1

Analytical Dynamics of Discrete Systems

The *variational approach* to mechanics is based on the concepts of energy and work and therefore provides a better understanding of mechanical phenomena. In some sense one can say that variational principles consider the system in a global sense, disregarding the specifities of the forces associated to kinematic constraints imposed on the system. It provides at the same time a very powerful tool for two main reasons:

- It considerably simplifies the analytical formulation of the motion equations for a complex mechanical system.
- It gives rise to approximate numerical methods for the solution of both discrete and continuous systems in the most natural manner, as will be later explained in Chapters 5 and 6.

The objective of this chapter is to recall to the reader, how the fundamental Newton's equations for dynamics can be effectively applied to general systems on which kinematic constraints are imposed. First we recall the concept of virtual displacements and the principle of virtual work for a single mass point (Section 1.1), then for a system of particles (Section 1.2), explaining how it leads to a global dynamic description of a system where unknown reaction forces associated to kinematic constraints do not appear. The intimately related concepts of kinematic constraints and generalized coordinates (or degrees of freedom) are also discussed in those sections. Note that in this book we will not discuss the dynamics of rigid bodies. Although a rigid body can be seen as a collection of constrained point masses and therefore a special case of the systems treated here, analyzing the dynamics of a rigid body and building models of multiple rigid components is a topic in itself that will not be handled here since the main scope of the book is vibrational behaviour of flexible systems.[1]

In Section 1.3 the dynamics of systems is described in an even more abstract way, using the energy concepts to show that the virtual work principle can be written as a variational principle, namely Hamilton's principle. It is then shown that the equations of motion can be

[1] For a thorough discussion of analytical dynamics for systems of rigid bodies, the reader is referred to (Lur'é 1968, Meirovitch 1970, Meirovitch 1980, Whittaker 1965, Goldstein 1986, Géradin and Cardona 2001).

Mechanical Vibrations: Theory and Application to Structural Dynamics, Third Edition.
Michel Géradin and Daniel J. Rixen.

derived from the energy description of the system: the Lagrange equations. Those equations are outlined for the general case of nonconservative systems in Section 1.4.[2]

Section 1.5 deals with the generalization of the Lagrange equations to systems undergoing impulsive loading (i.e. shocks, impact). Finally, Section 1.6 provides an introduction to the dynamics of constrained systems. It is shown that the method of Lagrange multipliers provides an efficient way to extend Lagrange's equations to systems described by coordinates for which constraints need to be explicitly accounted for.

Definitions

The list below complements the general definitions given in the book introduction, but remains local to Chapter 1.

F_s	gyroscopic forces.
N	number of particles.
P_s	*s-th* generalized impulse.
R	number of constraints.
$U_{ik}(q_s, t)$	*i-th* displacement component of *k-th* particle in terms of generalized coordinates.
$\mathcal{V}_d$	dislocation potential.
X_i, X_{ik}	*i-th* component of force on (*k-th*) particle.
m, m_k	mass of (*k-th*) particle.
u_i, u_{ik}	*i-th* displacement component of (*k-th*) particle.
x_{ik}	reference configuration of *k-th* particle.
λ_r	Lagrange multiplier.
ξ_{ik}	*i-th* instantaneous configuration of *k-th* particle.

1.1 Principle of virtual work for a particle

1.1.1 Nonconstrained particle

Let us consider a particle of mass m, submitted to a force field $\boldsymbol{X}$ of components X_i. The dynamic equilibrium of the particle can be expressed in d'Alembert's form:

$$m\ddot{u}_i - X_i = 0 \qquad i = 1, 2, 3 \tag{1.1}$$

where u_i represents the displacement of the particle.

Let us next imagine that the particle follows during the time interval $[t_1, t_2]$ a motion trajectory u_i^* distinct from the real one u_i. This allows us to define the virtual displacement of the

[2] The point of view adopted in this chapter is far from being the only possible one. It would also be possible to apply Kane's method (Kane and Levinson 1980) which implements the concept of generalized speeds (quasi-velocity coordinates) as a way to represent motion, similar to what the concept of generalized coordinates does for the configuration. Kane's implementation focuses on the motion aspects of dynamic systems rather than only on the system configuration. It therefore provides a suitable framework for treating nonholonomic constraints of differential type (see Section 1.2.2).

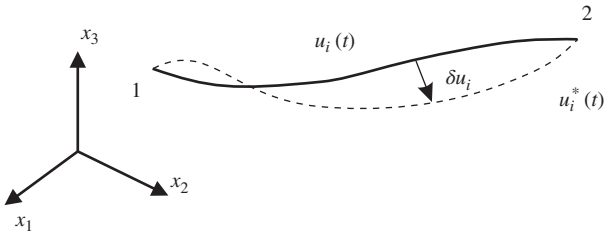


Figure 1.1 Virtual displacement of a particle.

particle by the relationship (Figure 1.1):

$$\delta u_i = u_i^* - u_i \tag{1.2}$$

By its very definition, the virtual displacement δu_i is arbitrary for $t_1 < t < t_2$. However, let us suppose that the varied trajectory and the real one both pass through the same points at the ends of the time interval. The end conditions then take the form:

$$\delta u_i(t_1) = \delta u_i(t_2) = 0 \tag{1.3}$$

An immediate consequence of definition (1.2) is that the variation operator δ commutes with the time-derivative operator d/dt since

$$\frac{d}{dt}(\delta u_i) = \frac{d}{dt}(u_i^* - u_i) = \dot{u}_i^* - \dot{u}_i = \delta \dot{u}_i \tag{1.4}$$

Let us next multiply the dynamic equilibrium equations (1.1) by the associated virtual displacement and sum over the components. The virtual work expression results:

$$\sum_{i=1}^{3} (m\ddot{u}_i - X_i)\delta u_i = 0 \tag{1.5}$$

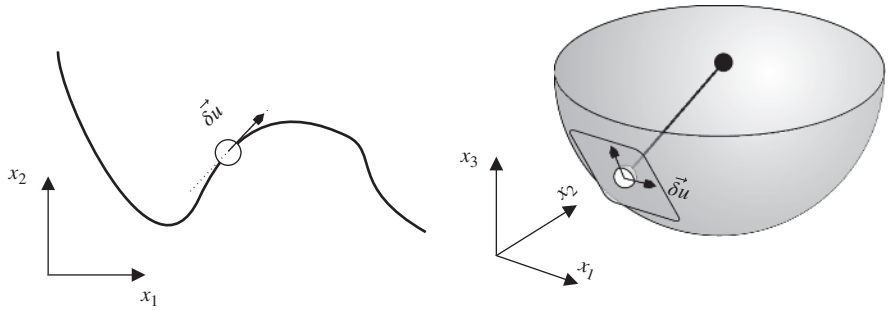
which shows that

> *The virtual work produced by the forces acting on the particle during a virtual displacement δu_i is equal to zero.*

1.1.2 Constrained particle

Equation (1.5) represents the scalar product between the forces acting on the particle and the virtual displacement $\overrightarrow{\delta u}$. It thus represents the *projection of the equilibrium along the direction* $\overrightarrow{\delta u}$. If (1.5) is satisfied for all variations δu_i, then the trajectory $u_i(t)$ satisfies the dynamic equilibrium in all directions.

If no kinematical constraint is imposed onto the particle, namely if no restriction is imposed on its displacement, the trajectory of the material point is determined by the equilibrium in all directions. But when kinematic constraints are specified for the particle, there exist reaction forces in addition to the applied forces. These reaction forces are inherent to the constraining



a. A particle on a plane curve b. The spherical pendulum

Figure 1.2 Kinematically admissible virtual displacements.

mechanism and ensure that the imposed kinematical constraints are satisfied. Those reaction forces are not known in advance since they depend on the motion itself.

Figure 1.2 describes a particle constrained to move along a curve and a spherical pendulum where a particle is constrained to have a constant distance with respect to a fixed point. On one hand, the presence of reaction forces acting in the direction of the constraint generally renders the equilibrium description more complex since those unknown forces must be determined along the entire trajectory such that kinematical constraints are satisfied. On the other hand, solving the equilibrium equations in the direction constrained by the kinematical conditions is not useful since, in that direction, the trajectory is prescribed by the constraint and thus known.

In the system described in Figure 1.2.a only the motion along the direction tangent to the curve needs to be determined. In doing so, the reaction forces, which act in the direction normal to the curve, do not participate to the motion and thus need not be determined: the position of the particle in the direction normal to the curve is obviously imposed by the constraint and does not require solving the equilibrium equation in that direction. In the same way, if the equilibrium of the particle of the spherical pendulum is expressed in the plane tangent to the sphere (Figure 1.2.b), only the forces actually applied participate in the determination of the trajectory.

Let us therefore decide that, in the presence of kinematical constraints, we consider only virtual displacements δu_i *compatible with the constraints* or, in other words, *kinematically admissible*. Equation (1.5) then describes the projection of the dynamic equilibrium in the space compatible with the constraints, namely in directions orthogonal to unknown reaction forces. The form (1.5) thus involves only effectively applied forces and stipulates that

> *The virtual work produced by the* ***effective*** *forces acting on the particle during a virtual displacement* δu_i ***compatible with the constraints*** *is equal to zero.*

This is the virtual *work principle* for a constrained particle. Conversely, Equation (1.5) indicates that

> *If the trajectory* u_i *of the particle is such that the* ***effectively applied*** *forces produce no virtual work for any virtual displacement* ***compatible with the constraints****, the equilibrium is then satisfied.*

The usefulness of the virtual work principle comes from the fact that it allows us to express the dynamic equilibrium in a simple manner along the directions compatible with the constraints. This will be explained next in more detail for a system of several particles.

1.2 Extension to a system of particles

1.2.1 Virtual work principle for N particles

Every particle k of a system of N particles with mass m_k satisfies the dynamic equilibrium:

$$m_k \ddot{u}_{ik} - X_{ik} = 0 \qquad \begin{matrix} i = 1, 2, 3 \\ k = 1, \ldots, N \end{matrix} \tag{1.6}$$

where X_{ik} are the force components representing the known external forces and where R_{ik} are the unknown reactions resulting from the kinematic constraints imposed on the system.

For every particle k, one considers virtual displacements δu_{ik} such as:

$$\delta u_{ik} = {u_{ik}}^* - u_{ik} \qquad i = 1, 2, 3 \tag{1.7}$$

$$\delta u_{ik}(t_1) = \delta u_{ik}(t_2) = 0 \qquad k = 1, \ldots, N \tag{1.8}$$

The virtual work principle is obtained by projecting the dynamic equilibrium equations on the virtual displacements and by summing up over the particles:

$$\sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{ik} - X_{ik} - R_{ik}) \delta u_{ik} = 0 \tag{1.9}$$

As for the case of one particle, we decide to consider only virtual displacement *compatible with the constraints*. Hence, the virtual displacements must satisfy constraints imposed on one particle as well as constraints imposed between particles. The situation where two points are rigidly linked to one another is an important example of such constraints. In that case, the reaction forces linking the particles are equal and opposite (Figure 1.3):

$$\vec{R}_1 + \vec{R}_2 = 0$$

and the virtual work associated to a virtual displacement $(\vec{\delta u}_1, \vec{\delta u}_2)$ is:

$$\delta\tau = \sum_{i=1}^{3} (R_{i1} \delta u_{i1} + R_{i2} \delta u_{i2}) = \vec{R}_1 \cdot \vec{\delta u}_1 + \vec{R}_2 \cdot \vec{\delta u}_2 = \vec{R}_1 \cdot (\vec{\delta u}_1 - \vec{\delta u}_2) = 0$$

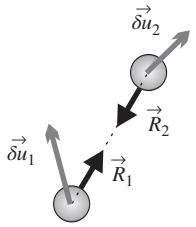


Figure 1.3 Virtual work of constraints between particles.

since the compatible virtual displacements must be equal in the direction of the rigid link. Projecting the equations onto the kinematically admissible displacements thus consists in summing up the equilibrium equations in the constrained direction such that the unknown linking forces vanish.

Since the reaction forces do vanish when projecting the equations of motion onto kinematically admissible displacement directions, the virtual work principle (1.9) is written as

$$\boxed{\sum_{k=1}^{N}\sum_{i=1}^{3}(m_k\ddot{u}_{ik} - X_{ik})\delta u_{ik} = 0} \tag{1.10}$$

So as for a single particle (Section 1.1.2), it can be stated that

> *The virtual work of the forces effectively applied onto a system of particles is zero with respect to any kinematically compatible virtual displacement* ***if and only if*** *the system is in dynamic equilibrium.*

Again, the principle of virtual work corresponds to the projection of the equilibrium equations in the directions compatible with the kinematical constraints. The resulting equations are then easier to solve since the constraining forces are no longer unknowns for the problem.

1.2.2 The kinematic constraints

Without kinematic constraints, the state of the system would be completely defined by the $3N$ displacement components u_{ik} since, starting from a reference configuration x_{ik}, they represent the instantaneous configuration:

$$\xi_{ik}(t) = x_{ik} + u_{ik}(x_{jk}, t) \qquad \begin{matrix} i,j = 1,2,3 \\ k = 1, \ldots, N \end{matrix} \tag{1.11}$$

The system is then said to possess $3N$ degrees of freedom.

In most mechanical systems, however, the particles are submitted to kinematic constraints, which restrain their motion and define dependency relationships between particles.

Holonomic constraints

The *holonomic* constraints are defined by implicit relationships of type:

$$f(\xi_{ik}, t) = 0 \tag{1.12}$$

If there is no explicit dependence with respect to time, the constraints are said to be *scleronomic*. They are *rheonomic* otherwise.

> *A holonomic constraint reduces by one the number of degrees of freedom of the system.*

Example 1.1

Let us consider the case of two mass particles connected by a rigid link of length ℓ. Their instantaneous positions ξ_{i1} and ξ_{i2} verify the relationship:

$$f(\xi_{ik}, t) = \sum_{i=1}^{3} (\xi_{i2} - \xi_{i1})^2 - \ell^2 = 0$$

The variation of the constraint with respect to virtual displacements can be expressed by:

$$\delta f = \sum_{i=1}^{3} ((\xi_{i1} - \xi_{i2})\delta u_{i1} - (\xi_{i1} - \xi_{i2})\delta u_{i2}) = 0$$

indicating that virtual displacements must satisfy (see also Figure 1.3):

$$(\vec{\xi}_1 - \vec{\xi}_2) \cdot \vec{\delta u}_1 = (\vec{\xi}_1 - \vec{\xi}_2) \cdot \vec{\delta u}_2$$

Hence virtual displacements must be equal in the direction of the link, and a system of two particles that are rigidly linked has $6 - 1 = 5$ degrees of freedom.

Nonholonomic constraints

A constraint is said *nonholonomic* if it cannot be put in the form (1.12). In particular, non-holonomic constraints often take the form of differential relationships:

$$f(\dot{\xi}_{ik}, \xi_{ik}, t) = 0 \tag{1.1}$$

Such relationships are generally not integrable, and therefore *they do not allow reduction of the number of degrees of freedom of the system*.

Example 1.2

Let us consider the case of the centre of a rigid wheel with radius r constrained to roll without sliding on a plane (Figure1.4).

It is also assumed that the rotation axis remains parallel to the plane. The second material point considered in the description is a reference point located on the circumference of the

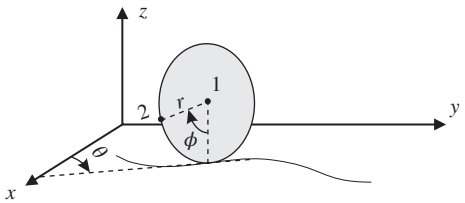


Figure 1.4 Nonholonomic constraint.

wheel. The rolling-without-sliding condition can be expressed by the constraints:

$$\dot{x}_1 + r\dot{\phi}\cos\theta = 0 \tag{E1.2.a}$$

$$\dot{y}_1 - r\dot{\phi}\sin\theta = 0 \tag{E1.2.b}$$

where the angles ϕ and θ are such as:

$$x_2 - x_1 = r\sin\phi\cos\theta \tag{E1.2.c}$$

$$y_2 - y_1 = -r\sin\phi\sin\theta \tag{E1.2.d}$$

The four relationships above yield two implicit nonholonomic constraints between x_i and y_i. Finally, the wheel kinematics imposes the constraint:

$$z_2 - z_1 = -r\cos\phi \tag{E1.2.e}$$

$$z_1 = r. \tag{E1.2.f}$$

The system is thus described in terms of the eight variables:

$$x_1, y_1, z_1, x_2, y_2, z_2, \phi, \theta$$

and is submitted to:

- *the two nonholonomic constraints* (E1.2.a) *and* (E1.2.b)
- *the four holonomic constraints* (E1.2.c), (E1.2.d), (E1.2.e) *and* (E1.2.f).

The wheel degrees of freedom are restrained only by the holonomic constraints and therefore four independent variables are left: two translations in the rolling plane and two rotations. The non-holonomic constraints act as behaviour constraints: they do not restrict the possible configurations of the system but simply the way to reach them.

If we introduce the additional holonomic constraint that the orientation θ is fixed to a constant θ_{fixed}, namely $\theta = \theta_{fixed}$ one observes that the nonholonomic constraints (E1.2.a) *and* (E1.2.b) *now become integrable and one can write:*

$$x_1 + r\phi\cos\theta_{fixed} = x_1(\phi = 0)$$

$$y_1 - r\phi\sin\theta_{fixed} = y_1(\phi = 0)$$

In this case all constraints are holonomic and the system has only one degree of freedom: it can for instance be seen that ϕ describes the configuration in a unique way.

1.2.3 Concept of generalized displacements

If R holonomic kinematic constraints exist between the $3N$ displacement components of the system, the number of degrees of freedom is then reduced to $3N - R$. It is then necessary to define $n = 3N - R$ configuration parameters, or *generalized coordinates*, denoted $(q_1, \ldots, q_n)$ in terms of which the displacements of the system particles are expressed in the form:

$$u_{ik}(x_{jk}, t) = U_{ik}(q_1, \ldots, q_n, t) \tag{1.14}$$

When only holonomic constraints are applied to the system, the generalized coordinates q_s remain independent and may be varied in an arbitrary manner without violating the kinematic constraints. The virtual displacements δu_{ik} compatible with the holonomic constraints may be expressed in the form:

$$\delta u_{ik} = \sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial q_s} \delta q_s \tag{1.15}$$

The virtual work equation becomes:

$$\sum_{s=1}^{n} \left[\sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{ik} - X_{ik}) \frac{\partial U_{ik}}{\partial q_s} \right] \delta q_s = 0 \tag{1.16}$$

The coefficients $\frac{\partial U_{ik}}{\partial q_s}$ define the displacement directions of mass k when the generalized coordinate q_s is varied. The variations δq_s are totally independent by definition, meaning that they can be chosen arbitrarily without violating any kinematic constraint. The identity (1.16) being satisfied for any virtual displacement, it follows that each associated term in the virtual work (1.16) principle must be zero. These terms correspond to the equilibrium projected onto the direction of the generalized coordinate q_s and written as:

$$\boxed{\sum_{k=1}^{N} \sum_{i=1}^{3} \left(m_k \frac{d^2 U_{ik}(q_1, \ldots, q_n, t)}{dt^2} - X_{ik} \right) \frac{\partial U_{ik}}{\partial q_s} = 0 \qquad s = 1, \cdots n} \tag{1.17}$$

The second term in this equation corresponds to the *generalized force* conjugate to the degree of freedom q_s:

$$Q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} X_{ik} \frac{\partial U_{ik}}{\partial q_s} \tag{1.18}$$

The first term in (1.17) has the meaning of a generalized inertia force; its structure is obtained in the next section in terms of generalized coordinates.

Example 1.3

Figure 1.5 depicts a simple two-dimensional pendulum. The system has $2 - 1 = 1$ *degree of freedom and we choose* θ *as generalized coordinate so that Equation (1.14) for the*

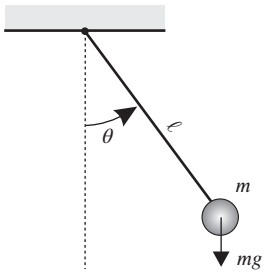


Figure 1.5 The simple pendulum.

pendulum is:

$$u_1 = \ell \cos\theta - \ell$$
$$u_2 = \ell \sin\theta$$

and the compatible virtual displacements are:

$$\delta u_1 = (-\ell \sin\theta)\, \delta\theta$$
$$\delta u_2 = (\ell \cos\theta)\, \delta\theta$$

which defines the direction orthogonal to the rigid link. The virtual work equation is then written as:

$$(m\ddot{u}_1 - mg)(-\ell \sin\theta) + (m\ddot{u}_2)(\ell \cos\theta) = 0 \qquad \text{(E1.3.a)}$$

The accelerations can be expressed as:

$$\ddot{u}_1 = -\ell\ddot{\theta} \sin\theta - \ell\dot{\theta}^2 \cos\theta$$
$$\ddot{u}_2 = \ell\ddot{\theta} \cos\theta - \ell\dot{\theta}^2 \sin\theta$$

and replacing in (E1.3.a)*, one finds the equation of motion:*

$$m\ell^2\ddot{\theta} + mg\ell \sin\theta = 0 \qquad \text{(E1.3.b)}$$

Example 1.4

Let us consider the double pendulum of Figure 1.6. The system is made of two mass particles. The motion is restricted to 2-D motion, so that its kinematics is described by the four instantaneous position components ξ_{ik}. The two holonomic constraints applied to the system express the length invariance of the members:

$$\xi_{11}^2 + \xi_{21}^2 = \ell_1^2$$
$$(\xi_{12} - \xi_{11})^2 + (\xi_{22} - \xi_{21})^2 = \ell_2^2$$

The system kinematics may thus be described in terms of $4 - 2 = 2$ generalized coordinates. As a straightforward choice one may adopt the two rotation angles of the pendulum θ_1 and θ_2

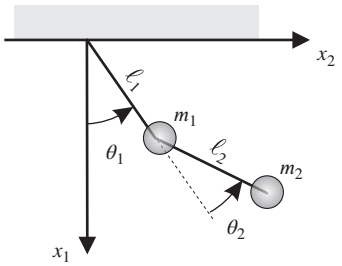


Figure 1.6 The double pendulum.

(the second angle being measured relatively to the first):

$$\xi_{11} = \ell_1 \cos\theta_1 \tag{E1.4.a}$$

$$\xi_{21} = \ell_1 \sin\theta_1 \tag{E1.4.b}$$

$$\xi_{12} = \xi_{11} + \ell_2 \cos(\theta_1 + \theta_2) = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) \tag{E1.4.c}$$

$$\xi_{22} = \xi_{21} + \ell_2 \sin(\theta_1 + \theta_2) = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) \tag{E1.4.d}$$

The equations of motion can then be written using the virtual work principle. This will not be done here (see Section 1.7.1).

1.3 Hamilton's principle for conservative systems and Lagrange equations

Hamilton's principle (Hamilton 1834) is no more than a time-integrated form of the virtual work principle obtained by transforming the expression:

$$\int_{t_1}^{t_2} \left[\sum_{k=1}^{N} \sum_{i=1}^{3} (-m_k \ddot{u}_{ik} + X_{ik}) \delta u_{ik} \right] dt = 0 \tag{1.19}$$

where δu_{ik} are arbitrary but compatible virtual displacements which verify the end conditions (1.8).

First, let us assume that the applied forces X_{ik} can be derived from a potential $\mathcal{V}$. By definition the potential is such that:

$$X_{ik} = -\frac{\partial \mathcal{V}}{\partial u_{ik}} \tag{1.20}$$

So the virtual work of the forces can be expressed in the form:

$$\begin{aligned} \sum_{k=1}^{N} \sum_{i=1}^{3} X_{ik} \delta u_{ik} &= -\sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial \mathcal{V}}{\partial u_{ik}} \sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial q_s} \delta q_s \\ &= -\sum_{s=1}^{n} \frac{\partial \mathcal{V}}{\partial q_s} \delta q_s = \sum_{s=1}^{n} Q_s \delta q_s = -\delta \mathcal{V} \end{aligned}$$

It is thus seen that the generalized forces are derived from the potential $\mathcal{V}$ by the relationship:

$$Q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} X_{ik} \frac{\partial U_{ik}}{\partial q_s} = -\frac{\partial \mathcal{V}}{\partial q_s} \tag{1.21}$$

Next the term associated with inertia forces is transformed by noting that:

$$\begin{aligned} \frac{d}{dt}(m_k \dot{u}_{ik} \delta u_{ik}) &= m_k \ddot{u}_{ik} \delta u_{ik} + m_k \dot{u}_{ik} \delta \dot{u}_{ik} \\ &= m_k \ddot{u}_{ik} \delta u_{ik} + \delta \left(\frac{1}{2} m_k \dot{u}_{ik} \dot{u}_{ik} \right) \end{aligned}$$

Owing to the definition of the kinetic energy $\mathcal{T}$ of the system:

$$\mathcal{T} = \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \dot{u}_{ik}\dot{u}_{ik} \tag{1.22}$$

(1.19) may be rewritten in the form:

$$\left[-\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \dot{u}_{ik}\delta u_{ik}\right]_{t_1}^{t_2} + \delta\int_{t_1}^{t_2}(\mathcal{T}-\mathcal{V})dt = 0 \tag{1.23}$$

in which the time boundary term can be eliminated by taking account of the end conditions (1.8).

The functional (1.23) can be expressed in terms of the generalized coordinates q_s by noticing that:

$$\dot{u}_{ik} = \frac{\partial U_{ik}}{\partial t} + \sum_{s=1}^{n}\frac{\partial U_{ik}}{\partial q_s}\dot{q}_s \tag{1.24}$$

and therefore, that $\mathcal{T}$ and $\mathcal{V}$ respectively take the forms:

$$\mathcal{T} = \mathcal{T}(q,\dot{q},t) \qquad\qquad \mathcal{V} = \mathcal{V}(q,t) \tag{1.25}$$

By making use of Equations (1.8) and (1.15), the boundary conditions may also be written:

$$\delta q_s(t_1) = \delta q_s(t_2) = 0 \tag{1.26}$$

Hamilton's principle for a conservative system may thus be stated in the following form:

> *The real trajectory of the system is such that the integral*
>
> $$\int_{t_1}^{t_2}(\mathcal{T}-\mathcal{V})\,dt$$
>
> *remains stationary with respect to any compatible virtual displacement, arbitrary between both instants t_1 and t_2 but vanishing at the ends of the interval.*

$$\boxed{\begin{array}{c}\delta\displaystyle\int_{t_1}^{t_2}(\mathcal{T}-\mathcal{V})dt = 0\\ \delta q(t_1) = \delta q(t_2) = 0\end{array}} \tag{1.27}$$

Starting from expression (1.27) of Hamilton's principle, the system equations of motion are easily obtained in terms of generalized coordinates: owing to (1.25) one may write:

$$\delta\mathcal{T} = \sum_{s=1}^{n}\left(\frac{\partial\mathcal{T}}{\partial q_s}\delta q_s + \frac{\partial\mathcal{T}}{\partial\dot{q}_s}\delta\dot{q}_s\right)$$

giving the more explicit form of (1.27):

$$\int_{t_1}^{t_2} \sum_{s=1}^{n} \left[\left(\frac{\partial \mathcal{T}}{\partial q_s} + Q_s \right) \delta q_s + \frac{\partial \mathcal{T}}{\partial \dot{q}_s} \delta \dot{q}_s \right] dt = 0$$

in which the second term can be integrated by parts:

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{T}}{\partial \dot{q}_s} \quad \delta \dot{q}_s \, dt = \left[\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \delta q_s \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) \delta q_s \, dt$$

Taking into account the boundary conditions, the following expression equivalent to Hamilton's principle results:

$$\int_{t_1}^{t_2} \sum_{s=1}^{n} \left[-\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s \right] \delta q_s \, dt = 0 \tag{1.28}$$

The variation δq_s being arbitrary on the whole time interval, the motion equations result in the form obtained by *Lagrange* (Lagrange 1788):

$$\boxed{-\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s = 0 \qquad s = 1, \ldots, n} \tag{1.29}$$

The Lagrange equations (1.29) are a set of n equations for the n unknown degrees of freedom q_s and are equivalent to the equations obtained from the virtual work principle (1.17). The advantage of the Lagrange form is however that, once the kinetic energy $\mathcal{T}$ has been expressed in terms of the degrees of freedom q_s, the inertia terms are directly obtained in terms of q_s and their derivatives, whereas the virtual work Equations (1.17) contain $\ddot{u}_{ik}$ that still needs to be expressed in terms q_s (see for instance the last step in Example 1.3).

The first two terms in (1.29) represent the generalized inertia forces associated with the generalized coordinates q_s. Their structure will be detailed in the next paragraph after taking account of the kinematic constraints.

Although here the discussion was made assuming that the effective applied forces derive from a potential, the Lagrange equations (1.29) are also valid if the applied forces do not derive from a potential. For forces deriving from a potential the generalized forces Q_s can be computed by (1.21), otherwise one has to use the fundamental definition (1.18). Their classification will be given in a later section.

Example 1.5

Consider the simple two-dimensional pendulum described in Figure 1.7.a. As found before (see Example 1.3) we can choose the angle θ as degree of freedom so that:

$$\begin{aligned} u_1 &= \ell \cos\theta - \ell \\ u_2 &= \ell \sin\theta \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \dot{u}_1 &= (-\ell \sin\theta)\, \dot{\theta} \\ \dot{u}_2 &= (\ell \cos\theta)\, \dot{\theta} \end{aligned}$$

The kinetic and potential energy of the system are then written for θ:

$$\mathcal{T} = \frac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2) = \frac{1}{2}m\ell^2\dot{\theta}^2$$
$$\mathcal{V} = -mgu_1 = mg\ell(1 - \cos\theta)$$

Observe that, by definition, the potential energy is such that $-\partial V/\partial u_1$ yields the applied force in direction of u_1. One computes:

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\theta}} = \frac{d}{dt}(m\ell^2\dot{\theta}) = m\ell^2\ddot{\theta}$$
$$\frac{\partial \mathcal{T}}{\partial \theta} = 0$$
$$Q = -\frac{\partial \mathcal{V}}{\partial \theta} = -mg\ell\sin\theta$$

The Lagrange equation thus yields the expected pendulum equation:

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$$

Example 1.6

Consider now the pendulum in Figure 1.7.b, where a constant acceleration is given to the attachment point so that $u_{2,att} = \frac{a}{2}t^2$, where a is a given constant. With this rheonomic/holonomic constraint the system has one degree of freedom and we again choose θ to describe the system so that:

$$\begin{aligned} u_1 &= \ell\cos\theta - \ell \\ u_2 &= \ell\sin\theta + \tfrac{a}{2}t^2 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \dot{u}_1 &= (-\ell\sin\theta)\,\dot{\theta} \\ \dot{u}_2 &= (\ell\cos\theta)\,\dot{\theta} + at \end{aligned}$$

and the energies are computed as:

$$\mathcal{T} = \frac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2) = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m(at)^2 + mat(\ell\cos\theta)\,\dot{\theta}$$
$$\mathcal{V} = -mgu_1 = mg\ell(1 - \cos\theta)$$

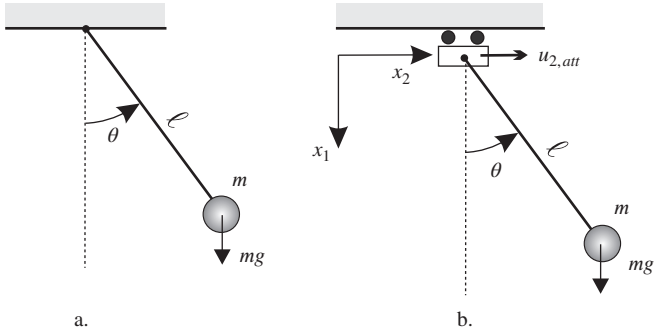


Figure 1.7 The simple pendulum with scleronomic (a) and with rheonomic constraint (b).

The terms in the Lagrange equations are obtained as:

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\theta}} = \frac{d}{dt}(m\ell^2\dot{\theta} + at\ell\cos\theta) = m\ell^2\ddot{\theta} + ma\ell\cos\theta - mat\ell\dot{\theta}\sin\theta$$

$$\frac{\partial \mathcal{T}}{\partial \theta} = -mat(\ell\sin\theta)\,\dot{\theta}$$

$$Q = -\frac{\partial \mathcal{V}}{\partial \theta} = -mg\ell\sin\theta$$

The Lagrange equation yields:

$$m\ell^2\ddot{\theta} + ma\ell\cos\theta + mg\ell\sin\theta = 0$$

Note that this equation can also be written as:

$$m\ell^2\ddot{\theta} + m\ell\sqrt{g^2+a^2}\sin(\theta+\alpha) = 0 \qquad \textit{where } \alpha = \arctan\frac{a}{g}$$

indicating that the problem can be considered as a pendulum in a combined acceleration field of the gravity and the imposed acceleration on the support. It also shows that motion of the support at constant velocity does not affect the system behaviour.

1.3.1 Structure of kinetic energy and classification of inertia forces

Let us substitute in the general kinetic energy expression (1.22) the velocities expressed in terms of generalized coordinates (1.24). The kinetic energy is then split naturally in three contributions:

$$\mathcal{T}(q,\dot{q},t) = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 \tag{1.30}$$

where $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are respectively homogeneous forms of degree 0, 1 or 2 in the generalized velocities $\dot{q}_s$.

– The first term:

$$\mathcal{T}_0 = \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{3} m_k\left(\frac{\partial U_{ik}}{\partial t}\right)^2 = \mathcal{T}_0(q,t) \tag{1.31}$$

is obviously the *transport kinetic energy* of the system since it corresponds to the situation where the degrees of freedom $q_1, \ldots, q_n$ are frozen.

– The second term:

$$\mathcal{T}_1 = \sum_{s=1}^{n}\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial U_{ik}}{\partial t} m_k \frac{\partial U_{ik}}{\partial q_s}\dot{q}_s \tag{1.32}$$

is the *mutual kinetic energy*.

– The third term:

$$\mathcal{T}_2 = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \frac{\partial U_{ik}}{\partial q_s}\frac{\partial U_{ik}}{\partial q_r}\dot{q}_s\dot{q}_r \tag{1.33}$$

is the *relative kinetic energy* since it corresponds to what is left when the explicit dependence of velocities $\dot{u}_{ik}$ with respect to time is suppressed.

Let us note that frequent use will be made of the following expressions for $\mathcal{T}_1$ and $\mathcal{T}_2$:

$$\mathcal{T}_1 = \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}_1}{\partial \dot{q}_s} \qquad \mathcal{T}_2 = \frac{1}{2} \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}_2}{\partial \dot{q}_s} \tag{1.34}$$

which result immediately from Euler's theorem on homogeneous functions.

Euler's theorem on homogeneous functions

If $f(x_1, \ldots, x_n)$ is homogeneous of degree m in the variables $(x_1, \ldots, x_n)$ the following equality is satisfied:

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf \tag{1.35}$$

The proof holds by observing that if f is homogeneous of degree m, it may be written in the form:

$$f(x_1, \ldots, x_n) = x_1^m \, g\left(1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$$

Noting $y_i = x_i/x_1$, one obtains:

$$x_1 \frac{\partial f}{\partial x_1} = m x_1^m g - x_1^{m+1} \sum_{i=2}^{n} \frac{x_i}{x_1^2} \frac{\partial g}{\partial y_i}$$

$$x_i \frac{\partial f}{\partial x_i} = x_i x_1^{m-1} \frac{\partial g}{\partial y_i} \qquad i \neq 1$$

The proof of Euler's theorem holds by summing up all the above relations.

Let us make use of the decomposition (1.30) to interpret the term describing the generalized inertia forces in Lagrange's equations (1.29):

$$\begin{aligned} -\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} &= -\frac{d}{dt}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s} + \frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) + \frac{\left(\partial \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2\right)}{\partial q_s} \\ &= -\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right) - \sum_{r=1}^{n}\left[\frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r}\dot{q}_r\right] - \frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) \\ &\quad + \frac{\partial(\mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2)}{\partial q_s} \end{aligned} \tag{1.36}$$

– The *transport inertia forces* are those obtained by setting $\dot{q}_s = 0$. One obtains:

$$-\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}_0}{\partial q_s} \tag{1.37}$$

– The *relative inertia forces* are those obtained by assuming that the constraints do not depend explicitly on time ($\partial U_{ik}/\partial t = 0$). The remaining terms are the $\mathcal{T}_2$ terms:

$$-\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}_2}{\partial q_s} \tag{1.38}$$

– The *complementary inertia forces* contain the missing terms:

$$F_s = -\sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial \mathcal{T}_1}{\partial q_s}$$

Making use of (1.34) they can be put in the equivalent form:

$$F_s = \sum_{r=1}^{n} \dot{q}_r \left[\frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 \mathcal{T}_1}{\partial q_r \partial \dot{q}_s} \right] = \sum_{r=1}^{n} \dot{q}_r g_{rs} \tag{1.39}$$

where the coefficients:

$$g_{rs} = \frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 \mathcal{T}_1}{\partial q_r \partial \dot{q}_s} = -g_{sr} \tag{1.40}$$

do not depend on the velocities $\dot{q}_s$, but only on the generalized displacements and time. The complementary inertia forces have the nature of *Coriolis* or *gyroscopic forces*. As a consequence of the skew symmetry of the coefficients (1.40), the associated instantaneous power is equal to zero:

$$\sum_{s=1}^{n} F_s \dot{q}_s = 0 \tag{1.41}$$

1.3.2 Energy conservation in a system with scleronomic constraints

When the kinematic constraints are independent of time, the kinetic energy (1.30) reduces to the sole term $\mathcal{T}_2$ and therefore becomes a homogeneous quadratic form of the generalized velocities. Owing to (1.34) one may write:

$$2\mathcal{T} = \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}}{\partial \dot{q}_s}$$

or, after differentiation:

$$2\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} \ddot{q}_s \frac{\partial \mathcal{T}}{\partial \dot{q}_s} + \sum_{s=1}^{n} \dot{q}_s \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) \tag{1.42}$$

On the other hand, since $\mathcal{T} = \mathcal{T}(q, \dot{q})$, one may also write

$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} \ddot{q}_s \frac{\partial \mathcal{T}}{\partial \dot{q}_s} + \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}}{\partial q_s} \tag{1.43}$$

Therefore, by subtracting (1.43) from (1.42) and making use of Lagrange equations (1.29),

$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} \dot{q}_s \left[\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) - \frac{\partial \mathcal{T}}{\partial q_s} \right] = \sum_{s=1}^{n} \dot{q}_s Q_s \tag{1.44}$$

Since in the conservative case the forces Q_s depend also on a potential,

$$\sum_{s=1}^{n} \dot{q}_s Q_s = \sum_{s=1}^{n} \dot{q}_s \left(-\frac{\partial \mathcal{V}}{\partial q_s} \right) = -\frac{d\mathcal{V}}{dt}$$

one obtains

$$\boxed{\frac{d}{dt}(\mathcal{T}+\mathcal{V})=0} \tag{1.45}$$

The corresponding energy integral:

$$\mathcal{T}+\mathcal{V}=\mathcal{E} \tag{1.46}$$

plays a fundamental role in the theory governing the linear oscillations of a stable system about its equilibrium position.

Example 1.7

The vibrational behaviour of a pendulum (Figure 1.5) can easily be described in terms of total energy conservation.

The kinetic and potential energies corresponding to an angular displacement θ are:

$$\mathcal{T}=\frac{1}{2}m\ell^2\dot{\theta}^2 \qquad \mathcal{V}=mg\ell(1-\cos\theta) \tag{E1.7.a}$$

The total energy conservation law yields:

$$\mathcal{E}=\mathcal{T}+\mathcal{V} \tag{E1.7.b}$$

where $\mathcal{E}$ is the energy input into the system.

Figure 1.8.a shows the potential energy of the system as a function of the angular displacement. The horizontal lines represent various initial energy levels $\mathcal{E}_i$ and we call $\mathcal{E}_3$ the maximum possible potential energy, namely $\mathcal{V}=2mg\ell$.

If the total energy level of the system is lower than $\mathcal{E}_3$, the system can reach a state where $\mathcal{E}_3=\mathcal{V}$ and where the kinetic energy is null. In that configuration the system has zero velocity and has reached its maximum angular displacement θ_{max}, solution of:

$$\mathcal{E}=mg\ell(1-\cos\theta_{max}) \tag{E1.7.c}$$

The kinetic energy is obtained by subtracting from $\mathcal{E}$ the potential energy:

$$\mathcal{T}=\mathcal{E}-\mathcal{V}$$

and in the case $\mathcal{E}<\mathcal{E}_3$ we can use (E1.7.c) *together with* (E1.7.a) *to obtain:*

$$\dot{\theta}=\pm\sqrt{\frac{2g}{\ell}(\cos\theta-\cos\theta_{max})} \tag{E1.7.d}$$

In a phase-space representation $(\theta,\dot{\theta})$ this relation yields closed trajectories as represented by the dotted curves in Figure 1.8.b.

If the total energy level is $\mathcal{T}>\mathcal{E}_3$, the system can reach its maximum potential position with a non-zero velocity since $\mathcal{E}-\mathcal{V}=\mathcal{T}>0$ for the entire motion. The velocity is then found from the energy conservation as:

$$\mathcal{T}=\mathcal{E}-\mathcal{V}$$

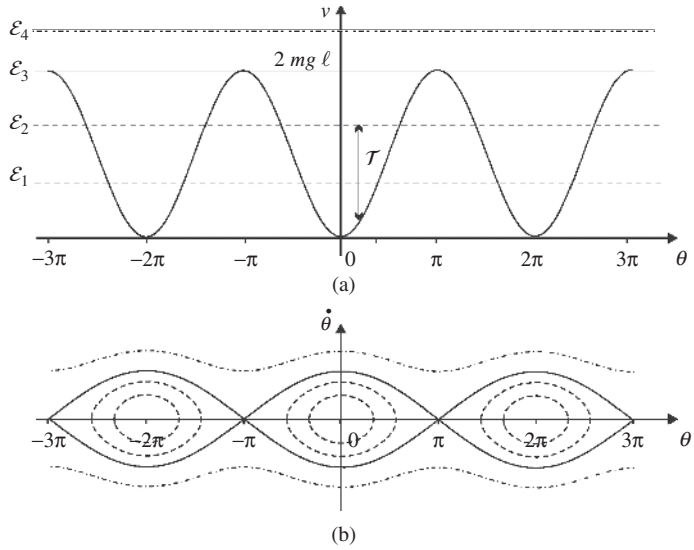


Figure 1.8 The nonlinear pendulum: (a) potential energy, (b) phase space diagram $\dot{\theta} = f(\theta)$.

and in the case $\mathcal{E} < \mathcal{E}_3$ *we can use* (E1.7.c) *together with* (E1.7.a) *to obtain:*

$$\dot{\theta} = \pm\sqrt{\frac{2g}{\ell}\left(\cos\theta - 1 + \frac{\mathcal{E}}{mg\ell}\right)} \tag{E1.7.e}$$

This leads to nonclosed trajectories in the phase-space as indicated by the dash-dotted line in Figure 1.8.b.

To summarize, three regions can be distinguished:

- *For* $\mathcal{E} < \mathcal{E}_3$*, the motion is oscillatory and the corresponding trajectories in the phase space are closed regular curves (of the ellipse type if* $\cos\theta \simeq 1 - \frac{\theta^2}{2}$*).*
- *For* $\mathcal{E} = \mathcal{E}_3$*, the positions* $\theta = \pm(2n+1)\pi$*,* $(n = 0, 1, \ldots)$ *are bifurcation points of the solution.*
- *For* $\mathcal{E} > \mathcal{E}_3$*, the motion is no longer oscillatory but the pendulum undergoes complete rotation with variable rotation speed.*

In the case $\mathcal{E} < \mathcal{E}_3$*, making use of Equation* (E1.7.d)*, the oscillation period can be computed if we notice that from the definition of velocity:*

$$\dot{\theta} = \frac{d\theta}{dt}$$

the time required to undergo a displacement $d\theta$ *is:*

$$dt = \frac{d\theta}{\dot{\theta}} = \pm\sqrt{\frac{\ell}{2g(\cos\theta - \cos\theta_{max})}}\quad d\theta$$

Due to symmetry of the phase-plane trajectory, the period – which is the time corresponding to the path of one closed trajectory – is given by:

$$T = 4\sqrt{\frac{\ell}{2g}} \int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{max}}} \tag{E1.7.f}$$

This last integral can be transformed into a known elliptic integral via the following change of variable:

$$\sin\frac{\theta}{2} = \sin\frac{\theta_{max}}{2}\sin\phi$$

where $[0, \theta_{max}] \rightarrow [0, \frac{\pi}{2}]$. *We find successively:*

$$d\theta = \frac{2\sin\frac{\theta_{max}}{2}\cos\phi}{\sqrt{1 - \sin^2\frac{\theta_{max}}{2}\sin^2\phi}} d\phi$$

and

$$\sqrt{\cos\theta - \cos\theta_{max}} = \sqrt{2}\sin\frac{\theta_{max}}{2}\cos\phi$$

From the results above, the expression of the period becomes:

$$T = 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2\frac{\theta_{max}}{2}\sin^2\phi}} \tag{E1.7.g}$$

Expanding the integral (Abramowitz and Stegun 1970) into:

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \frac{\pi}{2}\left[1 + \frac{k^2}{4} + \ldots\right]$$

allows us to write the period as a function of the angular displacement amplitude:

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{\theta_{max}^2}{16} + \ldots\right] \tag{E1.7.h}$$

This shows that the period is a quadratic increasing function of the amplitude.

1.3.3 Classification of generalized forces

A distinction can be made between *internal* and *external* forces to the system. In both cases they are said to be *conservative* if the associated virtual work is recoverable.

Internal forces

Among the internal forces, the distinction can be made between the linking forces, those associated with elastic deformation and those resulting from a dissipation mechanism.

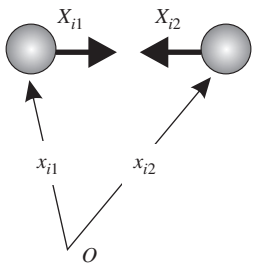


Figure 1.9 Linking forces.

Linking forces

The linking forces appear in a rigid connection between two particles. As already discussed in Section 1.2.1 they are such as the system of forces is in equilibrium (Figure 1.9):

$$X_{i1} + X_{i2} = 0 \tag{1.47}$$

The virtual work associated with the virtual displacement $(\delta u_{i1}, \delta u_{i2})$ is:

$$\begin{aligned}\delta\tau &= \sum_{i=1}^{3}(X_{i1}\delta u_{i1} + X_{i2}\delta u_{i2}) \\ &= \sum_{i=1}^{3}[X_{i1}(\delta u_{i1} - \delta u_{i2})] \\ &= 0\end{aligned}$$

since the nonzero relative virtual displacements are not compatible with the constraints. Hence it can be deduced that *the linking forces do not contribute to the generalized forces* acting on the global system. Their absence from the evaluation of the generalized forces is one of the attractive aspects of Lagrangian mechanics.

Elastic forces

An elastic body can be defined as a body for which any produced work is stored in a recoverable form, thus giving rise to a variation of internal energy:

$$\delta\mathcal{V}_{int} = \sum_{i=1}^{3}\sum_{k=1}^{N}\frac{\partial\mathcal{V}_{int}}{\partial u_{ik}}\delta u_{ik} = -\delta\tau$$

where τ is the virtual work of internal forces. It can be expressed in terms of generalized displacements:

$$\mathcal{V}_{int} = \mathcal{V}_{int}(q,t)$$

$$\delta\tau = \sum_{s=1}^{n} Q_s\delta q_s = -\delta\mathcal{V}_{int}$$

with the generalized forces of elastic origin

$$Q_s = -\frac{\partial \mathcal{V}_{int}}{\partial q_s} \tag{1.48}$$

Dissipation forces

A dissipation (or dissipative) force may be characterized by the fact that it remains parallel and in opposite direction to the velocity vector and is a function of its modulus. Therefore, a dissipation force acting on a mass particle k may be expressed in the form:

$$\boldsymbol{X}_k = -C_k f_k(v_k)\,\frac{\boldsymbol{v}_k}{v_k}$$

or, in terms of components:

$$X_{ik} = -C_k f_k(v_k)\,\frac{v_{ik}}{v_k} \tag{1.49}$$

where

- C_k is a constant
- $f_k(v_k)$ is the function expressing velocity dependence
- v_k is the absolute velocity of particle k:

$$v_k = |\boldsymbol{v}_k| = \sqrt{\sum_{i=1}^{3} v_{ik}^2} = \sqrt{\sum_{i=1}^{3} \dot{u}_{ik}^2}$$

The virtual work of the dissipation forces acting on the system is:

$$\sum_{s=1}^{n} Q_s \delta q_s = \sum_{i=1}^{3}\sum_{k=1}^{N} X_{ik}\delta u_{ik}$$

$$= \sum_{i=1}^{3}\sum_{k=1}^{N}\sum_{s=1}^{n} X_{ik}\frac{\partial u_{ik}}{\partial q_s}\delta q_s$$

yielding:

$$Q_s = -\sum_{i=1}^{3}\sum_{k=1}^{N} C_k f_k(v_k)\frac{v_{ik}}{v_k}\frac{\partial u_{ik}}{\partial q_s} \tag{1.50}$$

By noticing that:

$$v_{ik} = \frac{du_{ik}}{dt} = \frac{\partial u_{ik}}{\partial t} + \sum_{r=1}^{n}\frac{\partial u_{ik}}{\partial q_r}\dot{q}_r$$

$$\frac{\partial v_{ik}}{\partial \dot{q}_s} = \frac{\partial u_{ik}}{\partial q_s} \tag{1.51}$$

one may write:

$$Q_s = -\sum_{i=1}^{3}\sum_{k=1}^{N} C_k f_k(v_k)\frac{v_{ik}}{v_k}\frac{\partial v_{ik}}{\partial \dot{q}_s}$$

$$= -\sum_{k=1}^{N} C_k \frac{f_k(v_k)}{v_k} \frac{\partial}{\partial \dot{q}_s} \left[\frac{1}{2} \sum_{i=1}^{3} v_{ik}^2 \right]$$

$$= -\sum_{k=1}^{N} C_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_s} \tag{1.52}$$

Let us next introduce the dissipation function $\mathcal{D}$ as:

$$\mathcal{D} = \sum_{k=1}^{N} \int_0^{v_k} C_k f_k(\gamma) d\gamma \tag{1.53}$$

and thus:

$$Q_s = -\frac{\partial \mathcal{D}}{\partial \dot{q}_s} \tag{1.54}$$

The dissipated power takes the form:

$$\mathrm{P} = \sum_{s=1}^{n} Q_s \dot{q}_s = -\sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{D}}{\partial \dot{q}_s}$$

By assuming that the dissipation function $\mathcal{D}$ is homogeneous of order m in the generalized velocities, one gets from (1.53) the energy dissipation equation

$$\boxed{\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = -m\mathcal{D}} \tag{1.55}$$

The order m of the dissipation function and thus the order $m - 1$ of the generalized dissipation forces Q_s describes the physical dissipation mode:

$m = 1$	$\leftrightarrow$	dry friction
$m = 2$	$\leftrightarrow$	viscous damping
$m = 3$	$\leftrightarrow$	aerodynamic drag

Let us finally note that the dissipation forces, although classified here as internal forces, may have external origins too.

External forces

Conservative forces

When the external forces are conservative, their virtual work remains zero during a cycle:

$$\delta\tau = \oint Q_s \delta q_s = 0$$

and a potential of external forces $\mathcal{V}_{ext}(q, t)$ can be introduced such as:

$$Q_s = -\frac{\partial \mathcal{V}_{ext}}{\partial q_s} \tag{1.56}$$

Nonconservative forces

When the external forces are of the nonconservative type, the evaluation of the corresponding generalized forces is achieved by making use of a virtual work equation:

$$\delta\tau = \sum_{s=1}^{n} Q_s \delta q_s = \sum_{i=1}^{3}\sum_{k=1}^{N} X_{ik}\delta u_{ik}$$
$$= \sum_{i=1}^{3}\sum_{k=1}^{N}\sum_{s=1}^{n} X_{ik}\frac{\partial u_{ik}}{\partial q_s}\delta q_s$$

and thus:

$$Q_s = \sum_{i=1}^{3}\sum_{k=1}^{N} X_{ik}\frac{\partial u_{ik}}{\partial q_s} \tag{1.57}$$

Taking into account the nonconservative external forces, the power balance of a system can be written in the more general form:

$$\boxed{\frac{d}{dt}(\mathcal{T}+\mathcal{V}) = -m\mathcal{D} + \sum_{s=1}^{n} Q_s\dot{q}_s} \tag{1.58}$$

1.4 Lagrange equations in the general case

In the general case of a nonconservative system with rheonomic constraints, the Lagrange equations of motion may be explicitly expressed in the form:

$$\boxed{-\frac{d}{dt}\left(\frac{\partial\mathcal{T}}{\partial\dot{q}_s}\right) + \frac{\partial\mathcal{T}}{\partial q_s} - \frac{\partial\mathcal{V}}{\partial q_s} - \frac{\partial\mathcal{D}}{\partial\dot{q}_s} + Q_s(t) = 0 \qquad s = 1, \ldots, n} \tag{1.59}$$

or, by explicitly introducing the relative inertia forces:

$$\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{T}_2}{\partial\dot{q}_s}\right) - \frac{\partial\mathcal{T}_2}{\partial q_s} = Q_s(t) - \frac{\partial\mathcal{V}^*}{\partial q_s} - \frac{\partial\mathcal{D}}{\partial\dot{q}_s} + F_s - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{T}_1}{\partial\dot{q}_s}\right) \qquad s = 1, \ldots, n} \tag{1.60}$$

where

$Q_s(t)$ ..... nonconservative external generalized forces
$\mathcal{V} = \mathcal{V}_{ext} + \mathcal{V}_{int}$ ..... total potential
$\mathcal{V}^* = \mathcal{V} - \mathcal{T}_0$ ..... potential modified by the transport kinetic energy
$F_s = \sum_{r=1}^{n} \dot{q}_r G_{rs}$ ..... generalized gyroscopic forces
$\mathcal{D}$ ..... dissipation function

Example 1.8

Let us derive the equations of motion for the system of Figure 1.10 made of a wheel of rotating inertia I inside which a mass m is attached through a system of springs and a viscous damper.

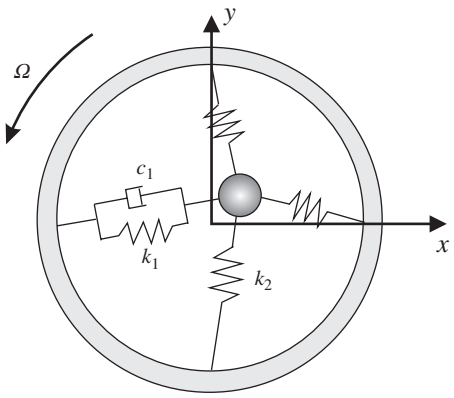


Figure 1.10 Rotating system.

Assuming a coordinate system (x, y) *rotating at a constant rotation speed* Ω *and attached to the wheel, mass m has the following absolute velocity components:*

$$\boldsymbol{v} = \left[\dot{x} - \Omega y, \dot{y} + \Omega x\right]$$

Hence the kinetic energy is given by:

$$\mathcal{T} = \frac{1}{2}m\left[(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2\right] + \frac{1}{2}I\Omega^2$$

Assuming that displacements x and y remain small, the potential energy of the springs is equal to

$$\mathcal{V} = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$$

while the dissipation function of the damper takes the form:

$$\mathcal{D} = \frac{1}{2}c_1\dot{x}^2$$

By applying the Lagrange equations (1.59), the equations of motion are obtained in the form:

$$m\ddot{x} - 2m\Omega\dot{y} - m\Omega^2x + c_1\dot{x} + k_1x = 0$$

$$m\ddot{y} + 2m\Omega\dot{x} - m\Omega^2y + k_2y = 0$$

They can also be put in the matrix form:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + (\boldsymbol{C} + \boldsymbol{G})\dot{\boldsymbol{q}} + (\boldsymbol{K} - \Omega^2\boldsymbol{M})\boldsymbol{q} = 0 \qquad \text{(E1.8.a)}$$

where the vector of generalized displacements $\boldsymbol{q}^T = \left[x\ y\right]$ *and the mass, damping, stiffness and gyroscopic coupling matrices are defined below:*

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{K}^* = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \qquad \boldsymbol{G} = \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix}$$

The meaning of the different terms is more apparent when using the Lagrange equations in the form (1.60):

$$\mathcal{T}_0 = \frac{1}{2}m\Omega^2(x^2+y^2) = \frac{1}{2}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}m\Omega^2 & 0\\ 0 & m\Omega^2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \frac{\Omega^2}{2}\boldsymbol{q}^T\boldsymbol{M}\boldsymbol{q} \tag{E1.8.b}$$

$$\mathcal{T}_1 = m\Omega(x\dot{y}-\dot{x}y) = \begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}\begin{bmatrix}0 & -m\Omega\\ m\Omega & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{G}\boldsymbol{q} \tag{E1.8.c}$$

$$\mathcal{T}_2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}\begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} \tag{E1.8.d}$$

$$\mathcal{V} = \frac{1}{2}(k_1x^2+k_2y^2) = \frac{1}{2}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q} \tag{E1.8.e}$$

$$\mathcal{D} = \frac{1}{2}c_1\dot{x}^2 = \frac{1}{2}\begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}\begin{bmatrix}c_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{C}\dot{\boldsymbol{q}} \tag{E1.8.f}$$

By subtracting (E1.8.b) *from* (E1.8.e)*, the modified potential takes the form:*

$$\mathcal{V}^* = \mathcal{V} - \mathcal{T}_0 = \frac{1}{2}\boldsymbol{q}^T(\boldsymbol{K}-\Omega^2\boldsymbol{M})\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}^*\boldsymbol{q}$$

The mass and damping matrices of the system are clearly positive definite. The effective stiffness matrix $\boldsymbol{K}^*$*, however, loses its positive definite character when* $\Omega^2 > \min(k_1/m, k_2/m)$*. The stability behaviour of the system is thus controlled by its rotating speed, as will be shown in Section 2.11.*

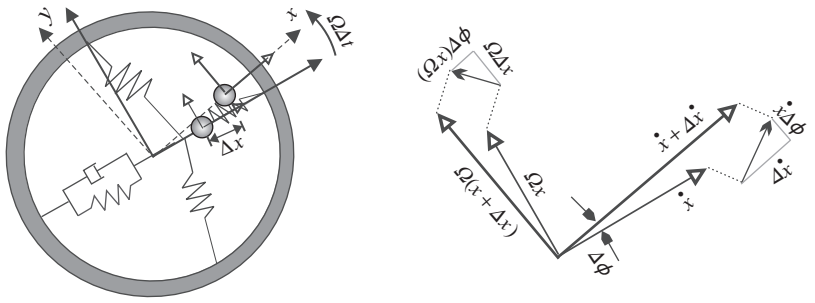
Remark 1.1 *Although the physical meaning of centrifugal forces* $-\Omega^2\boldsymbol{M}\boldsymbol{q}$ *in Equation* (E1.8.a) *is usually easily understood, this is not the case of gyroscopic forces* $\boldsymbol{G}\dot{\boldsymbol{q}}$ *(also called Coriolis forces) which are more difficult to explain. The gyroscopic forces are in fact fictitious forces in the sense that they appear in the equation of motion of the system when they are expressed with respect to a moving frame as it is the case here.*

In order to have a physical insight, let us assume that in the inertia wheel system, the mass m remains on the rotating axis x and moves with a varying velocity $\dot{x}$ *as described in Figure 1.11. We will graphically show that such a motion in fact corresponds to absolute accelerations given to mass m which will prove the presence of centrifugal and gyroscopic forces.*

Figure 1.11.a shows the configuration and absolute velocities of the system at a given time and at a small time interval Δt *later. Let the time interval the axis has rotated be an amount* $\Delta\phi = \Omega\Delta t$*. The velocity along the x axis has changed because of the relative acceleration* $\ddot{x}$ *along x and the velocity along y has changed because its position on the x axis has changed by* $\Delta x = \dot{x}\Delta t$*. Also, observe that in that time interval the direction of the axes x and y has changed.*

If we now analyze the variation of the absolute velocity (Figure 1.11.b), we observe that:

- *the velocity change along x results in the absolute acceleration components:*
  - *i.* $\frac{\Delta\dot{x}}{\Delta t} = \ddot{x}$ *along x due to the relative velocity variation,*
  - *ii.* $\frac{\dot{x}\Delta\phi}{\Delta t} = \dot{x}\Omega$ *along y due to the change of direction of the x axis,*



a. Constant speed along x b. Absolute accelerations measured in rotating frame

Figure 1.11 Physical interpretation of gyroscopic forces.

- *the velocity change along y results in the absolute acceleration components:*
  i. $-\frac{(x\Omega)\Delta\phi}{\Delta t} = -x\Omega^2$ *along x due to the change of direction of the y axis,*
  ii. $\frac{\Delta x\Omega}{\Delta t} = \dot{x}\Omega$ *along y due to the change of position of the mass along x in the time interval.*

Clearly, $(i.)$ *is the relative acceleration generating the relative inertia forces,* $(ii. + iii.)$ *are the Coriolis accelerations generating the gyroscopic forces and* $(iv.)$ *is the centripetal acceleration generating the centrifugal forces.*

1.5 Lagrange equations for impulsive loading

1.5.1 Impulsive loading of a mass particle

Let us consider a particle undergoing a displacement u_i. Hamilton's principle takes the form:

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{3} \left[\delta \left(\frac{1}{2} m \dot{u}_i \dot{u}_i \right) + X_i \delta u_i \right] dt = 0 \tag{1.61}$$

$$\text{with} \qquad \delta u_i(t_1) = \delta u_i(t_2) = 0$$

and it is assumed that $\dot{u}_i$ and $\delta\dot{u}_i$ are piecewise continuous.

In representing the momentum of the particle, the possible discontinuity of the velocity field $\dot{u}_i$ for certain t values does not allow performing the time derivative of the momentum $m\dot{u}_i$. Therefore, let us integrate by parts the second term of principle (1.61), in order to let the force integral over the time interval appear:

$$\begin{aligned} \delta I &= \left[\sum_{i=1}^{3} \delta u_i \int_{t_1}^{t} X_i(t')dt' \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{i=1}^{3} \left[m\dot{u}_i - \int_{t_1}^{t} X_i(t')dt' \right] \delta\dot{u}_i \, dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{3} M_i \delta\dot{u}_i \, dt = 0 \end{aligned} \tag{1.62}$$

For the sake of conciseness let us define:

$$M_i = m\dot{u}_i - \int_{t_1}^{t} X_i(t')dt' \tag{1.63}$$

and in order to obtain the variational derivatives of this modified form of Hamilton's principle let us demonstrate first *du Bois-Reymond's theorem* (Courant and Hilbert 1953):

$$\begin{gathered}\int_{t_1}^{t_2} \sum_{i=1}^{3} M_i \delta\dot{u}_i \; dt = 0 \\ \text{for piecewise continuous } \; \delta\dot{u}_i \; \text{ and } \; \delta u_i(t_1) = \delta u_i(t_2) = 0 \\ \Longleftrightarrow \\ M_i = C_i \\ \text{where } \; C_i \; \text{ are constants}\end{gathered} \tag{1.64}$$

The proof goes as follows:

Sufficient condition: if $M_i = C_i$, then

$$\begin{aligned}\int_{t_1}^{t_2} \sum_{i=1}^{3} M_i \delta\dot{u}_i \; dt &= \sum_{i=1}^{3} C_i \int_{t_1}^{t_2} \delta\dot{u}_i \; dt \\ &= \left[\sum_{i=1}^{3} C_i \delta u_i\right]_{t_1}^{t_2} = 0\end{aligned}$$

Necessary condition: since

$$\int_{t_1}^{t_2} \sum_{i=1}^{3} M_i \delta\dot{u}_i \; dt = 0$$

for any virtual velocity $\delta\dot{u}_i$ piecewise continuous verifying $\delta u_i(t_1) = \delta u_i(t_2) = 0$, let us take:

$$\delta u_i = \int_{t_1}^{t} (M_i - C_i) \; dt' = \int_{t_1}^{t} M_i(t') \; dt' - C_i(t - t_1)$$

with

$$C_i = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} M_i(t') \; dt'$$

The corresponding virtual velocity is:

$$\delta\dot{u}_i = M_i - C_i$$

and one may write:

$$\begin{aligned}\int_{t_1}^{t_2} \sum_{i=1}^{3} M_i \delta \dot{u}_i \, dt &= \int_{t_1}^{t_2} \sum_{i=1}^{3} M_i \delta \dot{u}_i \, dt - \int_{t_1}^{t_2} \sum_{i=1}^{3} C_i \delta \dot{u}_i \, dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{3} (M_i - C_i) \delta \dot{u}_i \, dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{3} (M_i - C_i)^2 dt \\ &= 0\end{aligned}$$

Therefore $M_i = C_i$ which concludes the proof.

By making use of du Bois-Reymond's theorem, Hamilton's principle in the form (1.62) provides the equations of motion:

$$m\dot{u}_i(t) - \int_{t_1}^{t} X_i(t') \, dt' = C_i \qquad i = 1, 2, 3 \tag{1.65}$$

where $C_i = m\dot{u}_i(t_1)$ are the components of the momentum at time t_1, hence:

$$m(\dot{u}_i(t) - \dot{u}_i(t_1)) = \int_{t_1}^{t} X_i(t') \, dt' \qquad i = 1, 2, 3 \tag{1.66}$$

This last form expresses that *the variation of the momentum of the particle over the time interval* $[t_1, t]$ *is equal to the impulse of the external forces impressed on the particle during the same time interval*.

In many shock and impact problems, the application time of the loading compared to the time scale at which phenomena are observed is so small that it is valid to consider that the external force is of impulse type, i.e. applied during an infinitesimal time interval but impressing on the system a finite impulse:

$$P_i = \int_{t_-}^{t_+} X_i(t) \, dt \tag{1.67}$$

In the impulsive case, Equation (1.66) becomes:

$$\boxed{m\dot{u}_i(t_+) - m\dot{u}_i(t_-) = P_i \quad i = 1, 2, 3} \tag{1.68}$$

showing that a velocity discontinuity such as represented by Figure 1.12 results from an impulsive loading equal to P_i/m.

Equations (1.68) are the only applicable ones in the impulsive case since the instantaneous force X_i becomes infinite during the shock. Alternatively this result is classically derived from the time-integration of Newton's equation, taking the limit for an infinite force on a zero time interval. In the next section we show that the same mathematical reasoning as presented here for a particle can be followed in order to find the impulse equations in terms of generalized coordinates.

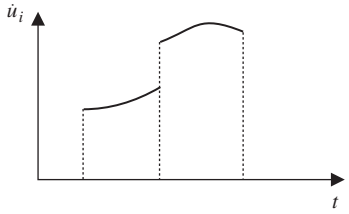


Figure 1.12 Velocity discontinuity.

1.5.2 Impulsive loading for a system of particles

For a system of N mass particles described by n generalized coordinates q_s, Hamilton's principle takes the form:

$$\delta I = \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \delta \dot{q}_s + \left(\frac{\partial (\mathcal{T} - \mathcal{V}}{\partial q_s} + Q_s - \frac{\partial \mathcal{D}}{\partial \dot{q}_s} \right) \delta q_s \right] dt = 0 \tag{1.69}$$

with

$$\delta q_s(t_1) = \delta q_s(t_2) = 0 \tag{1.70}$$

Let us integrate the second term by parts in order to let the time integral of the forces appear:

$$\delta I = \left[\sum_{s=1}^{n} \delta q_s \int_{t_1}^{t} \left(\frac{\partial (\mathcal{T} - \mathcal{V})}{\partial q_s} + Q_s - \frac{\partial \mathcal{D}}{\partial \dot{q}_s} \right) dt' \right]_{t_1}^{t_2}$$
$$+ \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[\frac{\partial \mathcal{T}}{\partial \dot{q}_s} - \int_{t_1}^{t} \left(\frac{\partial (\mathcal{T} - \mathcal{V})}{\partial q_s} + Q_s - \frac{\partial \mathcal{D}}{\partial \dot{q}_s} \right) dt' \right] \delta \dot{q}_s \, dt$$

Hence, by setting:

$$M_s = \frac{\partial \mathcal{T}}{\partial \dot{q}_s} - \int_{t_1}^{t} \left(\frac{\partial (\mathcal{T} - \mathcal{V})}{\partial q_s} + Q_s - \frac{\partial \mathcal{D}}{\partial \dot{q}_s} \right) dt' \tag{1.71}$$

one obtains:

$$\delta I = \int_{t_1}^{t_2} \sum_{s=1}^{n} M_s \delta \dot{q}_s \, dt = 0$$

As previously, du Bois-Reymond's theorem (Courant and Hilbert 1953) can be stated in the form:

$$\int_{t_1}^{t_2} \sum_{s=1}^{n} M_s \delta \dot{q}_s dt = 0$$
$$\text{with } \delta q_s(t_1) = \delta q_s(t_2) = 0 \text{ and } \delta \dot{q}_s \text{ piecewise continuous}$$
$$\Longleftrightarrow \tag{1.72}$$
$$M_s = C_s$$
$$\text{with } C_s \text{ constants}$$

The demonstration is similar to the case of one mass particle. Hence, the generalization of Equations (1.65) to (1.68) is written:

$$\frac{\partial \mathcal{T}}{\partial \dot{q}_s} - \int_{t_1}^{t} \left(\frac{\partial (\mathcal{T} - \mathcal{V})}{\partial q_s} + Q_s - \frac{\partial \mathcal{D}}{\partial \dot{q}_s} \right) dt' = C_s \qquad s = 1, \ldots, n \tag{1.73}$$

$$\frac{\partial \mathcal{T}}{\partial \dot{q}_s} - \left[\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right]_{t_1} = \int_{t_1}^{t} \left(\frac{\partial (\mathcal{T} - \mathcal{V})}{\partial q_s} + Q_s - \frac{\partial \mathcal{D}}{\partial \dot{q}_s} \right) dt' \qquad s = 1, \ldots, n \tag{1.74}$$

In the case of impulsive loading, one may write:

$$P_s = \int_{t_-}^{t_+} Q_s dt' \tag{1.75}$$

and

$$\boxed{\left[\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right]_{t_+} - \left[\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right]_{t_-} = P_s \qquad s = 1, \ldots, n} \tag{1.76}$$

since $\frac{\partial (\mathcal{T} - \mathcal{V})}{\partial q_s} - \frac{\partial \mathcal{D}}{\partial \dot{q}_s}$ remains finite between t_- and t_+.

This result is the generalization of (1.68) to a system of particles. Let us finally note that the equality relationships (1.74) to (1.76) between the change of momentum and the impulse impressed on the system could also be deduced from a time integration of the Lagrange equations (1.29), taking the limit for an infinite force on a zero time interval.

Example 1.9

Let us consider the system of Figure 1.13 made of two masses m_1 and m_2 lying on a horizontal plane. m_2 is initially at rest, while m_1 moves with constant velocity $v_1 = V$ and hits mass m_2 at time $t = t^-$. We are interested in v_1^+ and v_2^+, the velocities after the shock (at time $t = t^+$).

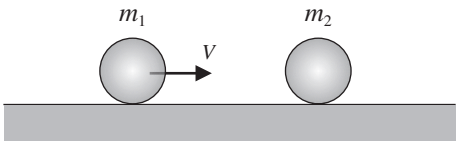


Figure 1.13 Collision of two masses.

A first relation is obtained by expressing conservation of the total energy of the system, which in this case reduces to the kinetic energy:

$$\mathcal{T} = \frac{1}{2} m_1 (v_1^-)^2 + \frac{1}{2} m_2 (v_2^-)^2 = \frac{1}{2} m_1 V^2 = \frac{1}{2} m_1 (v_1^+)^2 + \frac{1}{2} m_2 (v_2^+)^2 \tag{E1.9.a}$$

The second relation results from the computation of the jump in velocities through the shock. The latter are obtained from the Lagrange equations (1.76) which read in this case:

$$\left[\frac{\partial \mathcal{T}}{\partial v_i} \right]_{t_+} - \left[\frac{\partial \mathcal{T}}{\partial v_i} \right]_{t_-} = P_i \qquad i = 1, 2$$

and yield:

$$m_1 v_1^+ - m_1 v_1^- = P_1$$
$$m_2 v_2^+ - m_2 v_2^- = P_2$$

P_1 and P_2 being the internal impulses generated by the shock, they are in equilibrium:

$$P_1 + P_2 = 0$$

and therefore we get the equation of momentum conservation:

$$m_1 v_1^+ + m_2 v_2^+ = m_1 V \tag{E1.9.b}$$

This leads to:

$$v_2^+ = \frac{m_1}{m_2}(V - v_1^+) \tag{E1.9.c}$$

and substituting into Equation (E1.9.a) *yields after simplification:*

$$V^2(m_1 - m_2) + (v_1^+)^2(m_1 + m_2) - 2m_1 V v_1^+ = 0$$

Solving for v_1^+ and substituting into (E1.9.c) *provides the only feasible solution:*

$$v_1^+ = \frac{m_1 - m_2}{m_1 + m_2} V \qquad\qquad v_2^+ = \frac{2m_1}{m_1 + m_2} V \tag{E1.9.d}$$

It can be observed from (E1.9.d) *that after the shock:*

- *if $m_1 > m_2$, both masses move in the same direction as initial velocity V.*
- *if $m_1 = m_2$, there is exchange of velocities between masses ($v_1^+ = 0$, $v_2^+ = V$).*
- *If $m_2 > m_1$, the masses move in opposite directions.*

1.6 Dynamics of constrained systems

In the way they have been formulated in Section 1.3 Lagrange equations involve only the forces effectively applied onto the systems thanks to the choice of kinematically admissible coordinates q_s. They thus express the equilibrium in a subspace orthogonal to the constraints so that reaction forces do not appear in the equations of motion. In some cases it is nevertheless useful or easier to make the reaction forces appear explicitly in the expression of the equilibrium and to choose generalized coordinates that do not satisfy some of the kinematic constraints. The only constraints considered here are the holonomic ones. A discussion on the treatment of nonholonomic constraints can be found for instance in (Géradin and Cardona 2001, Lanczos 1949).

Let us suppose that a subset of m kinematic constraints (1.12) is not explicitly satisfied by the choice of the generalized coordinates q_s. Substitution of (1.11) and (1.14) into the holonomic constraints (1.12) allows expressing the constraints for the generalized coordinates:

$$f_r(x_{ik} + U_{ik}(q_s, t)) = f_r(q_s, t) = 0 \qquad\qquad r = 1, \ldots m \tag{1.77}$$

Their variation:

$$\delta f_r = \sum_{s=1}^{n} \frac{\partial f_r}{\partial q_s} \delta q_s = 0 \qquad r = 1, \ldots m \tag{1.78}$$

indicates that the constraints are still verified if the δq_s define a motion orthogonal to the direction determined by $\frac{\partial f_r}{\partial q_s}$ in the space of the generalized coordinates. Thus, the derivatives determine the directions of the reaction forces and can be expressed by:

$$R_{rs} = \lambda_r \frac{\partial f_r}{\partial q_s} \tag{1.79}$$

where λ_r denotes the intensity of the reaction associated to constraint f_r. It is called a *Lagrange multiplier*.

Using (1.78) and (1.79), the virtual work of the reaction forces is:

$$\sum_{r=1}^{m} \sum_{s=1}^{n} R_{rs} \delta q_s = \sum_{r=1}^{m} \lambda_r \sum_{s=1}^{n} \frac{\partial f_r}{\partial q_s} \delta q_s = \sum_{r=1}^{m} \lambda_r \delta f_r \tag{1.80}$$

The latter can be added to the virtual work expression (1.28) obtained from Hamilton's principle:

$$\int_{t_1}^{t_2} \sum_{s=1}^{n} \left[-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s + \sum_{r=1}^{m} \lambda_r \frac{\partial f_r}{\partial q_s} \right] \delta q_s \, dt = 0 \tag{1.81}$$

It provides the Lagrange equations of motion in terms of generalized coordinates q_s which do not satisfy the constraints:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) - \frac{\partial \mathcal{T}}{\partial q_s} = Q_s + \sum_{r=1}^{m} \lambda_r \frac{\partial f_r}{\partial q_s}$$

where the Lagrange multipliers λ_r (i.e. the reaction force intensities) are determined to satisfy the conditions (1.77). The Lagrange equations together with the complementary conditions form a system of $n + m$ equations with $n + m$ unknowns:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) - \dfrac{\partial \mathcal{T}}{\partial q_s} - Q_s - \displaystyle\sum_{r=1}^{m} \lambda_r \frac{\partial f_r}{\partial q_s} = 0 & s = 1, \ldots n \\ f_r(q_s, t) = 0 & r = 1, \ldots m \end{cases} \tag{1.82}$$

These Lagrange equations can also be deduced from Hamilton's principle if one introduces an additional potential representing the work produced by the constraining forces, i.e. the dislocation potential $\mathcal{V}_d$:

$$\mathcal{V}_d = -\sum_{r=1}^{m} \lambda_r f_r(q_s, t) \tag{1.83}$$

This potential is then added to the potential of the system and it can be verified that Hamilton's principle (1.27) yields the equilibrium and compatibility equations (1.82) when stating the stationarity with respect to q_s and λ_r respectively.

Example 1.10

Let us reconsider the simple pendulum described in Figure 1.5, but now let us try to write the constrained equations of motion using the Cartesian displacements as unknowns. The constraint on the system can be written as:

$$f = \sqrt{(\ell + u_1)^2 + u_2^2} - \ell = 0 \tag{E1.10.a}$$

Taking the derivatives of the constrain with respect to u_1 and u_2, one finds the direction of the reaction forces, namely

$$\vec{R} = \lambda \begin{bmatrix} \dfrac{\partial f}{\partial u_1} \\ \dfrac{\partial f}{\partial u_2} \end{bmatrix} = \lambda \begin{bmatrix} \dfrac{(\ell + u_1)}{\sqrt{(\ell + u_1)^2 + u_2^2}} \\ \dfrac{u_2}{\sqrt{(\ell + u_1)^2 + u_2^2}} \end{bmatrix} = \lambda \begin{bmatrix} \dfrac{(\ell + u_1)}{\ell} \\ \dfrac{u_2}{\ell} \end{bmatrix}$$

The constrained equations can thus be written as:

$$\begin{cases} m\ddot{u}_1 - mg - \lambda \dfrac{(\ell + u_1)}{\ell} = 0 \\ m\ddot{u}_2 - \lambda \dfrac{u_2}{\ell} = 0 \\ \sqrt{(\ell + u_1)^2 + u_2^2} - \ell = 0 \end{cases}$$

If we choose the degree of freedom θ such that:

$$u_1 = \ell \cos\theta - \ell$$
$$u_2 = \ell \sin\theta$$

the constraint (E1.10.a) *is always satisfied. Using these relations the reaction forces can be expressed as:*

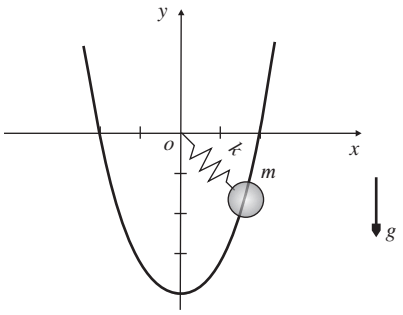
$$\vec{R} = \lambda \begin{bmatrix} \dfrac{(\ell + u_1)}{\ell} \\ \dfrac{u_2}{\ell} \end{bmatrix} = \lambda \begin{bmatrix} \dfrac{\ell \cos\theta}{\ell} \\ \dfrac{\ell \sin\theta}{\ell} \end{bmatrix} = \lambda \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

and as discussed in Example 1.3, projecting this force in the direction compatible with the constraint leads to the unconstrained equation of motion for the pendulum.

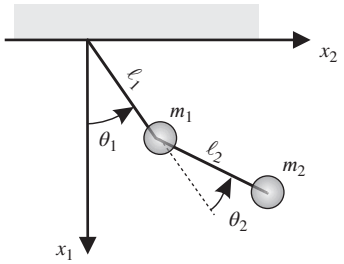
1.7 Exercises

1.7.1 Solved exercises

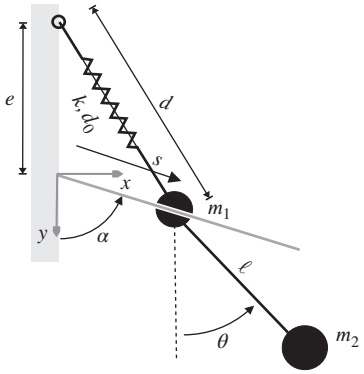
Problem 1.1 Prove that a system of N particles that are rigidly linked to one another (thus forming a rigid body in space) can be described by 6 degrees of freedom when $N > 2$.



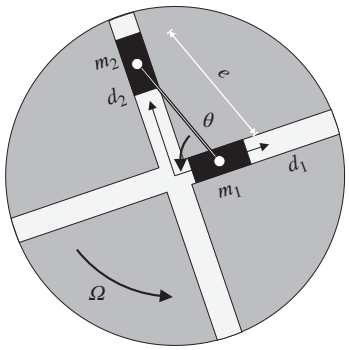
a. A mass on a parabolic curve



b. The double pendulum



c. The guided pendulum



d. Rotating system with 2 sliding masses

Figure 1.14 Solved exercises.

Solution

Every individual particle has 3 degrees of freedom if it is not rigidly linked. Since the particles form a rigid body, the distance between any pair of particles is constant. If the system is made of 2 particles ($N = 2$), one constraint can be defined. If a particle is then added to the system of 2 particles, 2 additional constraints must be defined in order to ensure that the third particle has a constant distance with respect to the 2 previous ones. For any additional particle in the system, 3 additional nonredundant constraints must be defined in order to ensure that the particle is rigidly connected to the system. Hence, for a system of N particles rigidly linked, $1 + 2 + 3(N - 3)$ nonredundant holonomic constraints can be defined so that the system has $3N - (1 + 2 + 3(N - 3)) = 6$ degrees of freedom.

Problem 1.2 In the system described in Figure 1.14.a a mass moves on a parabolic curve described by the equation:

$$y(x) = \frac{x^2}{\ell} - 4\ell$$

where ℓ is a given length. A linear spring with stiffness k is attached to the mass and fixed to the point $(0, 0)$. The spring has a zero natural length. The gravity acts in the direction $-y$.

Choose x as the degree of freedom of the system. Write the potential and kinetic energies of the system and derive the Lagrange equations.

Solution

Calling u the deformation of the spring we find:

$$\mathcal{V} = \frac{1}{2}ku^2 + mgy = \frac{1}{2}k(x^2 + y(x)^2) + mgy(x) \tag{P1.2.a}$$

$$\mathcal{T} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(\dot{x}^2 + 4\frac{x^2}{\ell^2}\dot{x}^2\right) = \frac{1}{2}m\dot{x}^2\left(1 + 4\frac{x^2}{\ell^2}\right) \tag{P1.2.b}$$

The terms of the Lagrange equation (1.59) then write:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m\dot{x}\left(1 + 4\frac{x^2}{\ell^2}\right)\right) = m\ddot{x}\left(1 + 4\frac{x^2}{\ell^2}\right) + \frac{8mx}{\ell^2}\dot{x}^2 \tag{P1.2.c}$$

$$\frac{\partial \mathcal{T}}{\partial x} = \frac{4mx\dot{x}^2}{\ell^2} \tag{P1.2.d}$$

$$\frac{\partial \mathcal{V}}{\partial x} = k\left(x + y\frac{\partial y}{\partial x}\right) + mg\frac{\partial y}{\partial x} = kx\left(2\frac{x^2}{\ell^2} - 7\right) + 2mg\frac{x}{\ell} \tag{P1.2.e}$$

and there are no nonconservative forces in this system. The equation of motion is thus:

$$m\left(1 + 4\frac{x^2}{\ell^2}\right)\ddot{x} + \frac{4mx}{\ell^2}\dot{x}^2 + kx\left(2\frac{x^2}{\ell^2} - 7\right) + 2mg\frac{x}{\ell} = 0 \tag{P1.2.f}$$

Problem 1.3 Let us consider the double pendulum undergoing 2-D motion in a gravity field g as already introduced in Example 1.4. Using as generalized coordinates the relative angular displacements as shown on Figure 1.14.b, you are asked:

- To express the position coordinates ξ_{ik} of both masses in terms of the generalized coordinates θ_1 and θ_2 as displayed on Figure 1.14.b.
- To express the Cartesian velocities of both masses.
- To express the potential and kinetic energies of the system.
- To develop the system equations of motion in Lagrange form.

Solution

The positions of the masses were given in the Example 1.4, Equations (E1.4.a–E1.4.d) and the absolute velocities are computed as we compute the velocities:

$$\dot{u}_{11} = -\ell_1\dot{\theta}_1 \sin\theta_1$$

$$\dot{u}_{21} = \ell_1\dot{\theta}_1 \cos\theta_1$$

$$\dot{u}_{12} = -\ell_1\dot{\theta}_1 \sin\theta_1 - \ell_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2)$$

$$\dot{u}_{22} = \ell_1\dot{\theta}_1 \cos\theta_1 + \ell_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2)$$

The kinetic energy is then expressed as:

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}(m_1(\dot{u}_{11}^2 + \dot{u}_{21}^2) + m_2(\dot{u}_{12}^2 + \dot{u}_{22}^2)) \\ &= \frac{1}{2}(m_1\ell_1^2\dot{\theta}_1^2 + m_2(\ell_1^2\dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2\ell_1\ell_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2)) \qquad \text{(P1.3.a)}\end{aligned}$$

The potential energy of the system due to gravity forces is:

$$\mathcal{V} = -m_1 g u_{11} - m_2 g u_{12}$$

which, in terms of the generalized coordinates is:

$$\mathcal{V} = m_1 g\ell_1(1 - \cos\theta_1) + m_2 g(\ell_1(1 - \cos\theta_1) + \ell_2(1 - \cos(\theta_1 + \theta_2))) \qquad \text{(P1.3.b)}$$

Let us then compute the different terms of the Lagrange equations (1.59), noting that no non-conservative forces are present:
For $s = 1$, namely $q_s = \theta_1$,

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{\theta}_1} &= \frac{d}{dt}(m_1\ell_1^2\dot{\theta}_1 + m_2(\ell_1^2\dot{\theta}_1 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2) + \ell_1\ell_2(2\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2)) \\ &= m_1\ell_1^2\ddot{\theta}_1 + m_2(\ell_1^2\ddot{\theta}_1 + \ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + \ell_1\ell_2(2\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 - \ell_1\ell_2(2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2\sin\theta_2) \\ \frac{\partial\mathcal{T}}{\partial\theta_1} &= 0 \\ \frac{\partial\mathcal{V}}{\partial\theta_1} &= m_1 g\ell_1\sin\theta_1 + m_2 g(\ell_1\sin\theta_1 + \ell_2\sin(\theta_1 + \theta_2))\end{aligned}$$

For $s = 2$, namely $q_s = \theta_2$,

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{\theta}_2} &= \frac{d}{dt}(m_2(\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2) + \ell_1\ell_2\dot{\theta}_1\cos\theta_2)) \\ &= m_2(\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + \ell_1\ell_2\ddot{\theta}_1\cos\theta_2 - \ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2) \\ \frac{\partial\mathcal{T}}{\partial\theta_2} &= -m_2\ell_1\ell_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 \\ \frac{\partial\mathcal{V}}{\partial\theta_2} &= m_2 g\ell_2\sin(\theta_1 + \theta_2)\end{aligned}$$

Substituting these results in the Lagrange equations:

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{T}}{\partial\theta_1} + \frac{\partial\mathcal{V}}{\partial\theta_1} = 0 \qquad \text{(P1.3.c)}$$

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{\theta}_2} - \frac{\partial\mathcal{T}}{\partial\theta_2} + \frac{\partial\mathcal{V}}{\partial\theta_2} = 0 \qquad \text{(P1.3.d)}$$

then yields the equations of motion.

Problem 1.4 Let us consider the guided double pendulum of Figure 1.14.c where a mass m_1 is prescribed to move without friction on a bar making an angle α with respect to the vertical direction. The spring of stiffness k is attached to a fixed point and its undeformed length is d_0. Gravity g is acting along y. Using the degrees of freedom s and θ, write the energies of the system and find its equations of motion.

Solution

The positions of mass m_1 and m_2 are given by:

$$x_1 = s\sin\alpha \qquad x_2 = x_1 + \ell\sin\theta$$
$$y_1 = s\cos\alpha \qquad y_2 = y_1 + \ell\cos\theta$$

and the absolute velocities are then:

$$\dot{u}_{x1} = \dot{s}\sin\alpha \qquad \dot{u}_{x2} = \dot{u}_{x1} + \ell\dot{\theta}\cos\theta$$
$$\dot{u}_{y1} = \dot{s}\cos\alpha \qquad \dot{u}_{y2} = \dot{u}_{y1} - \ell\dot{\theta}\sin\theta$$

The kinetic and potential energies are then computed, yielding:

$$\mathcal{T} = \frac{1}{2}m_1\dot{s}^2 + \frac{1}{2}m_2(\dot{s}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{s}\dot{\theta}\sin(\alpha-\theta)) \tag{P1.4.a}$$

$$\mathcal{V} = -m_1 g s\cos\alpha - m_2 g(s\cos\alpha + \ell\cos\theta) + \frac{1}{2}k(d-d_0)^2 \tag{P1.4.b}$$

where

$$\begin{aligned} d^2 &= (e + s\cos\alpha)^2 + (s\sin\alpha)^2 \\ &= e^2 + s^2 + 2se\cos\alpha \end{aligned} \tag{P1.4.c}$$

The terms in the Lagrange equations (1.59) (all forces being conservative) are obtained as follows:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{s}} = (m_1+m_2)\ddot{s} + m_2\ell\ddot{\theta}\sin(\alpha-\theta) \\ &\qquad\qquad -m_2\ell\dot{\theta}^2\cos(\alpha-\theta) \\ &\frac{\partial\mathcal{T}}{\partial s} = 0 \\ &\frac{\partial\mathcal{V}}{\partial s} = -m_1 g\cos\alpha - m_2 g\cos\alpha + k(d-d_0)\frac{\partial d}{\partial s} \end{aligned} \qquad \begin{aligned} &\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{\theta}} = m_2\ell^2\ddot{\theta} + m_2\ell\sin(\alpha-\theta)\ddot{s} \\ &\qquad\qquad -m_2\ell\dot{s}\cos(\alpha-\theta)\dot{\theta} \\ &\frac{\partial\mathcal{T}}{\partial\theta} = -m_2\ell\dot{s}\dot{\theta}\cos(\alpha-\theta) \\ &\frac{\partial\mathcal{V}}{\partial\theta} = m_2 g\ell\sin\theta \end{aligned} \tag{P1.4.d}$$

Finally the Lagrange equations are written:

$$\begin{cases} \left(m_1+m_2\right)\ddot{s} + m_2\ell\ddot{\theta}\sin(\alpha-\theta) - m_2\ell\dot{\theta}^2\cos(\alpha-\theta) \\ \qquad -(m_1+m_2)g\cos\alpha + k(d-d_0)\frac{s+e\cos\alpha}{d} = 0 \\ m_2\ell^2\ddot{\theta} + m_2\ell\sin(\alpha-\theta)\ddot{s} + m_2 g\ell\sin\theta = 0 \end{cases} \tag{P1.4.e}$$

Problem 1.5 Consider the rotating system of Figure 1.14.d. A wheel is rotating with constant angular velocity Ω imposed to the system. Two masses m_1 and m_2 slide without friction into the orthogonal grooves. The mass points are linked through a rigid bar of length e. The inertia of the rigid bar is neglected.

- Find the equations of motion in terms of the displacements d_1 and d_2 of the masses along the grooves.
- If $m_1 = m_2 = m$ and if no force is applied to the system, show that
  1. the reaction force in the rigid link is also constant and the bar is always under traction;
  2. the velocity $\dot{\theta}$, namely the angular velocity of the bar relative to the wheel remains constant.

Solution

Let us define the angular position of the disk:

$$\phi(t) = \Omega t$$

with respect to an inertial frame centred on the rotation axis of the system. In that system, the masses have the instantaneous coordinates:

$$\begin{aligned} x_1 &= d_1 \cos\phi & x_2 &= -d_2 \sin\phi \\ y_1 &= d_1 \sin\phi & y_2 &= d_2 \cos\phi \end{aligned} \tag{P1.5.a}$$

and absolute velocities:

$$\begin{aligned} \dot{x}_1 &= \dot{d}_1 \cos\phi - d_1 \Omega \sin\phi & \dot{x}_2 &= -\dot{d}_2 \sin\phi - d_2 \Omega \cos\phi \\ \dot{y}_1 &= \dot{d}_1 \sin\phi + d_1 \Omega \cos\phi & \dot{y}_2 &= \dot{d}_2 \cos\phi - d_2 \Omega \sin\phi \end{aligned} \tag{P1.5.b}$$

The local displacements d_1 and d_2 are constrained by the fixed length e requiring that:

$$\sqrt{d_1^2 + d_2^2} = e \tag{P1.5.c}$$

This condition could be satisfied by choosing a single generalized coordinate. If one would choose for instance θ, the angle of the rod relative to the wheel, one would write:

$$d_1 = e \cos\theta \qquad d_2 = e \sin\theta \tag{P1.5.d}$$

and substitute these equations in (P1.5.b). However in this exercise we consider the two degrees of freedom d_1 and d_2, and impose the condition (P1.5.c) explicitly on the system using a Lagrange multiplier (Section 1.6).
Let us compute the kinetic energy:

$$\mathcal{T} = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \tag{P1.5.e}$$

or, after substitution of (P1.5.b),

$$\mathcal{T} = \frac{1}{2} m_1 (\dot{d}_1^2 + d_1^2 \Omega^2) + \frac{1}{2} m_2 (\dot{d}_2^2 + d_2^2 \Omega^2) \tag{P1.5.f}$$

In the absence of external force, the dynamics of the system is governed by the Lagrangian obtained from the addition to (P1.5.f) of the inextensibility constraint (P1.5.c) with a Lagrangian multiplier λ:

$$\mathcal{L}(d_1, d_2, \phi, \lambda) = \frac{1}{2} m_1(\dot{d}_1^2 + d_1^2\Omega^2) + \frac{1}{2} m_2(\dot{d}_2^2 + d_2^2\Omega^2) + \lambda(\sqrt{d_1^2 + d_2^2} - e) \tag{P1.5.g}$$

Taking the variations with respect to δd_1, δd_2, and $\delta\lambda$ one obtains the equations of the system (the constrained Lagrange equations (1.82)):

$$-m_1\ddot{d}_1 + \left(m_1\Omega^2 + \frac{\lambda}{e}\right) d_1 = 0 \tag{P1.5.h}$$

$$-m_2\ddot{d}_2 + \left(m_2\Omega^2 + \frac{\lambda}{e}\right) d_2 = 0 \tag{P1.5.i}$$

$$\sqrt{d_1^2 + d_2^2} - e = 0 \tag{P1.5.j}$$

the last equation being obviously equivalent to the constraint (P1.5.c).
In case the masses are equal, namely $m_1 = m_2 = m$, an elegant way to compute the Lagrange multipliers is obtained by multiplying Equations (P1.5.j) and (P1.5.j) by d_1 and d_2 respectively and summing up

$$-m(d_1\ddot{d}_1 + d_2\ddot{d}_2) + \left(m\Omega^2 + \frac{\lambda}{e}\right)(d_1^2 + d_2^2) = 0$$

Accounting for the constraint (P1.5.j) one finds:

$$\lambda = me(d_1\ddot{d}_1 + d_2\ddot{d}_2 - \Omega^2) \tag{P1.5.k}$$

Writing the constraint (P1.5.j):

$$d_1^2 + d_2^2 = e^2$$

and taking the time derivative twice we get:

$$\dot{d}_1 d_1 + \dot{d}_2 d_2 = 0 \quad \text{and} \quad \ddot{d}_1 d_1 + \ddot{d}_2 d_2 = -(\dot{d}_1^2 + \dot{d}_2^2) = -e^2\dot{\theta}^2 \tag{P1.5.l}$$

where the last equality was obtained using the time derivatives of (P1.5.d). Thus, from Equation (P1.5.k), the multiplier is obtained as:

$$\lambda = -me(\Omega^2 + \dot{\theta}^2) \tag{P1.5.m}$$

It is always negative, thus according to (P1.5.h–P1.5.i) always in the direction opposite to the centrifugal force, showing that the connecting bar is always in traction.
Let us now find the Lagrange equation corresponding to the minimum coordinate θ and first write the expression (P1.5.f) of the kinetic energy using (P1.5.d),

$$\mathcal{T} = \frac{1}{2} me^2(\dot{\theta}^2 + \Omega^2) \tag{P1.5.n}$$

and writing the Lagrange equation in this case yields:

$$me^2\ddot{\theta} = 0 \tag{P1.5.o}$$

showing that when masses are equal the angular velocity is constant, and therefore also the constraining force λ according to (P1.5.m).

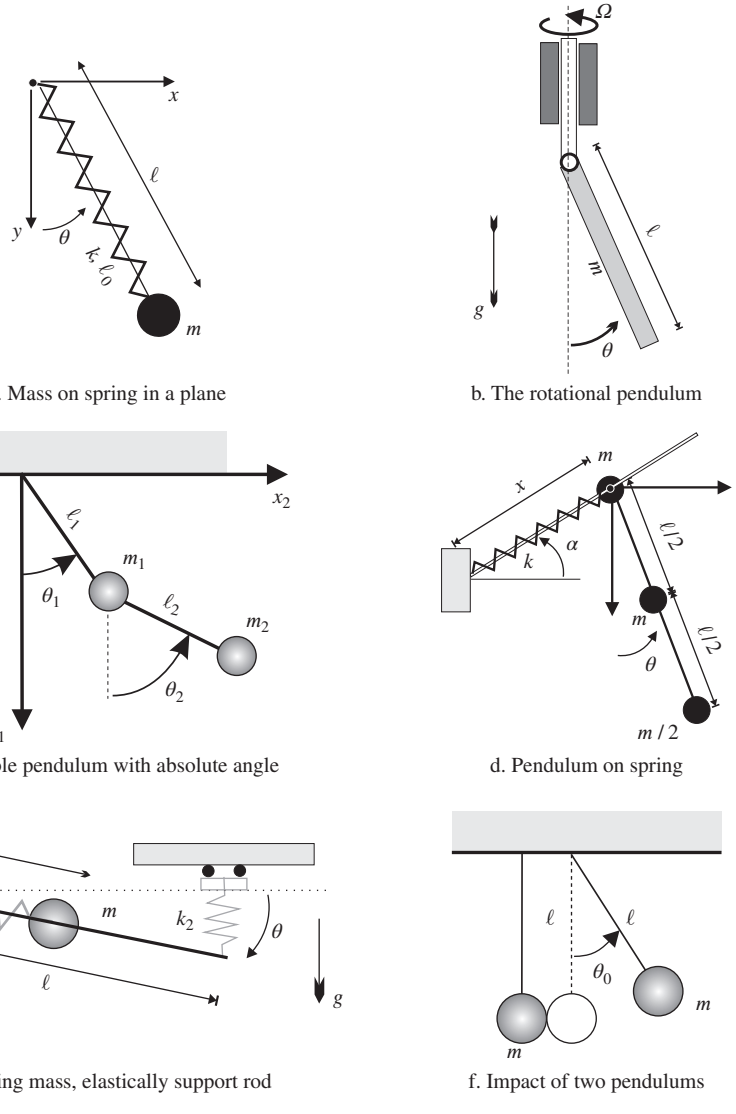


Figure 1.15 Selected exercises.

1.7.2 Selected exercises

Problem 1.6 For the mass on a linear spring depicted in Figure 1.15.a write the potential and kinetic energies using the angle θ and the length ℓ of the spring as generalized coordinates. Gravity is acting along direction y. The spring has a natural length equal to ℓ_0 and the system is assumed to move in the plane (x, y). Find the equations of motions using the Lagrange equations.

Problem 1.7 Let us consider a bar hinged on a vertical rotating shaft as described in Figure 1.15.b. The bar behaves like a pendulum in the gravity field g. The rotation speed Ω is constant and given. The bar has a uniform mass per unit length of m and a length ℓ. You are asked to

1. Using the angle θ as degree of freedom, determine the absolute velocity of a point on the bar as a function of the rotation speed Ω.
2. Compute the kinetic and potential energy of the entire bar.
3. Find the equation of motion using the Lagrange formalism.

Problem 1.8 Repeat the exercise of the double pendulum described in Problem 1.3 but now using absolute coordinates as displayed on Figure 1.15.c. Compare the equations of motion obtained with both sets of generalized coordinates.

Problem 1.9 A mass m slides without friction on a rod positioned at a fixed angle α with respect to the horizontal direction (Figure 1.15.d). The mass is fixed to a nonlinear spring developing a force kx^3 where x is its extension. The spring is aligned with the rod. To the sliding mass a single pendulum is attached. The pendulum consists of a massless rod and two masses attached to it: a mass m at a distance $\ell/2$ and a mass $m/2$ at a distance ℓ. Gravity acts in the vertical direction.

Write the kinetic and potential energy of the system using $\boldsymbol{x}$ and θ as degrees of freedom, then find the equations of motion of the system.

Problem 1.10 The mass m shown in Figure 1.15.e slides without friction on a massless rod of length ℓ. The mass is attached to a linear spring k_1 aligned with the rod (natural length a). A second linear spring k_2 is attached to the end of the rod (zero natural length). This spring is attached to a massless slider so that it remains vertical. Gravity acts in the vertical direction.

Using the generalized coordinates s and θ, respectively the position of the mass on the rod and the angle between the horizontal direction and the rod, write the kinetic and potential energies and derive the equations of motion. What would the equations of motion be if, in parallel to spring k_2, a viscous damper d would be present?

Problem 1.11 Let us consider the system of Figure 1.15.f made of 2 identical pendulums of mass m, length ℓ and moving in a gravity field g.

They are aligned so that there is no reaction force in the equilibrium position $\theta_1 = \theta_2 = 0$. Assuming that one of the pendulums is displaced from its equilibrium position and released from an initial angle θ_0, you are asked to:

- Determine the collision time t^- and the angular velocity of the moving pendulum just before the shock.
- Determine the angular velocities of both pendulums just after the shock.
- Sketch the trajectories of both pendulums in the phase plane $(\theta,\ \dot{\theta})$.

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