m)},\mathbf{z}_1^{(m)}) \tag{1.48}$$

where

$$\mathbf{z}^{(m)} = (\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)})^{\mathrm{T}}, \dot{\mathbf{z}}^{(m)} = \mathbf{z}_1^{(m)}$$
$$\mathbf{g}^{(m)} = \left(\mathbf{F}_0^{(m)}, -2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{c}}^{(m)} + \frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{b}^{(m)} + \mathbf{F}_1^{(m)},\right.$$
$$\left. 2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{b}}^{(m)} + \frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{c}^{(m)} + \mathbf{F}_2^{(m)}\right)^{\mathrm{T}}. \tag{1.49}$$

An equivalent system of Equation (1.48) is

$$\dot{\mathbf{y}}^{(m)} = \mathbf{f}^{(m)}(\mathbf{y}^{(m)}) \tag{1.50}$$

where

$$\mathbf{y}^{(m)} = (\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)})^{\mathrm{T}} \text{ and } \mathbf{f}^{(m)} = (\mathbf{z}_1^{(m)}, \mathbf{g}^{(m)})^{\mathrm{T}}. \tag{1.51}$$

*If equilibrium $\mathbf{y}^{(m)*}$ of Equation (1.50) (i.e., $\mathbf{f}^{(m)}(\mathbf{y}^{(m)*}) = \mathbf{0}$) exists, then the approximate solution of period-*m *motion exists as in Equation (1.44). In vicinity of equilibrium $\mathbf{y}^{(m)*}$, with $\mathbf{y}^{(m)} = \mathbf{y}^{(m)*} + \Delta\mathbf{y}^{(m)}$, the linearized equation of Equation (1.50) is*

$$\Delta\dot{\mathbf{y}}^{(m)} = D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*})\Delta\mathbf{y}^{(m)} \tag{1.52}$$

and the eigenvalue analysis of the equilibrium $\mathbf{y}^*$ is given by

$$|D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*}) - \lambda\mathbf{I}_{2n(2N+1)\times 2n(2N+1)}| = 0 \tag{1.53}$$

*where $D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*}) = \partial\mathbf{f}^{(m)}(\mathbf{y}^{(m)})/\partial\mathbf{y}^{(m)}|_{\mathbf{y}^{(m)*}}$. Thus, the stability and bifurcation period-*m *motion can be classified by the eigenvalues of $D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*})$ with*

$$(n_1, n_2, n_3 | n_4, n_5, n_6). \tag{1.54}$$

i. *If all eigenvalues of the equilibrium possess negative real parts, the approximate periodic solution is stable.*
ii. *If at least one of the eigenvalues of the equilibrium possesses a positive real part, the approximate periodic solution is unstable.*
iii. *The boundaries between stable and unstable equilibriums with higher order singularity give bifurcation and stability conditions with higher order singularity.*

Proof. The proof can be referred to Luo (2012a, 2013, 2014). ■

With periodic excitation, the dynamical systems can be stated as follows:

Theorem 1.5 *Consider a periodically forced, nonlinear vibration system as*

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t, \mathbf{p}) \in \mathscr{R}^n \tag{1.55}$$

*where $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t, \mathbf{p})$ is a C^r – continuous nonlinear function vector ($r \geq 1$) with forcing period $T = 2\pi/\Omega$. If such a vibration system has a period-*m *motion $\mathbf{x}^{(m)}(t)$ with finite norm $\|\mathbf{x}^{(m)}\|$*

and period $T = 2\pi/\Omega$, there is a generalized coordinate transformation with $\theta = \Omega t$ for the periodic motion of Equation (1.55) in the form of

$$\mathbf{x}^{(m)}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{\infty} \mathbf{b}_{k/m}(t)\cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t)\sin\left(\frac{k}{m}\theta\right) \tag{1.56}$$

with

$$\begin{aligned} \mathbf{a}_0^{(m)} &= (a_{01}^{(m)}, a_{02}^{(m)}, \ldots, a_{0n}^{(m)})^{\mathrm{T}}, \\ \mathbf{b}_{k/m} &= (b_{k/m1}, b_{k/m2}, \ldots, b_{k/mn})^{\mathrm{T}}, \\ \mathbf{c}_{k/m} &= (c_{k/m1}, c_{k/m2}, \ldots, c_{k/mn})^{\mathrm{T}} \end{aligned} \tag{1.57}$$

and

$$\begin{aligned} &\|\mathbf{x}^{(m)}\| = \|\mathbf{a}_0^{(m)}\| + \sum_{k=1}^{\infty} \|\mathbf{A}_{k/m}\|, \text{ and } \lim_{k\to\infty} \|\mathbf{A}_{k/m}\| = 0 \text{ but not uniform} \\ &\text{with } \mathbf{A}_{k/m} = (A_{k/m1}, A_{k/m2} \ldots, A_{k/mn})^{\mathrm{T}} \\ &\text{and } A_{k/mj} = \sqrt{b_{k/mj}^2 + c_{k/mj}^2}\ (j = 1, 2, \ldots, n). \end{aligned} \tag{1.58}$$

*For $\|\mathbf{x}^{(m)}(t) - \mathbf{x}^{(m)*}(t)\| < \varepsilon$ with a prescribed small $\varepsilon > 0$, the infinite term transformation $\mathbf{x}^{(m)}(t)$ of period-*m *motion of Equation (1.55), given by Equation (1.56), can be approximated by a finite term transformation $\mathbf{x}^{(m)*}(t)$ as*

$$\mathbf{x}^{(m)*}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{N} \mathbf{b}_{k/m}(t)\cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t)\sin\left(\frac{k}{m}\theta\right) \tag{1.59}$$

and the generalized coordinates are determined by

$$\begin{aligned} \ddot{\mathbf{a}}_0^{(m)} &= \mathbf{F}_0^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}), \\ \ddot{\mathbf{b}}^{(m)} &= -2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{c}}^{(m)} + \frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{b}^{(m)} + \mathbf{F}_1^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}), \\ \ddot{\mathbf{c}}^{(m)} &= 2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{b}}^{(m)} + \frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{c}^{(m)} + \mathbf{F}_2^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \end{aligned} \tag{1.60}$$

where for $N = 1, 2, \ldots, \infty$

$$\begin{aligned} \mathbf{k}_1 &= diag(\mathbf{I}_{n\times n}, 2\mathbf{I}_{n\times n}, \ldots, N\mathbf{I}_{n\times n}), \\ \mathbf{k}_2 &= diag(\mathbf{I}_{n\times n}, 2^2\mathbf{I}_{n\times n}, \ldots, N^2\mathbf{I}_{n\times n}), \\ \mathbf{b}^{(m)} &= (\mathbf{b}_{1/m}, \mathbf{b}_{2/m}, \ldots, \mathbf{b}_{N/m})^{\mathrm{T}}, \\ \mathbf{c}^{(m)} &= (\mathbf{c}_{1/m}, \mathbf{c}_{2/m}, \ldots, \mathbf{c}_{N/m})^{\mathrm{T}}, \\ \mathbf{F}_1^{(m)} &= (\mathbf{F}_{11}^{(m)}, \mathbf{F}_{12}^{(m)}, \ldots, \mathbf{F}_{1N}^{(m)})^{\mathrm{T}} \\ \mathbf{F}_2^{(m)} &= (\mathbf{F}_{21}^{(m)}, \mathbf{F}_{22}^{(m)}, \ldots, \mathbf{F}_{2N}^{(m)})^{\mathrm{T}}; \end{aligned} \tag{1.61}$$

and for $k = 1, 2, \ldots, N$

$$\begin{aligned}
&\mathbf{F}_0^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \\
&\quad = \frac{1}{2m\pi}\int_0^{2m\pi} \mathbf{F}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, t, \mathbf{p})d\theta; \\
&\mathbf{F}_{1k}^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \\
&\quad = \frac{1}{m\pi}\int_0^{2m\pi} \mathbf{F}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, t, \mathbf{p})\cos\left(\frac{k}{m}\theta\right)d\theta, \\
&\mathbf{F}_{2k}^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \\
&\quad = \frac{1}{m\pi}\int_0^{2m\pi} \mathbf{F}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, t, \mathbf{p})\sin\left(\frac{k}{m}\theta\right)d\theta.
\end{aligned} \tag{1.62}$$

The state-space form of Equation (1.61) is

$$\dot{\mathbf{z}}^{(m)} = \mathbf{z}_1^{(m)} \text{ and } \dot{\mathbf{z}}_1^{(m)} = \mathbf{g}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \tag{1.63}$$

where

$$\begin{aligned}
&\mathbf{z}^{(m)} = (\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)})^{\mathrm{T}}, \dot{\mathbf{z}}^{(m)} = \mathbf{z}_1^{(m)}, \\
&\mathbf{g}^{(m)} = \left(\mathbf{F}_0^{(m)}, -2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{c}}^{(m)} + \frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{b}^{(m)} + \mathbf{F}_1^{(m)},\right. \\
&\qquad\left. 2\frac{\Omega}{m}\mathbf{k}_1\dot{\mathbf{b}}^{(m)} + \frac{\Omega^2}{m^2}\mathbf{k}_2\mathbf{c}^{(m)} + \mathbf{F}_2^{(m)}\right)^{\mathrm{T}}.
\end{aligned} \tag{1.64}$$

An equivalent system of Equation (1.63) is

$$\dot{\mathbf{y}}^{(m)} = \mathbf{f}^{(m)}(\mathbf{y}^{(m)}) \tag{1.65}$$

where

$$\mathbf{y}^{(m)} = (\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)})^{\mathrm{T}} \text{ and } \mathbf{f}^{(m)} = (\mathbf{z}_1^{(m)}, \mathbf{g}^{(m)})^{\mathrm{T}}. \tag{1.66}$$

If equilibrium $\mathbf{y}^{(m)*}$ *of Equation (1.65) exists (i.e.,* $\mathbf{f}^{(m)}(\mathbf{y}^{(m)*}) = \mathbf{0}$*), then the approximate solution of period-*m *motion exists as in Equation (1.59). In vicinity of equilibrium* $\mathbf{y}^{(m)*}$*, with* $\mathbf{y}^{(m)} = \mathbf{y}^{(m)*} + \Delta\mathbf{y}^{(m)}$*, the linearized equation of Equation (1.65) is*

$$\Delta\dot{\mathbf{y}}^{(m)} = D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*})\Delta\mathbf{y}^{(m)} \tag{1.67}$$

and the eigenvalue analysis of equilibrium $\mathbf{y}^*$ *is given by*

$$|D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*}) - \lambda\mathbf{I}_{2n(2N+1)\times 2n(2N+1)}| = 0 \tag{1.68}$$

where $D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*}) = \partial\mathbf{f}^{(m)}(\mathbf{y}^{(m)})/\partial\mathbf{y}^{(m)}|_{\mathbf{y}^{(m)*}}$*. The stability and bifurcation of period-*m *motion can be classified by eigenvalues of* $D\mathbf{f}^{(m)}(\mathbf{y}^{(m)*})$ *are classified by*

$$(n_1, n_2, n_3 | n_4, n_5, n_6). \tag{1.69}$$

i. *If all eigenvalues of the equilibrium possess negative real parts, the approximate periodic solution is stable.*

ii. *If at least one of the eigenvalues of the equilibrium possesses positive real part, the approximate periodic solution is unstable.*
iii. *The boundaries between stable and unstable equilibriums with higher order singularity give bifurcation and stability conditions with higher order singularity.*

Proof. The proof can be referred to Luo (2012a, 2013, 2014). ■

As in the aforementioned two theorems for period-*m* motions, the analytical solution structures of quasi-periodic motions in nonlinear vibration systems will be presented from Luo (2014) as follows.

Theorem 1.6 *Consider a nonlinear vibration system as*

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) \in \mathscr{R}^n \tag{1.70}$$

where $\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{p})$ *is a* C^r *– continuous nonlinear function vector* $(r \geq 1)$.

A. *If such a dynamical system has a period-*m *motion* $\mathbf{x}^{(m)}(t)$ *with finite norm* $\|\mathbf{x}^{(m)}\|$ *and period* $T = 2\pi/\Omega$, *there is a generalized coordinate transformation with* $\theta = \Omega t$ *for the periodic motion of Equation (1.70) in a form of*

$$\mathbf{x}^{(m)}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{\infty} \mathbf{b}_{k/m}(t) \cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t) \sin\left(\frac{k}{m}\theta\right) \tag{1.71}$$

with

$$\begin{aligned}
&\mathbf{a}_1^{(0)} \equiv \mathbf{a}_0^{(m)} = (a_{01}^{(m)}, a_{02}^{(m)}, \ldots, a_{0n}^{(m)})^{\mathrm{T}},\\
&\mathbf{a}_2^{(k)} \equiv \mathbf{b}_{k/m} = (b_{k/m1}, b_{k/m2}, \ldots, b_{k/mn})^{\mathrm{T}},\\
&\mathbf{a}_3^{(k)} \equiv \mathbf{c}_{k/m} = (c_{k/m1}, c_{k/m2}, \ldots, c_{k/mn})^{\mathrm{T}}
\end{aligned} \tag{1.72}$$

which, under $\|\mathbf{x}^{(m)}(t) - \mathbf{x}^{(m)*}(t)\| < \varepsilon$ *with a prescribed small* $\varepsilon > 0$, *can be approximated by a finite term transformation* $\mathbf{x}^{(m)*}(t)$

$$\mathbf{x}^{(m)*}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{N_0} \mathbf{b}_{k/m}(t) \cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t) \sin\left(\frac{k}{m}\theta\right) \tag{1.73}$$

and the generalized coordinates are determined by

$$\ddot{\mathbf{a}}_{s_0} = \mathbf{g}_{s_0}(\mathbf{a}_{s_0}, \dot{\mathbf{a}}_{s_0}, \mathbf{p}) \tag{1.74}$$

where

$$\begin{aligned}
&\mathbf{k}_0^{(1)} = diag(\mathbf{I}_{n\times n}, 2\mathbf{I}_{n\times n}, \ldots, N_0\mathbf{I}_{n\times n}),\\
&\mathbf{k}_0^{(2)} = diag(\mathbf{I}_{n\times n}, 2^2\mathbf{I}_{n\times n}, \ldots, N_0^2\mathbf{I}_{n\times n}),\\
&\mathbf{a}_1^{(0)} \equiv \mathbf{a}_0^{(m)}, \mathbf{a}_2^{(k)} \equiv \mathbf{b}_{k/m}, \mathbf{a}_3^{(k)} \equiv \mathbf{c}_{k/m};\\
&\mathbf{a}_1 = \mathbf{a}_1^{(0)},\\
&\mathbf{a}_2 = (\mathbf{a}_2^{(1)}, \mathbf{a}_2^{(2)}, \ldots, \mathbf{a}_2^{(N)})^{\mathrm{T}} \equiv \mathbf{b}^{(m)},\\
&\mathbf{a}_3 = (\mathbf{a}_3^{(1)}, \mathbf{a}_3^{(2)}, \ldots, \mathbf{a}_3^{(N)})^{\mathrm{T}} \equiv \mathbf{c}^{(m)},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{F}_1 = \mathbf{F}_0^{(m)},\\
&\mathbf{F}_2 = (\mathbf{F}_{11}^{(m)}, \mathbf{F}_{12}^{(m)}, \ldots, \mathbf{F}_{1N_0}^{(m)})^{\mathrm{T}},\\
&\mathbf{F}_3 = (\mathbf{F}_{21}^{(m)}, \mathbf{F}_{22}^{(m)}, \ldots, \mathbf{F}_{2N_0}^{(m)})^{\mathrm{T}};\\
&\mathbf{a}_{s_0} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)^{\mathrm{T}},\\
&\mathbf{g}_{s_0} = \left(\mathbf{F}_1^{(m)}, -2\frac{\Omega}{m}\mathbf{k}_0^{(1)}\dot{\mathbf{a}}_3 + \frac{\Omega^2}{m^2}\mathbf{k}_0^{(2)}\mathbf{a}_2 + \mathbf{F}_2,\right.\\
&\qquad\left. 2\frac{\Omega}{m}\mathbf{k}_0^{(1)}\dot{\mathbf{a}}_2 + \frac{\Omega^2}{m^2}\mathbf{k}_0^{(2)}\mathbf{a}_3 + \mathbf{F}_3\right)^{\mathrm{T}}\\
&\text{for } N_0 = 1, 2, \ldots, \infty;
\end{aligned} \tag{1.75}$$

and

$$\begin{aligned}
&\mathbf{F}_0^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}) = \frac{1}{2m\pi}\int_0^{2m\pi} \mathbf{f}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, \mathbf{p})d\theta;\\
&\mathbf{F}_{1k}^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}) = \frac{1}{m\pi}\int_0^{2m\pi} \mathbf{f}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, \mathbf{p})\cos\left(\frac{k}{m}\theta\right)d\theta,\\
&\mathbf{F}_{2k}^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}) = \frac{1}{m\pi}\int_0^{2m\pi} \mathbf{f}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, \mathbf{p})\sin\left(\frac{k}{m}\theta\right)d\theta\\
&\text{for } k = 1, 2, \ldots, N_0.
\end{aligned} \tag{1.76}$$

B. *If after the* kth *Hopf bifurcation with* $p_k\omega_k = \omega_{k-1}$ $(k = 1, 2, \ldots)$ *and* $\omega_0 = \Omega/m$, *there is a dynamical system of coefficients as*

$$\ddot{\mathbf{a}}_{s_0s_1\ldots s_k} = \mathbf{g}_{s_0s_1\ldots s_k}(\mathbf{a}_{s_0s_1\ldots s_k}, \dot{\mathbf{a}}_{s_0s_1\ldots s_k}, \mathbf{p}) \tag{1.77}$$

where

$$\begin{aligned}
&\mathbf{a}_{s_0s_1\ldots s_k} = (\mathbf{a}_{s_0s_1\ldots s_{k-1}1}, \mathbf{a}_{s_0s_1\ldots s_{k-1}2}, \mathbf{a}_{s_0s_1\ldots s_{k-1}3})^{\mathrm{T}},\\
&\mathbf{g}_{s_0s_1\ldots s_k} = (\mathbf{F}_{s_0s_1\ldots s_{k-1}1}, -2\omega_k\mathbf{k}_k^{(1)}\dot{\mathbf{a}}_{s_0s_1\ldots s_{k-1}3} + \omega_k^2\mathbf{k}_k^{(2)}\mathbf{a}_{s_0s_1\ldots s_{k-1}2} + \mathbf{F}_{s_0s_1\ldots s_{k-1}2},\\
&\qquad 2\omega_k\mathbf{k}_k^{(1)}\dot{\mathbf{a}}_{s_0s_1\ldots s_{k-1}2} + \omega_k^2\mathbf{k}_k^{(2)}\mathbf{a}_{s_1s_2\ldots s_{k-1}3} + \mathbf{F}_{s_1s_2\ldots s_{k-1}3})^{\mathrm{T}},\\
&\mathbf{k}_k^{(1)} = diag(\mathbf{I}_{n_{k-1}\times n_{k-1}}, 2\mathbf{I}_{n_{k-1}\times n_{k-1}}, \ldots, N_k\mathbf{I}_{n_{k-1}\times n_{k-1}}),\\
&\mathbf{k}_k^{(2)} = diag(\mathbf{I}_{n_{k-1}\times n_{k-1}}, 2^2\mathbf{I}_{n_{k-1}\times n_{k-1}}, \ldots, N_k^2\mathbf{I}_{n_{k-1}\times n_{k-1}})\\
&n_{k-1} = n(2N_0 + 1)(2N_1 + 1)\ldots(2N_{k-1} + 1)
\end{aligned} \tag{1.78}$$

with a periodic solution as

$$\begin{aligned}
\mathbf{a}_{s_0s_1\ldots s_k} = \mathbf{a}_{s_0s_1\ldots s_k1}^{(0)}(t) + \sum_{l_{k+1}=1}^{\infty} &\mathbf{a}_{s_0s_1\ldots s_k2}^{(l_{k+1})}(t)\cos(l_{k+1}\theta_{k+1})\\
&+ \mathbf{a}_{s_0s_1\ldots s_k3}^{(l_{k+1})}(t)\sin(l_{k+1}\theta_{k+1})
\end{aligned} \tag{1.79}$$

with

$$
\begin{aligned}
& s_i = 1, 2, 3\ (i = 0, 1, 2, \ldots, k), \\
& \mathbf{a}_{s_0 s_1 \ldots s_k 1} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_k 1}, \\
& \mathbf{a}_{s_0 s_1 \ldots s_k 2} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 2}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 2})^{\mathrm{T}}, \\
& \mathbf{a}_{s_0 s_1 \ldots s_k 3} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 3}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 3}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}}; \\
& \mathbf{a}_{s_0 s_1 \ldots s_{k-1} 1} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_{k-1} 1}, \\
& \mathbf{a}_{s_0 s_1 \ldots s_{k-1} 2} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_{k-1} 2}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_{k-1} 2}, \ldots, \mathbf{a}^{(N_k)}_{s_0 s_1 \ldots s_{k-1} 2})^{\mathrm{T}}, \\
& \mathbf{a}_{s_0 s_1 \ldots s_{k-1} 3} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_{k-1} 3}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_{k-1} 3}, \ldots, \mathbf{a}^{(N_k)}_{s_0 s_1 \ldots s_{k-1} 3})^{\mathrm{T}}; \\
& \vdots \\
& \mathbf{a}_1 = \mathbf{a}^{(0)}_1, \\
& \mathbf{a}_2 = (\mathbf{a}^{(1)}_2, \mathbf{a}^{(2)}_2, \ldots, \mathbf{a}^{(N_0)}_2)^{\mathrm{T}}, \\
& \mathbf{a}_3 = (\mathbf{a}^{(1)}_3, \mathbf{a}^{(2)}_3, \ldots, \mathbf{a}^{(N_0)}_3)^{\mathrm{T}};
\end{aligned} \tag{1.80}
$$

which, under $\|\mathbf{a}_{s_0 s_1 \ldots s_k}(t) - \mathbf{a}^*_{s_0 s_1 \ldots s_k}(t)\| < \varepsilon$ *with a prescribed small* $\varepsilon > 0$, *can be approximated by a finite term transformation* $\mathbf{a}^*_{s_0 s_1 \ldots s_k}(t)$

$$
\begin{aligned}
\mathbf{a}^*_{s_0 s_1 \ldots s_k} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_k 1}(t) + \sum_{l_{k+1}=1}^{N_{k+1}} & \mathbf{a}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 2}(t) \cos(l_{k+1}\theta_{k+1}) \\
& + \mathbf{a}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 3}(t) \sin(l_{k+1}\theta_{k+1})
\end{aligned} \tag{1.81}
$$

and the generalized coordinates are determined by

$$
\ddot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}} = \mathbf{g}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p}) \tag{1.82}
$$

where

$$
\begin{aligned}
\mathbf{a}_{s_0 s_1 \ldots s_{k+1}} &= (\mathbf{a}_{s_0 s_1 \ldots s_k 1}, \mathbf{a}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}}, \\
\mathbf{g}_{s_0 s_1 \ldots s_{k+1}} &= (\mathbf{F}_{s_1 s_2 \ldots s_k 1}, -2\omega_{k+1}\mathbf{k}^{(1)}_{k+1}\dot{\mathbf{a}}_{s_1 s_2 \ldots s_k 3} + \omega^2_{k+1}\mathbf{k}^{(2)}_{k+1}\mathbf{a}_{s_1 s_2 \ldots s_k 2} + \mathbf{F}_{s_1 s_2 \ldots s_k 2}, \\
&\qquad 2\omega_{k+1}\mathbf{k}^{(1)}_{k+1}\dot{\mathbf{a}}_{s_1 s_2 \ldots s_k 2} + \omega^2_{k+1}\mathbf{k}^{(2)}_{k+1}\mathbf{a}_{s_1 s_2 \ldots s_k 3} + \mathbf{F}_{s_1 s_2 \ldots s_k 3})^{\mathrm{T}};
\end{aligned} \tag{1.83}
$$

and

$$
\begin{aligned}
\mathbf{k}^{(1)}_{k+1} &= diag(\mathbf{I}_{n_k \times n_k}, 2\mathbf{I}_{n_k \times n_k}, \ldots, N_{k+1}\mathbf{I}_{n_k \times n_k}), \\
\mathbf{k}^{(2)}_{k+1} &= diag(\mathbf{I}_{n_k \times n_k}, 2^2\mathbf{I}_{n_k \times n_k}, \ldots, N^2_{k+1}\mathbf{I}_{n_k \times n_k}) \\
n_k &= n(2N_0 + 1)(2N_1 + 1) \ldots (2N_k + 1);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{a}_{s_0 s_1 \ldots s_k 1} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_k 1},\\
&\mathbf{a}_{s_0 s_1 \ldots s_k 2} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 2}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 2})^{\mathrm{T}},\\
&\mathbf{a}_{s_0 s_1 \ldots s_k 3} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 3}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 3}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}};\\
&\mathbf{a}_{s_0 s_1 \ldots s_{k+1}} = (\mathbf{a}_{s_0 s_1 \ldots s_k 1}, \mathbf{a}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}};\\
&\mathbf{F}_{s_0 s_1 \ldots s_k 1} = \mathbf{F}^{(0)}_{s_0 s_1 \ldots s_k 1},\\
&\mathbf{F}_{s_0 s_1 \ldots s_k 2} = (\mathbf{F}^{(1)}_{s_0 s_1 \ldots s_k 2}, \mathbf{F}^{(2)}_{s_0 s_1 \ldots s_k 2}, \ldots, \mathbf{F}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 2})^{\mathrm{T}},\\
&\mathbf{F}_{s_0 s_1 \ldots s_k 3} = (\mathbf{F}^{(1)}_{s_0 s_1 \ldots s_k 3}, \mathbf{F}^{(2)}_{s_0 s_1 \ldots s_k 3}, \ldots, \mathbf{F}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}}\\
&\text{for } N_{k+1} = 1, 2, \ldots, \infty;
\end{aligned} \tag{1.84}
$$

and

$$
\begin{aligned}
&\mathbf{F}_{s_0 s_1 \ldots s_k 1}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})\\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}^*_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}^*_{s_0 s_1 \ldots s_k}, \mathbf{p}) d\theta_{k+1};\\
&\mathbf{F}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 2}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})\\
&\quad = \frac{1}{\pi}\int_0^{2\pi} \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}^*_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}^*_{s_0 s_1 \ldots s_k}, \mathbf{p}) \cos(l_{k+1}\theta_{k+1}) d\theta_{k+1},\\
&\mathbf{F}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 3}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})\\
&\quad = \frac{1}{\pi}\int_0^{2\pi} \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}^*_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}^*_{s_0 s_1 \ldots s_k}, \mathbf{p}) \sin(l_{k+1}\theta_{k+1}) d\theta_{k+1}\\
&\text{for } l_{k+1} = 1, 2, \ldots, N_{k+1}.
\end{aligned} \tag{1.85}
$$

C. *Equation (1.82) becomes*

$$
\dot{\mathbf{z}}_{s_0 s_1 \ldots s_{k+1}} = \mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}_{s_0 s_1 \ldots s_{k+1}}) \tag{1.86}
$$

where

$$
\begin{aligned}
\mathbf{z}_{s_0 s_1 \ldots s_{k+1}} &= (\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}})^{\mathrm{T}},\\
\mathbf{f}_{s_0 s_1 \ldots s_{k+1}} &= (\dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{g}_{s_0 s_1 \ldots s_{k+1}})^{\mathrm{T}}.
\end{aligned} \tag{1.87}
$$

If equilibrium $\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}$ *of Equation (1.86) (i.e.,* $\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}) = \mathbf{0}$*) exists, then the approximate solution of the periodic motion of the* kth *generalized coordinates for the period-*m *motion exists as in Equation (1.81). In the vicinity of equilibrium* $\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}$*, with*

$$
\mathbf{z}_{s_0 s_1 \ldots s_{k+1}} = \mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}} + \Delta\mathbf{z}_{s_0 s_1 \ldots s_{k+1}}, \tag{1.88}
$$

the linearized equation of Equation (1.86) is

$$\Delta\dot{\mathbf{z}}_{s_0 s_1 \ldots s_{k+1}} = D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}})\Delta\mathbf{z}_{s_0 s_1 \ldots s_{k+1}} \tag{1.89}$$

and the eigenvalue analysis of equilibrium $\mathbf{z}^*$ *is given by*

$$|D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}) - \lambda \mathbf{I}_{2n_k(2N_{k+1}+1)\times 2n_k(2N_{k+1}+1)}| = 0 \tag{1.90}$$

where

$$D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}) = \left.\frac{\partial \mathbf{f}_{s_0 s_1 \ldots s_{k+1}}\left(\mathbf{z}_{s_0 s_1 \ldots s_{k+1}}\right)}{\partial \mathbf{z}_{s_0 s_1 \ldots s_{k+1}}}\right|_{\mathbf{z}^*_{s_1 s_2 \ldots s_{k+1}}}. \tag{1.91}$$

The stability and bifurcation of such a periodic motion of the kth *generalized coordinates can be classified by the eigenvalues of* $D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}})$ *with*

$$(n_1, n_2, n_3 | n_4, n_5, n_6). \tag{1.92}$$

i. *If all eigenvalues of the equilibrium possess negative real parts, the approximate quasi-periodic solution is stable.*
ii. *If at least one of the eigenvalues of the equilibrium possesses positive real part, the approximate quasi-periodic solution is unstable.*
iii. *The boundaries between stable and unstable equilibriums with higher order singularity give bifurcation and stability conditions with higher order singularity.*

D. *For the* kth *order Hopf bifurcation of period*-m *motion, a relation exists as*

$$p_k \omega_k = \omega_{k-1}. \tag{1.93}$$

i. *If* p_k *is an irrational number, the* kth-*order Hopf bifurcation of the period*-m *motion is called the quasi-period-*p_k *Hopf bifurcation, and the corresponding solution of the* kth *generalized coordinates is* p_k*-quasi-periodic to the system of the* $(k-1)$th *generalized coordinates.*
ii. *If* $p_k = 2$, *the* kth-*order Hopf bifurcation of the period*-m *motion is called a period-doubling Hopf bifurcation (or a period-2 Hopf bifurcation), and the corresponding solution of the* kth *generalized coordinates is period-doubling to the system of the* $(k-1)$th *generalized coordinates.*
iii. *If* $p_k = q$ *with an integer q, the* kth-*order Hopf bifurcation of the period*-m *motion is called a period-q Hopf bifurcation, and the corresponding solution of the* kth *generalized coordinates is of q-times period to the system of the* $(k-1)$th *generalized coordinates.*
iv. *If* $p_k = p/q$ *(p, q are irreducible integer), the* kth-*order Hopf bifurcation of the period*-m *motion is called a period-*p/q *Hopf bifurcation, and the corresponding solution of the* kth *generalized coordinates is* p/q*-periodic to the system of the* $(k-1)$th *generalized coordinates.*

Proof. The proof of this theorem can be referred to Luo (2014). ■

Similarly, for periodically forced vibration systems, the analytical solution of quasi-periodic motions can be presented as follows.

Theorem 1.7 *Consider a periodically forced, nonlinear vibration system as*

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t, \mathbf{p}) \in \mathscr{R}^n \tag{1.94}$$

where $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t, \mathbf{p})$ *is a* C^r *– continuous nonlinear function vector* $(r \geq 1)$ *with forcing period* $T = 2\pi/\Omega$.

A. *If such a vibration system has a period-*m *motion* $\mathbf{x}^{(m)}(t)$ *with finite norm* $\|\mathbf{x}^{(m)}\|$, *there is a generalized coordinate transformation with* $\theta = \Omega t$ *for the period-*m *motion of Equation (1.94) in a form of*

$$\mathbf{x}^{(m)}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{\infty} \mathbf{b}_{k/m}(t)\cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t)\sin\left(\frac{k}{m}\theta\right) \tag{1.95}$$

with

$$\begin{aligned}
\mathbf{a}_1^{(0)} &\equiv \mathbf{a}_0^{(m)} = (a_{01}^{(m)}, a_{02}^{(m)}, \ldots, a_{0n}^{(m)})^{\mathrm{T}}, \\
\mathbf{a}_2^{(k)} &\equiv \mathbf{b}_{k/m} = (b_{k/m1}, b_{k/m2}, \ldots, b_{k/mn})^{\mathrm{T}}, \\
\mathbf{a}_3^{(k)} &\equiv \mathbf{c}_{k/m} = (c_{k/m1}, c_{k/m2}, \ldots, c_{k/mn})^{\mathrm{T}}
\end{aligned} \tag{1.96}$$

which, under $\|\mathbf{x}^{(m)}(t) - \mathbf{x}^{(m)*}(t)\| < \varepsilon$ *with a prescribed small* $\varepsilon > 0$, *can be approximated by a finite term transformation* $\mathbf{x}^{(m)*}(t)$

$$\mathbf{x}^{(m)*}(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{N} \mathbf{b}_{k/m}(t)\cos\left(\frac{k}{m}\theta\right) + \mathbf{c}_{k/m}(t)\sin\left(\frac{k}{m}\theta\right) \tag{1.97}$$

and the generalized coordinates are determined by

$$\ddot{\mathbf{a}}_{s_0} = \mathbf{g}_{s_0}(\mathbf{a}_{s_0}, \dot{\mathbf{a}}_{s_0}, \mathbf{p}) \tag{1.98}$$

where

$$\begin{aligned}
\mathbf{k}_0^{(1)} &= diag(\mathbf{I}_{n\times n}, 2\mathbf{I}_{n\times n}, \ldots, N_0\mathbf{I}_{n\times n}), \\
\mathbf{k}_0^{(2)} &= diag(\mathbf{I}_{n\times n}, 2^2\mathbf{I}_{n\times n}, \ldots, N_0^2\mathbf{I}_{n\times n}), \\
\mathbf{a}_1^{(0)} &\equiv \mathbf{a}_0^{(m)}, \mathbf{a}_2^{(k)} \equiv \mathbf{b}_{k/m}, \mathbf{a}_3^{(k)} \equiv \mathbf{c}_{k/m}; \\
\mathbf{a}_1 &= \mathbf{a}_1^{(0)}, \\
\mathbf{a}_2 &= (\mathbf{a}_2^{(1)}, \mathbf{a}_2^{(2)}, \ldots, \mathbf{a}_2^{(N_0)})^{\mathrm{T}} \equiv \mathbf{b}^{(m)}, \\
\mathbf{a}_3 &= (\mathbf{a}_3^{(1)}, \mathbf{a}_3^{(2)}, \ldots, \mathbf{a}_3^{(N_0)})^{\mathrm{T}} \equiv \mathbf{c}^{(m)}, \\
\mathbf{F}_2 &= (\mathbf{F}_{11}^{(m)}, \mathbf{F}_{12}^{(m)}, \ldots, \mathbf{F}_{1N_0}^{(m)})^{\mathrm{T}}, \\
\mathbf{F}_3 &= (\mathbf{F}_{21}^{(m)}, \mathbf{F}_{22}^{(m)}, \ldots, \mathbf{F}_{2N_0}^{(m)})^{\mathrm{T}}; \\
\mathbf{a}_{s_0} &= (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)^{\mathrm{T}},
\end{aligned}$$

$$\mathbf{g}_{s_0} = \left(\mathbf{F}_1^{(m)}, -2\frac{\Omega}{m}\mathbf{k}_0^{(1)}\dot{\mathbf{a}}_3 + \frac{\Omega^2}{m^2}\mathbf{k}_0^{(2)}\mathbf{a}_2 + \mathbf{F}_2,\right.$$
$$\left. 2\frac{\Omega}{m}\mathbf{k}_0^{(1)}\dot{\mathbf{a}}_2 + \frac{\Omega^2}{m^2}\mathbf{k}_0^{(2)}\mathbf{a}_3 + \mathbf{F}_3\right)^{\mathrm{T}}$$
$$\text{for } N_0 = 1, 2, \ldots, \infty; \tag{1.99}$$

and

$$\begin{aligned}
&\mathbf{F}_0^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \\
&\quad = \frac{1}{2m\pi}\int_0^{2m\pi} \mathbf{F}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, t, \mathbf{p})d\theta; \\
&\mathbf{F}_{1k}^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \\
&\quad = \frac{1}{m\pi}\int_0^{2m\pi} \mathbf{F}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, t, \mathbf{p})\cos\left(\frac{k}{m}\theta\right)d\theta, \\
&\mathbf{F}_{2k}^{(m)}(\mathbf{a}_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{\mathbf{a}}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) \\
&\quad = \frac{1}{m\pi}\int_0^{2m\pi} \mathbf{F}(\mathbf{x}^{(m)*}, \dot{\mathbf{x}}^{(m)*}, t, \mathbf{p})\sin\left(\frac{k}{m}\theta\right)d\theta.
\end{aligned} \tag{1.100}$$

B. *For the* k*th Hopf bifurcation with* $p_k\omega_k = \omega_{k-1}$ $(k = 1, 2, \ldots)$ *and* $\omega_0 = \Omega/m$, *there is a dynamical system of coefficients as*

$$\ddot{\mathbf{a}}_{s_0 s_1 \ldots s_k} = \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_k}, \mathbf{p}) \tag{1.101}$$

where

$$\begin{aligned}
&\mathbf{a}_{s_0 s_1 \ldots s_k} = (\mathbf{a}_{s_0 s_1 \ldots s_{k-1} 1}, \mathbf{a}_{s_0 s_1 \ldots s_{k-1} 2}, \mathbf{a}_{s_0 s_1 \ldots s_{k-1} 3})^{\mathrm{T}}, \\
&\mathbf{g}_{s_0 s_1 \ldots s_k} = (\mathbf{F}_{s_1 s_2 \ldots s_{k-1} 1}, -2\omega_k \mathbf{k}_k^{(1)}\dot{\mathbf{a}}_{s_1 s_2 \ldots s_k 3} + \omega_k^2 \mathbf{k}_k^{(2)}\mathbf{a}_{s_1 s_2 \ldots s_{k-1} 2} + \mathbf{F}_{s_1 s_2 \ldots s_{k-1} 2}, \\
&\qquad 2\omega_k \mathbf{k}_k^{(1)}\dot{\mathbf{a}}_{s_1 s_2 \ldots s_k 2} + \omega_k^2 \mathbf{k}_k^{(2)}\mathbf{a}_{s_1 s_2 \ldots s_{k-1} 3} + \mathbf{F}_{s_1 s_2 \ldots s_{k-1} 3})^{\mathrm{T}}, \\
&\mathbf{k}_k^{(1)} = diag(\mathbf{I}_{n_{k-1}\times n_{k-1}}, 2\mathbf{I}_{n_{k-1}\times n_{k-1}}, \ldots, N_k \mathbf{I}_{n_{k-1}\times n_{k-1}}), \\
&\mathbf{k}_k^{(2)} = diag(\mathbf{I}_{n_{k-1}\times n_{k-1}}, 2^2\mathbf{I}_{n_{k-1}\times n_{k-1}}, \ldots, N_k^2 \mathbf{I}_{n_{k-1}\times n_{k-1}}), \\
&n_{k-1} = n(2N_0 + 1)(2N_1 + 1)\ldots(2N_{k-1} + 1)
\end{aligned} \tag{1.102}$$

with a periodic solution as

$$\begin{aligned}
\mathbf{a}_{s_0 s_1 \ldots s_k} = \mathbf{a}_{s_0 s_1 \ldots s_k 1}^{(0)}(t) + \sum_{l_{k+1}=1}^{\infty} \mathbf{a}_{s_0 s_1 \ldots s_k 2}^{(l_{k+1})}(t)\cos(l_{k+1}\theta_{k+1}) \\
+ \mathbf{a}_{s_0 s_1 \ldots s_k 3}^{(l_{k+1})}(t)\sin(l_{k+1}\theta_{k+1})
\end{aligned} \tag{1.103}$$

with

$$
\begin{aligned}
&s_i = 1,2,3\ (i = 0,1,2,\ldots,k),\\
&\mathbf{a}_{s_0 s_1 \ldots s_k 1} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_k 1},\\
&\mathbf{a}_{s_0 s_1 \ldots s_k 2} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 2}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 2})^{\mathrm{T}},\\
&\mathbf{a}_{s_0 s_1 \ldots s_k 3} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 3}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 3}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}};\\
&\mathbf{a}_{s_0 s_1 \ldots s_{k-1} 1} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_{k-1} 1},\\
&\mathbf{a}_{s_0 s_1 \ldots s_{k-1} 2} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_{k-1} 2}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_{k-1} 2}, \ldots, \mathbf{a}^{(N_k)}_{s_0 s_1 \ldots s_{k-1} 2})^{\mathrm{T}},\\
&\mathbf{a}_{s_0 s_1 \ldots s_{k-1} 3} = (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_{k-1} 3}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_{k-1} 3}, \ldots, \mathbf{a}^{(N_k)}_{s_0 s_1 \ldots s_{k-1} 3})^{\mathrm{T}};\\
&\vdots\\
&\mathbf{a}_1 = \mathbf{a}^{(0)}_1,\\
&\mathbf{a}_2 = (\mathbf{a}^{(1)}_2, \mathbf{a}^{(2)}_2, \ldots, \mathbf{a}^{(N_0)}_2)^{\mathrm{T}},\\
&\mathbf{a}_3 = (\mathbf{a}^{(1)}_3, \mathbf{a}^{(2)}_3, \ldots, \mathbf{a}^{(N_0)}_3)^{\mathrm{T}};
\end{aligned}
\tag{1.104}
$$

which, under $\|\mathbf{a}_{s_0 s_1 \ldots s_k}(t) - \mathbf{a}^*_{s_0 s_1 \ldots s_k}(t)\| < \varepsilon$ *with a prescribed small* $\varepsilon > 0$, *can be approximated by a finite term transformation* $\mathbf{a}^*_{s_0 s_1 \ldots s_k}(t)$

$$
\begin{aligned}
\mathbf{a}^*_{s_0 s_1 \ldots s_k} = {} & \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_k 1}(t) + \sum_{l_{k+1}=1}^{N_{k+1}} \mathbf{a}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 2}(t)\cos(l_{k+1}\theta_{k+1})\\
& + \mathbf{a}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 3}(t)\sin(l_{k+1}\theta_{k+1})
\end{aligned}
\tag{1.105}
$$

and the generalized coordinates are determined by

$$
\ddot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}} = \mathbf{g}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})
\tag{1.106}
$$

where

$$
\begin{aligned}
\mathbf{a}_{s_0 s_1 \ldots s_{k+1}} = {} & (\mathbf{a}_{s_0 s_1 \ldots s_k 1}, \mathbf{a}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}},\\
\mathbf{g}_{s_0 s_1 \ldots s_{k+1}} = {} & (\mathbf{F}_{s_0 s_1 \ldots s_k 1}, -2\omega_{k+1}\mathbf{k}^{(1)}_{k+1}\dot{\mathbf{a}}_{s_0 s_1 \ldots s_k 3} + \omega^2_{k+1}\mathbf{k}^{(2)}_{k+1}\mathbf{a}_{s_0 s_1 \ldots s_k 2} + \mathbf{F}_{s_0 s_1 \ldots s_k 2},\\
& 2\omega_{k+1}\mathbf{k}^{(1)}_{k+1}\dot{\mathbf{a}}_{s_0 s_1 \ldots s_k 2} + \omega^2_{k+1}\mathbf{k}^{(2)}_{k+1}\mathbf{a}_{s_0 s_1 \ldots s_k 3} + \mathbf{F}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}};
\end{aligned}
\tag{1.107}
$$

and

$$
\begin{aligned}
&\mathbf{k}^{(1)}_{k+1} = diag(\mathbf{I}_{n_k\times n_k}, 2\mathbf{I}_{n_k\times n_k}, \ldots, N_{k+1}\mathbf{I}_{n_k\times n_k}),\\
&\mathbf{k}^{(2)}_{k+1} = diag(\mathbf{I}_{n_k\times n_k}, 2^2\mathbf{I}_{n_k\times n_k}, \ldots, N^2_{k+1}\mathbf{I}_{n_k\times n_k})\\
&n_k = n(2N_0+1)(2N_1+1)\ldots(2N_k+1);\\
&\mathbf{a}_{s_0 s_1 \ldots s_k 1} = \mathbf{a}^{(0)}_{s_0 s_1 \ldots s_k 1},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a}_{s_0 s_1 \ldots s_k 2} &= (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 2}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 2})^{\mathrm{T}},\\
\mathbf{a}_{s_0 s_1 \ldots s_k 3} &= (\mathbf{a}^{(1)}_{s_0 s_1 \ldots s_k 3}, \mathbf{a}^{(2)}_{s_0 s_1 \ldots s_k 3}, \ldots, \mathbf{a}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}};\\
\mathbf{a}_{s_0 s_1 \ldots s_{k+1}} &= (\mathbf{a}_{s_0 s_1 \ldots s_k 1}, \mathbf{a}_{s_0 s_1 \ldots s_k 2}, \mathbf{a}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}};\\
\mathbf{F}_{s_0 s_1 \ldots s_k 1} &= \mathbf{F}^{(0)}_{s_0 s_1 \ldots s_k 1},\\
\mathbf{F}_{s_0 s_1 \ldots s_k 2} &= (\mathbf{F}^{(1)}_{s_0 s_1 \ldots s_k 2}, \mathbf{F}^{(2)}_{s_0 s_1 \ldots s_k 2}, \ldots, \mathbf{F}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 2})^{\mathrm{T}},\\
\mathbf{F}_{s_0 s_1 \ldots s_k 3} &= (\mathbf{F}^{(1)}_{s_0 s_1 \ldots s_k 3}, \mathbf{F}^{(2)}_{s_0 s_1 \ldots s_k 3}, \ldots, \mathbf{F}^{(N_{k+1})}_{s_0 s_1 \ldots s_k 3})^{\mathrm{T}}\\
&\text{for } N_{k+1} = 1, 2, \ldots, \infty; \qquad (1.108)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{F}_{s_0 s_1 \ldots s_k 1}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})\\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}^*_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}^*_{s_1 s_2 \ldots s_k}, \mathbf{p})d\theta_{k+1};\\
&\mathbf{F}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 2}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})\\
&\quad = \frac{1}{\pi}\int_0^{2\pi} \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}^*_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}^*_{s_1 s_2 \ldots s_k}, \mathbf{p})\cos(l_{k+1}\theta_{k+1})d\theta_{k+1},\\
&\mathbf{F}^{(l_{k+1})}_{s_0 s_1 \ldots s_k 3}(\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{p})\\
&\quad = \frac{1}{\pi}\int_0^{2\pi} \mathbf{g}_{s_0 s_1 \ldots s_k}(\mathbf{a}^*_{s_0 s_1 \ldots s_k}, \dot{\mathbf{a}}^*_{s_1 s_2 \ldots s_k}, \mathbf{p})\sin(l_{k+1}\theta_{k+1})d\theta_{k+1}\\
&\text{for } l_{k+1} = 1, 2, \ldots, N_{k+1}. \qquad (1.109)
\end{aligned}
$$

C. *Equation (1.106) becomes*

$$
\dot{\mathbf{z}}_{s_0 s_1 \ldots s_{k+1}} = \mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}_{s_0 s_1 \ldots s_{k+1}}) \qquad (1.110)
$$

where

$$
\begin{aligned}
\mathbf{z}_{s_0 s_1 \ldots s_{k+1}} &= (\mathbf{a}_{s_0 s_1 \ldots s_{k+1}}, \dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}})^{\mathrm{T}},\\
\mathbf{f}_{s_0 s_1 \ldots s_{k+1}} &= (\dot{\mathbf{a}}_{s_0 s_1 \ldots s_{k+1}}, \mathbf{g}_{s_0 s_1 \ldots s_{k+1}})^{\mathrm{T}}. \qquad (1.111)
\end{aligned}
$$

If equilibrium $\mathbf{z}^*_{s_1 s_2 \ldots s_{k+1}}$ *of Equation (1.110) (i.e.,* $\mathbf{f}_{s_1 s_2 \ldots s_{k+1}}(\mathbf{z}^*_{s_1 s_2 \ldots s_{k+1}}) = \mathbf{0}$*) exists, then the approximate solution of the periodic motion of the* kth *generalized coordinates for the period*-m *motion exists as in Equation (1.105). In the vicinity of equilibrium* $\mathbf{z}^*_{s_1 s_2 \ldots s_{k+1}}$, *with*

$$
\mathbf{z}_{s_0 s_1 \ldots s_{k+1}} = \mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}} + \Delta\mathbf{z}_{s_0 s_1 \ldots s_{k+1}}, \qquad (1.112)
$$

the linearized equation of Equation (1.110) is

$$\Delta\dot{\mathbf{z}}_{s_0 s_1 \ldots s_{k+1}} = D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}})\Delta\mathbf{z}_{s_0 s_1 \ldots s_{k+1}} \tag{1.113}$$

and the eigenvalue analysis of equilibrium $\mathbf{z}^*$ *is given by*

$$|D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}) - \lambda\mathbf{I}_{2n_k(2N_{k+1}+1)\times 2n_k(2N_{k+1}+1)}| = 0 \tag{1.114}$$

where

$$D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}) = \left.\frac{\partial \mathbf{f}_{s_0 s_1 \ldots s_{k+1}}\left(\mathbf{z}_{s_0 s_1 \ldots s_{k+1}}\right)}{\partial \mathbf{z}_{s_0 s_1 \ldots s_{k+1}}}\right|_{\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}}}. \tag{1.115}$$

The stability and bifurcation of such a periodic motion of the kth *generalized coordinates can be classified by the eigenvalues of* $D\mathbf{f}_{s_0 s_1 \ldots s_{k+1}}(\mathbf{z}^*_{s_0 s_1 \ldots s_{k+1}})$ *with*

$$(n_1, n_2, n_3 | n_4, n_5, n_6). \tag{1.116}$$

i. If all eigenvalues of the equilibrium possess negative real parts, the approximate quasi-periodic solution is stable.
ii. If at least one of the eigenvalues of the equilibrium possesses positive real part, the approximate quasi-periodic solution is unstable.
iii. The boundaries between stable and unstable equilibriums with higher order singularity give bifurcation and stability conditions with higher order singularity.

D. For the kth *order Hopf bifurcation of period*-m *motion, a relation exists as*

$$p_k \omega_k = \omega_{k-1}. \tag{1.117}$$

i. If p_k *is an irrational number, the* kth*-order Hopf bifurcation of the period*-m *motion is called the quasi-period-*p_k *Hopf bifurcation, and the corresponding solution of the* kth *generalized coordinates is* p_k*-quasi-periodic to the system of the* $(k-1)$th *generalized coordinates.*
ii. If $p_k = 2$, *the* kth*-order Hopf bifurcation of the period*-m *motion is called a period-doubling Hopf bifurcation (or a period-2 Hopf bifurcation), and the corresponding solution of the* kth *generalized coordinates is period-doubling to the system of the* $(k-1)$th *generalized coordinates.*
iii. If $p_k = q$ *with an integer* q, *the* kth*-order Hopf bifurcation of the period*-m *motion is called a period-q Hopf bifurcation, and the corresponding solution of the* kth *generalized coordinates is of q-times period to the system of the* $(k-1)$th *generalized coordinates.*
iv. If $p_k = p/q$ *(*p, q *are irreducible integer), the* kth*-order Hopf bifurcation of the period*-m *motion is called a period-*p/q *Hopf bifurcation, and the corresponding solution of the* kth *generalized coordinates is* p/q*-periodic to the system of the* $(k-1)$th *generalized coordinates.*

Proof. The proof of this theorem can be referred to Luo (2014). ■

The general theory for the general nonlinear dynamical systems was found in Luo (2012a, 2014), and the analytical solutions for nonlinear dynamical systems with time-delay was presented as well. The generalized harmonic balance method is different from the traditional harmonic balance method. This generalized harmonic balance method provides a theoretic framework to analytically express all possible periodic motions in nonlinear dynamical systems. The procedure for different periodic solutions in different dynamical systems is *of the same*, as presented in Luo (2012a, 2013, 2014). However, the analytical expressions for different periodic solutions in the same dynamical systems are *distinguishing*, which should be obtained through the different, transformed, nonlinear dynamical systems. For instance, the period-1, period-2, and period-*m* solutions possess the completely different solution expressions. Even for the same period-*m* solutions with different parameters and/or locations of initial conditions, the analytical solutions in the same nonlinear system are completely different. One needs to work on them to obtain the *complete* pictures (*dynamics*) of stable and unstable periodic solutions plus chaos.

The detailed mathematical theory of the generalized harmonic balance method with the vigorous proof was presented in Luo (2012a, 2013, 2014). In fact, this method provides a finite-harmonic-term transformation with different time scales to obtain an autonomous nonlinear system of coefficients in the Fourier series form with finite harmonic terms. The dynamical behaviors of such an autonomous nonlinear system will determine the solution behaviors of original dynamical systems. For periodic solutions, the Fourier series forms of the finite harmonic terms are convergent. For transient solutions, such Fourier series forms of the finite harmonic terms may not be convergent. For different periodic solutions in a nonlinear dynamical system, the Fourier series solution forms are different, which are determined by how many finite harmonic terms with time-varying coefficients in the Fourier series form. To determine different periodic solutions in the same dynamical system and the corresponding dynamical behaviors, the different, transformed, nonlinear dynamical systems relative to the prescribed finite harmonic terms should be employed. Of course, periodic solutions in different dynamical systems are different, and the corresponding investigation should be carried out individually because the transformed, nonlinear dynamical systems are totally different. In summary, the generalized harmonic balance method provides a possibility of finding all possible periodic solutions plus chaos analytically. For the current stage, this method is the best way to analytically determine the complete dynamics of periodic solutions in nonlinear dynamical systems. In addition, the generalized harmonic balance method is also *a small-parameter-free method* to determine the periodic solutions in nonlinear dynamical systems.

Luo and Huang (2012a) the generalized harmonic balance method with finite terms to obtain the analytical solution of period-1 motion of the Duffing oscillator with a twin-well potential. Luo and Huang (2012b) presented a generalized harmonic balance method to find analytical solutions of period-*m* motions in such a Duffing oscillator. The analytical bifurcation trees of periodic motions in the Duffing oscillator to chaos are obtained (also see, Luo and Huang, 2012c,d, 2013a,b,c, 2014a). Such analytical bifurcation trees show the connection from periodic solution to chaos analytically. To better understand nonlinear behaviors in nonlinear dynamical systems, the analytical solutions for the bifurcation trees from period-1 motion to chaos in a periodically forced oscillator with quadratic nonlinearity were presented in Luo and Yu (2013a,b,c), and period-*m* motions in the periodically forced, van der Pol equation was presented in Luo and Lakeh (2013a). The analytical solutions for the van der Pol oscillator can be used to verify the conclusions in Cartwright and Littlewood (1945) and Levinson (1949).

The results for the quadratic nonlinear oscillator in Luo and Yu (2013a,b,c) analytically show the complicated period-1 motions and the corresponding bifurcation structures. In this book, the generalized harmonic balance method will be used to develop the analytical solutions.

1.2 Book Layout

This book consists of five chapters. Chapter 1 gave the brief literature review on analytical methods, including perturbation methods, the method of averaging, and generalized harmonic balance methods. Other chapters are briefly summarized as follows.

In Chapter 2, analytical bifurcation trees from period-*m* motions to chaos in periodically forced, Duffing oscillators will be presented. The analytical solutions of period-*m* motions in Duffing oscillators will be discussed because the Duffing oscillators are extensively applied in structural vibrations and physical problems. The bifurcation trees of period-1 motions to chaos for the Duffing oscillators will be discussed and the bifurcation trees of period-3 motions to chaos will also be presented for the Duffing oscillators. Different types of Duffing oscillators possess completely different bifurcation trees.

In Chapter 3, analytical solutions for period-*m* motions in periodically forced, self-excited oscillators will be presented in the Fourier series form with finite harmonic terms, and the stability and bifurcation of the corresponding period-*m* motions will be completed. The period-*m* motions in the periodically forced, van der Pol oscillator will be discussed, and the limit cycles for the van der Pol oscillator without any excitation will be discussed as well. The period-*m* motions are in independent periodic solution windows embedded in quasi-periodic and chaotic motions. The period-*m* motions for the van der Pol-Duffing oscillator will be presented, and bifurcation tree of period-*m* motion will be discussed. For a better understanding of complex period-*m* motions in such a van der Pol-Duffing oscillator, trajectories and amplitude spectrums will be illustrated numerically.

In Chapter 4, analytical solutions for period-*m* motions in parametrically forced, nonlinear oscillators are discussed. The bifurcation trees of periodic motions to chaos in a parametric oscillator with quadratic nonlinearity will be discussed analytically. Nonlinear behaviors of such periodic motions will be characterized through frequency-amplitude curves. This investigation shows that period-1 motions exist in parametric nonlinear systems and the corresponding bifurcation trees to chaos exist as well. In addition, analytical solutions for periodic motions in a Mathieu-Duffing oscillator are presented. The frequency-amplitude characteristics of asymmetric period-1 and symmetric period-2 motions will be discussed. Period-1 asymmetric and period-2 symmetric motions will be illustrated for a better understanding of periodic motions in the Mathieu-Duffing oscillator.

In Chapter 5, analytical solutions for period-*m* motions in a nonlinear rotor system will be discussed. This rotor system with two degrees of freedom is a simple rotor dynamical system and periodic excitations are from the rotor eccentricity. The analytical expressions of periodic solutions will be developed. The corresponding stability and bifurcation analyses of period-*m* motions will be carried out. Analytical bifurcation trees of period-1 motions to chaos will be presented. The Hopf bifurcation of periodic motion can cause not only the bifurcation tree but quasi-periodic motions. Displacement orbits of periodic motions in nonlinear rotor systems show motion complexity, and harmonic amplitude spectrums gives harmonic effects on periodic motions.