PART I

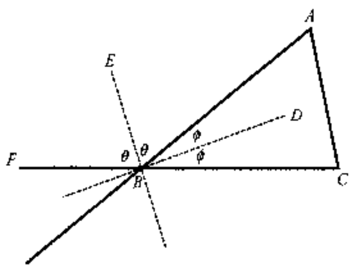
EUCLIDEAN GEOMETRY

CHAPTER 1

CONGRUENCY

1. Prove that the internal and external bisectors of the angles of a triangle are perpendicular.

   Solution. Let BD and BE be the angle bisectors, as shown in the diagram below.



   Then

$$\angle EBD = \angle EBA + \angle DBA = \frac{\angle FBA}{2} + \frac{\angle CBA}{2} = \frac{\angle FBA + CBA}{2} = 90.$$

Solutions Manual to Accompany Classical Geometry: Euclidean, Transformational, Inversive, and Projective, First Edition. By I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky.

3. Let P be a point inside $\triangle ABC$. Use the Triangle Inequality to prove that $AB + BC > AP + PC$.

Solution. Extend AP to meet BC at D. Using the Triangle Inequality,

$$AB + BD > AD = AP + PD$$

so that

$$AB + BD + DC > AP + PD + DC.$$

Since

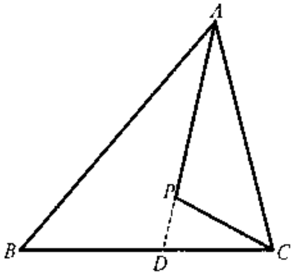
$$BD + DC = BC$$

and

$$PD + DC > PC,$$

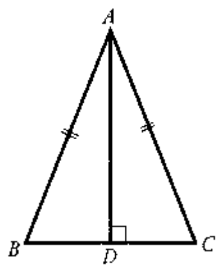
we have

$$AB + BC > AP + PC.$$



5. Given the isosceles triangle ABC with $AB = AC$, let D be the foot of the perpendicular from A to BC. Prove that AD bisects $\angle BAC$.

Solution. Referring to the diagram, the two right triangles ADB and ADC have a common side and equal hypotenuses, so they are congruent by **HSR**. Consequently, $\angle BAD \equiv \angle CAD$.

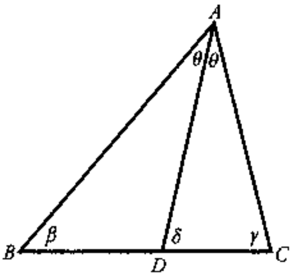


7. D is a point on BC such that AD is the bisector of $\angle A$. Show that

$$\angle ADC = 90 + \frac{\angle B - \angle C}{2}.$$

Solution. Referring to the diagram, $2\theta + \beta + \gamma = 180$, which implies that

$$\theta = 90 - \frac{\beta + \gamma}{2} .$$



From the Exterior Angle Theorem, we have

$$\delta = \theta + \beta,$$

so that

$$\delta = 90 - \frac{\beta + \gamma}{2} + \beta = 90 + \frac{\beta - \gamma}{2}.$$

9. Construct a right triangle given the hypotenuse and one side.

   Solution. We construct a right triangle ABC given the hypotenuse BC and the length c of side AC.

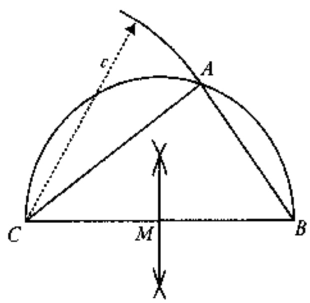
   Construction.

   (1) Construct the right bisector of BC, yielding M, the midpoint of BC.

   (2) With center M, draw a semicircle with diameter BC.

   (3) With center C and radius equal to c, draw an arc cutting the semicircle at A.

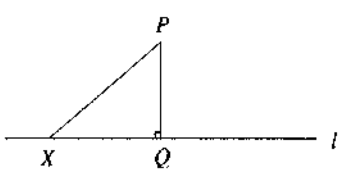
   Then ABC is the desired triangle.



   Justification. $\angle BAC$ is a right angle by Thales' Theorem.

11. Let Q be the foot of the perpendicular from a point P to a line l. Show that Q is the point on l that is closest to P.

    Solution. Let X be any point on l with $X \neq Q$, as in the figure below.



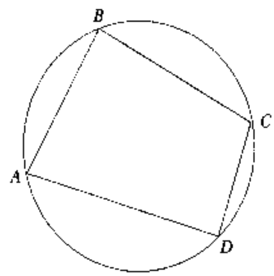
By Pythagoras' Theorem, we have

$$PX^2 = PQ^2 + XQ^2 > PQ^2,$$

and therefore $PX > PQ$.

13. Let $ABCD$ be a simple quadrilateral. Show that $ABCD$ is cyclic if and only if the opposite angles sum to $180°$.

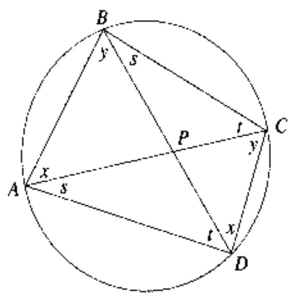
Solution. We will show that the simple quadilateral $ABCD$ can be inscribed in a circle if and only if $\angle A + \angle C = 180$ and $\angle B + \angle D = 180$.



Note that we only have to show that $\angle A + \angle C = 180$, since if this is true, then

$$\angle B + \angle D = 360 - (\angle A + \angle C) = 360 - 180 = 180.$$

Suppose first that the quadrilateral $ABCD$ is cyclic. Draw the diagonals AC and BD and let P be the intersection of the diagonals, then use Thales' Theorem to get the angles as shown.

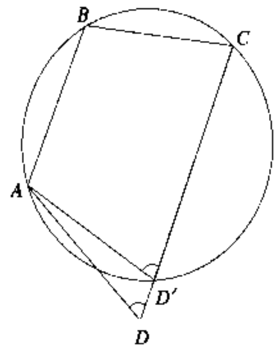


Since the sum of the internal angles in $\triangle ABC$ is 180, then

$$x + y + s + t = (x + s) + (y + t) = 180.$$

That is, $\angle A + \angle C = 180$ and $\angle B + \angle D = 180$, so that opposite angles are supplementary.

Conversely, suppose that $\angle A + \angle C = 180$ (and therefore that $\angle B + \angle D = 180$ also) and let the circle shown on the following page be the circumcircle of $\triangle ABC$.



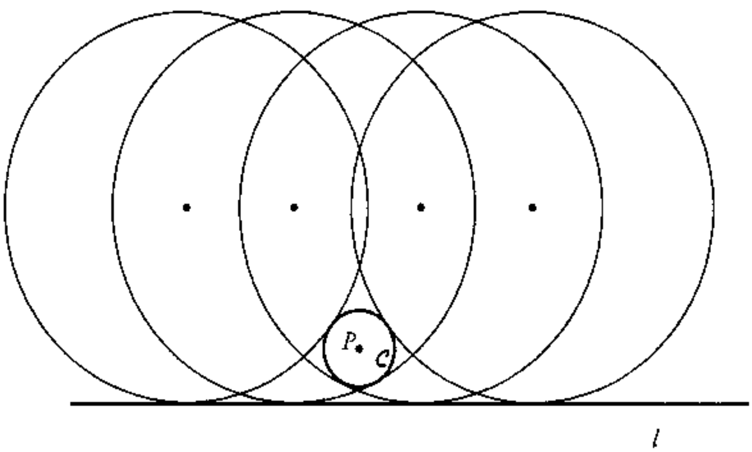
If the quadrilateral $ABCD$ is not cyclic, then the point D does not lie on this circumcircle. Assume that D lies outside the circle and let D' be the point where the line segment CD hits the circle. Since $ABCD'$ is a cyclic quadrilateral, $\angle B + \angle D' = 180$ and therefore $\angle D = \angle D'$, which contradicts the External Angle Inequality in $\triangle AD'D$.

If the point D is inside the circle, a similar argument leads to a contradiction of the External Angle Inequality.

Thus, if $\angle A + \angle C = 180$ and $\angle B + \angle D = 180$, then quadrilateral $ABCD$ is cyclic.

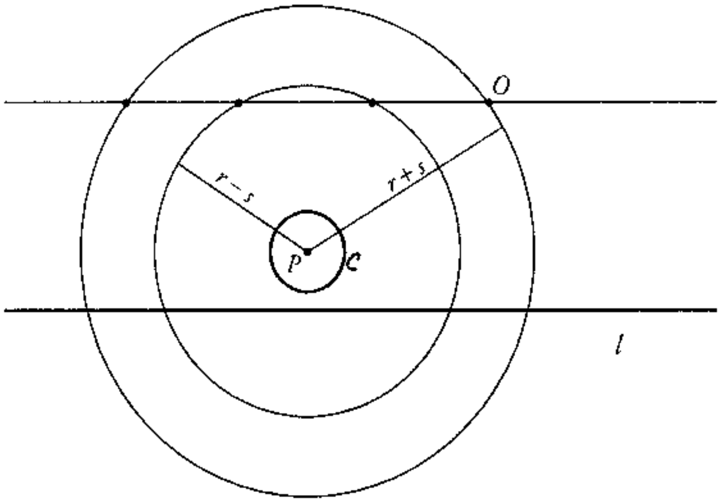
15. Given a circle $\mathcal{C}(P, s)$, a line l disjoint from $\mathcal{C}(P, s)$, and a radius r, $(r > s)$, construct a circle of radius r tangent to both $\mathcal{C}(P, s)$ and l.

Note: The analysis figure indicates that there are four solutions.



Solution. We see that the centers of the circles lie on the following constructible loci:

- a line parallel to l at distance r from l
- a circle $\mathcal{C}(P, r+s)$ or a circle $\mathcal{C}(P, r-s)$



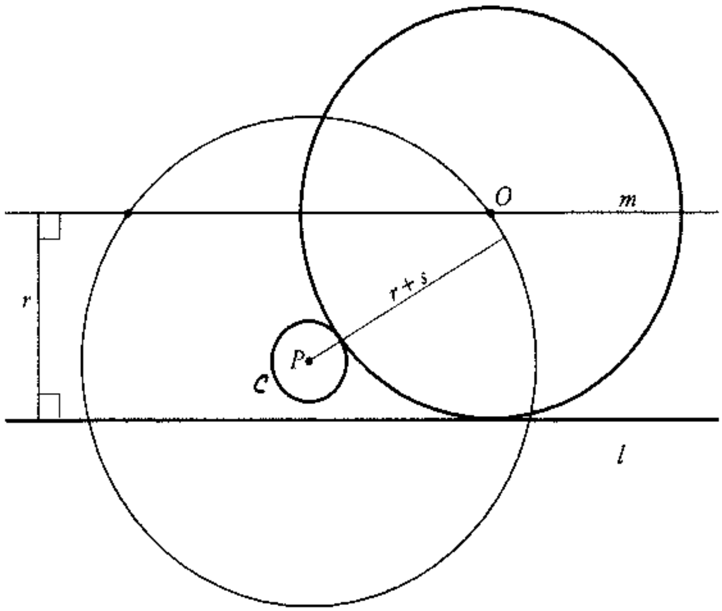
Since we are given the radius, the construction is reduced to finding the centers of the desired circles. We show how to construct one of them.

(1) Construct a line m parallel to l at distance r from l on the same side of l as $\mathcal{C}(P, s)$.

(2) Construct $\mathcal{C}(P, r+s)$.

(3) Let $O = m \cap \mathcal{C}(P, r+s)$. Note that if m and $\mathcal{C}(P, r+s)$ do not intersect, there is no solution.

(4) Construct $\mathcal{C}(O, r)$.


O
m
r
r + s
P
C
l