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General Information

1.1 Introduction

In the classification of mechanical structures, somewhere between one-dimensional (1D) bar structures and three-dimensional (3D) solid structures, a class of two-dimensional (2D) plates and shells (thin-walled flat and curved structures) can be distinguished. The attention is focused on a deformable solid body, which is limited by two surfaces (top and bottom) and lateral surfaces, see Figure 1.1. The distance between the top and bottom surfaces, identified as the thickness, is small compared to the other dimensions of the body (e.g. radius of curvature or span), measured referring to the so-called primary surface (2D physical model), most often taken as the middle surface defined as equidistant from the top and bottom surfaces.

The following, generally accepted nomenclature is going to be used throughout the book:

- shells = thin-walled curved shells
- curved membranes = special shells that have no bending rigidity
- plates = thin plane structures that have some subclasses:
 - flat membranes = plates with load in the middle plane, sometimes also called panels
 - plates under bending = plates with transverse load (normal to the middle plane), sometimes also called slabs

In the general description, for all these classes we will use the name ‘shell structures’ or, in brief, ‘shells’. In other words, we understand that shell structures can be flat.

Scientists, teachers, students, engineers and even the authors of software are interested in the mechanics of plates and shells. Due to the variety of potential users, the following variants of the mechanical theory have evolved:

- general advanced tensorial shell theory
- technical (engineering) shell theory

The scope of this book is limited to the case of linear constitutive and kinematic equations.

The theory is the basis for the construction of appropriate mathematical models (sets of differential and algebraic equations) and is associated with the calculation method that can be used to solve general or particular mechanical problems.

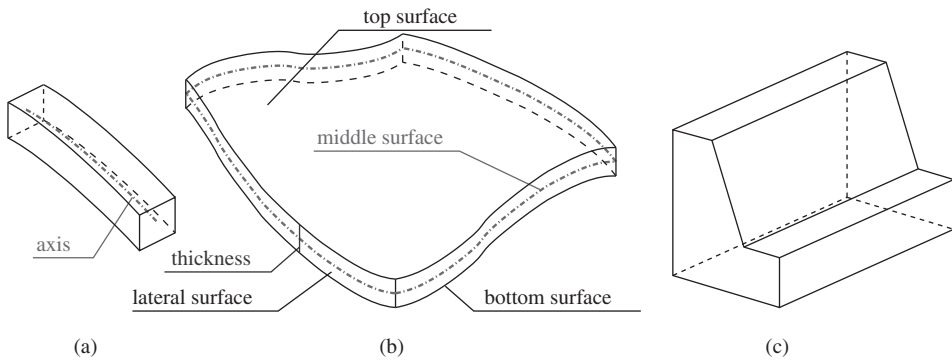


Figure 1.1 Structures: (a) bar (1D), (b) surface (2D) and (c) solid (3D)

In Section 1.2, encyclopedic information on the development of theories describing elastic plates and shells is included.

The description of shell structures, which makes them different from bar (1D) and solid (3D) bodies, must contain the following aspects:

- information on the coordinate systems and geometry of representative surfaces
- specification of kinematic constraints related to the mode of deformation
- definitions of so-called generalized strains with respect to the middle surface
- definitions of resultant forces and moments on the middle surface
- characteristics of fundamental stress and strain states

Detailed discussion is given in Sections 1.3 and 1.4.

A classification of plates and shells can be performed taking into account the slenderness (thickness to span ratio), the shape of the middle surface, the definitions and assumptions presented further in Section 1.4 and the character of stress distribution along the thickness, related to the stress state. In Section 1.5 and in Box 1.1, we present the classification of surface structures according to these aspects.

Thin-walled shell structures of various types are very important structural elements. Examples of shell structures can be encountered in civil and mechanical engineering (slabs, vaults, roofs, domes, chimneys, cooling towers, pipes, tanks, containers, pressure vessels), shipbuilding (ship hulls, submarine hulls) and in the vehicle and aerospace industries (car bodies and tyres, wings and fuselages of aeroplanes).

From an engineering point of view, it is necessary to predict different modes of behaviour of plates and shells under applied loading. In the case of a (flat) plate subjected to a transverse load, static equilibrium is preserved by the action of bending and twisting moments and transverse shear forces. On the other hand, the (curved) shell structure is able to carry the load inducing membrane tension or compression, distributed uniformly throughout the thickness (it is an optimal case from the viewpoint the material strength). This feature of shell structures makes them more economical and stiffer in comparison to plates.

Familiarity with the technical shell theory is necessary for engineers who are responsible for the safety of structures and are supposed to take into account various safety factors using computer-aided design.

Box 1.1 Summary of classification of shell structures

Thin plates	for	$\frac{h}{L_{\min}} < \frac{1}{10}$
Moderately thick plates	for	$\frac{1}{10} \leq \frac{h}{L_{\min}}$
Thin shells	for	$\frac{h}{R_{\min}} \leq \frac{1}{20}$
Moderately thick shells	for	$\frac{1}{20} < \frac{h}{R_{\min}} \leq \frac{1}{6}$
Thick shells	for	$\frac{1}{6} < \frac{h}{R_{\min}}$
Shallow shell	for	$\frac{f}{L} < \frac{1}{5}$

h – thickness of plate or shell

L – characteristic dimension of plate or shell

L_{\min} – the smallest dimension in the middle plane of a plate

R_{\min} – smaller of two principal radii of curvature

f – distance of shell from the horizontal plane, of its projection, that is rise

w – representative deflection

Geometrically linear theory of plates

with small deflections for $|w| < \frac{h}{5}$

von Kármán theory of plates

with moderately large deflections for $|w| \approx h$

Geometrically nonlinear theory of plates

with large deflections for $|w| > 5h$

As emphasized in Ramm and Wall (2004), shell structures exhibit the strong influence of initial geometry, slenderness, type of loading and boundary conditions on the deformation and load carrying capacity. Small variations or even imperfections of these parameters can change the structural response significantly and, in particular, cause loss of stability.

Shells are characterized by an advantageous ratio of stiffness to weight, which makes them suitable for lightweight and long-spanned structures. Moreover, optimal shells are designed to carry predominantly membrane forces with minimum bending effects. It is therefore extremely important to understand the principal mechanical features of plates and shells before using computer-aided design involving numerical simulation.

1.2 Review of Theories Describing Elastic Plates and Shells

The general description of the historical development of plate and shell theory, as well as details of specific theories are referred to in a lot of books and monographs. Here, the authors do not try to present the developmental trends of this branch of mechanics, even limiting interest to the theory of elastic plates and shells undergoing small deformations.

The beginnings of the linear theory of plates and shells date back to the nineteenth century, however, the vibration problem of bells was considered by Leonhard Euler in 1764. The name of Sophie Germain is associated with the theory of plates: in 1811 she submitted work on plates for a contest announced by the French Academy of Sciences.

Following the two encyclopaedic elaborations:

- Mechanics of Elastic Plates and Shells, vol. 8 in *Technical Mechanics* (Borkowski et al. 2001)
- Models and Finite Elements for Thin-walled Structures, Chapter 3 by Bischoff et al. in vol. 2 of *The Encyclopedia of Computational Mechanics* (Stein et al. 2004)

the authors of this book would like to mention the names of researchers associated with the theories of plates and shells from three different, consecutive periods (listing names in alphabetical order):

- nineteenth century:
A. Cauchy, S. Germain, A.E. Green, G. Kirchhoff, A.H. Love, S.D. Poisson and L. Rayleigh
- first half of the twentieth century:
E. Cosserat and F. Cosserat, A.L. Gol'denveizer, Th. von Kármán, S. Lévy, A.I. Lur'e and E. Reissner
- second half of the twentieth century:
Y. Başar, B. Budiansky, L.H. Donnell, J.L. Ericksen, W. Flügge, J.M. Gere, K. Girkmann, K.Z. Golimov, R. Harte, Z. Kączkowski, W.T. Koiter, W.B. Krätzig, H. Kraus, R.D. Mindlin, K.M. Mushtari, P.M. Naghdi, F.I. Niordson, W. Nowacki, V.V. Novozhilov, W. Pietraszkiewicz, E. Ramm, J.L. Sanders, J.G. Simmonds, I. Szabó, S.P. Timoshenko, C. Truesdell, V.Z. Vlasov, W. Wunderlich, S. Woinowski-Krieger, Cz. Woźniak and W. Zerna

We also mention some previous works relevant to the subject of this book, dividing them into:

- books dealing with the basis of mechanics: Timoshenko and Goodier (1951), Fung (1965), Washizu (1975), Reddy (1986), Borkowski et al. (2001) and Stein et al. (2004)
- monographs related to the theories of plates and shells: Girkmann (1956), Timoshenko and Woinowski-Krieger (1959), Kolkunov (1972), Nowacki (1980), Niordson (1985), Noor et al. (1989), Waszczyszyn and Radwańska (1995), Reddy (1999), Başar and Krätzig (2001), Borkowski et al. (2001), Reddy (2007), Radwańska (2009), Wiśniewski (2010) and Oñate (2013)

The general formulation of the theory of thin-walled structures is determined by their specific geometry with one dimension (thickness) much smaller in comparison to the other two dimensions. There are two essential concepts that can be used to formulate the mathematical description of the problem.

One possibility is to start from the equations of three-dimensional continuum, describing a body with a specified geometry. Applying a power series representation of certain quantities as a function of coordinate z (measured in the direction of a thickness) the reduction to a two-dimensional theory is performed. Using a specified number of terms of this representation a 2D problem with varying accuracy of approximation is obtained.

Alternatively, one can adopt suitable kinematic assumptions and treat a thin-walled structure as a two-dimensional continuum representation of a substitute problem, (see Borkowski et al. 2001). This option is associated with direct methods of formulating two-dimensional models of plates and shells, based on appropriate static and kinematic hypotheses. The approximation in this theory is that the deformed state of the shell is determined entirely by the configuration of its middle surface.

Beside the two approaches based on three-dimensional continuum mechanics or two-dimensional surface-based theories we mention a so-called Cosserat surface concept, see for instance Chapter 3 in vol. 2 of Stein et al. (2004). This approach is an extension of classical continuum formulation by adding information about the orientation of a material point equipped with rotational degrees of freedom.

Among the developed theories for shells a few specific approaches can be distinguished:

- general theory applying any parametrization of the curved middle surface
- theory that uses the orthogonal parameterization of the middle surface based on principal curvature coordinates
- general membrane-bending shell theory with or without the consideration of transverse shear deformation
- theories for particular cases of shells (e.g. for cylindrical or spherical shells of revolution)
- theory of plates
- theory of flat membranes

The full set of equations of the linear theory of shells, which contains Kirchhoff plate equations as a special case, are given in pages 173–174 of Love (1944). This theory is called the Kirchhoff–Love (K–L) theory of first approximation or order. In theory based on assumptions of K–L the effects of transverse shear and normal strains in the thickness direction are neglected. The weakening of these assumptions leads to enhanced variants of the equations, the so-called second and third approximations. This involves more complex forms of measurement of deformation and construction of constitutive equations. In fact, the first approximation theory is mathematically and physically incorrect. When the kinematic equations and constitutive equations, used in this approach, are substituted into the sixth equilibrium equation (expressing equilibrium of moments around the normal to the middle surface), the equation is not satisfied. The sixth equilibrium equation guarantees that all strains vanish for small rigid-body rotations of the shell.

The inconsistency of Love's first order theory was removed in the improved theory for thin shells by Sanders (1959), formulating the equations in principal curvature coordinates. For this new improved theory modified equilibrium equations, strain-displacement relations and boundary conditions were derived using the principle of virtual work. The detailed information about the basics of theory of Sanders is presented in Chapter 3.

Koiter checked and corrected Love's theory (see Koiter 1960). An assessment of the order of magnitude of the terms in Love's strain-energy expression was carried out. Appropriate consistent stress-strain relations for stress resultants and equilibrium equations in tensorial form were presented. In the theory the sixth equation of equilibrium is satisfied identically.

In work of Budiansky and Sanders (1963) the equations of the 'best' first-order linear elastic shell theory were formulated for shells of arbitrary shape in a coordinate system related to the middle surface using general tensor notation.

In the broad literature a variety of kinematic and constitutive equations can be found, because different simplifications were used in their derivation. The summary of various descriptions of the strain state and kinematic relations (even for linear analysis) is also presented in the work by Lewiński (1980). The following four essential features of the improved first approximation shell theory are cited here from this paper:

- matrices of generalized strains (membrane strains and changes of curvature) and stress resultants (forces and moments) are symmetric
- constitutive equations are decoupled
- the sixth equation of equilibrium is identically satisfied
- a rigid motion of the shell does not cause strains or stresses

Now, a little information about the classical three-parameter Sanders thin shell theory is given, because this formulation is applied in our book. The equations are considered to be the most suitable with respect to both theoretical and numerical applications. In the geometry description the orthogonality of the coordinate lines implies that the first metric tensor is diagonal, and the surface is described by only two Lamé parameters and two radii of curvature (or curvatures themselves), see Subsection 1.3.2. The following fields are used in the shell problem description: translation(s), rotation(s), generalized strain(s) and stress resultants, all defined with respect to the two-dimensional middle surface. In this three-parameter thin shell theory three translations u_1, u_2, w are adopted as independent variables in the description of the deformation (see Subsection 1.4.1).

The five-parameter theory is used to describe moderately thick shells with five independent generalized displacements: three translations u_1, u_2, w and two rotations ϑ_1, ϑ_2 (see Subsection 1.4.2).

At this point the assumptions adopted in this book are specified:

- translations, rotations and strains are assumed to be small enough for nonlinear components in the kinematic and equilibrium equations to be omitted (thus taking into account only the first order terms)
- the initial undeformed configuration of the structure is the reference configuration
- the material is treated as isotropic linearly elastic, described by Hooke's constitutive equations, that is to define the material only two parameters are used: Young's modulus and Poisson's ratio

A more advanced tensorial formulation of the theory of shell structures can be found, for instance, in Bařar and Krätzig (2001). The theoretical foundations there are coupled with:

- local formulation using differential and algebraic equations
- global formulations employing energy theorems and variational principles for plates and shells

In Chapter 3 of Vol. 2 of Stein et al. (2004), entitled ‘Models and Finite Elements for Thin-walled Structures’, both the mathematical and mechanical foundations of the theory of plates and shells and the description of FE formulations are presented. The chapter includes an extensive derivation of kinematic equations and strains, constitutive equations and stresses as well as the parametrization of displacements and rotations, both in linear and nonlinear range. The long list of references contains 211 items from 1833 to 2003.

Most recent efforts of scientists are aimed at the analysis of:

- anisotropic, composite (in particular layered) shells
- shells undergoing large deformations (with varying magnitude of displacements, rotations and strains)
- shell in inelastic (in particular plastic) states

However, these issues are beyond the scope of this book. The reader is referred to the following works on nonlinear theories of plates and shells: Woźniak (1966), Pietraszkiewicz (1977, 1979, 2001), Crisfield (1982), Hinton et al. (1982), Kleiber (1985), Borkowski et al. (2001), Wiśniewski (2010), de Borst et al. (2012).

1.3 Description of Geometry for 2D Formulation

The description of the geometry of 2D surfaces is based on the works by Waszczyszyn and Radwańska (1995) and Radwańska (2009).

1.3.1 Coordinate Systems, Middle Surface, Cross Section, Principal Coordinate Lines

The analysis of thin and moderately thick shell structures is most often performed with respect to the middle surface, that is to a geometrically two-dimensional object; only thick shells are treated as three-dimensional bodies.

The geometry of a shell structure is defined when the shape of the middle surface, the boundary contour and the thickness distribution have been specified. In the theoretical consideration we assume for simplicity that the thickness is constant.

Two families of curves are introduced. They are parametrized with so-called curvilinear coordinates ξ_1, ξ_2 , see Figure 1.2a, used for an explicit definition of the position of a point on the surface. In most cases a general curvilinear coordinate system will be employed, and further a discussion of particular cases will be provided, for instance the cylindrical (x, θ) , spherical (φ, θ) or Cartesian (x, y) coordinate systems will be applied. In the two-dimensional description of shells analogous pairs of variables (e.g. R_α) or pairs of formulae (e.g. $ds_\alpha = A_\alpha d\xi_\alpha$) will be used, where the Greek index α represents numbers 1 or 2.

On the middle surface the so-called principal curvature lines related to principal curvature radii are specified. Many equations formulated for particular shells refer to these principal (extreme) curvature lines.

At any point P on the middle surface a cross section can be defined. We consider two normal section planes Π_1 and Π_2 , see Figure 1.2b. These planes are perpendicular to each other and their intersections with the middle surface generate arc segments of unit length $ds_\alpha = 1$. We emphasize that the intersection of the planes Π_α is a

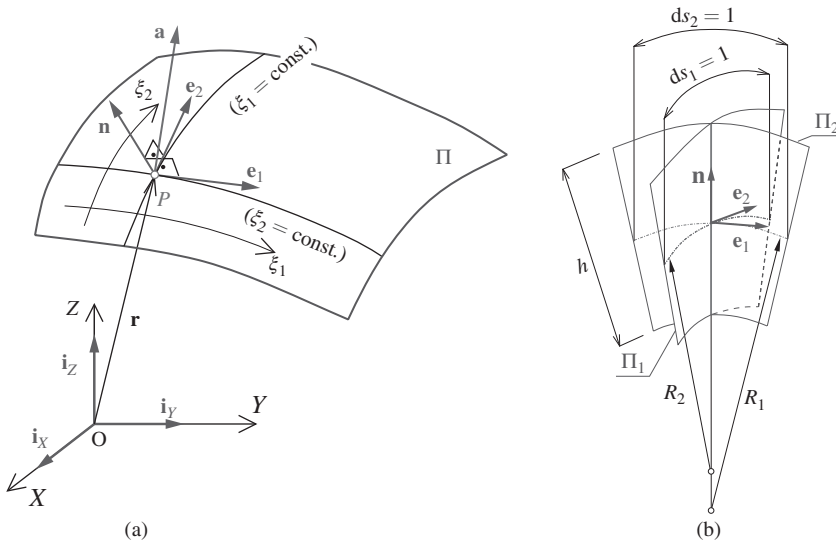


Figure 1.2 (a) A middle surface with curvilinear coordinates ξ_α and local base vectors $\mathbf{e}_\alpha, \mathbf{n}$ at point P , (b) straight fibre – intersection of planes $\Pi_\alpha, \alpha = 1, 2$. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

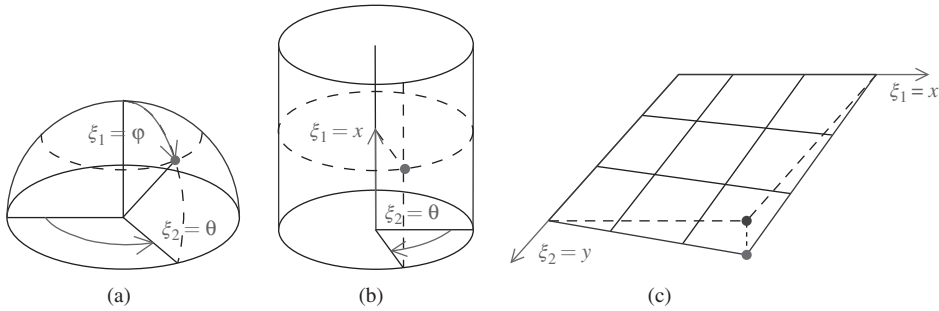


Figure 1.3 Three surfaces: (a) spherical, (b) cylindrical and (c) shallow hyperbolic, corresponding to appropriate coordinate systems

straight fibre (the so-called director), see Figure 1.2b. Its behaviour during deformation is precisely described according to the so-called kinematic hypothesis of Kirchhoff–Love (see Subsection 1.4.1) or Mindlin–Reissner (see Subsection 1.4.2).

Surface coordinates ξ_α are used to identify three common types of surface (see Figure 1.3):

- (a) spherical surface described in a spherical coordinate system (φ, θ)
- (b) cylindrical surface in a cylindrical system (x, θ)
- (c) shallow ruled hyperbolic surface in a Cartesian system (x, y)

1.3.2 Geometry of Middle Surface

For point P on the middle surface Π the connection between global Cartesian coordinates X, Y, Z and local curvilinear coordinates ξ_1, ξ_2 is expressed by the relation

$$\mathbf{r} = X\mathbf{i}_X + Y\mathbf{i}_Y + Z\mathbf{i}_Z \tag{1.1}$$

where:

$$X = f_1(\xi_1, \xi_2), \quad Y = f_2(\xi_1, \xi_2), \quad Z = f_3(\xi_1, \xi_2) \quad (1.2)$$

On the middle surface, a two-dimensional segment $P - P_1 - M - P_2$ is identified, resulting from the intersection of four lines $\xi_1 = \text{const.}$, $\xi_1 + d\xi_1 = \text{const.}$, $\xi_2 = \text{const.}$, $\xi_2 + d\xi_2 = \text{const.}$ (see Figure 1.4a). Next, curve l is considered. The curve, parametrized by coordinate λ , passes through points P and M that are located on the elementary surface subdomain, with lengths of sides ds_α , $\alpha = 1, 2$ measured by so-called Lamé parameters A_α , which are magnitudes of the tangential vectors $\mathbf{r}_{,\alpha}$:

$$ds_\alpha = A_\alpha d\xi_\alpha, \quad A_\alpha = |\mathbf{r}_{,\alpha}| = |\mathbf{g}_\alpha|, \quad (\cdot)_{,\alpha} = \frac{\partial(\cdot)}{\partial \xi_\alpha}, \quad \alpha = 1, 2 \quad (1.3)$$

$$\mathbf{r} = \mathbf{r}[\xi_1(\lambda), \xi_2(\lambda)], \quad d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial \xi_1} \frac{d\xi_1}{d\lambda} + \frac{\partial \mathbf{r}}{\partial \xi_2} \frac{d\xi_2}{d\lambda} \right) d\lambda = \mathbf{r}_{,1} d\xi_1 + \mathbf{r}_{,2} d\xi_2 \quad (1.4)$$

The length of the arch between points P and M on line l is calculated using the formula

$$\begin{aligned} (ds)^2 &= \mathbf{r}_{,1} \cdot \mathbf{r}_{,1} (d\xi_1)^2 + 2\mathbf{r}_{,1} \cdot \mathbf{r}_{,2} d\xi_1 d\xi_2 + \mathbf{r}_{,2} \cdot \mathbf{r}_{,2} (d\xi_2)^2 \\ &= (A_1)^2 (d\xi_1)^2 + 2A_1 A_2 \cos(\mathbf{g}_1, \mathbf{g}_2) d\xi_1 d\xi_2 + (A_2)^2 (d\xi_2)^2 \end{aligned} \quad (1.5)$$

The product of tangential vectors \mathbf{g}_α defines the first (I) metric tensor $g_{\alpha\beta}$

$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta} \quad (1.6)$$

Moreover, the I fundamental quadratic form of the surface is derived

$$(ds)^2 = g_{11} (d\xi_1)^2 + 2g_{12} d\xi_1 d\xi_2 + g_{22} (d\xi_2)^2 \quad (1.7)$$

Next, base vectors with unit length (versors) \mathbf{e}_α , \mathbf{n} are obtained:

$$\mathbf{e}_\alpha = \frac{\mathbf{r}_{,\alpha}}{A_\alpha} = \frac{\mathbf{g}_\alpha}{A_\alpha}, \quad \mathbf{n} = \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \quad (1.8)$$

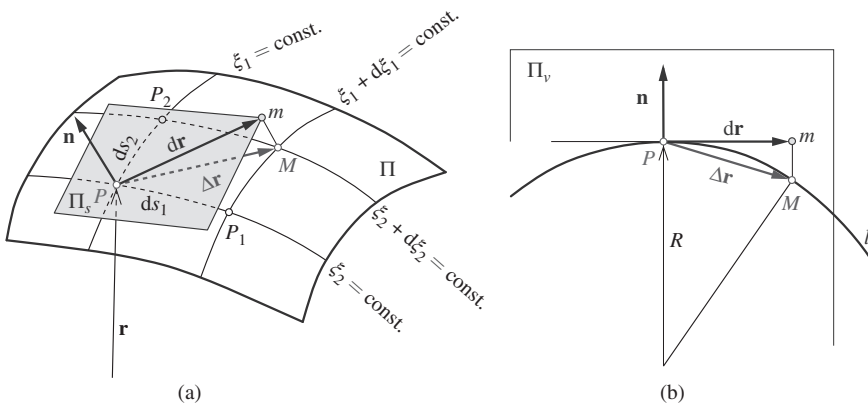


Figure 1.4 Description of objects: (a) on middle surface Π and in plane Π_s , (b) in plane Π_v . Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

where \times denotes the vector product of two vectors. The components of load $\hat{\mathbf{p}}$ and displacement vectors \mathbf{u} can be defined using the local base versors ($\mathbf{e}_\alpha, \mathbf{n}$):

$$\hat{\mathbf{p}} = \hat{p}_1 \mathbf{e}_1 + \hat{p}_2 \mathbf{e}_2 + \hat{p}_n \mathbf{n} \quad (1.9)$$

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + w \mathbf{n} \quad (1.10)$$

The measure of the middle surface curvature for a shell, denoted by m , can be calculated as the length of projection of vector $\Delta \mathbf{r}$ on direction \mathbf{n} , see Figure 1.4b

$$m = \mathbf{n} \cdot \Delta \mathbf{r} = \mathbf{n} \cdot \left(d\mathbf{r} + \frac{1}{2} d^2 \mathbf{r} + \dots \right) = \frac{1}{2} \mathbf{n} \cdot d^2 \mathbf{r} + \dots \quad (1.11)$$

To this end the second (II) metric tensor $b_{\alpha\beta}$

$$b_{\alpha\beta} = \mathbf{r}_{,\alpha\beta} \cdot \mathbf{n} = -\mathbf{r}_{,\alpha} \cdot \mathbf{n}_{,\beta} \quad (1.12)$$

and the II fundamental form of the surface

$$2m = b_{11} (d\xi_1)^2 + 2b_{12} d\xi_1 d\xi_2 + b_{22} (d\xi_2)^2 \quad (1.13)$$

are defined. For line l its curvature radius R and curvature k are calculated as

$$\frac{1}{R} \equiv k = \lim_{|\Delta \mathbf{r}| \rightarrow 0} \frac{2m}{|\Delta \mathbf{r}|^2} = \mathbf{n} \cdot \frac{d^2 \mathbf{r}}{ds^2} \quad (1.14)$$

In relation to the so-called principal coordinate lines, for which $g_{12} = b_{12} = 0$, two extreme principal curvature radii $R_{\alpha\alpha}$, as well as two characteristic parameters, mean curvature H and so-called Gaussian curvature K , are calculated using the formulae:

$$k_{\alpha\alpha} = -\frac{1}{R_{\alpha\alpha}} = \frac{b_{\alpha\alpha}}{g_{\alpha\alpha}} = \frac{b_{\alpha\alpha}}{(A_\alpha)^2} \quad (1.15)$$

$$k^2 - 2Hk + K = 0, \quad H = \frac{1}{2}(k_1 + k_2), \quad K = k_1 k_2 \quad (1.16)$$

1.3.3 Geometry of Surface Equidistant from Middle Surface

Similar to point P on the middle surface Π (see Figure 1.5), we consider point $P^{(z)}$ on surface $\Pi^{(z)}$, equidistant from the middle surface. The position vector $\mathbf{r}^{(z)}$ of point $P^{(z)}$ is the sum of position vector \mathbf{r} of point P and vector $z\mathbf{n}$:

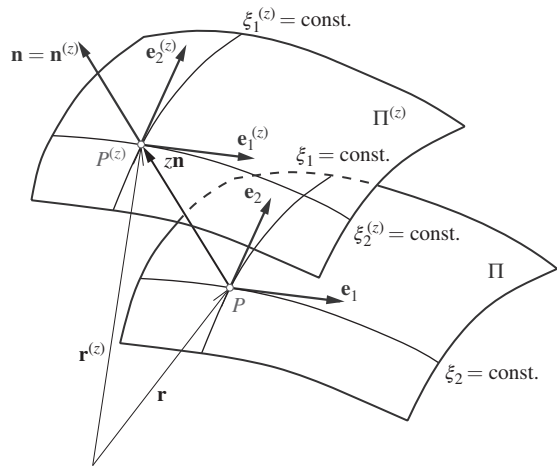
$$\mathbf{r}^{(z)} = \mathbf{r} + z\mathbf{n}, \quad -\frac{h}{2} \leq z \leq \frac{h}{2} \quad (1.17)$$

The following objects $ds_\alpha^{(z)}, \mathbf{e}_\alpha^{(z)}, A_\alpha^{(z)}, R_\alpha^{(z)}, \alpha = 1, 2$, are introduced for the equidistant surface. They are associated with analogous objects for the middle surface by linear functions of coordinate z :

$$ds_\alpha^{(z)} = A_\alpha^{(z)} d\xi_\alpha, \quad \mathbf{e}_\alpha^{(z)} = \frac{1}{A_\alpha^{(z)}} \mathbf{r}_{,\alpha}^{(z)}, \quad \mathbf{n}^{(z)} \equiv \mathbf{n} \quad (1.18)$$

$$A_\alpha^{(z)} = A_\alpha \left(1 + \frac{z}{R_\alpha} \right), \quad R_\alpha^{(z)} = R_\alpha \left(1 + \frac{z}{R_\alpha} \right) \quad (1.19)$$

Figure 1.5 Middle surface Π and equidistant surface $\Pi^{(z)}$. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.



1.3.4 Geometry of Selected Surfaces

We will now present three typical coordinate systems and three selected surfaces as well as scalar, vector and tensor quantities, useful in the description of a surface identified with the middle surface of a shell structure. We will specify base vectors and the first metric tensor. Omitting detailed derivations, we will provide formulae used for the description of geometry of these surface. For more information on the subject, the reader is referred, for instance, to Bařar and Krätzig (2001).

1.3.4.1 Spherical Surface

A spherical surface is located in a 3D space with a Cartesian coordinate system ($x^1 = x, x^2 = y, x^3 = z$). On this surface point P is considered, whose position is defined using two spherical surface coordinates $\xi_1 = \varphi, \xi_2 = \theta$ and radius $R_1 = R_2 = R$ (see Figure 1.6).

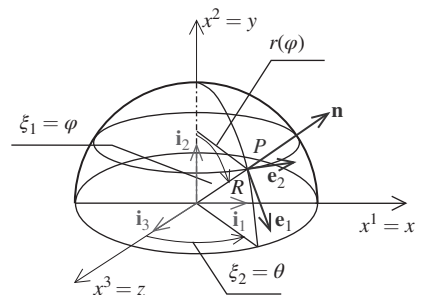
In the global system of axes $x^i, i = 1, 2, 3$, the position vector \mathbf{r} of point P is written first with Cartesian coordinates x^i , and next using two spherical coordinates ξ_α

$$\mathbf{r} = x^i \mathbf{i}_i = R \sin \varphi \sin \theta \mathbf{i}_1 + R \cos \varphi \mathbf{i}_2 + R \sin \varphi \cos \theta \mathbf{i}_3 \tag{1.20}$$

Base vectors ($\mathbf{e}_\alpha, \mathbf{n}$) are derived from the formulae:

$$\begin{aligned} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} &= R \begin{bmatrix} \cos \varphi \sin \theta \mathbf{i}_1 - \sin \varphi \mathbf{i}_2 + \cos \varphi \cos \theta \mathbf{i}_3 \\ \sin \varphi \cos \theta \mathbf{i}_1 + 0 \mathbf{i}_2 - \sin \varphi \sin \theta \mathbf{i}_3 \end{bmatrix} \\ \mathbf{e}_\alpha &= \frac{1}{R} \mathbf{g}_\alpha, \quad \mathbf{n} = \sin \varphi \sin \theta \mathbf{i}_1 + \cos \varphi \mathbf{i}_2 + \sin \varphi \cos \theta \mathbf{i}_3 \end{aligned} \tag{1.21}$$

Figure 1.6 Spherical surface



The following formulae are used in the description of a sphere:

- Lamé parameters:

$$A_1 = |\mathbf{g}_1| = R, \quad A_2 = |\mathbf{g}_2| = R \sin \varphi \quad (1.22)$$

- first metric tensor

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \varphi \end{bmatrix} \quad (1.23)$$

- principal curvature radii

$$R_1 = R_2 = R \quad (1.24)$$

- Gaussian and mean curvatures:

$$K = \frac{1}{R^2}, \quad H = \frac{1}{R} \quad (1.25)$$

1.3.4.2 Cylindrical Surface

A cylindrical surface, for which the symmetry axis is identical to axis $x^1 = x$ of the Cartesian coordinate system (x, y, z) , is shown in Figure 1.7. The position of point P from the cylindrical surface is specified using three Cartesian coordinates x^i , which are related to two cylindrical surface coordinates $\xi_1 = x$ and $\xi_2 = \theta$ and radius R .

The main formulae for the calculation of characteristic parameters, vectors and tensor are (the names are as for the previous surface):

$$\mathbf{r} = x^i \mathbf{i}_i = x \mathbf{i}_1 + R \sin \theta \mathbf{i}_2 + R \cos \theta \mathbf{i}_3 \quad (1.26)$$

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} 1 \mathbf{i}_1 \\ R \cos \theta \mathbf{i}_2 - R \sin \theta \mathbf{i}_3 \end{bmatrix} \quad (1.27)$$

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 \mathbf{i}_1 \\ \cos \theta \mathbf{i}_2 - \sin \theta \mathbf{i}_3 \end{bmatrix}$$

$$\mathbf{n} = \sin \theta \mathbf{i}_2 + \cos \theta \mathbf{i}_3$$

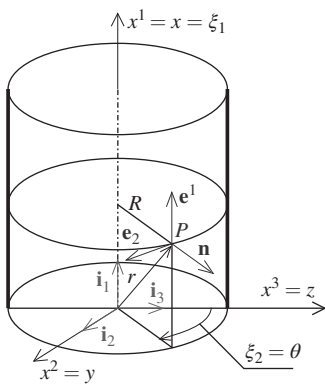


Figure 1.7 Cylindrical surface

$$A_1 = |\mathbf{g}_1| = 1, \quad A_2 = |\mathbf{g}_2| = R \quad (1.28)$$

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix} \quad (1.29)$$

$$R_1 = \infty, \quad R_2 = R \quad (1.30)$$

$$K = 0, \quad H = \frac{1}{2R} \quad (1.31)$$

1.3.4.3 Hyperbolic Paraboloid

The surface called the hyperbolic paraboloid is defined over a rectangle with dimensions $2a \times 2b$ on plane $x^3 = z = 0$ with two Cartesian coordinates $\xi_1 = x$, $\xi_2 = y$. The surface (see Figure 1.8) is defined by the equation:

$$z(x, y) = kxy, \quad k = \frac{f}{ab}, \quad m = z_{,y} = kx, \quad n = z_{,x} = ky \quad (1.32)$$

The characteristic formulae used to describe the surface in question are:

$$\mathbf{r} = x^i \mathbf{i}_i = x \mathbf{i}_1 + y \mathbf{i}_2 + kxy \mathbf{i}_3 \quad (1.33)$$

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} 1 \mathbf{i}_1 + n \mathbf{i}_3 \\ 1 \mathbf{i}_2 + m \mathbf{i}_3 \end{bmatrix} \quad (1.34)$$

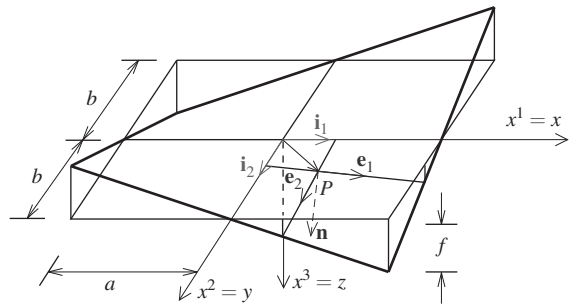
$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \frac{1}{\sqrt{1+m^2+n^2}} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}$$

$$\mathbf{n} = \frac{1}{\sqrt{1+m^2+n^2}} (-n \mathbf{i}_1 - m \mathbf{i}_2 + \mathbf{i}_3)$$

$$A_1 = |\mathbf{g}_1| \approx 1, \quad A_2 = |\mathbf{g}_2| \approx 1 \quad (1.35)$$

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1+n^2 & mn \\ mn & 1+m^2 \end{bmatrix} \quad (1.36)$$

Figure 1.8 Hyperbolic paraboloid



1.4 Definitions and Assumptions for 2D Formulation

1.4.1 Generalized Displacements and Strains Consistent with the Kinematic Hypothesis of Three-Parameter Kirchhoff–Love Shell Theory

The Kirchhoff–Love (K–L) kinematic hypothesis, adopted for thin shell structures, can be formulated in the following manner:

A straight fibre, located at the intersection of two cross-sectional planes, normal to the undeformed (initial) middle surface of a shell, after application of external actions remains straight and normal to the deformed (current) middle surface and has an unchanged length.

To describe the fields of generalized displacements and strains it is necessary to use two surfaces $\Pi, \Pi^{(z)}$ in the initial configuration, as well as two analogous surfaces $\Pi^*, \Pi^{*(z)}$, marked by * and related to the current configuration (after deformation).

In the description of kinematics two middle surfaces Π and Π^* (in initial and current configurations, respectively) are used (see Figure 1.9).

In the analysis of the current configuration the following vectors are distinguished: position vector \mathbf{r} , displacement (translation) vector \mathbf{u} and rotation vector \mathfrak{D} , whose components are related to the local base $(\mathbf{e}_\alpha, \mathbf{n})$ from the initial middle surface Π :

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u} \tag{1.37}$$

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + w \mathbf{n} \tag{1.38}$$

$$\mathfrak{D} = -\vartheta_2 \mathbf{e}_1 + \vartheta_1 \mathbf{e}_2 + \vartheta_n \mathbf{n} = \varphi_1 \mathbf{e}_1 + \varphi_2 \mathbf{e}_2 + \varphi_n \mathbf{n} \tag{1.39}$$

For the rotation vector we can use two types of components: $\vartheta_\alpha, \vartheta_n$ or $\varphi_\alpha, \varphi_n$ (see Figure 1.10).

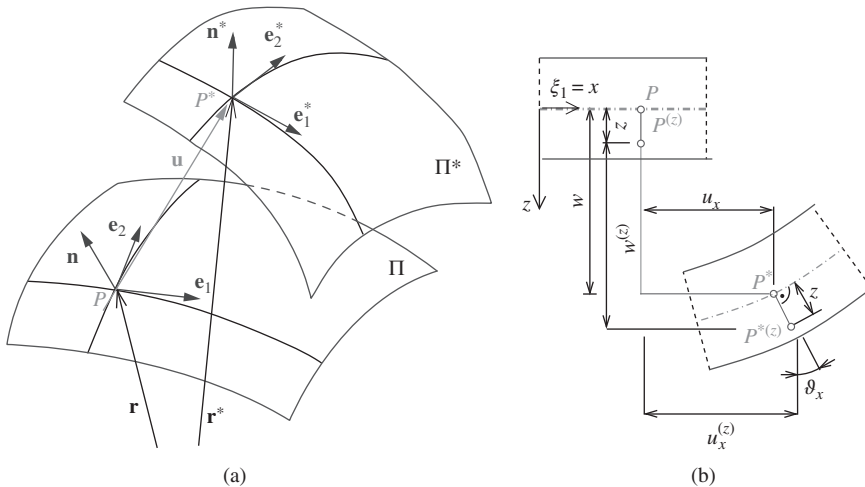
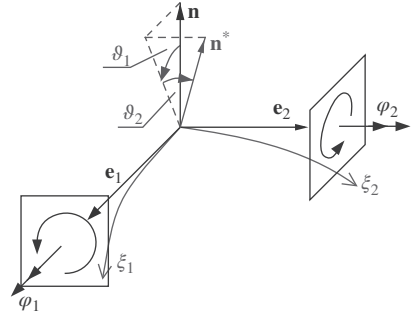


Figure 1.9 (a) Middle surfaces Π and Π^* (before and after deformation), (b) graphical interpretation of kinematic K–L hypothesis for the special case of a flat shell (plate) on plane (ξ_1, z) ; analogous section can be shown for plane (ξ_2, z) . Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

Figure 1.10 Description of rotations of vector \mathbf{n} normal to middle surface. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.



During deformation of the middle surface the orthogonal unit base $(\mathbf{e}_\alpha, \mathbf{n})$ changes into a different base $(\mathbf{e}_\alpha^*, \mathbf{n}^*)$, in general nonorthogonal:

$$\begin{aligned} \mathbf{e}_\alpha^* &= \frac{1}{A_\alpha^*} \mathbf{r}_{,\alpha}^* \approx \mathbf{e}_\alpha + \delta \mathbf{e}_\alpha \\ &= \mathbf{e}_\alpha + \frac{1}{A_\alpha} \left(u_{\beta,\alpha} - \frac{A_{\alpha,\beta}}{A_\beta} u_\alpha \right) \mathbf{e}_\beta + \frac{1}{A_\alpha} \left(w_{,\alpha} - \frac{A_{\alpha}}{R_\alpha} u_\alpha \right) \mathbf{n} \end{aligned} \quad (1.40)$$

$$\mathbf{n}^* = \mathbf{e}_1^* \times \mathbf{e}_2^* = \mathbf{n} + \delta \mathbf{n} = \mathbf{n} - \vartheta_2 \mathbf{e}_1 + \vartheta_1 \mathbf{e}_2 \quad (1.41)$$

The formula (1.41) expresses the change of normal vector \mathbf{n} into new vector \mathbf{n}^* by means of two rotations ϑ_α (see Figure 1.10).

The displacements at point $P^{(z)}$ on surface $\Pi^{(z)}$, equidistant from the middle surface Π , are calculated on the basis of translations u_α, w and rotations ϑ_α , defined at point P on the middle surface Π :

$$u_1^{(z)} = u_1 + z \vartheta_1, \quad u_2^{(z)} = u_2 + z \vartheta_2, \quad w^{(z)} = w \quad (1.42)$$

The K–L kinematic constraints imply the following relations between two rotations ϑ_α and three translations u_α, w :

$$\vartheta_1 = -\frac{1}{A_1} \frac{\partial w}{\partial \xi_1} + \frac{u_1}{R_1}, \quad \vartheta_2 = -\frac{1}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{R_2} \quad (1.43)$$

$$\vartheta_\alpha = -\frac{1}{A_\alpha} \frac{\partial w}{\partial \xi_\alpha} + \frac{u_\alpha}{R_\alpha}, \quad \alpha = 1, 2$$

Equations (1.43) show the possibility of using a shortened notation of two analogous formulae to describe two-dimensional shell structures.

The third rotation ϑ_n , around the normal, is related to the translations by the following equation

$$\vartheta_n = \frac{1}{2} \left[\left(\frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} - \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} \right) - \left(\frac{A_{1,2} u_1}{A_1 A_2} - \frac{A_{2,1} u_2}{A_1 A_2} \right) \right] \quad (1.44)$$

The name three-parameter theory of shell structures originates from the fact that only three translations of points from the middle surface, written in a vector

$$\mathbf{u} = [u_1, u_2, w]^T \quad (1.45)$$

and treated as independent components, suffice to describe generalized displacements of the shell (three translations and three rotations).

Further equations, related to the unchanging length of the straight fibre and its perpendicularity to the current middle surface, express the information concerning zero values of normal strains along the thickness and transverse shear strains:

$$\epsilon_{zz}^{(z)} = \frac{\partial w^{(z)}}{\partial z} = 0, \quad \gamma_{\alpha z} = \vartheta_\alpha + \left[\frac{1}{A_\alpha} \frac{\partial w}{\partial \xi_\alpha} - \frac{u_\alpha}{R_\alpha} \right] = 0, \quad \alpha = 1, 2 \quad (1.46)$$

Subsequently, the definitions of generalized strains and resultant forces referred to the middle surface are presented. It should be emphasized here that transverse shear forces t_α are treated as passive forces as a consequence of the constraints in K–L theory, in which transverse shear strains $\gamma_{\alpha z}$ equal zero.

1.4.2 Generalized Displacements and Strains Consistent with the Kinematic Hypothesis of Five-Parameter Mindlin–Reissner Shell Theory

Unlike the three-parameter K–L theory, in five-parameter Mindlin–Reissner (M–R) theory orthogonality of the straight fibre to the deformed middle surface is not imposed (see Figure 1.11) and a vector of five independent generalized displacements, three translations u_1, u_2, w and two rotations ϑ_1, ϑ_2 , is introduced

$$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2]^T \quad (1.47)$$

In the kinematic equations for moderately thick plates and shells, three translations and two rotations appear as independent variables. In this approach, next to membrane and bending strains, nonzero averaged transverse shear strains are taken into account. In the M–R theory the transverse shear forces are included, they appear in constitutive relations that are written in terms of appropriate strains, forces and the local stiffness operator. The equations of the M–R theory for plates will be given in Section 8.5.

1.4.3 Force and Moment Resultants Related to Middle Surface

In the K–L theory, beside the kinematic hypothesis, a static hypothesis is introduced, which simplifies the continuum equations:

In comparison with the other stress tensor components stress σ_z is so small that for all points of a thin shell it can be ignored in the constitutive relations, so that

$$\sigma_z(\xi_1, \xi_2, z) \equiv 0 \quad (1.48)$$

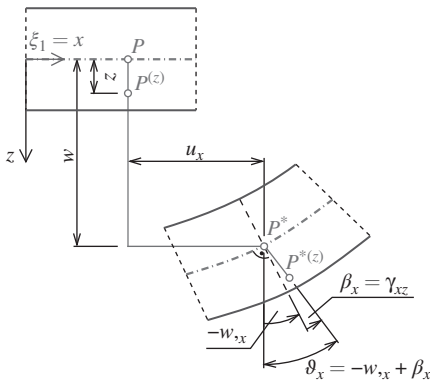


Figure 1.11 Position of a straight fibre before and after deformation of a plate according to the kinematic M–R hypothesis; analogical section can be shown for plane (ξ_2, z)

As a consequence of the static K–L hypothesis further simplifications are made in the description of thin shell structures.

At any point on the middle surface Π , one of two sections obtained using plane Π_α with the normal versor \mathbf{e}_α is shown in Figure 1.12, and the following stress vector is revealed at the intersection

$$\sigma_\alpha(\xi_1, \xi_2, z) = \sigma_{\alpha 1} \mathbf{e}_1 + \sigma_{\alpha 2} \mathbf{e}_2 + \sigma_{\alpha n} \mathbf{n}, \quad \alpha = 1, 2 \tag{1.49}$$

Next, integrating along the thickness, the intensities of resultant forces \mathbf{f}_α [kN/m] and moments \mathbf{m}_α [kNm/m] with adequate components, visible in section with normal \mathbf{e}_α , are first calculated for $\alpha = 1$:

$$\mathbf{f}_1 = f_{11} \mathbf{e}_1 + f_{12} \mathbf{e}_2 + f_{1n} \mathbf{n}, \quad \mathbf{m}_1 = m_{11} \mathbf{e}_2 - m_{12} \mathbf{e}_1 \tag{1.50}$$

where:

$$\begin{aligned} f_{11} = n_{11} &= \int_{-h/2}^{+h/2} \sigma_{11} \left(1 + \frac{z}{R_2} \right) dz, & f_{12} = n_{12} &= \int_{-h/2}^{+h/2} \sigma_{12} \left(1 + \frac{z}{R_2} \right) dz \\ f_{1n} = t_1 &= \int_{-h/2}^{+h/2} \sigma_{1z} \left(1 + \frac{z}{R_2} \right) dz \\ m_{11} &= \int_{-h/2}^{+h/2} \sigma_{11} z \left(1 + \frac{z}{R_2} \right) dz, & m_{12} &= \int_{-h/2}^{+h/2} \sigma_{12} z \left(1 + \frac{z}{R_2} \right) dz \end{aligned} \tag{1.51}$$

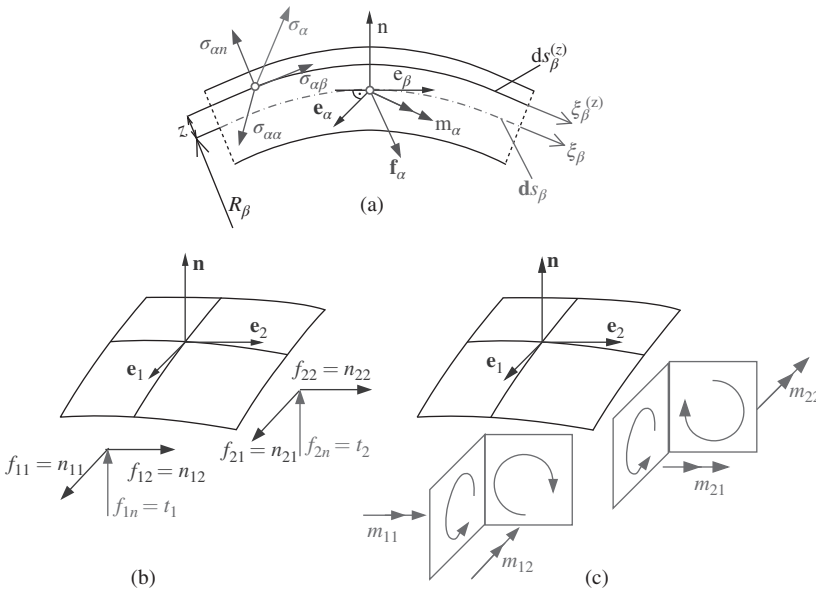


Figure 1.12 (a) Section of a shell with characteristic stresses and vectors of: force \mathbf{f}_α and moment \mathbf{m}_α , (b) and (c) elementary surface segments with all resultant forces revealed on section lines in the membrane-bending state. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

The derivations are repeated for $\alpha = 2$:

$$\mathbf{f}_2 = f_{21}\mathbf{e}_1 + f_{22}\mathbf{e}_2 + f_{2n}\mathbf{n}, \quad \mathbf{m}_2 = -m_{22}\mathbf{e}_1 + m_{21}\mathbf{e}_2 \tag{1.52}$$

where:

$$\begin{aligned} f_{22} = n_{22} &= \int_{-h/2}^{+h/2} \sigma_{22} \left(1 + \frac{z}{R_1} \right) dz, & f_{21} = n_{21} &= \int_{-h/2}^{+h/2} \sigma_{21} \left(1 + \frac{z}{R_1} \right) dz \\ f_{2n} = t_2 &= \int_{-h/2}^{+h/2} \sigma_{2z} \left(1 + \frac{z}{R_1} \right) dz \\ m_{22} &= \int_{-h/2}^{+h/2} \sigma_{22} z \left(1 + \frac{z}{R_1} \right) dz, & m_{21} &= \int_{-h/2}^{+h/2} \sigma_{21} z \left(1 + \frac{z}{R_1} \right) dz \end{aligned} \tag{1.53}$$

In these equations a set of quantities are defined for shells. They are membrane forces: normal n_{11}, n_{22} and tangential n_{12}, n_{21} , moments: bending m_{11}, m_{22} and twisting m_{12}, m_{21} , as well as transverse shear forces t_1, t_2 . In the definitions a factor $(1 + z/R_\alpha)$ appears, resulting from the relation between the lengths of arc segments from the middle and equidistant surfaces

$$ds_\alpha^{(z)} = ds_\alpha (1 + z/R_\alpha) \tag{1.54}$$

In the case of thin shell structures or weakly curved shells ($R_\alpha \rightarrow \infty$), when $z/R_\alpha \ll 1$, these definitions can be reduced, omitting the factor $(1 + z/R_\alpha) \approx 1$ and then the following equalities are valid $n_{12} = n_{21}, m_{12} = m_{21}$.

1.4.4 Generalized Strains in Middle Surface

In the current configuration (see Figure 1.13) Lamé parameters $A_\alpha^{*(z)}$ and base versors $\mathbf{e}_\alpha^{*(z)}$ are calculated for surface $\Pi^{*(z)}$ equidistant from the middle surface Π^* . The normal

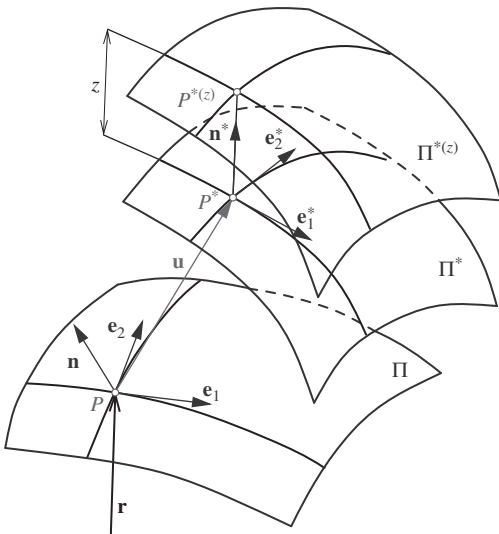


Figure 1.13 Surfaces $\Pi, \Pi^*, \Pi^{*(z)}$ defined for description of deformation. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

and shear strains are expressed in terms of the generalized strains from the middle surface using the linear function of coordinate z :

$$\varepsilon_{\alpha\alpha}^{(z)} = \frac{A_\alpha^{*(z)}}{A_\alpha^{(z)}} - 1 \approx \varepsilon_{\alpha\alpha} + \kappa_{\alpha\alpha} z, \quad \gamma_{12}^{(z)} = \mathbf{e}_1^{*(z)} \cdot \mathbf{e}_1^{*(z)} \approx \gamma_{12} + \chi_{12} z \quad (1.55)$$

The symbol \approx means that necessary simplifications were made when the relations were formulated, resulting from the assumption $(1 + z/R_\alpha)^{-1} \approx 1 - z/R_\alpha$ and the omission of higher order terms.

Further, we quote, without derivation, the kinematic equations proposed by Sanders for the three-parameter thin shell theory of Sanders (1959):

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} u_2 + \frac{w}{R_1} \\ \varepsilon_{22} &= \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} u_1 + \frac{w}{R_2} \\ \gamma_{12} &= \frac{A_2}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{u_2}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{u_1}{A_1} \right) \\ \kappa_{11} &= -\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} \right) - \frac{1}{A_1 A_2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2} \right) \frac{\partial A_1}{\partial \xi_2} \\ \kappa_{22} &= -\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2} \right) - \frac{1}{A_1 A_2} \left(\frac{1}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} \right) \frac{\partial A_2}{\partial \xi_1} \\ \chi_{12} &= -\frac{A_2}{A_1} \frac{\partial}{\partial \xi_1} \left[\frac{1}{A_2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2} \right) \right] - \frac{A_1}{A_2} \frac{\partial}{\partial \xi_2} \left[\frac{1}{A_1} \left(\frac{1}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} \right) \right] \\ &\quad + \frac{1}{2A_1 A_2} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \left[\frac{\partial(A_2 u_2)}{\partial \xi_1} - \frac{\partial(A_1 u_1)}{\partial \xi_2} \right] \end{aligned} \quad (1.56)$$

They are general equations, which after the introduction of specified coordinates ξ_α , Lamé parameters A_α and principal curvature radii R_α ($\alpha = 1, 2$), can be suitably modified in order to describe selected types of shells.

1.5 Classification of Shell Structures

To propose a classification of shells the two geometrical parameters are used:

- h – thickness of plate or shell
- L – characteristic dimension of plate or shell $L = \min(L_{\min}, R_{\min})$, where L_{\min} – the smallest dimension in the middle plane of a plate, R_{\min} – smaller of two principal radii of curvature of the middle surface in a shell

In next subsections the following properties of shell structures are discussed:

- geometry (shape)
- measure of slenderness h/L
- types of stress state
- magnitude of considered displacements

The classification is summarized in Box 1.1.

1.5.1 Curved, Shallow and Flat Shell Structures

Shell structures, represented by two-dimensional middle surfaces, are divided into curved, shallow and flat, see Figure 1.14.

When the middle surface is curved, we deal with thin-walled curved shell structures and the following cases are possible:

- shells curved in one direction, for example cylindrical or conical shells, for which Gaussian curvature $K = 1/(R_1 R_2) = 0$
- doubly curved shells, for example a spherical shell with $K > 0$ or rotational hyperbolic shell with $K < 0$
- shells with a ruled surface, for example a hyperbolic paraboloid shell
- shells of completely arbitrary shapes

Flat structures represented by a two-dimensional middle plane have two subclasses with respect to load type:

- flat membranes, that is plates with load (and deformation) in the middle plane, with no transverse displacements
- plates with transverse load, that is load normal to the middle plane, exhibiting transverse displacements (deflections)

Shallow shells are characterized by rise f , in other words by the surface deviation from its projection on the horizontal plane. The shell is considered shallow when $\frac{f}{L} < \frac{1}{5}$ and then the Cartesian set of coordinates can be used for the approximate description of geometry, but in the corresponding equations curvature radii R_x and R_y , resulting from the surface definition $z(x, y)$, are taken into account.

1.5.2 Thin, Moderately Thick, Thick Structures

The structure can be treated as thin-walled if $\frac{h}{L} \ll 1$. As regards the thickness the following limitations are adopted:

- thin flat structures (plates and membranes) $\frac{h}{L_{\min}} < \frac{1}{10}$
- moderately thick flat structures $\frac{1}{10} \leq \frac{h}{L_{\min}}$
- thin curved structures (shells and membranes) $\frac{h}{R_{\min}} \leq \frac{1}{20}$
- moderately thick curved structures $\frac{1}{20} < \frac{h}{R_{\min}} \leq \frac{1}{6}$
- thick shells $\frac{1}{6} < \frac{h}{R_{\min}}$

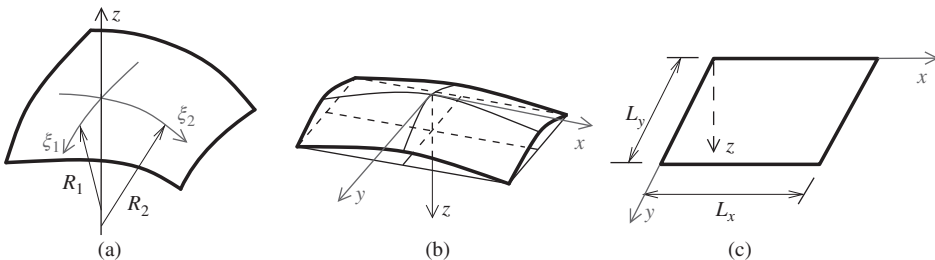


Figure 1.14 Middle surfaces of structures with different shapes: (a) curved, (b) shallow and (c) flat

1.5.3 Plates and Shells with Different Stress Distributions Along Thickness

If normal and shear stress distributions are uniform in the thickness direction, it is a state without bending and shear effects, understood as the membrane state in flat membranes and curved membrane shells.

If normal and in-plane shear stresses change linearly along the thickness, then we deal with a plate or shell in the bending state.

In thin plates and shells the effects related to transverse shear are insignificant and disregarded. Non-deformability of the thin structures in the thickness direction is additionally assumed.

In the case of moderately thick plates and shells, the effects related to transverse shear stresses (varying quadratically along the thickness) are taken into account next to the bending effects.

1.5.4 Range of Validity of Geometrically Linear and Nonlinear Theories for Plates and Shells

While defining the generalized strains in the middle surface and formulating the kinematic equations some simplifications are made, resulting from the approximation

$$(1 + z/R_\alpha)^{-1} \approx 1 - z/R_\alpha \quad (1.57)$$

The reader looking for a deeper knowledge of the theories of plates and shells can find a classification of geometrically nonlinear theories for elastic thin shell structures, for example in Pietraszkiewicz (1979). Taking as a starting point for analysis general nonlinear equilibrium and kinematic equations, many steps of simplification are proposed depending on the expected magnitudes of strains, translations and rotations. In a geometrically nonlinear theory of shells (or plates) the following components are present in the kinematic equations: (i) linear components related to three displacements u_α ($\alpha = 1, 2$) and w , (ii) nonlinear components dependent only on normal displacement w and (iii) nonlinear components related to three displacements u_α, w .

The geometrically linear theory for bending plates (with linear kinematic equations) is applied when deflections (translations normal to the middle plane) are expected of the order $|w| < h/5$. In the case of deflections of the order of the thickness $|w| \approx h$, the previously mentioned kinematic equations must be extended by components nonlinear only with respect to w . The behaviour of a plate in membrane-bending state is described by the equations from the von Kármán theory of plates with moderately large deflections. Expecting deflections of the order $|w| > 5h$, the kinematic equations are augmented by additional nonlinear terms dependent on u_α next to deflection w . These ranges of applicability of respective plate theories are listed in the second part of Box 1.1.

In the works devoted to geometrically nonlinear theories of shells, among others Pietraszkiewicz (1979), the classification of particular cases is presented taking into account the estimation of rotation angles, which provides the theories of: (i) small rotations, (ii) moderate rotations and (iii) large rotations, (iv) finite rotations and strains.

The considerations and examples connected with geometrically as well as physically nonlinear theories are beyond the scope of this book. Summarizing, the information contained in Subsections 1.5.1–1.5.4 allows one to apply an appropriate theory for a considered problem and to choose a suitable solution method.

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