1

Strategic-form two-player games

Introduction

Our analysis of game problems begins with the case of two-player strategic-form (equivalently, normal-form) games. The basic notions of game theory comprise **Players, Strategies and Payoffs**. In the sequel, denote players by I and II. A normal-form game is organized in the following way. Player I chooses a certain strategy x from a set X, while player II simultaneously chooses some strategy y from a set Y. In fact, the sets X and Y may possess any structure (a finite set of values, a subset of R^n, a set of measurable functions, etc.). As a result, players I and II obtain the payoffs $H_1(x, y)$ and $H_2(x, y)$, respectively.

Definition 1.1 *A normal-form game is an object*

$$\Gamma =< I, II, X, Y, H_1, H_2 >,$$

where X, Y designate the sets of strategies of players I and II, whereas H_1, H_2 indicate their payoff functions, $H_i : X \times Y \to R, i = 1, 2$.

Each player selects his strategy regardless of the opponent's choice and strives for maximizing his own payoff. However, a player's payoff depends both on his strategy and the behavior of the opponent. This aspect makes the specifics of game theory.

How should one comprehend the solution of a game? There exist several approaches to construct solutions in game theory. Some of them will be discussed below. First, let us consider the notion of a Nash equilibrium as a central concept in game theory.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

Definition 1.2 *A Nash equilibrium in a game* Γ *is a set of strategies* (x^*, y^*) *meeting the conditions*

$$\begin{aligned} H_1(x, y^*) &\le H_1(x^*, y^*), \\ H_2(x^*, y) &\le H_2(x^*, y^*) \end{aligned} \tag{1.1}$$

for arbitrary strategies x, y *of the players.*

Inequalities (1.1) imply that, as the players deviate from a Nash equilibrium, their payoffs do decrease. Hence, deviations from the equilibrium appear non-beneficial to any player. Interestingly, there may exist no Nash equilibria. Therefore, a major issue in game problems concerns their existence. Suppose that a Nash equilibrium exists; in this case, we say that the payoffs $H_1^* = H_1(x^*, y^*),\ H_2^* = H_2(x^*, y^*)$ are optimal. A set of strategies (x, y) is often called a **strategy profile**.

1.1 The Cournot duopoly

We mention the Cournot duopoly [1838] among pioneering game models that gained wide popularity in economic research. The term "duopoly" corresponds to a two-player game.

Imagine two companies, *I* and *II*, manufacturing some quantities of a same product (q_1 and q_2, respectively). In this model, the quantities represent the strategies of the players. The market price of the product equals an initial price p after deduction of the total quantity $Q = q_1 + q_2$. And so, the unit price constitutes $(p - Q)$. Let c be the unit cost such that $c < p$. Consequently, the players' payoffs take the form

$$H_1(q_1, q_2) = (p - q_1 - q_2)q_1 - cq_1, \quad H_2(q_1, q_2) = (p - q_1 - q_2)q_2 - cq_2. \tag{1.2}$$

In the current notation, the game is defined by $\Gamma = < I, II, Q_1 = [0, \infty), Q_2 = [0, \infty), H_1, H_2 >$.

Nash equilibrium evaluation (see formula (1.1)) calls for solving two problems, *viz.*, $\max_{q_1} H_1(q_1, q_2^*)$ and $\max_{q_2} H_2(q_1^*, q_2)$. Moreover, we have to demonstrate that the maxima are attained at $q_1 = q_1^*,\ q_2 = q_2^*$. The quadratic functions $H_1(q_1, q_2^*)$ and $H_2(q_1^*, q_2)$ get maximized by

$$\begin{aligned} q_1 &= \frac{1}{2}\left(p - c - q_2^*\right) \\ q_2 &= \frac{1}{2}\left(p - c - q_1^*\right). \end{aligned}$$

Naturally, these quantities must be non-negative, which dictates that

$$q_i^* \le p - c, \ \ i = 1, 2. \tag{1.3}$$

By resolving the derived system of equations in q_1^*, q_2^*, we find

$$q_1^* = q_2^* = \frac{p - c}{3}$$

that satisfy the conditions (1.3). And the optimal payoffs become

$$H_1^* = H_2^* = \frac{(p-c)^2}{9}.$$

1.2 Continuous improvement procedure

Imagine that player I knows the strategy q_2 of player II. Then his **best response** lies in the strategy q_1 yielding the maximal payoff $H_1(q_1, q_2)$. Recall that $H_1(q_1, q_2)$ is a concave parabola possessing its vertex at the point

$$q_1 = \frac{1}{2}(p - c - q_2). \tag{2.1}$$

We denote the best response function by $q_1 = R(q_2) = \frac{1}{2}(p - c - q_2)$. Similarly, if the strategy q_1 of player I becomes known to player II, his best response is the strategy q_2 corresponding to the maximal payoff $H_2(q_1, q_2)$. In other words,

$$q_2 = R(q_1) = \frac{1}{2}(p - c - q_1). \tag{2.2}$$

Draw the lines of the best responses (2.1)–(2.2) on the plane (q_1, q_2) (see Figure 1.1). For any initial strategy q_2^0, construct the sequence of the best responses

$$q_2^{(0)} \to q_1^{(1)} = R(q_2^{(0)}) \to q_2^{(1)} = R(q_1^{(1)}) \to \cdots \to q_1^{(n)} = R(q_2^{(n-1)}) \to q_2^{(n)} = R(q_1^n) \to \dots$$

The sequence (q_1^n, q_2^n) is said to be the best response sequence. Such iterative procedure agrees with the behavior of sellers on a market (each of them modifies his strategy depending on the actions of the competitors). According to Figure 1.1, the best response sequence of the players tends to an equilibrium for any initial strategy $q_2^{(0)}$. However, we emphasize that the best response sequence does not necessarily brings a Nash equilibrium.

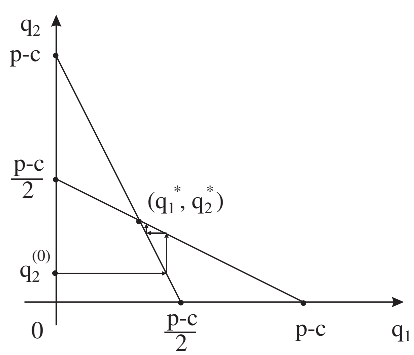

Figure 1.1 The Cournot duopoly.

1.3 The Bertrand duopoly

Another two-player game which models market pricing concerns the Bertrand duopoly [1883].

Consider two companies, I and II, manufacturing products A and B, respectively. Here the players choose product prices as their strategies. Assume that company I declares the unit prices of c_1, while company II declares the unit prices of c_2.

As the result of prices quotation, one observes the demands for each product on the market, i.e., $Q_1(c_1, c_2) = q - c_1 + kc_2$ and $Q_2(c_1, c_2) = q - c_2 + kc_1$. The symbol q means an initial demand, and the coefficient k reflects the interchangeability of products A and B.

By analogy to the Cournot model, the unit cost will be specified by c. Consequently, the players' payoffs acquire the form

$$H_1(c_1, c_2) = (q - c_1 + kc_2)(c_1 - c), \quad H_2(c_1, c_2) = (q - c_2 + kc_1)(c_2 - c).$$

The game is completely defined by: $\Gamma =< I, II, Q_1 = [0, \infty), Q_2 = [0, \infty), H_1, H_2 >$.

Fix the strategy c_1 of player I. Then the best response of player II consists in the strategy c_2 guaranteeing the maximal payoff $\max_{c_2} H_2(c_1, c_2)$. Since $H_2(c_1, c_2)$ forms a concave parabola, its vertex is at the point

$$c_2 = \frac{1}{2}(q + kc_1 + c). \tag{3.1}$$

Similarly, if the strategy c_2 of player II is fixed, the best response of player I becomes the strategy c_1 ensuring the maximal payoff $\max_{c_1} H_1(c_1, c_2)$. We easily find

$$c_1 = \frac{1}{2}(q + kc_2 + c). \tag{3.2}$$

There exists a unique solution to the system of equations (3.1)–(3.2):

$$c_1^* = c_2^* = \frac{q + c}{2 - k}.$$

We seek for positive solutions; therefore, $k < 2$.

The resulting solution represents a Nash equilibrium. Indeed, the best response of player II to the strategy c_1^* lies in the strategy c_2^*; and vice versa, the best response of player I to the strategy c_2^* makes the strategy c_1^*.

The optimal payoffs of the players in the equilibrium are given by

$$H_1^* = H_2^* = \left[\frac{q - c(1 - k)}{2 - k}\right]^2.$$

Draw the lines of the best responses (3.1)–(3.2) on the plane (c_1, c_2) (see Figure 1.2). Denote by $R(c_1), R(c_2)$ the right-hand sides of (3.1) and (3.2). For any initial strategy c_2^0, construct the best response sequence

$$c_2^{(0)} \to c_1^{(1)} = R(c_2^{0)}) \to c_2^{(1)} = R(c_1^{(1)}) \to \cdots \to c_1^{(n)} = R(c_2^{(n-1)}) \to c_2^{(n)} = R(c_1^n) \to \cdots$$

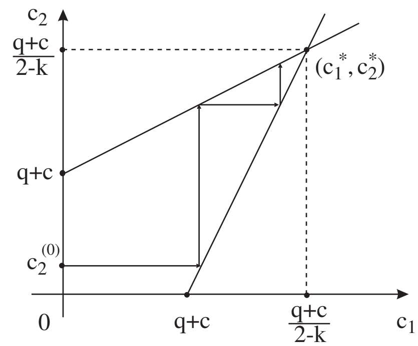

Figure 1.2 The Bertrand duopoly.

Figure 1.2 demonstrates the following. The best response sequence tends to the equilibrium (c_1^*, c_2^*) for any initial strategy $c_2^{(0)}$.

1.4 The Hotelling duopoly

This two-player game introduced by Hotelling [1929] also belongs to pricing problems but takes account of the location of companies on a market. Consider a linear market (see Figure 1.3) representing the unit segment $[0, 1]$. There exist two companies, I and II, located at points x_1 and x_2. Each company quotes its price for the same product (the parameters c_1 and c_2, respectively). Subsequently, each customer situated at point x compares his costs to visit each company, $L_i(x) = c_i + |x - x_i|, i = 1, 2$, and chooses the one corresponding to smaller costs. Within the framework of the Hotelling model, the costs $L(x)$ can be interpreted as the product price supplemented by transport costs. And all customers are decomposed into two sets, $[0, x)$ and $(x, 1]$. The former prefer company I, whereas the latter choose company II. The boundary of these sets x follows from the equality $L_1(x) = L_2(x)$:

$$x = \frac{x_1 + x_2}{2} + \frac{c_2 - c_1}{2}.$$

In this case, we understand the payoffs as the incomes of the players, i.e.,

$$H_1(c_1, c_2) = c_1 x = c_1 \left[\frac{x_1 + x_2}{2} + \frac{c_2 - c_1}{2}\right], \tag{4.1}$$

$$H_2(c_1, c_2) = c_2(1 - x) = c_2 \left[1 - \frac{x_1 + x_2}{2} - \frac{c_2 - c_1}{2}\right]. \tag{4.2}$$

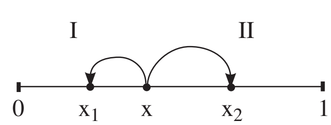

Figure 1.3 The Hotelling duopoly on a segment.

A Nash equilibrium (c_1^*, c_2^*) satisfies the equations $\frac{\partial H_1(c_1,c_2^*)}{\partial c_1} = 0, \ \frac{\partial H_2(c_1^*,c_2)}{\partial c_2} = 0.$ And so,

$$\frac{\partial H_1(c_1,c_2)}{\partial c_1} = \frac{c_2 - c_1}{2} + \frac{x_1 + x_2}{2} - \frac{c_1}{2} = 0,$$
$$\frac{\partial H_2(c_1,c_2)}{\partial c_2} = 1 - \frac{c_2 - c_1}{2} - \frac{x_1 + x_2}{2} - \frac{c_2}{2} = 0.$$

Summing up the above equations yields

$$c_1^* + c_2^* = 2,$$

which leads to the equilibrium prices

$$c_1^* = \frac{2 + x_1 + x_2}{3}, \ c_2^* = \frac{4 - x_1 - x_2}{3}.$$

Substitute the equilibrium prices into (4.1)–(4.2) to get the equilibrium payoffs:

$$H_1(c_1^*, c_2^*) = \frac{[2 + x_1 + x_2]^2}{18}, \ H_2(c_1^*, c_2^*) = \frac{[4 - x_1 - x_2]^2}{18}.$$

Just like in the previous case, here the payoff functions (4.1)–(4.2) are concave parabolas. Hence, the strategy improvement procedure tends to the equilibrium.

1.5 The Hotelling duopoly in 2D space

The preceding section proceeded from the idea that a market forms a linear segment. Actually, a market makes a set in 2D space. Let a city be a unit circle S with a uniform distribution of customers (see Figure 1.4). For the sake of simplicity, suppose that companies I and II are located at diametrally opposite points $(-1, 0)$ and $(1, 0)$. Each company announces a certain product price $c_i, \ i = 1, 2$. Without loss of generality, we believe that $c_1 < c_2$.

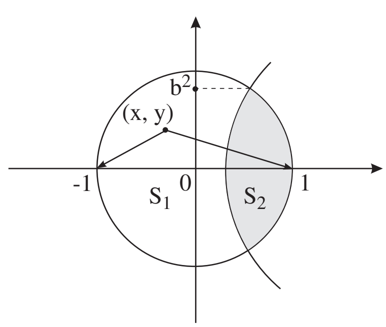

Figure 1.4 The Hotelling duopoly in 2D space.

A customer situated at point $(x, y) \in S$ compares the costs to visit the companies. Denote by $\rho_1(x, y) = \sqrt{(x+1)^2 + y^2}$ and $\rho_2(x, y) = \sqrt{(x-1)^2 + y^2}$ the distance to each company. Again, the total costs comprise a product price and transport costs: $L_i(x, y) = c_i + \rho_i(x, y),\ i = 1, 2$. The set of all customers is divided into two subsets, S_1 and S_2, whose boundary meets the equation

$$c_1 + \sqrt{(x+1)^2 + y^2} = c_2 + \sqrt{(x-1)^2 + y^2}.$$

After trivial manipulations, one obtains

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$a = (c_2 - c_1)/2,\ b = \sqrt{1 - a^2}. \tag{5.1}$$

Therefore, the boundary of the sets S_1 and S_2 represents a hyperbola. The players' payoffs take the form

$$H_1(c_1, c_2) = c_1 s_1,\ H_2(c_1, c_2) = c_2 s_2,$$

with $s_i (i = 1, 2)$ meaning the areas occupied by appropriate sets.

As far as $s_1 + s_2 = \pi$, it suffices to evaluate s_2. Using Figure 1.4, we have

$$\begin{aligned} s_2 &= \frac{\pi}{2} - 2\left[a \int_0^{b^2} \sqrt{1 + \frac{y^2}{b^2}} dy + \int_{b^2}^1 \sqrt{1 - y^2} dy\right] \\ &= \frac{\pi}{2} - 2\left[ab \int_0^{b} \sqrt{1 + y^2} dy + \int_{b^2}^1 \sqrt{1 - y^2} dy\right]. \end{aligned} \tag{5.2}$$

The Nash equilibrium (c_1^*, c_2^*) of this game follows from the conditions

$$\frac{\partial H_1(c_1, c_2)}{\partial c_1} = \pi - s_2 - c_1 \frac{\partial s_2}{\partial c_1} = 0, \tag{5.3}$$

$$\frac{\partial H_2(c_1, c_2)}{\partial c_2} = s_2 + c_2 \frac{\partial s_2}{\partial c_2} = 0. \tag{5.4}$$

Revert to formula (5.2) to derive

$$\frac{\partial s_2}{\partial c_1} = \frac{b^2 - a^2}{b} \int_0^b \sqrt{1 + y^2} dy + a^2 \sqrt{1 + b^2}. \tag{5.5}$$

By virtue of $\frac{\partial a}{\partial c_1} = -\frac{\partial a}{\partial c_2}$, we arrive at

$$\frac{\partial s_2}{\partial c_2} = -\frac{\partial s_2}{\partial c_1}. \tag{5.6}$$

The function $s_2(c_1, c_2)$ strictly increases with respect to the argument c_1. This fact is immediate from an important observation. If player *I* quotes a higher price, then the customer from S_2 (characterized by greater costs to visit company *I* in comparison with company *II*) still benefits by visiting company *II*.

To proceed, let us evaluate the equilibrium in this game.

Owing to (5.6), the expressions (5.3)–(5.4) yield

$$s_2 \left(1 + \frac{c_1}{c_2}\right) = \pi.$$

And so, if $c_1 < c_2$, then s_2 must exceed $\pi/2$. Meanwhile, this contradicts the following idea. Imagine that the price declared by company *I* appears smaller than the one offered by the opponent; in this case, the set of customers preferring this company (S_1) becomes larger than S_2, i.e., $s_2 < \pi/2$. Therefore, the solution to the system (5.3)–(5.4) (if any) exists only under $c_1 = c_2$. Generally speaking, this conclusion also follows from the symmetry of the problem.

Thus, we look for the solution in the class of identical prices: $c_1 = c_2$. Then $s_1 = s_2 = \pi/2$ and the notation $a = 0,\ b = 1$ from (5.5) brings to

$$\frac{\partial s_2}{\partial c_1} = \int_0^1 \sqrt{1 + y^2} dy = \frac{1}{2}\left[\sqrt{2} + \ln(1 + \sqrt{2})\right].$$

Formulas (5.3)–(5.4) lead to the equilibrium prices

$$c_1^* = c_2^* = \frac{\pi}{\sqrt{2} + \ln(1 + \sqrt{2})} \approx 1.3685.$$

1.6 The Stackelberg duopoly

Up to here, we have studied two-player games with equal rights of the opponents (they choose decisions simultaneously). The Stackelberg duopoly [1934] deals with a certain hierarchy of players. Notably, player *I* chooses his decision first, and then player *II* does. Player *I* is called a **leader**, and player *II* is called a **follower**.

Definition 1.3 *A Stackelberg equilibrium in a game* Γ *is a set of strategies* (x^*, y^*) *such that* $y^* = R(x^*)$ *represents the best response of player II to the strategy* x^* *which solves the problem*

$$H_1(x^*, y^*) = \max_x H_1(x, R(x)).$$

Therefore, in a Stackelberg equilibrium, a leader knows that a *follower* chooses the best response to any strategy and easily finds the strategy x^* maximizing his payoff.

Now, analyze the Stackelberg model within the Cournot duopoly. There exist two companies, *I* and *II*, manufacturing a same product. At step 1, company *I* announces its product output q_1. Subsequently, company *II* chooses its strategy q_2.

Recall the outcomes of Section 1.1; the best response of player *II* to the strategy q_1 is the strategy $q_2 = R(q_1) = (p - c - q_1)/2$. Knowing this, player *I* maximizes his payoff

$$H_1(q_1, R(q_1)) = q_1(p - c - q_1 - R(q_1)) = q_1(p - c - q_1)/2.$$

Clearly, the optimal strategy of this player lies in

$$q_1^* = \frac{p - c}{2}.$$

Accordingly, the optimal strategy of player *II* makes up

$$q_2^* = \frac{p - c}{4}.$$

The equilibrium payoffs of the players equal

$$H_1^* = \frac{(p - c)^2}{8},$$
$$H_2^* = \frac{(p - c)^2}{16}.$$

Obviously, the leader gains twice as much as the *follower* does.

1.7 Convex games

Nash equilibria do exist in all games discussed above. Generally speaking, the class of games admitting no equilibria appears much wider. The current section focuses on this issue. For the time being, note that the existence of Nash equilibria in the duopolies relates to the form of payoff functions (all economic examples considered employ continuous concave functions).

Definition 1.4 *A function $H(x)$ is called concave (convex) on a set $X \subseteq R^n$, if for any $x, y \in X$ and $\alpha \in [0, 1]$ the inequality $H(\alpha x + (1 - \alpha)y) \geq (\leq)\alpha H(x) + (1 - \alpha)H(y)$ holds true.*

Interestingly, this definition directly implies the following result. Concave functions also meet the inequality

$$H\left(\sum_{i=1}^{p} \alpha_i x_i\right) \geq \sum_{i=1}^{p} \alpha_i H(x_i)$$

for any convex combination of the points $x_i \in X,\ i = 1, \dots, p$, where $\alpha_i \geq 0,\ i = 1, \dots, p$ and $\sum \alpha_i = 1$.

The Nash theorem [1951] forms a central statement regarding equilibrium existence in such games. Prior to introducing this theorem, we prove an auxiliary result to-be-viewed as an alternative definition of a Nash equilibrium.

Lemma 1.1 *A Nash equilibrium exists in a game* $\Gamma =< I, II, X, Y, H_1, H_2 >$ *iff there is a set of strategies* (x^*, y^*) *such that*

$$\max_{x,y} \left\{ H_1(x, y^*) + H_2(x^*, y) \right\} = H_1(x^*, y^*) + H_2(x^*, y^*). \tag{7.1}$$

Proof: The necessity part. Suppose that a Nash equilibrium (x^*, y^*) exists. According to Definition 1.2, for arbitrary (x, y) we have

$$H_1(x, y^*) \le H_1(x^*, y^*), \quad H_2(x^*, y) \le H_2(x^*, y^*)\,.$$

Summing these inequalities up yields

$$H_1(x, y^*) + H_2(x^*, y) \le H_1(x^*, y^*) + H_2(x^*, y^*) \tag{7.2}$$

for arbitrary strategies x, y of the players. And the expression (7.1) is immediate.

The sufficiency part. Assume that there exists a pair (x^*, y^*) satisfying (7.1) and, hence, (7.2). By choosing $x = x^*$ and, subsequently, $y = y^*$ in inequality (7.2), we arrive at the conditions (1.1) that define a Nash equilibrium. The proof of Lemma 1.1 is finished.

Lemma 1.1 allows to use the conditions (7.1) or (7.2) instead of equilibrium verification in formula (1.1).

Theorem 1.1 *Consider a two-player game* $\Gamma =< I, II, X, Y, H_1, H_2 >$*. Let the sets of strategies* X, Y *be compact convex sets in the space* R^n*, and the payoffs* $H_1(x, y),\ H_2(x, y)$ *represent continuous convex functions in x and y, respectively. Then the game possesses a Nash equilibrium.*

Proof: We apply the *ex contrario* principle. Suppose that no Nash equilibria actually exist. In this case, the above lemma requires that, for any pair of strategies (x, y), there is (x', y') violating the condition (7.2), i.e.,

$$H_1(x', y) + H_2(x, y') > H_1(x, y) + H_2(x, y).$$

Take the sets

$$S_{(x',y')} = \left\{ (x, y) : H_1(x', y) + H_2(x, y') > H_1(x, y) + H_2(x, y) \right\},$$

representing open sets due to the continuity of the functions $H_1(x, y)$ and $H_2(x, y)$. The whole space of strategies $X \times Y$ is covered by the sets $S_{(x',y')}$, i.e., $\bigcup_{(x',y') \in X \times Y} S_{(x',y')} = X \times Y$. Owing to the compactness of $X \times Y$, one can separate out a finite subcovering

$$\bigcup_{i=1,\dots,p} S_{(x_i,y_i)} = X \times Y.$$

For each $i = 1, \dots, p$, denote

$$\varphi_i(x, y) = \left[H_1(x_i, y) + H_2(x, y_i) - (H_1(x, y) + H_2(x, y)) \right]^+, \tag{7.3}$$

where $a^+ = \max\{a, 0\}$. All functions $\varphi_i(x, y)$ enjoy non-negativity; moreover, at least, for a single $i = 1, \dots, p$ the function $\varphi_i(x, y)$ is positive according to the definition of $S_{(x_i,y_i)}$. Hence, it appears that $\sum_{i=1}^{p} \varphi_i(x, y) > 0,\ \forall (x, y)$.

Now, we define the mapping $\varphi(x, y) : X \times Y \to X \times Y$ by

$$\varphi(x, y) = \left(\sum_{i=1}^{p} \alpha_i(x, y) x_i, \sum_{i=1}^{p} \alpha_i(x, y) y_i \right),$$

where

$$\alpha_i(x, y) = \frac{\varphi_i(x, y)}{\sum_{i=1}^{p} \varphi_i(x, y)},\quad i = 1, \dots, p,\ \ \sum_{i=1}^{p} \alpha_i(x, y) = 1.$$

The functions $H_1(x, y),\ H_2(x, y)$ are continuous, whence it follows that the mapping $\varphi(x, y)$ turns out continuous. By the premise, X and Y form convex sets; consequently, the convex combinations $\sum_{i=1}^{p} \alpha_i x_i \in X$, $\sum_{i=1}^{p} \alpha_i y_i \in Y$. Thus, $\varphi(x, y)$ makes a self-mapping of the convex compact set $X \times Y$. The Brouwer fixed point theorem states that this mapping has a fixed point $(\bar{x}, \bar{y})$ such that $\varphi(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$, or

$$\bar{x} = \sum_{i=1}^{p} \alpha_i(\bar{x}, \bar{y}) x_i,\ \ \bar{y} = \sum_{i=1}^{p} \alpha_i(\bar{x}, \bar{y}) y_i.$$

Recall that the functions $H_1(x, y)$ and $H_2(x, y)$ are concave in x and y, respectively. And so, we naturally arrive at

$$\begin{aligned} H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}) &= H_1\left(\sum_{i=1}^{p} \alpha_i x_i, \bar{y} \right) + H_2\left(\bar{x}, \sum_{i=1}^{p} \alpha_i y_i \right) \\ &\geq \sum_{i=1}^{p} \alpha_i H_1(x_i, \bar{y}) + \sum_{i=1}^{p} \alpha_i H_2(\bar{x}, y_i). \end{aligned} \tag{7.4}$$

On the other hand, by the definition $\alpha_i(x, y)$ is positive simultaneously with $\varphi_i(x, y)$. For a positive function $\varphi_i(\bar{x}, \bar{y})$ (there exists at least one such function), one obtains (see (7.3))

$$H_1(x_i, \bar{y}) + H_2(\bar{x}, y_i) > H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}). \tag{7.5}$$

Indexes j corresponding to $\alpha_j(\bar{x}, \bar{y}) = 0$ satisfy the inequality

$$\alpha_j(\bar{x}, \bar{y}) \left(H_1(x_j, \bar{y}) + H_2(\bar{x}, y_j) \right) > \alpha_j(\bar{x}, \bar{y}) \left(H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}) \right). \tag{7.6}$$

Multiply the expression (7.5) by $\alpha_i(\bar{x}, \bar{y})$ and sum up with (7.6) over all indexes $i, j = 1, \dots, p$. These manipulations yield the inequality

$$\sum_{i=1}^{p} \alpha_i H_1(x_i, \bar{y}) + \sum_{i=1}^{p} \alpha_i H_2(\bar{x}, y_i) > H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}),$$

which evidently contradicts (7.4). And the conclusion regarding the existence of a Nash equilibrium in convex games follows immediately. This concludes the proof of Theorem 1.1.

1.8 Some examples of bimatrix games

Consider a two-player game $\Gamma =< I, II, M, N, A, B >$, where players have finite sets of strategies, $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$, respectively. Their payoffs are defined by matrices A and B. In this game, player I chooses row i, whereas player II chooses column j; and their payoffs are accordingly specified by $a(i, j)$ and $b(i, j)$. Such games will be called **bimatrix games**. The following examples show that Nash equilibria may exist or not exist in such games.

Prisoners' dilemma. Two prisoners are arrested on suspicion of a crime. Each of them chooses between two actions, *viz.*, admitting the crime (strategy "Yes") and remaining silent (strategy "No"). The payoff matrices take the form

$$A = \begin{matrix} \\ \text{Yes} \\ \text{No} \end{matrix} \begin{matrix} \begin{matrix} \text{Yes} & \text{No} \end{matrix} \\ \begin{pmatrix} -6 & 0 \\ -10 & -1 \end{pmatrix} \end{matrix} \qquad B = \begin{matrix} \\ \text{Yes} \\ \text{No} \end{matrix} \begin{matrix} \begin{matrix} \text{Yes} & \text{No} \end{matrix} \\ \begin{pmatrix} -6 & -10 \\ 0 & -1 \end{pmatrix} \end{matrix}.$$

Therefore, if the prisoners admit the crime, they sustain a punishment of 6 years. When both remain silent, they sustain a small punishment of 1 year. However, admitting the crime seems very beneficial (if one prisoner admits the crime and the other does not, the former is set at liberty and the latter sustains a major punishment of 10 years). Clearly, a Nash equilibrium lies in the strategy profile (Yes, Yes), where players' payoffs constitute $(-6, -6)$. Indeed, by deviating from this strategy, a player gains -10. Prisoners' dilemma has become popular in game theory due to the following features. It models a Nash equilibrium leading to guaranteed payoffs (however, being appreciably worse than payoffs in the case of coordinated actions of the players).

Battle of sexes. This game involves two players, a "husband" and a "wife." They decide how to pass away a weekend. Each spouse chooses between two strategies, "boxing" and "theater." Depending on their choice, the payoffs are defined by the matrices

$$A = \begin{matrix} \\ \text{Boxing} \\ \text{Theater} \end{matrix} \begin{matrix} \begin{matrix} \text{Boxing} & \text{Theater} \end{matrix} \\ \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix} \qquad B = \begin{matrix} \\ \text{Boxing} \\ \text{Theater} \end{matrix} \begin{matrix} \begin{matrix} \text{Boxing} & \text{Theater} \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \end{matrix}.$$

In the previous game, we have obtained a single Nash equilibrium. Contrariwise, the battle of sexes admits two equilibria (actually, there exist three Nash equilibria—see the discussion below). The list of Nash equilibria includes the strategy profiles (Boxing, Boxing) and (Theater, Theater), but spouses have different payoffs. One gains 1, whereas the other gains 4.

The Hawk-Dove game. This game is often involved to model the behavior of different animals; it proceeds from the following assumption. While assimilating some resource V (e.g., a territory), each individual chooses between two strategies, namely, aggressive strategy (Hawk) or passive strategy (Dove). In their rivalry, Hawk always captures the whole of the resource from Dove. When two Doves meet, they share the resource equally. And finally, both individuals with aggressive strategy struggle for the resource. In this case, an individual obtains the resource with the identical probability of $1/2$, but both Hawks suffer from the losses of c. Let us present the corresponding payoff matrices:

$$A = \begin{array}{c} \\ \text{Hawk} \\ \text{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{pmatrix} \frac{1}{2}V - c & V \\ 0 & V/2 \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ \text{Hawk} \\ \text{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{pmatrix} \frac{1}{2}V - c & 0 \\ V & V/2 \end{pmatrix} \end{array}.$$

Depending on the relationship between the available quantity of the resource and the losses, one obtains a game of the above types. If the losses c are smaller than $V/2$, prisoners' dilemma arises immediately (a single equilibrium, where the optimal strategy is Hawk for both players). At the same time, the condition $c \geq V/2$ corresponds to the battle of sexes (two equilibria, (Hawk, Dove) and (Dove, Hawk)).

The Stone-Scissors-Paper game. In this game, two players assimilate 1 USD by simultaneously announcing one of the following words: "Stone," "Scissors," and "Paper." The payoff is defined according to the rule: Stone breaks Scissors, Scissors cut Paper, and Paper wraps up Stone. And so, the players' payoffs are expressed by the matrices

$$A = \begin{array}{c} \\ \text{Stone} \\ \text{Scissors} \\ \text{Paper} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Stone} & \text{Scissors} & \text{Paper} \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ \text{Stone} \\ \text{Scissors} \\ \text{Paper} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Stone} & \text{Scissors} & \text{Paper} \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}.$$

Unfortunately, the Stone-Scissors-Paper game admits no Nash equilibria among the considered strategies. It is impossible to suggest a strategy profile such that a player would not benefit by unilateral deviation from his strategy.

1.9 Randomization

In the preceding examples, we have observed the following fact. There may exist no equilibria in finite games. The "way out" concerns randomization. For instance, recall the Stone-Scissors-Paper game; obviously, one should announce a strategy randomly, and an opponent would not guess it. Let us extend the class of strategies and seek for a Nash equilibrium among probabilistic distributions defined on the sets $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$.

Definition 1.5 *A mixed strategy of player I is a vector* $x = (x_1, x_2, \dots, x_m)$, *where* $x_i \geq 0$, $i = 1, \dots, m$ *and* $\sum_{i=1}^{m} x_i = 1$. *Similarly, introduce a mixed strategy of player II as* $y = (y_1, y_2, \dots, y_n)$, *where* $y_j \geq 0$, $j = 1, \dots, n$ *and* $\sum_{i=1}^{n} y_j = 1$.

Therefore, x_i (y_j) represents a probability that player I (II) chooses strategy i (j, respectively). In contrast to new strategies, we call $i \in M$, $j \in N$ by pure strategies. Note that pure strategy i corresponds to the mixed strategy $x = (0, \dots, 0, 1, 0, ...0)$, where 1 occupies position i (in the sequel, we simply write $x = i$ for compactness). Denote by X (Y) the set of mixed strategies of player I (player II, respectively). Those pure strategies adopted with a positive probability in a mixed strategy form the **support** or **spectrum** of the mixed strategy.

Now, any strategy profile (i, j) is realized with the probability $x_i y_j$. Hence, the expected payoffs of the players become

$$H_1(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i y_j, \quad H_2(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} b(i, j) x_i y_j. \tag{9.1}$$

Thus, the extension of the original discrete game acquires the form $\Gamma =< I, II, X, Y, H_1, H_2 >$, where players' strategies are probabilistic distributions of x and y, and the payoff functions have the bilinear representation (9.1). Interestingly, strategies x and y make simplexes $X = \{x : \sum_{i=1}^{m} x_i = 1, x_i \geq 0,\ i = 1, \dots, m\}$ and $Y = \{y : \sum_{j=1}^{n} y_j = 1, y_j \geq 0,\ j = 1, \dots, n\}$ in the spaces R^m and R^n, respectively.

The sets X and Y form convex polyhedra in R^m and R^n, and the payoff functions $H_1(x, y)$, $H_2(x, y)$ are linear in each variable. And so, the resulting game $\Gamma =< I, II, X, Y, H_1, H_2 >$ belongs to the class of convex games, and Theorem 1.1 is applicable.

Theorem 1.2 *Bimatrix games admit a Nash equilibrium in the class of mixed strategies.*

The Nash theorem establishes the existence of a Nash equilibrium, but offers no algorithm to evaluate it. In a series of cases, one can benefit by the following assertion.

Theorem 1.3 *A strategy profile (x^*, y^*) represents a Nash equilibrium iff for any pure strategies $i \in M$ and $j \in N$:*

$$H_1(i, y^*) \leq H_1(x^*, y^*), \ H_2(x^*, j) \leq H_2(x^*, y^*). \tag{9.2}$$

Proof: The necessity part is immediate from the definition of a Nash equilibrium. Indeed, the conditions (1.1) hold true for arbitrary strategies x and y (including pure strategies).

The sufficiency of the conditions (9.2) can be shown as follows. Multiply the first inequality $H_1(i, y^*) \leq H_1(x^*, y^*)$ by x_i and perform summation over all $i = 1, \dots, m$. These operations yield the condition $H_1(x, y^*) \leq H_1(x^*, y^*)$ for an arbitrary strategy x. Analogous reasoning applies to the second inequality in (9.2). The proof of Theorem 1.3 is completed.

Theorem 1.4 (on complementary slackness) *Let (x^*, y^*) be a Nash equilibrium strategy profile in a bimatrix game. If for some i: $x_i^* > 0$, then the equality $H_1(i, y^*) = H_1(x^*, y^*)$ takes place. Similarly, if for some j: $y_j^* > 0$, we have $H_2(x^*, j) = H_2(x^*, y^*)$.*

Proof is by *ex contrario*. Suppose that for a certain index i' such that $x_{i'}^* > 0$ one obtains $H_1(i', y^*) < H_1(x^*, y^*)$. Theorem 1.3 implies that the inequality $H_1(i, y^*) \leq H_1(x^*, y^*)$ is valid for the rest indexes $i \neq i'$. Therefore, we arrive at the system of inequalities

$$H_1(i, y^*) \leq H_1(x^*, y^*), \ i = 1, \dots, n, \tag{9.2$'$}$$

where inequality i' turns out strict. Multiply (9.2$'$) by x_i^* and perform summation to get the contradiction $H(x^*, y^*) < H(x^*, y^*)$. By analogy, one easily proves the second part of the theorem.

Theorem 1.4 claims that a Nash equilibrium involves only those pure strategies leading to the optimal payoff of a player. Such strategies are called **equalizing**.

Theorem 1.5 *A strategy profile (x^*, y^*) represents a mixed strategy Nash equilibrium profile iff there exist pure strategy subsets $M_0 \subseteq M$, $N_0 \subseteq N$ and values H_1, H_2 such that*

$$\sum_{j \in N_0} H_1(i,j) y_j^* \left\{ \begin{matrix} \equiv \\ \leq \end{matrix} \right\} H_1, \quad for \quad \left\{ \begin{matrix} i \in M_0 \\ i \notin M_0 \end{matrix} \right\} \tag{9.3}$$

$$\sum_{i \in M_0} H_2(i,j) x_i^* \left\{ \begin{matrix} \equiv \\ \leq \end{matrix} \right\} H_2, \quad for \quad \left\{ \begin{matrix} j \in N_0 \\ i \notin N_0 \end{matrix} \right\} \tag{9.4}$$

and

$$\sum_{i \in M_0} x_i^* = 1, \ \sum_{j \in N_0} y_j^* = 1. \tag{9.5}$$

Proof (the necessity part). Assume that (x^*, y^*) is an equilibrium in a bimatrix game. Set $H_1 = H_1(x^*, y^*)$, $H_2 = H_2(x^*, y^*)$ and $M_0 = \{i \in M : x_i^* > 0\}$, $N_0 = \{j \in N : y_j^* > 0\}$. Then the conditions (9.3)–(9.5) directly follow from Theorems 1.3 and 1.4.

The sufficiency part. Suppose that the conditions (9.3)–(9.5) hold true for a certain strategy profile (x^*, y^*). Formula (9.5) implies that (a) $x_i^* = 0$ for $i \notin M_0$ and (b) $y_j^* = 0$ for $j \notin N_0$. Multiply (9.3) by x_i^* and (9.4) by y_j^*, as well as perform summation over all $i \in M$ and $j \in N$, respectively. Such operations bring us to the equalities

$$H_1(x^*, y^*) = H_1, \ H_2(x^*, y^*) = H_2.$$

This result and Theorem 1.3 show that (x^*, y^*) is an equilibrium. The proof of Theorem 1.5 is concluded.

Theorem 1.5 can be used to evaluate Nash equilibria in bimatrix games. Imagine that we know the optimal strategy spectra M_0, N_0. It is possible to employ equalities from the conditions (9.3)–(9.5) and find the optimal mixed strategies x^*, y^* and the optimal payoffs H_1^*, H_2^* from the system of linear equations. However, this system can generate negative solutions (which contradicts the concept of mixed strategies). Then one should modify the spectra and go over them until an equilibrium appears. Theorem 1.5 demonstrates high efficiency, if all $x_i, i \in M$ and $y_j, j \in N$ have positive values in an equilibrium.

Definition 1.6 *An equilibrium strategy profile (x^*, y^*) is called completely mixed, if $x_i > 0$, $i \in M$ and $y_j > 0, j \in N$.*

Suppose that a bimatrix game admits a completely mixed equilibrium strategy profile (x, y). According to Theorem 1.5, it satisfies the system of linear equations

$$\begin{aligned} &\sum_{j \in N} H_1(i,j) y_j^* = H_1, i \in M \\ &\sum_{i \in M} H_2(i,j) x_i^* = H_2, j \in N \\ &\sum_{i \in M} x_i^* = 1, \ \sum_{j \in N} y_j^* = 1. \end{aligned} \tag{9.6}$$

Actually, the system (9.6) comprises $n + m + 2$ equations with $n + m + 2$ unknowns. Its solution gives a Nash equilibrium in a bimatrix game and the values of optimal payoffs.

1.10 Games 2 × 2

A series of bimatrix games can be treated via geometric considerations. The simplest case covers players choosing between two strategies. The mixed strategies of players *I* and *II* take the form $(x, 1-x)$ and $(y, 1-y)$, respectively. And their payoffs are defined by the matrices

$$A = \begin{matrix} & y & 1-y \\ \begin{matrix} x \\ 1-x \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \end{matrix} \quad B = \begin{matrix} & y & 1-y \\ \begin{matrix} x \\ 1-x \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}} \end{matrix}.$$

The mixed strategy payoffs of the players become

$$\begin{aligned} H_1(x, y) &= a_{11}xy + a_{12}x(1-y) + a_{21}(1-x)y + a_{22}(1-x)(1-y) \\ &= Axy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}, \end{aligned}$$

$$\begin{aligned} H_2(x, y) &= b_{11}xy + b_{12}x(1-y) + b_{21}(1-x)y + b_{22}(1-x)(1-y) \\ &= Bxy + (b_{12} - b_{22})x + (b_{21} - b_{22})y + b_{22}, \end{aligned}$$

where $A = a_{11} - a_{12} - a_{21} + a_{22},\ B = b_{11} - b_{12} - b_{21} + b_{22}$.

By virtue of Theorem 1.3, the equilibrium (x, y) follows from inequalities (9.2), i.e.,

$$H_1(0, y) \le H_1(x, y),\ H_1(1, y) \le H_1(x, y), \tag{10.1}$$

$$H_2(x, 0) \le H_2(x, y),\ H_2(x, 1) \le H_1(x, y). \tag{10.2}$$

Rewrite inequalities (10.1) as

$$\begin{aligned} (a_{21} - a_{22})y + a_{22} &\le Axy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}, \\ Ay + (a_{21} - a_{22})y + a_{12} &\le Axy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}, \end{aligned}$$

and, consequently,

$$(a_{22} - a_{12})x \le Axy, \tag{10.3}$$

$$Ay(1-x) \le (a_{22} - a_{12})(1-x). \tag{10.4}$$

Now, take the unit square $0 \le x \le 1, 0 \le y \le 1$ (see Figure 1.5) and draw the set of points (x, y) meeting the conditions (10.3)–(10.4).

If $x = 0$, then (10.3) is immediate, whereas the condition (10.4) implies the inequality $Ay \le a_{22} - a_{12}$. In the case of $x = 1$, the expression (10.4) is valid, and (10.3) leads to $Ay \ge a_{22} - a_{12}$. And finally, under $0 \le x \le 1$, the conditions (10.3)–(10.4) bring to $Ay = a_{22} - a_{12}$.

Similar analysis of inequalities (10.2) yields the following. If $y = 0$, then $Bx \le b_{22} - b_{21}$. In the case of $y = 1$, we have $Bx \ge b_{22} - b_{21}$. If $0 \le y \le 1$, then $Bx = b_{22} - b_{21}$.

Depending on the signs of A and B, these conditions result in different sets of feasible equilibria in a bimatrix game (zigzags inside the unit square).

Prisoners' dilemma. Recall the example studied above; here $A = B = -6 + 10 + 0 - 1 = 3$ and $a_{22} - a_{12} = b_{22} - b_{21} = -1$. Hence, an equilibrium represents the intersection of two lines $x = 1$ and $y = 1$ (see Figure 1.6). Therefore, the equilibrium is unique and takes the form $x = 1,\ y = 1$.

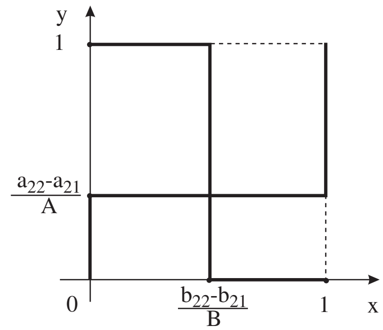

Figure 1.5 A zigzag in a bimatrix game.

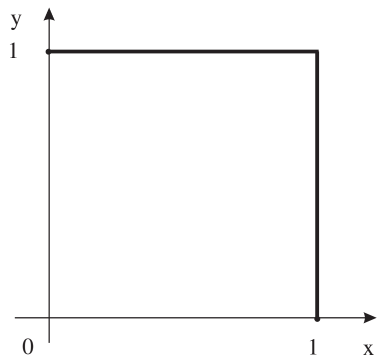

Figure 1.6 A unique equilibrium in the prisoners' dilemma game.

Battle of sexes. In this example, one obtains $A = B = 4 - 0 - 0 + 1 = 5$ and $a_{22} - a_{12} = 1$, $b_{22} - b_{21} = 4$. And so, the zigzag defining all equilibrium strategy profiles is shown in Figure 1.7. Obviously, the game under consideration includes three equilibria. Among them, two equilibria correspond to pure strategies ($x = 0$, $y = 1$), ($x = 1$, $y = 0$), while the third one has the mixed strategy type ($x = 4/5$, $y = 1/5$). The payoffs in these equilibria make up $(H_1^* = 4, H_2^* = 1)$, $(H_1^* = 1, H_2^* = 4)$ and $(H_1^* = 4/5, H_2^* = 4/5)$, respectively.

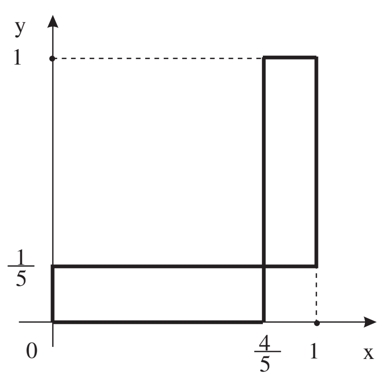

Figure 1.7 Three equilibria in the battle of sexes game.

The stated examples illustrate the following aspect. Depending on the shape of zigzags, bimatrix games may admit one, two, or three equilibria, or even the continuum of equilibria.

1.11 Games $2 \times n$ and $m \times 2$

Suppose that player I chooses between two strategies, whereas player II has n strategies available. Consequently, their payoffs are defined by the matrices

$$A = \begin{matrix} & y_1 & y_2 & \dots & y_n \\ x \\ 1-x \end{matrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix} \quad B = \begin{matrix} x \\ 1-x \end{matrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \end{pmatrix}.$$

In addition, assume that player I uses the mixed strategy $(x, 1-x)$. If player II chooses strategy j, his payoff equals $H_2(x, j) = b_{1,j}x + b_{2j}(1-x)$, $j = 1, \dots, n$.

We show these payoffs (linear functions) in Figure 1.8. According to Theorem 1.3, the equilibrium (x, y) corresponds to $\max_j H_2(x, j) = H_2(x, y)$. For any x, construct the maximal envelope $l(x) = \max_j H_2(x, j)$. As a matter of fact, $l(x)$ represents a jogged line composed of at most $n+1$ segments. Denote by $x_0 = 0, x_1, \dots, x_k = 1$, $k \le n+1$ the salient points of this envelope. Since the function $H_1(x, y)$ is linear in x, its maximum under a fixed strategy of player II is attained at the points x_i, $i = 0, \dots, k$. Hence, equilibria can be focused only in these points. Imagine that the point x_i results from intersection of the straight lines $H_2(x, j_1)$ and $H_2(x, j_2)$. This means that player II optimally plays the mix of his strategies j_1 and j_2 in response to the strategy x by player I. Thus, we obtain a game 2×2 with the payoff matrices

$$A = \begin{pmatrix} a_{1j_1} & a_{1j_2} \\ a_{2j_1} & a_{2j_2} \end{pmatrix} \quad B = \begin{pmatrix} b_{1j_1} & b_{1j_2} \\ b_{2j_1} & b_{2j_2} \end{pmatrix}.$$

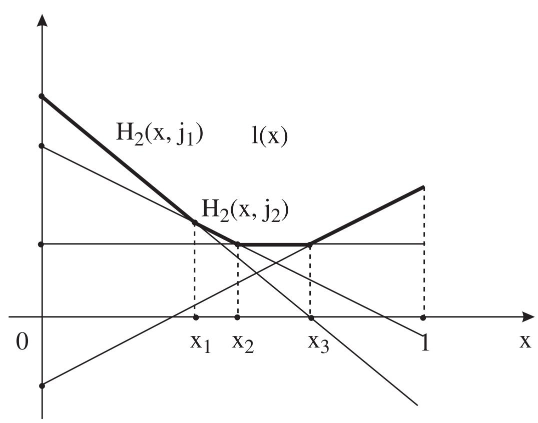

Figure 1.8 The maximal envelope l(x).

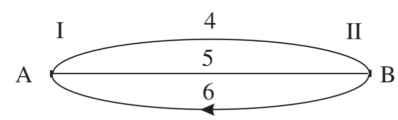

Figure 1.9 The road selection game.

It has been solved in the previous section. To verify the optimality of the strategy x_i, one can adhere to the following reasoning. The strategies x_i and the mix $(y, 1-y)$ of the strategies j_1 and j_2 form an equilibrium, if there exists $y, 0 \le y \le 1$ such that $H_1(1, y) = H_1(2, y)$. In this case, the payoff of player I is independent from x, and the best response of player II to the strategy x_i lies in mixing the strategies j_1 and j_2. Rewrite the last condition as

$$a_{1j_1}y + a_{1j_2}(1-y) = a_{2j_1}y + a_{2j_2}(1-y). \tag{11.1}$$

Let us consider this procedure using an example.

Road selection. Points A and B communicate through three roads. One of them is one-way road right-to-left (see Figure 1.9).

A car (player I) leaves A, and another car (player II) moves from B. The journey-time on these roads varies (4, 5, and 6 hours, respectively, for a single car on a road). If both players choose the same road, the journey-time doubles. Each player has to select a road for the journey.

And so, player I (player II) chooses between two strategies (among three strategies, respectively). The payoff matrices take the form

$$A = \begin{matrix} x \\ 1-x \end{matrix} \begin{pmatrix} -8 & -4 & -4 \\ -5 & -10 & -5 \end{pmatrix} \quad B = \begin{matrix} x \\ 1-x \end{matrix} \begin{pmatrix} -8 & -5 & -6 \\ -4 & -10 & -6 \end{pmatrix}.$$

Find the payoffs of player II: $H_2(x, 1) = -4x - 5$, $H_2(x, 2) = 5x - 10$, $H_2(x, 3) = -6$. Draw these functions on Figure 1.10 and the maximal envelope $l(x)$ (see the thick line).

The salient points of $l(x)$ are located at $x = 0, x = 0.5, x = 0.8$, and $x = 1$. The corresponding equilibria form $(x = 0, y = (1, 0, 0))$, $(x = 1, y = (0, 1, 0))$. The point $x = 1/2$ answers

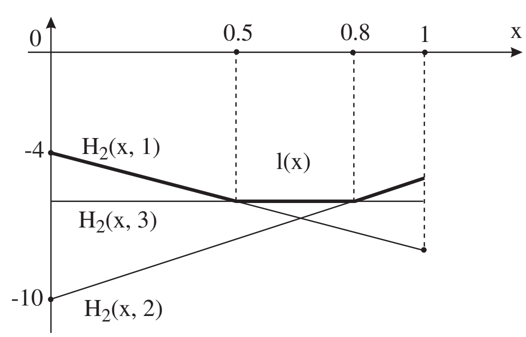

Figure 1.10 The maximal envelope in the road selection game.

for intersection of the functions $H_2(x, 1)$ and $H_2(x, 3)$. The condition (11.1) implies that $y = (1/4, 0, 3/4)$. However, this condition fails at the point $x = 0.8$. Therefore, the game in question admits three equilibrium solutions:

1. car I moves on the first road, and car II chooses the second road,
2. car I moves on the second road, and car II chooses the first road,
3. car I moves on the first or second road with identical probabilities, and car II chooses the first road with the probability of $1/4$ or the third road with the probability of $3/4$.

Interestingly, in the third equilibrium, the mean journey time of player I constitutes 5 h, whereas player II spends 6 h. It would seem that player II "has the cards" (owing to the additional option of using the third road). For instance, suppose that the third road is closed. In the worst case, player II requires just 5 h for the journey. This contradicting result is known as **the Braess paradox**. We will discuss it later. In fact, if player I informs the opponent that he chooses the road by coin-tossing, player II has nothing to do but to follow the third strategy above.

1.12 The Hotelling duopoly in 2D space with non-uniform distribution of buyers

Let us revert to the problem studied in Section 1.5. Consider the Hotelling duopoly in 2D space with non-uniform distribution of buyers in a city (according to some density function $f(x, y)$). As previously, we believe that a city represents a circle S having the radius of 1 (see Figure 1.11). It contains two companies (players I and II) located at different points P_1 and P_2. The players strive for defining optimal prices for their products depending on their location in the city.

Again, players I and II quote prices for their products (some quantities c_1 and c_2, respectively). A buyer located at a point $P \in S$ compares his costs (for the sake of simplicity, here we define them by $F(c_i, \rho(P, P_i)) = c_i + \rho^2$) and chooses the company with the minimal value.

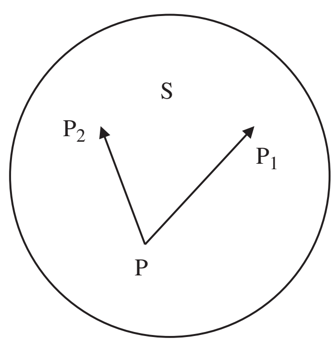

Figure 1.11 The Hotelling duopoly.

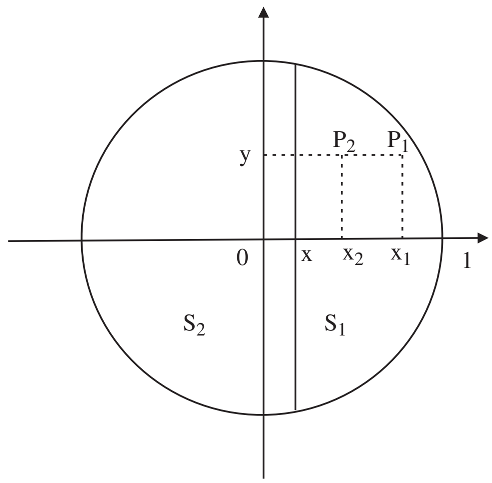

Figure 1.12 P_1, P_2 have the same ordinate y.

Therefore, all buyers in S get decomposed into two subsets (S_1 and S_2) according to their priorities of companies I and II. Then the payoffs of players I and II are specified by

$$H_1(c_1, c_2) = c_1\mu(S_1), \quad H_2(c_1, c_2) = c_2\mu(S_2), \tag{12.1}$$

where $\mu(S) = \int_S f(x, y)dxdy$ denotes the probabilistic measure of the set S. First, we endeavor to evaluate equilibrium prices under the uniform distribution of buyers.

Rotate the circle S such that the points P_1 and P_2 have the same ordinate y (see Figure 1.12). Designate the abscissas of P_1 and P_2 by x_1 and x_2, respectively. Without loss of generality, assume that $x_1 \geq x_2$.

Within the framework of the Hotelling scheme, the sets S_1 and S_2 form sectors of the circle divided by the straight line

$$c_1 + (x - x_1)^2 = c_2 + (x - x_2)^2,$$

which is parallel to the axis O_y with the coordinate

$$x = \frac{1}{2}(x_1 + x_2) + \frac{c_1 - c_2}{2(x_1 - x_2)}. \tag{12.2}$$

According to (12.1), the payoffs of the players in this game constitute

$$H_1(c_1, c_2) = c_1\left(\arccos x - x\sqrt{1 - x^2}\right)/\pi, \tag{12.3}$$

$$H_2(c_1, c_2) = c_2\left(\pi - \arccos x + x\sqrt{1 - x^2}\right)/\pi, \tag{12.4}$$

where x meets (12.2). We find the equilibrium prices via the equation $\frac{\partial H_1}{\partial c_1} = \frac{\partial H_2}{\partial c_2} = 0$.

Evaluate the derivative of (12.3) with respect to c_1:

$$\pi\frac{\partial H_1}{\partial c_1} = \arccos x - x\sqrt{1-x^2} + c_1\left[-\frac{1}{\sqrt{1-x^2}}\frac{1}{2(x_1-x_2)} - \sqrt{1-x^2}\frac{1}{2(x_1-x_2)} + \frac{2x^2}{2\sqrt{1-x^2}}\frac{1}{2(x_1-x_2)}\right].$$

Using the first-order necessary optimality conditions, we get

$$c_1 = (x_1-x_2)\left[\frac{\arccos x}{\sqrt{1-x^2}} - x\right]. \tag{12.5}$$

Similarly, the condition $\frac{\partial H_2}{\partial c_2} = 0$ brings to

$$c_2 = (x_1-x_2)\left[x + \frac{\pi - \arccos x}{\sqrt{1-x^2}}\right]. \tag{12.6}$$

Finally, the expressions (12.2), (12.5), and (12.6) imply that the equilibrium prices can be rewritten as

$$c_1 = \frac{x_1-x_2}{2}\left[\frac{\pi}{\sqrt{1-x^2}} - 2\left(\frac{x_1+x_2}{2} - x\right)\right], \tag{12.7}$$

$$c_2 = \frac{x_1-x_2}{2}\left[\frac{\pi}{\sqrt{1-x^2}} + 2\left(\frac{x_1+x_2}{2} - x\right)\right], \tag{12.8}$$

where

$$x = \frac{x_1+x_2}{4} - \frac{\pi/2 - \arccos x}{2\sqrt{1-x^2}}. \tag{12.9}$$

Remark 1.1 If $x_1 + x_2 = 0$, then $x = 0$ due to (12.2). Hence, $c_1 = c_2 = \pi x_1$ according to (12.5)–(12.6), and $H_1 = H_2 = \pi x_1/2$ according to (12.3)–(12.4). The maximal equilibrium prices are achieved under $x_1 = 1$ and $x_2 = -1$; they make up $c_1 = c_2 = \pi$. The optimal payoffs take the values of $H_1 = H_2 = \pi/2 \approx 1.570$. Thus, if buyers possess the uniform distribution in the circle, the companies should be located as far as possible from each other (in the optimal solution).

To proceed, suppose that buyers are distributed non-uniformly in the circle. Analyze the case when the density function in the polar coordinates acquires the form (see Figure 1.13)

$$f(r,\theta) = 3(1-r)/\pi, \quad 0 \le r \le 1, 0 \le \theta \le 2\pi. \tag{12.10}$$

Obviously, buyers lie closer to the city center.

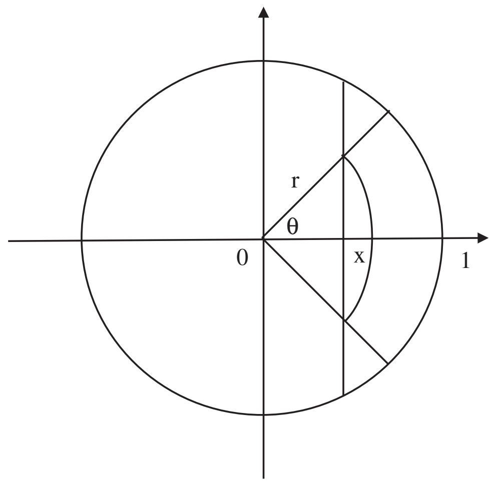

Figure 1.13 Duopoly in the polar coordinates.

Note that it suffices to consider the situation of $x_1 + x_2 \geq 0$ (otherwise, simply reverse the signs of x_1, x_2). The expected incomes of the players (12.1) are given by

$$H_1(c_1, c_2) = \frac{6}{\pi} c_1 A(x), \quad H_2(c_1, c_2) = c_2 \left(1 - \frac{6}{\pi} A(x)\right), \tag{12.11}$$

where

$$A(x) = \int_x^1 r(1-r) \arccos\left(\frac{x}{r}\right) dr = \frac{1}{6}\left[\arccos x - x\sqrt{1-x^2} - 2x \int_x^1 \sqrt{r^2 - x^2} dr\right]$$
$$= \frac{1}{6}\left[\arccos x - 2x\sqrt{1-x^2} - x^3 \log x + x^3 \log\left(1 + \sqrt{1-x^2}\right)\right],$$

such that

$$\frac{\pi}{6}\frac{\partial H_1}{\partial c_1} = A(x) + c_1 A'(x)\frac{\partial x}{\partial c_1} = A(x) - \frac{c_1}{2(x_1 - x_2)} \int_x^1 \sqrt{r^2 - x^2} dr,$$
$$\frac{\partial H_2}{\partial c_2} = 1 - \frac{6}{\pi} A(x) - c_2 \frac{6}{\pi} A'(x)\frac{\partial x}{\partial c_2} = 1 - \frac{6}{\pi} A(x) - \frac{6}{\pi}\frac{c_2}{2(x_1 - x_2)} \int_x^1 \sqrt{r^2 - x^2} dr,$$

since

$$A'(x) = -\int_x^1 \frac{r(1-r)}{\sqrt{r^2 - x^2}} dr = -\int_x^1 \sqrt{r^2 - x^2} dr. \tag{12.12}$$

The conditions $\frac{\partial H_1}{\partial c_1} = \frac{\partial H_2}{\partial c_2} = 0$ yield that

$$c_1 = 2(x_1 - x_2)A(x) / \int_x^1 \sqrt{r^2 - x^2} dr, \tag{12.13}$$

$$c_2 = 2(x_1 - x_2)\left(\frac{\pi}{6} - A(x)\right) / \int_x^1 \sqrt{r^2 - x^2} dr. \tag{12.14}$$

By substituting c_1 and c_2 into

$$x = \frac{1}{2}(x_1 + x_2) + \frac{c_1 - c_2}{2(x_1 - x_2)}, \tag{12.2'}$$

we arrive at

$$x - \frac{1}{2}(x_1 + x_2) = (2A(x) - \pi/6) / \int_x^1 \sqrt{r^2 - x^2} dr. \tag{12.15}$$

Remark 1.2 It follows from (12.12) that $A(x)$ represents a convex decreasing function such that $A(0) = \pi/12$ and $A(1) = 0$. The right-hand side of (12.15) is negative, which leads to

$$x \le (x_1 + x_2)/2.$$

Below we demonstrate that equation (12.15) admits a unique solution. Rewrite it as

$$B(x) = -\left[x - \frac{1}{2}(x_1 + x_2)\right] A'(x) - (2A(x) - \pi/6) = 0. \tag{12.16}$$

The derivative of the function $B(x)$, i.e.,

$$B'(x) = -3A'(x) - A''(x)\left(x - \frac{x_1 + x_2}{2}\right) = \int_x^1 \left[3\sqrt{r^2 - x^2} + \frac{x}{\sqrt{r^2 - x^2}}\left(\frac{x_1 + x_2}{2} - x\right)\right] dr,$$

possesses positive values exclusively. Hence, $B(x)$ increases within the interval $[0, \frac{x_1+x_2}{2}]$ such that $B(0) = -\frac{x_1+x_2}{4} < 0$ and $B(\frac{x_1+x_2}{2}) = \pi/6 - 2A(\frac{x_1+x_2}{2}) \ge 0$.

If $x_1 + x_2 = 0$, then $x = 0$ satisfies equation (12.15). Moreover, the conditions (12.13)–(12.14) lead to $c_1 = c_2 = \frac{2}{3}\pi x_1$, whereas formula (12.11) implies that $H_1 = H_2 = \frac{1}{3}\pi x_1$. Under $x_1 = 1, x_2 = -1$, we have $c_1 = c_2 = \frac{2}{3}\pi \approx 2.094$ and $H_1 = H_2 = \frac{1}{3}\pi \approx 1.047$.

1.13 Location problem in 2D space

In the preceding subsection, readers have observed the following fact. If the location points P_1 and P_2 are fixed, there exist equilibrium prices c_1 and c_2. In other words, c_1 and c_2 make some functions of x_1, x_2. In this context, an interesting question arises immediately. Are there equilibrium points x_1^*, x_2^* of location for these companies? Such problem often appears during infrastructure planning for regional socioeconomic systems. Consider the posed problem in the case of non-uniform distribution of companies.

Suppose that player II chooses some point $x_2 < 0$. Player I aims at finding a certain point x_1 which maximizes his income $H_1(c_1, c_2)$. Let us solve the equation $\frac{\partial H_1}{\partial x_1} = 0$. By virtue of (12.11),

$$\frac{\pi}{6}\frac{\partial H_1}{\partial x_1} = \frac{\partial c_1}{\partial x_1}A(x) + c_1 A'(x)\frac{\partial x}{\partial x_1} = 0. \tag{13.1}$$

Differentiation of (12.13) and (12.16) with respect to x_1 gives

$$\frac{1}{2}\frac{\partial c_1}{\partial x_1} = -\frac{A(x)}{A'(x)} - (x_1 - x_2)\left[1 - \frac{A'(x)A''(x)}{[A'(x)]^2}\right]\frac{\partial x}{\partial x_1} \tag{13.2}$$

and

$$-\left(\frac{\partial x}{\partial x_1} - \frac{1}{2}\right)A'(x) - \left[x - \frac{1}{2}(x_1 + x_2)\right]A''(x)\frac{\partial x}{\partial x_1} - 2A'(x)\frac{\partial x}{\partial x_1} = 0\,.$$

Therefore,

$$\frac{\partial x}{\partial x_1} = A'(x)\left[6A'(x) + 2\left(x - \frac{x_1 + x_2}{2}\right)A''(x)\right]^{-1}. \tag{13.3}$$

Equations (13.1)–(13.3) can serve for obtaining the optimal response x_1 of player I.

Owing to the symmetry of this problem, if an equilibrium exists, it has the form $(x_1, x_2 = -x_1)$. In this case, $x = 0$, $A(0) = \pi/12$, $A'(0) = -1/2$, $A''(0) = 0$. The expression (13.3) brings to

$$\frac{\partial x}{\partial x_1} = (-1/2)/(-3 + 0) = 1/6.$$

On the other hand, formula (13.2) yields

$$\frac{\partial c_1}{\partial x_1} = \frac{\pi}{3} - \frac{2}{3}x_1.$$

Substitute these results into (13.1) to derive

$$\left(\frac{\pi}{3} - \frac{2}{3}x_1\right)\frac{\pi}{12} + \left(\frac{2}{3}\pi x_1\right)\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{6} = 0,$$

and, finally,

$$x_1^* = \frac{\pi}{4}.$$

Thus, the optimal location points of the companies become $x_1^* = \pi/4, x_2^* = -\pi/4$; the corresponding equilibrium prices and incomes constitute $c_1 = c_2 = \pi^2/6$ and $H_1 = H_2 = \pi^2/12$, respectively.

Remark 1.3 Recall the case of the uniform distribution of buyers discussed earlier. Similar argumentation generates the following outcomes.

It appears from (12.3), (12.7), and (12.9) that

$$\pi\frac{\partial H_1}{\partial x_1} = \frac{\partial c_1}{\partial x_1}\left(\arccos x - x\sqrt{1-x^2}\right) - 2c_1\sqrt{1-x^2}\,\frac{\partial x}{\partial x_1},$$

$$\frac{\partial c_1}{\partial x_1} = \frac{\pi}{2\sqrt{1-x^2}} + x - x_1 + \frac{x_1 - x_2}{2}\left(2 + \pi x(1-x^2)^{-3/2}\right)\frac{\partial x}{\partial x_1},$$

$$\frac{\partial x}{\partial x_1} = \frac{1}{4}\left[1 + \frac{1}{2(1-x^2)} + \frac{x}{2(1-x^2)^{3/2}}\left(\frac{\pi}{2} - \arccos x\right)\right]^{-1}.$$

Consequently,

$$\pi\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = \left[\frac{\partial c_1}{\partial x_1}\right]_{x=0}\frac{\pi}{2} - 2[c_1]_{x=0}\left[\frac{\partial x}{\partial x_1}\right]_{x=0}$$

$$= \left(\frac{\pi}{2} - \frac{2x_1}{3}\right)\frac{\pi}{2} - 2\pi x_1\frac{1}{6} = \frac{\pi}{4}\left(\pi - \frac{8}{3}x_1\right) > 0, \quad \forall x_1 \in (0,1).$$

And so, the maximal incomes are attained at the points $x_1^* = -x_2^* = 1$. According to (12.3) and (12.7), these points correspond to

$$c_i^* = \pi \approx 3.1415 \text{ and } H_i^* = \pi/2 \approx 1.5708, \quad i = 1, 2. \tag{13.4}$$

Exercises

1. The crossroad problem.

   Two automobilists move along two mutually perpendicular roads and simultaneously meet at a crossroad. Each of them may stop (strategy I) or continue motion (strategy II).

   By assumption, a player would rather stop than suffer a catastrophe; on the other hand, a player would rather continue motion if the opponent stops. This conflict can be represented by a bimatrix game with the payoff matrix

   $$\begin{pmatrix} (1,1) & (1-\varepsilon, 2) \\ (2, 1-\varepsilon) & (0,0) \end{pmatrix}.$$

   Here $\varepsilon \geq 0$ is a number characterizing player's displeasure of his stop to let the opponent pass.

   Find pure strategy Nash equilibria and mixed strategy Nash equilibria in the crossroad problem.

2. Games 2×2. Evaluate Nash equilibria in the bimatrix games below:

$$A = \begin{pmatrix} -6 & 0 \\ -9 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & -9 \\ 0 & -1 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

3. Find a Nash equilibrium in a bimatrix game defined by

$$A = \begin{pmatrix} 3 & 6 & 8 \\ 4 & 3 & 2 \\ 7 & -5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 4 & 3 \\ 7 & 7 & 3 \\ 4 & 6 & 6 \end{pmatrix}.$$

4. Evaluate a Stackelberg equilibrium in a two-player game with the payoff functions

$$H_1(x_1, x_2) = bx_1(c - x_1 - x_2) - d\,,$$
$$H_2(x_1, x_2) = bx_2(c - x_1 - x_2) - d\,.$$

5. Consider a general bimatrix game and demonstrate that (x, y) is a mixed strategy equilibrium profile iff the following inequalities hold true:

$$(x-1)(ay-\alpha) \geq 0\,, \quad x(ay-\alpha) \geq 0\,,$$
$$(y-1)(bx-\beta) \geq 0\,, \quad y(bx-\beta) \geq 0\,,$$

where

$$a = a_{11} - a_{12} - a_{21} + a_{22}\,, \quad \alpha = a_{22} - a_{12}\,,$$
$$b = b_{11} - b_{12} - b_{21} + b_{22}\,, \quad \beta = a_{22} - a_{21}\,.$$

6. Prove the following result. If a bimatrix game admits a completely mixed Nash equilibrium strategy profile, then $n = m$.

7. Find an equilibrium in a game $2 \times n$ described by the payoff matrices

$$A = \begin{pmatrix} 2 & 0 & 5 \\ 2 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 7 & 8 \end{pmatrix}.$$

8. Find an equilibrium in a game $m \times 2$ described by the payoff matrices

$$A = \begin{pmatrix} 8 & 2 \\ 2 & 7 \\ 3 & 9 \\ 6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 8 & 4 \\ 7 & 2 \\ 2 & 9 \end{pmatrix}.$$

9. Consider the company allocation problem in 2D space. Evaluate equilibrium prices (p_1, p_2) under the cost function $F_2 = p^2 + \rho^2$.

10. Consider the company allocation problem in 2D space. Find the optimal allocation of companies (x_1, x_2) under the cost function $F_2 = p^2 + \rho^2$.