1

Elastic Waves

In a deformable solid medium, the disturbance to mechanical equilibrium is represented by the change in particle velocity and the corresponding changes in stress and strain states. When some parts of a solid are first disturbed, finite time durations are required for this disequilibrium to be felt by other parts of the body, due to the deformable properties of the body. This kind of propagation as a result of the disturbance in stress and strain through a solid body is termed a *stress wave*.

1.1 Elastic Wave in a Uniform Circular Bar

1.1.1 The Propagation of a Compressive Elastic Wave

Consider a uniform circular bar made of isotropic material, as shown in Figure 1.1. Let x denote the *longitudinal coordinate* measured from an origin O, which is fixed in the space; and let $u(x)$ denote the *displacement* undergone by a plane AB in the bar, which is initially at a distance x from O. Then $u + \frac{\partial u}{\partial x}\delta x$ is the *displacement* of plane $A'B'$ which is parallel to AB but is initially at a distance $x + \delta x$ from O.

A force applied rapidly at time $t = 0$, over the end plane at $x = 0$, will cause a *disturbance* to propagate elastically along the bar, so that a compressive normal stress, $-\sigma_0$, will pass through plane AB at time t.

It should be noted that the slender bar assumption is adopted here, i.e. the pulse length is at least six times the typical cross-sectional dimension of the bar. In this case, the strain and inertia in the *transverse* direction can be neglected. The gravitational force and damping of the material are also ignored in the following analysis.

The equilibrium of a representative element of the bar is illustrated in Figure 1.2. Here A_0 is the initial cross-section area of the bar, ρ_0 is the initial density of the material, and $-\sigma_0$ is the stress transmitted, with the negative sign reflecting the fact that the stress is compressive, as shown in the figure.

From Newton's second law, the equation of motion of the representative element is $-\frac{\partial \sigma_0}{\partial x}\delta x\, A_0 = \rho_0 A_0 \delta x\, \frac{\partial^2 u}{\partial t^2}$, which could be simplified as

$$\frac{\partial \sigma_0}{\partial x} = -\rho_0 \frac{\partial^2 u}{\partial t^2} \tag{1.1}$$

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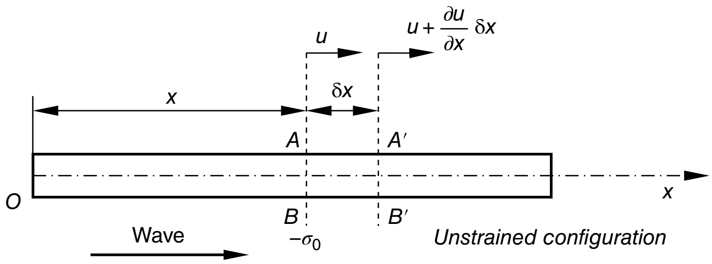

Figure 1.1 The propagation of a compressive elastic wave in a uniform circular bar.

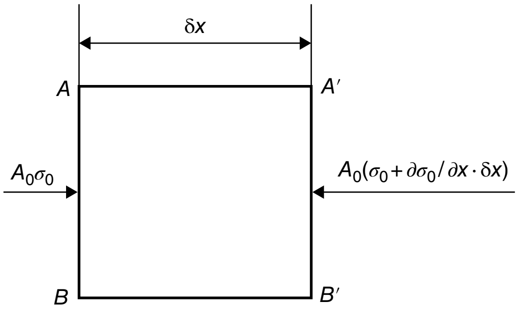

Figure 1.2 The equilibrium of a representative element of the bar.

To analyze the deformation of the representative element of length δx, it is clear that the strain of the element is

$$\varepsilon = \frac{\partial u}{\partial x} \tag{1.2}$$

Assuming the solid material has Young's modulus E, then, according to Hooke's law, in the linear elastic stage we have

$$-\sigma_0 = E\frac{\partial u}{\partial x} \tag{1.3}$$

The stress variation over the element is obtained from the partial differential of Eq. (1.3) with respect to x:

$$\frac{\partial \sigma_0}{\partial x} = -E\frac{\partial^2 u}{\partial x^2} \tag{1.4}$$

Substituting Eq. (1.1) into Eq. (1.4) leads to

$$\rho_0\frac{\partial^2 u}{\partial t^2} = E\frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

With the notation of $c_L = \sqrt{E/\rho_0}$, Eq. (1.5) is rewritten as

$$\frac{\partial^2 u}{\partial t^2} = c_L^2\frac{\partial^2 u}{\partial x^2} \tag{1.6}$$

Obviously, Eq. (1.6) is a typical one-dimensional (1D) wave equation of the following form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.7}$$

Considering the general solution of Eq. (1.7), $u(x, t)$, in the following form:

$$u(x,t) = f_1(x - ct) + f_2(x + ct) \tag{1.8}$$

and substituting Eq. (1.8) into the wave equation (Eq. 1.7), we find

$$\frac{\partial u}{\partial t} = -cf_1'(x-ct) + cf_2'(x+ct), \quad \frac{\partial^2 u}{\partial t^2} = c^2 f_1''(x-ct) + c^2 f_2''(x+ct)$$
$$\frac{\partial u}{\partial x} = f_1'(x-ct) + f_2'(x+ct), \quad \frac{\partial^2 u}{\partial x^2} = f_1''(x-ct) + f_2''(x+ct)$$

Thus it can be verified that Eq. (1.8) satisfies the wave equation, so it gives a general solution to Eq. (1.7). In order to understand the mechanical meaning of Eq. (1.8), only one term is studied here, $u(x,t) = f_1(x - ct)$, i.e., $f_2 = 0$ (see Figure 1.3).

At $t = t_1$, the particle at position $x = x_1$ has a displacement $u = s$, and at $t = t_2$, the particle at position $x = x_2$ also has a displacement $u = s$. Thus from Eq. (1.8), the displacement should satisfy $s = f_1(x_1 - ct_1) = f_1(x_2 - ct_2)$, which results in $x_1 - ct_1 = x_2 - ct_2$, leading to the following speed of wave propagation:

$$c = \frac{x_2 - x_1}{t_2 - t_1} \tag{1.9}$$

This confirms that for the wave propagation governed by Eq. (1.6), $c_L = \sqrt{E/\rho_0}$ precisely represents the speed of the longitudinal waves (compressive or tensile).

There are two terms on the right-hand side of the general solution of wave propagation, Eq. (1.8). The term $f_1(x - ct)$ denotes the wave traveling in the $+x$ direction, i.e., a forward-traveling wave, and the term $f_2(x + ct)$ denotes the wave traveling in the $-x$ direction, i.e., a backward-traveling wave. Both the traveling waves in Eq. (1.8), $f_1(x - ct)$ and $f_2(x + ct)$, have the following characteristics: the waves are traveling at a constant speed with no change in their shape or magnitude, i.e. the 1D longitudinal waves are *non-dispersive*.

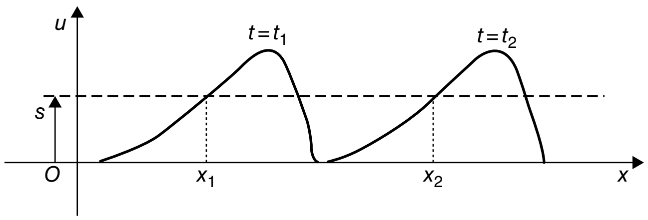

Figure 1.3 Particle displacements produced by a forward wave.

Table 1.1 Typical longitudinal wave speed in solid materials

	Steel	Aluminum	Glass	Polystyrene
E (GPa)	205	75	95	
ρ_0 (g/cm^3)	7.8	2.7	2.5	
c_L (m/s)	5100	5300	6200	2300

E, Young's modulus; ρ_0, density; c_L, wave speed.

The speed of longitudinal (compressive or tensile) waves in four typical materials are given in Table 1.1. It should be emphasized that the wave speed depends on both Young's modulus and density of the material. Therefore, although the density of aluminum is only about one-third of that of steel, the wave speeds are similar.

For a 1D longitudinal wave, there are two ways in which you can distinguish between a compressive and a tensile wave:

- *Look at the sign of the stress*. A compressive wave produces a negative normal stress and a tensile wave produces a positive stress.
- *Look at the directions of the particle velocity and the wave propagation*. For a compressive wave, the particle velocity is in the same direction as the wave propagation, whereas for a tensile wave, the particle velocity is in the opposite direction to the wave propagation.

1.2 Types of Elastic Wave

Different types of elastic wave can propagate in solids. These waves are classified according to the relationship between the motion of the particles and the direction of propagation of the waves and also according to the boundary conditions. The most common types of elastic wave in solids are:

- Longitudinal (irrotational) waves
- Transverse (shear) waves
- Surface (Rayleigh) waves
- Interfacial (Stoneley) waves
- Bending (flexural) waves (in beams and plates).

1.2.1 Longitudinal Waves

Longitudinal waves are those in which the particle velocity is parallel to the direction of travel of the wave. In particular, longitudinal waves are called compressional waves or compression waves, because they produce compression as the particle velocity and wave velocity are in the same direction. By contrast, a tensile wave produces tension as the particles and waves travel in opposite directions.

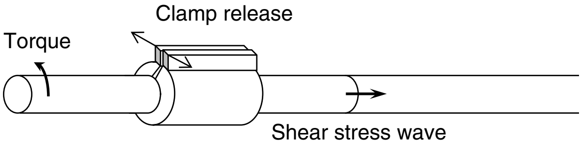

Figure 1.4 The transverse wave in a circular bar produced by suddenly releasing the clamp.

Longitudinal waves are also known as *irrotational waves*. In seismology, they are known as P-waves. This is because a longitudinal wave travels at the fastest speeds and so arrives at seismic stations first (**p**rimary waves); the rock moves forward and backward in the same direction as the wave is traveling (**p**ush–pull wave, **p**arallel to propagation), and the wave is able to **p**ass through solids and liquids. In infinite and semi-infinite media, longitudinal waves are also known as *dilatational waves*, due to the changes in the volume of the media.

1.2.2 Transverse Waves

Transverse waves are another common type of wave, in which the particle velocity is perpendicular to the wave's propagation. An example of a transverse wave is shown in Figure 1.4. In this arrangement a circular bar is clamped at a given position and a torque is applied to the left end of the bar. Thus there is shear stress on the left side of the clamp and zero stress on the right-hand side. When the clamp is suddenly released, the stress disturbance will propagate, i.e., a wave will travel from the left side to the right side of the bar. The particle velocity is within the cross-sectional plane of the bar, while the wave propagation direction is along the bar; hence, as the particle velocity is perpendicular to the wave velocity, this torsional wave is a transverse wave. The normal strains are all zero, with no resulting change in density, while the shear strains are non-zero, producing a change in shape. Thus, transverse waves are called *shear waves*, and are also known as *distortional* or *equivolumal waves*.

For an elastic material with shear modulus G and density ρ_0, the speed of the transverse wave is derived as $c_S = \sqrt{G/\rho_0}$, which is slower than that of the longitudinal wave in the same solid, as $\frac{c_S}{c_L} = \sqrt{\frac{G}{E}} = \frac{1}{\sqrt{2(1+\nu)}} < 1$.

In seismology, transverse waves are known as **S**-waves, because they do not travel as quickly as P-waves (**s**low wave) and will arrive at a seismic station second (**s**econdary wave); the rocks move from **s**ide to side (**s**hear wave) and the waves only travel through **s**olids.

1.2.3 Surface Wave (Rayleigh Wave)

Surface waves are analogous to gravitational waves on the surface of water. As shown in Figure 1.5, the material particles move up and down as well as back and forth, tracing elliptical paths. The surface wave is restricted to the region adjacent to the surface. The particle velocity decreases very rapidly (exponentially) as one moves away from

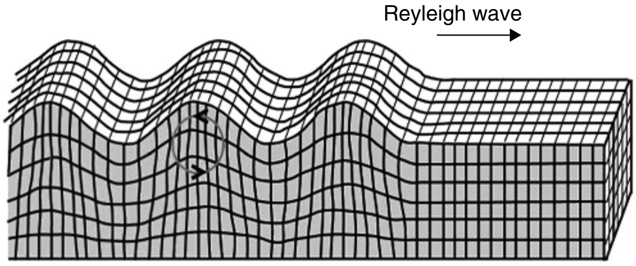

Figure 1.5 Diagram of a Rayleigh wave.

the surface. Surface waves in solids (called *Rayleigh waves*) are a particular case of interfacial waves where one of the materials has negligible density and elastic wave speed.

If a hammer hits the surface of a semi-infinite solid, several waves are propagated after the blow – among these the speed of the longitudinal wave (P-wave) is greater than that of the transverse wave (S-wave), and the surface wave (Rayleigh wave) only affects the solid for a finite distance under the surface.

1.2.4 Interfacial Waves

When two semi-infinite media with different material properties are in contact, special waves form at their interface in the case of a disturbance. The surface wave in a solid (Rayleigh wave) could be regarded as a special case of the interfacial wave – that is, the density and elastic wave in one of the contacting media, such as air, could be omitted.

1.2.5 Waves in Layered Media (Love Waves)

The earth is composed of layers with different properties, and so special wave patterns emerge. This was first studied by Love. As a result of Love waves, the horizontal component of displacement produced by earthquakes can be significantly larger than the vertical component, which is a behavior that is not consistent with Rayleigh waves.

1.2.6 Bending (Flexural) Waves

These waves involve propagation of flexure in 1D (beams and arches) or 2D (plates and shells) configurations. Made from a material of density ρ_0 and elastic modulus E, a straight beam with cross-sectional area A_0 and principal moment of inertia I is shown in Figure 1.6. A coordinate system with the x-axis along the beam length and the z-axis in the direction of deflection is adopted. When a bending moment M and shear force Q are applied, a transverse deflection w is produced in the beam.

By considering the equilibrium of a small element of length δx as shown in Figure 1.6 (b), the equation of motion of this element gives

$$-(\rho_0 A_0 \delta x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial Q}{\partial x}\delta x \tag{1.10}$$

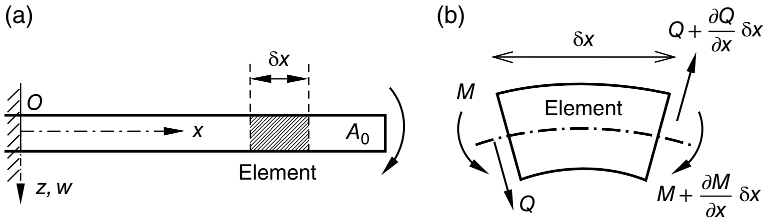

Figure 1.6 A straight beam under bending. (a) Beam under bending; (b) free body diagram of an element.

From the relation given by elastic mechanics,

$$EI\frac{\partial^3 w}{\partial x^3} = Q \tag{1.11}$$

The combination of Eqs (1.10) and (1.11) leads to the wave equation of a bending wave:

$$\rho_0 A_0 \frac{\partial^2 w}{\partial t^2} = -EI\frac{\partial^4 w}{\partial x^4} \tag{1.12}$$

Or in an equivalent form:

$$\frac{\partial^2 w}{\partial t^2} = -c_{\mathrm{L}}^2 k^2 \frac{\partial^4 w}{\partial x^4} \tag{1.13}$$

where $c_{\mathrm{L}} = \sqrt{E/\rho_0}$ is the speed of the longitudinal wave, and k denotes the radius of gyration of the cross-section about the neutral axis, i.e., $I = A_0 k^2$ holds.

Obviously, solutions in the form of $w(x, t) = f_1(x - ct)$ or $w(x, t) = f_1(x + ct)$, which are the general solutions of the regular wave equation, do not satisfy the bending wave equation (Eq. 1.13). This implies that flexural disturbance of arbitrary form always propagates with dispersion.

1.3 Reflection and Interaction of Waves

1.3.1 Mechanical Impedance

Let us focus on the longitudinal waves again. As shown in Sections 1.1 and 1.2, for a forward longitudinal wave, i.e., the wave moving in the positive x-direction, from the general solution of the wave equation, Eq. (1.8), the displacement of particle is

$$u(x,t) = f(x - ct) \tag{1.14}$$

Differentiating Eq. (1.14) with respect to time t leads to the particle velocity:

$$v_0 = \frac{\partial u}{\partial t} = -cf'(x,t) \tag{1.15}$$

A partial differentiation of Eq. (1.14) with respect to particle position x leads to the strain of this element:

$$\varepsilon = \frac{\partial u}{\partial x} = f'(x,t) \tag{1.16}$$

From the property of an elastic material, the stress on this element is given by

$$\sigma = -E\varepsilon = -Ef' = \frac{Ev_0}{c} \tag{1.17}$$

Using the expression of wave speed, $c = \sqrt{E/\rho_0}$, Eq.(1.17) can be rewritten as

$$\sigma = \frac{Ev_0}{c} = \rho_0 c v_0 = v_0\sqrt{E\rho_0} \tag{1.18}$$

where the quantity $\rho_0 c$ is termed the *mechanical impedance*, or *sonic/sound impedance*, of the material. This expression can also be applied to the propagation of the tensile wave:

$$v_0 = \frac{\sigma}{\sqrt{E\rho_0}} = \frac{c}{E}\sigma = \frac{\sigma}{\rho_0 c} \tag{1.19}$$

In Eq. (1.19), the particle velocity is related to the current stress. For example, for steel, if the stress is 100 MPa, then the particle velocity is

$$v_0 = \frac{c}{E}\sigma = \frac{5100\,\mathrm{m/s} \times 100\,\mathrm{MPa}}{205\,\mathrm{GPa}} \approx 2.49\,\mathrm{m/s} \tag{1.20}$$

The mechanical impedance of steel is

$$\rho_0 c = 7800\,\mathrm{kg/m^3} \times 5100\,\mathrm{m/s} \approx 4 \times 10^7\,\mathrm{Ns/m^3} \tag{1.21}$$

Wave speed and *mechanical impedance* are two very important concepts for stress waves. The wave speed indicates the velocity of the disturbance propagating in a deformable solid, while the mechanical impedance represents the degree of resistance of the deformable solid to the disturbance.

1.3.2 Waves When they Encounter a Boundary

We will briefly describe the interaction of waves when they encounter a boundary. Figure 1.7 shows the longitudinal waves that are reflected and refracted at the boundary as well as the two transverse waves that are generated at the interface. These effects, reflection and refraction, occur when the wave encounters a medium with different mechanical impedance, which is defined as the product of the medium density and its elastic wave speed. These refraction and reflection angles are given by a simple relationship of the form:

$$\frac{\sin\theta_1}{c_\mathrm{L}} = \frac{\sin\theta_2}{c_\mathrm{S}} = \frac{\sin\theta_3}{c_\mathrm{L}} = \frac{\sin\theta_4}{c'_\mathrm{L}} = \frac{\sin\theta_5}{c'_\mathrm{S}} \tag{1.22}$$

The interactions of a wave with an interface are very simple when the incidence is normal ($\theta_1 = 0$). In this case, a longitudinal wave refracts/transmits and reflects longitudinal

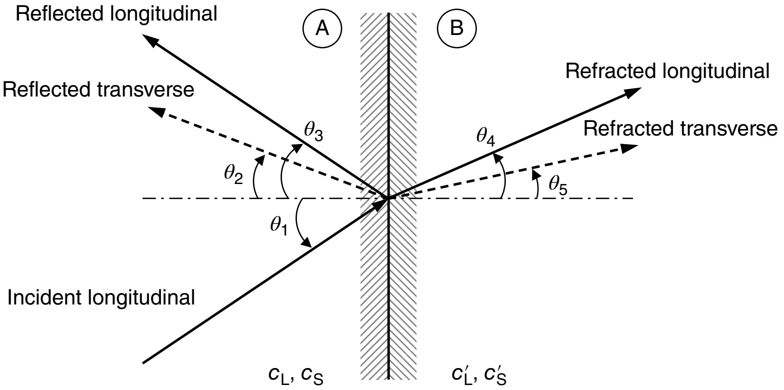

Figure 1.7 Reflection and refraction when a longitudinal wave encounters an interface.

waves, and a shear wave refracts/transmits shear waves. In this way, it becomes a 1D wave reflection and transmission problem.

1.3.3 Reflection and Transmission of 1D Longitudinal Waves

Figure 1.8(a) shows the fronts of a wave propagating along a cylinder in a medium in which the wave speed is c_A. The particle velocity is v and the stress is σ. Figure 1.8(b) illustrates the stresses at the interface related to incident, transmitted, and reflected

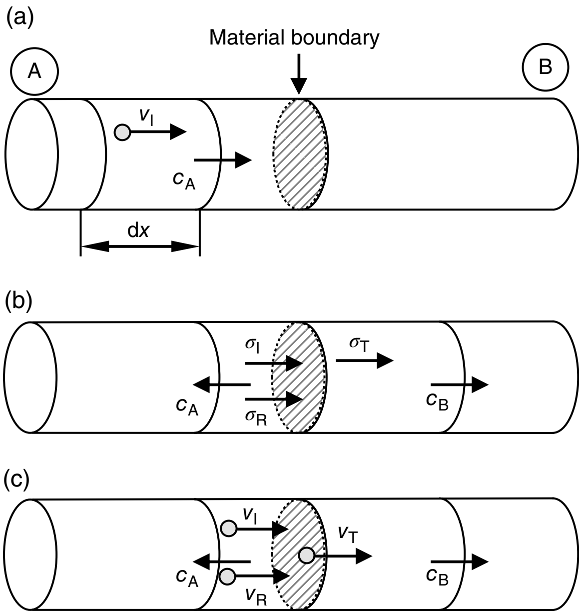

Figure 1.8 The reflection and transmission of a longitudinal wave in a one-dimensional cylinder.

waves. Figure 1.8(c) depicts the particle velocities generated by the incident, transmitted, and reflected waves. The amplitudes of the transmitted and reflected waves can be calculated from the densities, ρ_A, ρ_B, and wave velocities, c_A, c_B, of the two media.

Let subscripts I, T and R pertain to the incident, transmitted and reflected waves, respectively. From the force equilibrium at the interface, i.e., the A–B boundary of the bar,

$$\sigma_I + \sigma_R = \sigma_T \tag{1.23}$$

Then, from the material's continuity at the interface,

$$v_I + v_R = v_T \tag{1.24}$$

For the incident, transmitted, and reflected waves, employing the impedance relation between particle velocity and stress, i.e., Eq.(1.19),

$$v_I = \frac{\sigma_I}{\rho_A c_A}, \quad v_R = -\frac{\sigma_R}{\rho_A c_A}, \quad v_T = \frac{\sigma_T}{\rho_B c_B} \tag{1.25}$$

The stresses produced by the transmitted and reflected waves are obtained by combining Eqs (1.23)– (1.25):

$$\frac{\sigma_T}{\sigma_I} = \frac{2\rho_B c_B}{\rho_B c_B + \rho_A c_A}, \quad \frac{\sigma_R}{\sigma_I} = \frac{\rho_B c_B - \rho_A c_A}{\rho_B c_B + \rho_A c_A} \tag{1.26}$$

From Eq. (1.26), the amplitudes of transmitted and reflected waves are all determined by the mechanical impedances of materials. When the mechanical impedance of material B is larger, then $\rho_B c_B > \rho_A c_A$, $\sigma_R/\sigma_I > 0$, i.e., a wave with the same sign as the incident wave is reflected; whereas when the mechanical impedance of material A is larger, then $\rho_B c_B < \rho_A c_A$, $\sigma_R/\sigma_I < 0$, i.e., a wave with the opposite sign to the incident wave is reflected.

Similarly, the particle velocities produced by the transmitted and reflected waves can be calculated as

$$\frac{v_R}{v_I} = \frac{\rho_A c_A - \rho_B c_B}{\rho_A c_A + \rho_B c_B}, \quad \frac{v_T}{v_I} = \frac{2\rho_A c_A}{\rho_A c_A + \rho_B c_B} \tag{1.27}$$

Special Case 1: Reflection at a Free End

For stress waves being reflected at a free end, material B can be regarded as air with $\rho_B c_B = 0$. Substituting $\rho_B c_B = 0$ into Eq. (1.26) and (1.27) gives

$$\frac{\sigma_T}{\sigma_I} = 0, \quad \frac{\sigma_R}{\sigma_I} = -1, \quad \frac{v_R}{v_I} = 1, \quad \frac{v_T}{v_I} = 2 \tag{1.28}$$

The stress and particle velocity are depicted in Figure 1.9. The stress remains zero at the free end when an incident wave arrives, and the reflected wave at the free end has the opposite sign to the incident wave; thus, the reflected wave is a tensile wave for a compressive incident wave, and the reflected wave is a compressive wave for a tensile incident wave. The particle velocity is doubled at the free end; when the reflected wave passes, the particle velocity is the same as that when the incident wave passes.

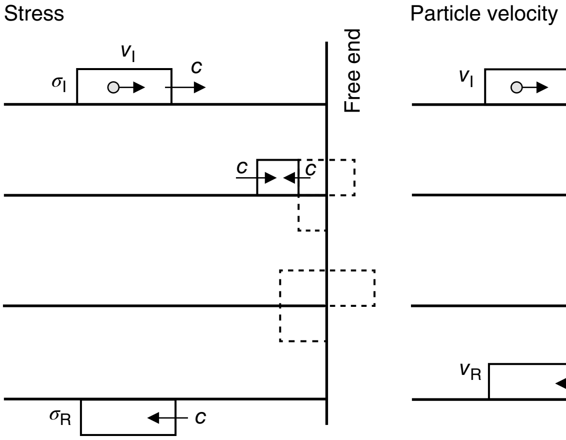

Figure 1.9 One-dimensional wave reflection at a free end.

Special Case 2: Reflection at a Fixed End

For stress wave being reflected at a fixed (clamped) end, material B could be regarded as a rigid body, $E_B = \infty$, and this leads to $\rho_B c_B = \infty$. Substituting $\rho_B c_B = \infty$ into Eq.(1.26) and (1.27) results in

$$\frac{\sigma_T}{\sigma_I} = 2, \quad \frac{\sigma_R}{\sigma_I} = 1, \quad \frac{v_R}{v_I} = -1, \quad \frac{v_T}{v_I} = 0 \tag{1.29}$$

The stress and particle velocity are depicted in Figure 1.10. The stress is doubled at a fixed end when an incident wave arrives, and the reflected wave at the fixed end has the same sign as the incident wave; thus, the reflected wave is also a compressive wave for a compressive incident wave, and the reflected wave is a tensile wave for a tensile incident wave. The particle velocity remains at zero at the fixed end; when the reflected wave

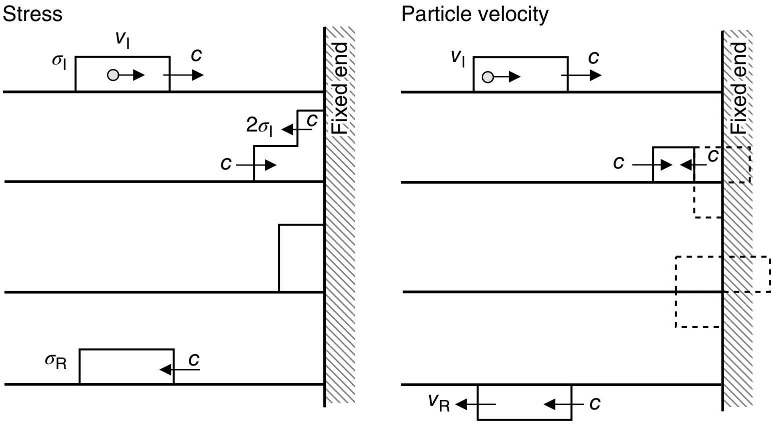

Figure 1.10 One-dimensional wave reflection at a fixed end.

passes, the particle velocity is in the opposite direction to when the incident wave passes, but the magnitudes are the same.

Examples of 1-D Wave Propagation and Interaction

Example 1 Formation of a Rectangular Pulse

Consider a semi-infinite long bar made of elastic material, as shown in Figure 1.11. At time t = 0, give the free end A an initial velocity v_0 towards the *right*, which creates a 1D compressive wave.

Then at an instant $t > 0$, the wave front reaches B with $AB = ct$, whilst the particle originally located at A moves to A' with $AA' = v_0 t$. In segment $A'B$, the stress created is compressive with magnitude $\sigma = \rho c v_0$, and all the particles in this segment have the same velocity v_0.

At $t = T$, give the free end A a velocity v_0 towards the *left*, which creates a 1D tensile wave.

At an instant $t > T$, the tensile wave front reaches D with $A'D = c(t-T)$, while the free end moves to A''. These two elastic waves superpose on each other, resulting in a rectangular stress pulse of length cT and magnitude $\sigma = \rho c v_0$. This pulse (in segment DB) is a compressive wave and moves towards the right at speed c. Apart from the particles within segment DB, the particles in the bar possess no stress and no velocity, so they are in their undisturbed state.

Example 2 The Interaction of a Compressive Pulse and a Tensile Pulse

Consider a slender bar A_1A_2 with a center point B, as shown in Figure 1.12. In the bar segment A_1B, a compressive pulse of stress σ is propagating from the left towards the right, while the particles within the pulse are traveling at velocity $+v_0$ (to the right).

At the same time, in the bar segment A_2B that is symmetrical to segment A_1B, a tensile wave with the same stress magnitude σ is propagating from the right towards the left, while the particles within the pulse are traveling at velocity $+v_0$ (to the right).

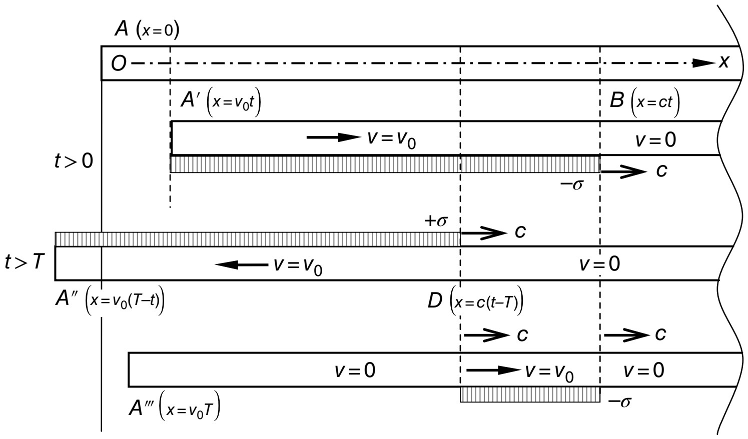

Figure 1.11 A rectangular pulse in a semi-infinite long bar.

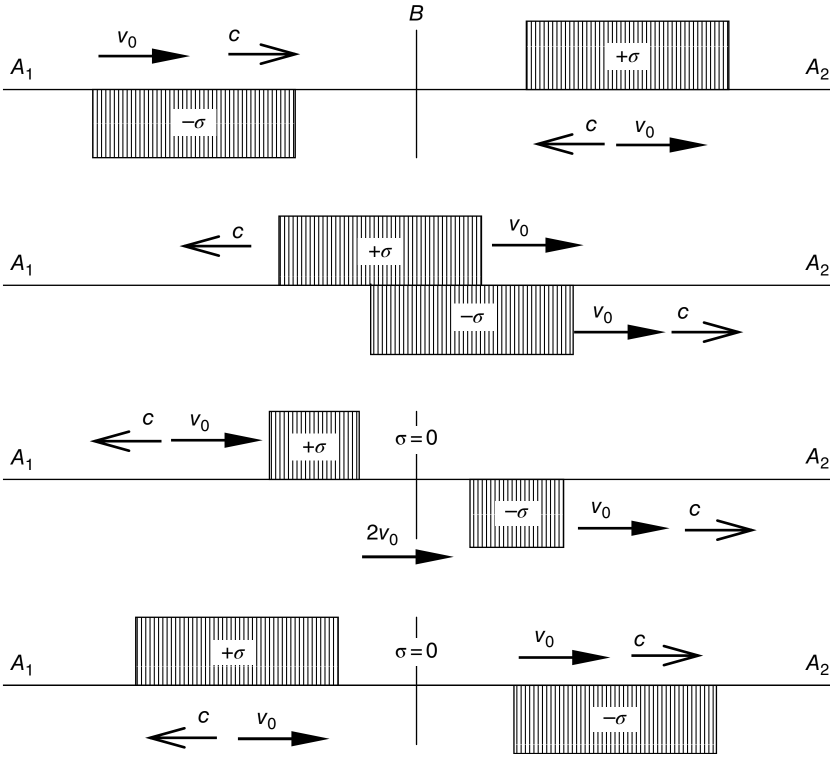

Figure 1.12 The interaction of a compressive pulse and a tensile pulse.

When the two pulses meet each other at cross-section B, the stress at B is reduced to zero, while the particle velocity is doubled, i.e., $2v_0$. After that, in the region where the two pulses overlap each other, the stress remains zero, while the particle velocity becomes $2v_0$, i.e., it is doubled.

After the two pulses are separated, a compressive pulse propagates in segment A_2B, while a tensile wave propagates in segment A_1B. Considering the fact that at any instant the stress at B remains at zero, imagine the bar is cut at B and becomes the free end of both bars A_1B and A_2B. Then the above propagation picture represents the wave reflection at the free end B, i.e., when a 1D wave is reflected at a free end, the stress changes its sign while the particle velocity is doubled.

Example 3 The Interaction of Two Tensile Pulses

Consider a slender bar A_1A_2 with center point D, as shown in Figure 1.13. In segments A_1D and A_2D, which are symmetrical to each other, two identical tensile pulses of stress σ are propagating head to head, while the particles within the pulses are traveling at velocity $+v_0$ (to the right) and $-v_0$ (to the left), respectively.

When the two pulses meet each other at cross-section D, the particle velocity at D is reduced to zero, while the stress becomes 2σ, i.e., it is doubled. After that, in the region

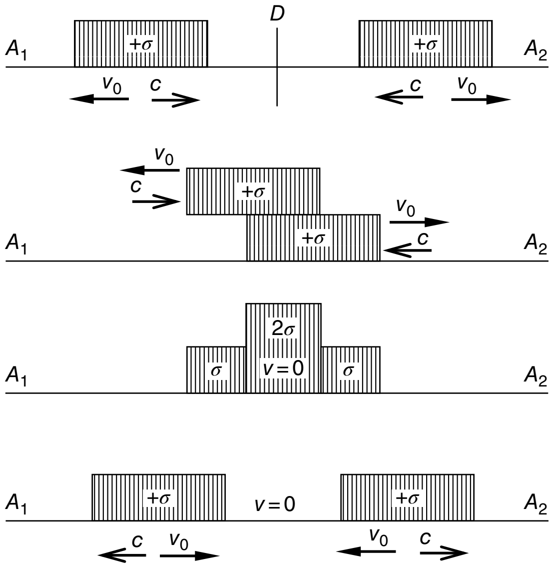

Figure 1.13 The interaction of two tensile pulses.

where the two pulses overlap each other, the particle velocity remains zero, whilst the stress becomes 2σ, i.e., it is doubled.

After the two pulses are separated, two tensile pulses continue to propagate in the respective segments, but the pulses propagate in the opposite direction to the incident pulses. Considering the fact that at any instant the particle velocity at D remains zero, imagine the bar is cut at D and becomes the fixed ends of both bars A_1D and A_2D. Then the above propagation picture represents the wave reflection at fixed end D; when a 1D wave is reflected at a fixed end, the particle velocity will remain zero there, while the stress is doubled. In the rest of the bars, both the wave propagation and the particle velocity change their signs.

Example 4 The Normal Collinear Collision of Two Identical Bars

Consider two bars of identical material and size traveling in opposite directions at the same speed, v_0, as shown in Figure 1.14. A collinear collision of these two bars occurs at $t = 0$. Immediately after the collision, compressive waves of stress $\sigma = \rho c v_0$ will propagate along the two bars. The velocity of the particles inside the region passed by the wave fronts becomes zero.

In Figure 1.14(b), the stress wave propagations and reflections are plotted in the plane of position and time ($x - t$), which is termed the *space–time diagram*, or the *Lagrange diagram*. As the speed of an elastic wave is constant, in the space–time diagram each wave is represented by a straight line of slope $1/c$. Thus, after the normal collinear collision, the two compression waves generated are represented by two straight lines starting from the origin O, with a slope of $1/c$, moving towards the upper-left and upper-right directions, respectively.

At time $t = L/c$, with L denoting the length of each bar, everywhere in the two bars has zero particle velocity, i.e., the two bars are at rest, but they still have compressive stress

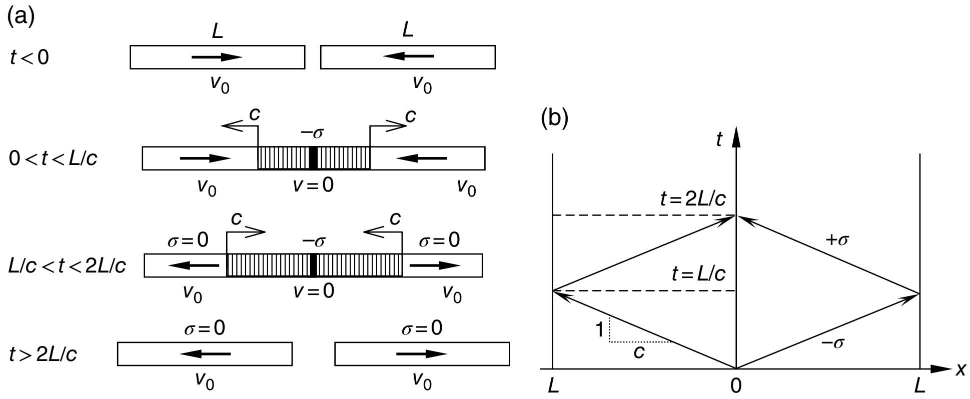

Figure 1.14 The normal collinear collision of two identical bars. (a) Stress distribution; (b) space–time diagram.

$\sigma = \rho c v_0$. These compressive pulses are reflected at the free ends and become tensile ones. When these two reflected tensile pulses travel inwards, in the regions behind the wavefronts the stresses reduce to zero, while the particles gain an outward velocity v_0. In the space–time diagram (Figure 1.14b), the two reflected tensile waves are represented by two straight lines starting from the boundary with a slope of $1/c$.

At time $t = 2L/c$, two reflected tensile wavefronts meet at the interface, $x = 0$, so that the particles at the interface also gain an outward velocity v_0; consequently, the two bars are separated. From the analysis of this example, it is evident that the space–time diagram (Lagrange diagram) is a powerful tool with which to display wave propagations and interactions.

For the bars of cross-sectional area A, the initial kinetic energy of one bar is $K_0 = AL\rho v_0^2/2$. At time $t = L/c$, the initial kinetic energy is converted entirely to the elastic strain energy of the bar, $W^e = AL\sigma^2/2E = AL\rho v_0^2/2$. After time $t = 2L/c$, however, the strain energy is converted entirely back to kinetic energy. There is no energy loss in the whole process, so the *coefficient of restitution* (COR) is $e = 1$ in this perfectly elastic collision case, which is unlikely to happen in real collisions.

In summary, first, the governing equation of elastic waves was derived in this chapter, and then various types of elastic wave were classified. Finally, the wave reflection and interaction were elaborated with several examples.

Questions 1

1 Please list the basic assumptions of 1D elastic wave theory, and point out its limitations.

2 For engineering applications, in which cases should the effects of elastic wave propagation be considered?

Problems 1

1.1 A long cylindrical rod is subjected to a suddenly applied torque at one of its ends. Show that the speed of the elastic torsional wave produced is given by $c_T = \sqrt{G/\rho_0}$, where G and ρ_0 are the shear modulus and density of the material, respectively.

1.2 Three identical bars lie along a straight line, as shown in the figure. The length of each bar is L. Initially, bars 2 and 3 contact each other, and bar 1 travels with velocity v_0 towards them. With the help of a t–x diagram, illustrate what happens after the collision at $t = 0$.