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DEFINITION OF A THIN PLATE

As a prelude and to place the engineering theory to be developed later in proper perspective, a simple test problem is analyzed in this chapter using the rigorous approach of the theory of elasticity, i.e. by considering the plate as a three-dimensional solid. The assumptions that lead to thin plate theory are brought out as natural inferences from the results of this three-dimensional analysis. A brief outline of the field variables and governing equations of elasticity is also included.

1.1 THE ELASTICITY APPROACH

The main difference between the elasticity approach and conventional engineering analysis based on the mechanics-of-materials approach is the inclusion of a suitable hypothesis regarding the geometry of deformation in the latter. For instance, the engineering theory of beams is based on Euler-Bernoulli hypothesis regarding the preservation of plane cross-sections during deformation; the corresponding analysis using the theory of elasticity does not require this hypothesis.

The theory of elasticity is based on the use of the concepts of equilibrium, continuum and a material constitutive relationship to analyze a structure, i.e. to obtain the displacements, strains and stresses at any point within it when the loads and support conditions are specified. The theory can be stated in a mathematical form in terms of the field equations, viz. the equations of equilibrium, the strain-displacement relations and the stress-strain law; these equations are given here with reference to Cartesian *x-y-z* axes for a *linearly* *elastic isotropic body undergoing small deformations in the absence of body forces*. The equations are in terms of the field variables, viz. the three displacements u, v and w along x, y and z directions, respectively, the nine components of the strain tensor, and the nine components of the stress tensor. Normal stresses and strains are usually denoted by σ_{ii} and ε_{ii}, respectively, while shear stresses and strains are denoted by τ_{ij} and γ_{ij}, with $i, j = x, y, z$. The first subscript in all these quantities denotes the direction of the normal to the plane under consideration, and the second subscript denotes the direction of the stress or strain. If the outward normal to a surface is in a positive (negative) coordinate direction, the corresponding stresses are taken as positive when acting in the positive (negative) coordinate directions. This is the sign convention commonly adopted and is as shown in Fig.1.1; a corresponding sign convention holds good for strains.

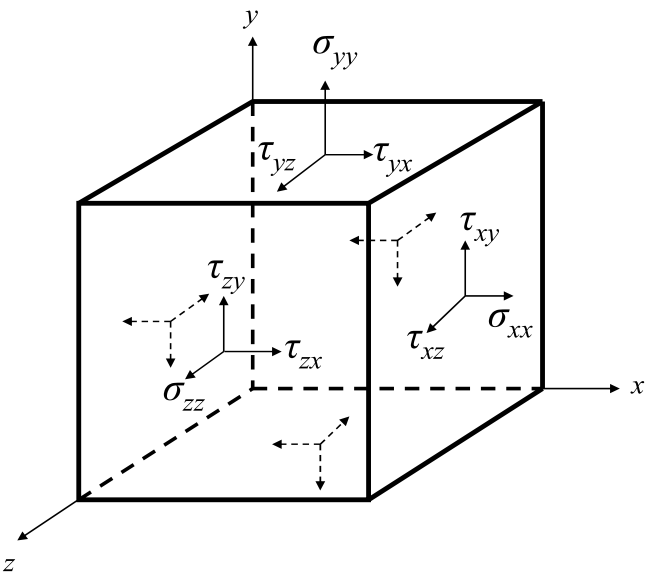

Fig. 1.1 Positive stress components

It should be noted that only six stress (and six strain) components are independent since $\tau_{ij} = \tau_{ji}$ (and $\gamma_{ij} = \gamma_{ji}$) by the principle of complementary shear. In this book, the normal stresses and strains are denoted using single subscripts, i.e. σ_i and ε_i instead of σ_{ii} and ε_{ii}.

The field equations are:

Equilibrium Equations:

$$\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} = 0$$
$$\tau_{xy,x} + \sigma_{y,y} + \tau_{yz,z} = 0$$
$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} = 0 \qquad (1.1)$$

Strain-Displacement Relations:

$$\varepsilon_x = u_{,x} \qquad \gamma_{yz} = v_{,z} + w_{,y}$$
$$\varepsilon_y = v_{,y} \qquad \gamma_{xz} = u_{,z} + w_{,x}$$
$$\varepsilon_z = w_{,z} \qquad \gamma_{xy} = u_{,y} + v_{,x} \qquad (1.2)$$

Stress-Strain Law:

$$\sigma_i = 2G\varepsilon_i + \lambda e, \qquad \tau_{ij} = G\gamma_{ij} \quad \text{with } i, j = x, y, z \qquad (1.3)$$

These field equations are supplemented by the boundary conditions, i.e. the mathematical description of the supports and the applied loads.

In the above equations, G and λ are Lamè's constants, e is the volumetric strain given by $\varepsilon_x + \varepsilon_y + \varepsilon_z$, and a subscript comma is used to denote differentiation. In terms of the Young's modulus E and Poisson's ratio μ, Lamè's constants are given by

$$\lambda = \frac{\mu E}{(1+\mu)(1-2\mu)} \qquad G = \frac{E}{2(1+\mu)} \qquad (1.4)$$

The solution of the elasticity problem can be carried out by using the *displacement approach*, wherein the displacements *u, v* and *w* are first solved for and then the strains and stresses are evaluated using Eqns.(1.2) & (1.3), or the *stress approach*, wherein the stresses are first obtained and the strains and displacements subsequently, or a *mixed approach.*

The field equations given above are sufficient for the displacement approach, while, in addition, certain conditions are required in the other approaches to ensure the continuum nature of the structure after deformation; these *compatibility* conditions will be introduced later in the book as and when a need for them arises.

1.2 A TEST PROBLEM

One of the problems amenable to exact analysis using the elasticity approach is considered here. An infinitely long prismatic body of rectangular cross-section as shown in Fig.1.2 is supported at the two longitudinal edges such that

$$w = \sigma_x = 0 \quad \text{at } x\text{=0 and } a, \text{ for all } y \text{ and } z \tag{1.5}$$

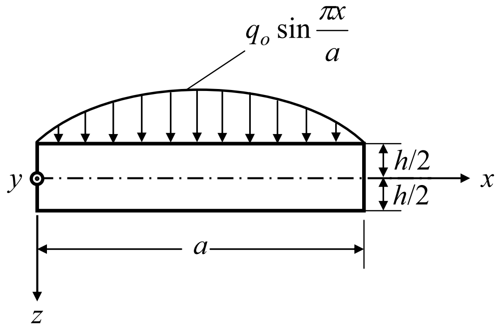

Fig.1.2 The test problem

This boundary condition, which we shall encounter later on also, corresponds to supporting the end planes (x=0 and x=a) by *shear diaphragms* attached to them – a shear diaphragm being defined as one that completely restrains displacements in its own plane, but fully permits the out-of-plane displacement. Obviously, this condition corresponds to a simple support in the limiting case of a thin plate.

The body is subjected to sinusoidal transverse loading applied on the top surface (Fig.1.2). If this load is assumed to be independent of y, then the body undergoes cylindrical bending such that the displacements, strains and stresses do not vary along y and the problem reduces to one of the cross-sectional plane. It is further assumed that the ends of the infinitely long strip are restrained in the axial direction so that v=0 everywhere, reducing the problem to one of plane strain.

Corresponding to the lateral surfaces, the boundary conditions are

$$\sigma_z = -q_o \sin\frac{\pi x}{a} \quad \& \quad \tau_{xz} = 0 \quad \text{at } z = -h/2\text{, for all } x \text{ and } y$$

$$\sigma_z = \tau_{xz} = 0 \quad \text{at } z = h/2\text{, for all } x \text{ and } y \tag{1.6}$$

The other equations to be satisfied are the field equations (1.1)–(1.3) simplified for the case of cylindrical bending as given below.

$$\begin{aligned} \sigma_{x,x} + \tau_{xz,z} &= 0 \\ \tau_{xz,x} + \sigma_{z,z} &= 0 \end{aligned} \tag{1.1a}$$

$$\begin{aligned} \varepsilon_x &= u_{,x} \\ \varepsilon_z &= w_{,z} \\ \gamma_{xz} &= u_{,z} + w_{,x} \end{aligned} \tag{1.2a}$$

$$\begin{aligned} \sigma_x &= 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_z) \\ \sigma_z &= 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_z) \\ \tau_{xz} &= G\gamma_{xz} \end{aligned} \tag{1.3a}$$

Following the displacement approach, the governing equations are obtained by the use of Eqns.(1.2a) & (1.3a) in Eqn.(1.1a) to yield

$$\begin{aligned} (2G+\lambda)u_{,xx} + Gu_{,zz} + (G+\lambda)w_{,xz} &= 0 \\ (G+\lambda)u_{,xz} + (2G+\lambda)w_{,zz} + Gw_{,xx} &= 0 \end{aligned} \tag{1.7}$$

A solution for these equations can be chosen as

$$\begin{aligned} u &= U(z)\cos\left(\frac{\pi x}{a}\right) \\ w &= W(z)\sin\left(\frac{\pi x}{a}\right) \end{aligned} \tag{1.8}$$

For this displacement field, σ_x will be of the form

$$\sigma_x = (..)\sin\left(\frac{\pi x}{a}\right) \tag{1.9}$$

It can be seen that the edge conditions (Eqn.(1.5)) are thus satisfied exactly. Substitution of Eqn.(1.8) into Eqn.(1.7) reduces the problem to the solution of the following coupled ordinary differential equations with constant coefficients:

$$\begin{aligned} -(2G+\lambda)\beta^2 U + GU'' + (G+\lambda)\beta W' = 0 \\ -(G+\lambda)\beta U' + (2G+\lambda)W'' - G\beta^2 W = 0 \end{aligned} \tag{1.10}$$

where $\beta = \dfrac{\pi}{a}$, and $(..)' = \dfrac{d(..)}{dz}$, $(..)'' = \dfrac{d^2(..)}{dz^2}$.

The solution of the above equations is straightforward and can be obtained as

$$\begin{aligned} U &= A\cosh\beta z + B\sinh\beta z + Cz\cosh\beta z + Dz\sinh\beta z \\ W &= [B - \frac{C(3-4\mu)}{\beta}]\cosh\beta z + [A - \frac{D(3-4\mu)}{\beta}]\sinh\beta z \\ &\quad + Dz\cosh\beta z + Cz\sinh\beta z \end{aligned} \tag{1.11}$$

where *A, B, C* and *D* are undetermined constants.

Rewriting the lateral surface conditions (Eqn.(1.6)) in terms of *U, W, U'*, *W',* etc., one obtains four equations which are sufficient to evaluate the constants *A* to *D*. This procedure yields

$$A = \frac{D(1 - 2\mu - \frac{\beta h}{2}\coth\frac{\beta h}{2})}{\beta}$$

$$B = \frac{C(1 - 2\mu - \frac{\beta h}{2}\tanh\frac{\beta h}{2})}{\beta}$$

$$C = \frac{D\coth\frac{\beta h}{2}\left(\frac{\beta h}{2} + \sinh\frac{\beta h}{2}\cosh\frac{\beta h}{2}\right)}{\left(\frac{\beta h}{2} - \sinh\frac{\beta h}{2}\cosh\frac{\beta h}{2}\right)}$$

$$D = \frac{q_o(1+\mu)\sinh\frac{\beta h}{2}}{E(\beta h + \sinh\beta h)} \tag{1.12}$$

Use of Eqns.(1.8), (1.11) and (1.12) yields the values of u and w at any point of the domain. The strains and stresses are then obtained from Eqns.(1.2a) and (1.3a), respectively. Thus the solution of the problem is complete.

1.3 THE CASE OF A THIN PLATE

It is convenient to group the stresses and strains occurring in the structure into those corresponding to bending (σ_x , ε_x), transverse shear (τ_{xz} , γ_{xz}), and thickness stretch/contraction (σ_z , ε_z). The study of these quantities for various span-to-thickness ratios would enable one to arrive at the assumptions that form the basis of the theory of plates. Such a study has to be carried out by looking at the contributions of these stresses and strains to the strain energy of deformation of the structure. This is because any approximation to the three-dimensional problem can be considered acceptable only if it leads to a reasonably good estimate of the strain energy, and the contributions of the different stress and strain components to this energy provide a correct estimate of their relative importance.

The strain energy U_e for the present test problem can be written as

$$U_e = \frac{1}{2}\int(\sigma_x\varepsilon_x + \tau_{xz}\gamma_{xz} + \sigma_z\varepsilon_z)dVol$$

$$= \frac{1}{2}\int\sigma_x\varepsilon_x dVol + \frac{1}{2}\int\tau_{xz}\gamma_{xz}\, dVol + \frac{1}{2}\int\sigma_z\varepsilon_z dVol$$

$$= U_b + U_s + U_t \tag{1.13}$$

where U_b is the strain energy corresponding to bending, U_s to transverse shear, and U_t to thickness stretch/contraction.

By substituting for the stresses and strains, and integrating over the domain, one obtains

$$\begin{aligned} U_b = \frac{q_o^2 a^2}{384\pi E\Gamma}&[3(1-\mu-2\mu^2)(\sinh 4\beta h - 2\sinh 2\beta h) \\ &+12\beta h(1-\mu-2\mu^2)(\cosh 2\beta h - 1) \\ &-12(\beta h)^2(1-\mu-2\mu^2)\sinh 2\beta h \\ &+8(\beta h)^3\{-10+2\mu+12\mu^2+(7+\mu-6\mu^2)\cosh 2\beta h\} \\ &-8(\beta h)^4(1+\mu)\sinh 2\beta h + 16(\beta h)^5(1+\mu)] \end{aligned} \tag{1.14a}$$

$$\begin{aligned} U_s = \frac{q_o^2 a^2(1+\mu)}{192\pi E\Gamma}&[(3\sinh 4\beta h - 6\sinh 2\beta h) + 12\beta h(\cosh 2\beta h - 1) \\ &-12(\beta h)^2\sinh 2\beta h - 8(\beta h)^3(2+\cosh 2\beta h) \\ &+8(\beta h)^4\sinh 2\beta h - 16(\beta h)^5] \end{aligned} \tag{1.14b}$$

$$\begin{aligned} U_t = \frac{q_o^2 a^2}{384\pi E\Gamma}&[3(5-\mu-6\mu^2)(\sinh 4\beta h - 2\sinh 2\beta h) \\ &+12\beta h(5-\mu-6\mu^2)(\cosh 2\beta h - 1) \\ &-12(\beta h)^2(5-\mu-6\mu^2)\sinh 2\beta h \\ &-8(\beta h)^3(5-\mu-6\mu^2)(2+\cosh 2\beta h) \\ &-8(\beta h)^4(1+\mu)\sinh 2\beta h + 16(1+\mu)(\beta h)^5] \end{aligned} \tag{1.14c}$$

where $\Gamma = [\,\sinh^2\beta h - (\beta h)^2\,]^2$.

A comparison of the relative magnitudes of these energies can be carried out as follows. Noting that βh $(=\pi h/a)$ is a small parameter for thin plates, the ratios (U_s/U_b) and (U_t/U_b) are written in the form of a power series in βh as

$$\begin{aligned}\frac{U_s}{U_b} &= \frac{1}{5(1-\mu)}(\beta h)^2 - \frac{(1-22\mu)}{1050(1-\mu)^2}(\beta h)^4 \\ &\quad - \frac{(11-4\mu-196\mu^2)}{94500(1-\mu)^3}(\beta h)^6 + \text{etc.}\end{aligned} \tag{1.15a}$$

$$\begin{aligned}\frac{U_t}{U_b} &= \frac{-\mu}{10(1-\mu)}(\beta h)^2 + \frac{(65-129\mu+43\mu^2)}{2100(1-\mu)^2}(\beta h)^4 \\ &\quad - \frac{(10-626\mu+1204\mu^2-399\mu^3)}{189000(1-\mu)^3}(\beta h)^6 + \text{etc.}\end{aligned} \tag{1.15b}$$

For the limiting case of $\beta h \to 0$, it is clear that both (U_s/U_b) and (U_t/U_b) tend to zero. Thus, when the thickness is very small, the analysis can be simplified by approximating the stresses and strains such that U_s and U_t are zero, and such a structure can then be termed a 'thin plate'.

The energies U_s and U_t are made zero by assuming that σ_z, γ_{xz} and ε_z are all zero. (Strictly speaking, it is not necessary that both σ_z and ε_z be zero to make U_t zero; further, the transverse shear stress τ_{xz}, though directly proportional to γ_{xz}, cannot be neglected as its presence is required to satisfy the *z*-direction equilibrium for any element cut out of the transversely loaded plate. These will be discussed at length in Chapter 3). This assumption, that the transverse shear strain, the transverse normal strain and the corresponding normal stress are all negligible, forms the basis for the development of the classical plate theory as detailed out in the next chapter.

SUMMARY

After a brief outline of the theory of elasticity, the problem of a simply supported rectangular strip under sinusoidally distributed transverse load has been solved rigorously. The strain energies corresponding to bending, transverse shear and transverse normal stretch/contraction have been compared. It has been shown for the case of a thin plate that the bending strain energy is predominant, and, in comparison with it, the other two energies tend to be negligibly small. On this basis, the neglect of the transverse shear strain, the transverse normal strain and the transverse normal stress, has been identified as the main hypothesis for the development of the classical plate theory.

CONCEPTUAL QUESTION

Refer to a book on Theory of Elasticity, and find out about the following:

(a) the principle of complementary shear;

(b) how λ and G can be expressed in terms of E and μ ;

(c) the two-dimensional case known as plane stress;

(d) the strain energy of deformation for a general state of stress.