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Vector Spaces, Subspaces, and Linear Transformations

The study of matrices is based on the concept of linear transformations between two vector spaces. It is therefore necessary to define what this concept means in order to understand the setup of a matrix. In this chapter, as well as in the remainder of the book, the set of all real numbers is denoted by R, and its elements are referred to as scalars. The set of all n-tuples of real numbers will be denoted by R^n ($n \geq 1$).

1.1 VECTOR SPACES

This section introduces the reader to ideas that are used extensively in many books on linear and matrix algebra. They involve extensions of the Euclidean geometry which are important in the current mathematical literature and are described here as a convenient introductory reference for the reader. We confine ourselves to real numbers and to vectors whose elements are real numbers.

1.1.1 Euclidean Space

A vector $(x_0, y_0)'$ of two elements can be thought of as representing a point in a two-dimensional Euclidean space using the familiar Cartesian x, y coordinates, as in Figure 1.1. Similarly, a vector $(x_0, y_0, z_0)'$ of three elements can represent a point in a three-dimensional Euclidean space, also shown in Figure 1.1. In general, a vector of n elements can be said to represent a point (an n-tuple) in what is called an n-dimensional Euclidean space. This is a special case of a wider concept called a vector space, which we now define.

Matrix Algebra Useful for Statistics, Second Edition. Shayle R. Searle and André I. Khuri.

Companion website: www.wiley.com/go/searle/matrixalgebra2e

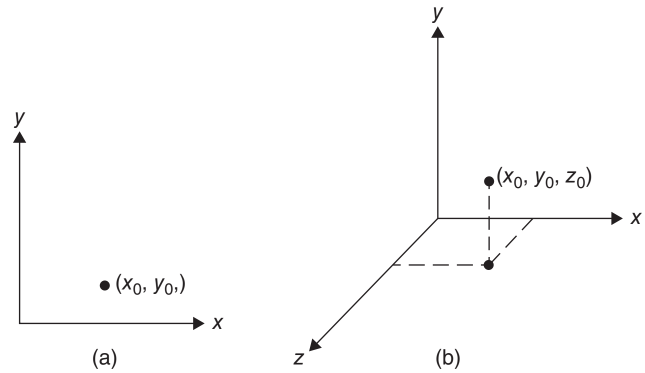

Figure 1.1 *(a) Two-Dimensional and (b) Three-Dimensional Euclidean Spaces.*

Definition 1.1 (Vector Spaces) *A vector space over R is a set of elements, denoted by V, which can be added or multiplied by scalars, in such a way that the sum of two elements of V is an element of V, and the product of an element of V by a scalar is an element of V. Furthermore, the following properties must be satisfied:*

(1) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ *for all* $\boldsymbol{u}$, $\boldsymbol{v}$ *in V.*
(2) $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ *for all* $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ *in V.*
(3) *There exists an element in V, called the zero element and is denoted by* $\mathbf{0}$*, such that* $\mathbf{0} + \boldsymbol{u} = \boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$ *for every* $\boldsymbol{u}$ *in V.*
(4) *For each* $\boldsymbol{u}$ *in V, there exists a unique element* $-\boldsymbol{u}$ *in V such that* $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = \mathbf{0}$.
(5) *For every* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in V and any scalar* α, $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$.
(6) $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}$ *for any scalars* α *and* β *and any* $\boldsymbol{u}$ *in V.*
(7) $\alpha(\beta\boldsymbol{u}) = (\alpha\beta)\boldsymbol{u}$ *for any scalars* α *and* β *and any* $\boldsymbol{u}$ *in V.*
(8) *For every* $\boldsymbol{u}$ *in V,* $\mathbf{1}\boldsymbol{u} = \boldsymbol{u}$*, where* $\mathbf{1}$ *is the number one, and* $0\boldsymbol{u} = \mathbf{0}$*, where* 0 *is the number zero.*

Vector spaces were first defined by the Italian mathematician Giuseppe Peano in 1888.

Example 1.1 *The Euclidean space* R^n *is a vector space whose elements are of the form* $(x_1, x_2, \ldots, x_n)$, $n \geq 1$. *For every pair of elements in* R^n *their sum is in* R^n*, and so is the product of a scalar and any elements that is in* R^n*. It is easy to verify that properties (1) through (8) in Definition 1.1 are satisfied. The zero element is* $(0, 0, \ldots, 0)$.

Example 1.2 *The set of all polynomials in x of of degree n or less of the form* $\sum_{i=0}^{n} a_i x^i$*, where the* a_i*'s are scalars, is a vector space: the sum of any two such polynomials is a polynomial of the same form, and so is the product of a scalar with a polynomial. For the zero element,* $a_i = 0$ *for* $\forall\ i$.

Example 1.3 *The set of all positive functions defined on the closed interval* $[-2, 2]$ *is not a vector space since multiplying any such function by a negative scalar produces a function that is not in that set.*

Definition 1.2 (Vector Subspace) *Let V be a vector space over R, and let W be a subset of V. Then W is said to be a vector subspace of V if it satisfies the following conditions:*

(1) *The sum of any two elements in W is an element of W.*
(2) *The product of any element in W by any scalar is an element in W.*
(3) *The zero element of V is also an element of W.*

It follows that for W to be a vector subspace of V, it must itself be a vector space. A vector subspace may consist of one element only, namely the zero element.

The set of all continuous functions defined on the closed interval $[a, b]$ is a vector subspace of all functions defined on the same interval. Also, the set of all points on the straight line $2x - 5y = 0$ is a vector subspace of R^2. However, the set of all points on any straight line in R^2 not going through the origin $(0, 0)$ is not a vector subspace.

Example 1.4 *Let V_1, V_2, and V_3 be the sets of vectors having the forms **x**, **y**, and **z**, respectively:*

$$\boldsymbol{x} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}, \quad \text{and} \quad \boldsymbol{z} = \begin{bmatrix} \gamma \\ 0 \\ \delta \end{bmatrix} \quad \text{for real } \alpha,\ \beta,\ \gamma,\ \text{and } \delta.$$

Then V_1, V_2, and V_3 each define a vector space, and they are all subspaces of R^3. Furthermore, V_1 and V_2 are each a subspace of V_3.

1.2 BASE OF A VECTOR SPACE

Suppose that every element in a vector space V can be expressed as a linear combination of a number of elements in V. The set consisting of such elements is said to *span* or *generate* the vector space V and is therefore called a *spanning set* for V.

Definition 1.3 *Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ be elements in a vector space V. If there exist scalars $a_1, a_2, \ldots, a_n$, not all equal to zero, such that $\sum_{i=1}^{n} a_i \boldsymbol{u}_i = \boldsymbol{0}$, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are said to be linearly dependent. If, however, $\sum_{i=1}^{n} a_i \boldsymbol{u}_i = \boldsymbol{0}$ is true only if all the a_i's are zero, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are said to be linearly independent.*

Note 1:
If $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are linearly independent, then none of them can be zero. To see this, if, for example, $\boldsymbol{u}_1 = \boldsymbol{0}$, then $a_1\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \cdots + 0\boldsymbol{u}_n = \boldsymbol{0}$ for any $a_1 \neq 0$, which implies that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are linearly dependent, a contradiction. It follows that any set of elements of V that contains the zero element $\boldsymbol{0}$ must be linearly dependent. Furthermore, if $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are linearly dependent, then at least one of them can be expressed as a linear combination of the remaining elements.

Example 1.5 *Consider the vectors*

$$\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{x}_4 = \begin{bmatrix} -6 \\ 12 \\ -18 \end{bmatrix}, \text{ and } \boldsymbol{x}_5 = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}.$$

It is clear that

$$2\boldsymbol{x}_1 + \boldsymbol{x}_4 = \begin{bmatrix} 6 \\ -12 \\ 18 \end{bmatrix} + \begin{bmatrix} -6 \\ 12 \\ -18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \tag{1.1}$$

that is,

$$a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_4 = \mathbf{0} \quad \text{for} \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

which is not zero. Therefore, $\boldsymbol{x}_1$ and $\boldsymbol{x}_4$ are linearly dependent. So also are $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$ because

$$2\boldsymbol{x}_1 + 3\boldsymbol{x}_2 - 3\boldsymbol{x}_3 = \begin{bmatrix} 6 \\ -12 \\ 18 \end{bmatrix} + \begin{bmatrix} 0 \\ 15 \\ -15 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{1.2}$$

In contrast to (1.1) and (1.2), consider

$$a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 = \begin{bmatrix} 3a_1 \\ -6a_1 \\ 9a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5a_2 \\ -5a_2 \end{bmatrix} = \begin{bmatrix} 3a_1 \\ -6a_1 + 5a_2 \\ 9a_1 - 5a_2 \end{bmatrix}. \tag{1.3}$$

There are no values a_1 and a_2 which make (1.3) a zero vector other than $a_1 = 0 = a_2$. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

Definition 1.4 *If the elements of a spanning set for a vector space V are linearly independent, then the set is said to be a basis for V. The number of elements in this basis is called the dimension of V and is denoted by dim(V).*

Note 2:
The reference in Definition 1.4 was to *a* basis and not the basis because for any vector space there are many bases. All bases of a space V have the same number of elements which is dim(V).

Example 1.6 *The vectors,*

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \textit{ and } \quad x_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

are all in R^3. Any two of them form a basis for the vector space whose typical vector is $(\alpha, \beta, 0)'$ for α and β real. The dimension of the space is 2. (The space in this case is, of course, a subspace of R^3.)

Note 3:
If $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ form a basis for V, and if $\boldsymbol{u}$ is a given element in V, then there exists a unique set of scalars, $a_1, a_2, \ldots, a_n$, such that $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$. (see Exercise 1.4).

1.3 LINEAR TRANSFORMATIONS

Linear transformations concerning two vector spaces are certain functions that map one vector space, U, into another vector space, V. More specifically, we have the following definition:

Definition 1.5 *Let U and V be two vector spaces over R. Suppose that T is a function defined on U whose values belong to V, that is, T maps U into V. Then, T is said to be a linear transformation on U into V if*

$$T(a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2) = a_1 T(\boldsymbol{u}_1) + a_2 T(\boldsymbol{u}_2),$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ in U and any scalars a_1, a_2.

For example, let T be a function from R^3 into R^3 such that $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3, x_3)$. It can be verified that T is a linear transformation.

Example 1.7 *In genetics, the three possible genotypes concerning a single locus on a chromosome at which there are only two alleles, G and g, are GG, Gg, and gg. Denoting these by g_1, g_2, and g_3, respectively, gene effects relative to these genotypes can be defined (e.g., Anderson and Kempthorne, 1954) in terms of a mean μ, a measure of gene substitution α, and a measure of dominance δ, such that*

$$\begin{aligned} \mu &= \tfrac{1}{4}g_1 + \tfrac{1}{2}g_2 + \tfrac{1}{4}g_3 \\ \alpha &= \tfrac{1}{4}g_1 \qquad\quad\;\; - \tfrac{1}{4}g_3 \\ \delta &= -\tfrac{1}{4}g_1 + \tfrac{1}{2}g_2 - \tfrac{1}{4}g_3 \end{aligned}$$

These equations represent a linear transformation of the vector $(g_1, g_2, g_3)'$ to $(\mu, \alpha, \delta)'$. In Chapter 2, it will be seen that such a transformation is determined by an array of numbers consisting of the coefficients of the g_i's in the above equations. This array is called a *matrix*.

1.3.1 The Range and Null Spaces of a Linear Transformation

Let T be a linear transformation that maps U into V, where U and V are two given vector spaces. The image of U under T, also called the *range* of T and is denoted by $\mathfrak{R}(T)$, is the set of all elements in V of the form $T(\boldsymbol{u})$ for $\boldsymbol{u}$ in U. The *null space*, or the *kernel*, of T consists of all elements $\boldsymbol{u}$ in U such that $T(\boldsymbol{u}) = \mathbf{0}_v$ where $\mathbf{0}_v$ is the zero element in V. This space is denoted by $\aleph(T)$. It is easy to show that $\mathfrak{R}(T)$ is a vector subspace of V and $\aleph(T)$ is a vector subspace of U (see Exercise 1.7). For example, let T be a linear transformation from R^3 into R^2 such that $T(x_1, x_2, x_3) = (x_1 - x_3, x_2 - x_1)$, then $\aleph(T)$ consists of all points in R^3 such that $x_1 - x_3 = 0$ and $x_2 - x_1 = 0$, or equivalently, $x_1 = x_2 = x_3$. These equations represent a straight line in R^3 passing through the origin.

Theorem 1.1 *Let T be a linear transformation from the vector space U into the vector space V. Let $n = dim(U)$. Then, $n = p + q$, where $p = dim[\aleph(T)]$ and $q = dim[\mathfrak{R}(T)]$.*

Proof. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p$ be a basis for $\aleph(T)$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_q$ be a basis for $\mathfrak{R}(T)$. Furthermore, let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_q$ be elements in U such that $T(\boldsymbol{w}_i) = \boldsymbol{v}_i$ for $i = 1, 2, \ldots, q$. Then,

1. $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p; \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_q$ are linearly independent.
2. The elements in (1) span U.

To show (1), suppose that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p; \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_q$ are not linearly independent, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q$ such that

$$\sum_{i=1}^{p} \alpha_i \boldsymbol{u}_i + \sum_{i=1}^{q} \beta_i \boldsymbol{w}_i = \mathbf{0}_u, \tag{1.4}$$

where $\mathbf{0}_u$ is the zero element in U. Mapping both sides of (1.4) under T, we get

$$\sum_{i=1}^{p} \alpha_i T(\boldsymbol{u}_i) + \sum_{i=1}^{q} \beta_i T(\boldsymbol{w}_i) = \mathbf{0}_v,$$

where $\mathbf{0}_v$ is the zero elements in V. Since the $\boldsymbol{u}_i$'s belong to the null space, then

$$\sum_{i=1}^{q} \beta_i T(\boldsymbol{w}_i) = \sum_{i=1}^{q} \beta_i \boldsymbol{v}_i = \mathbf{0}_v.$$

But, the $\boldsymbol{v}_i$'s are linearly independent, therefore, $\beta_i = 0$ for $i = 1, 2, \ldots, q$. From (1.4) it can be concluded that $\alpha_i = 0$ for $i = 1, 2, \ldots, p$ since the $\boldsymbol{u}_i$'s are linearly independent. It follows that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p; \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_q$ are linearly independent.

To show (2), let $\boldsymbol{u}$ be any element in U, and let $T(\boldsymbol{u}) = \boldsymbol{v}$. There exist scalars $a_1, a_2, \ldots, a_q$ such that $\boldsymbol{v} = \sum_{i=1}^{q} a_i \boldsymbol{v}_i$. Hence,

$$\begin{aligned} T(\boldsymbol{u}) &= \sum_{i=1}^{q} a_i T(\boldsymbol{w}_i) \\ &= T\left(\sum_{i=1}^{q} a_i \boldsymbol{w}_i\right). \end{aligned}$$

It follows that

$$T\left(\boldsymbol{u} - \sum_{i=1}^{q} a_i \boldsymbol{w}_i\right) = \mathbf{0}_v.$$

This indicates that $\boldsymbol{u} - \sum_{i=1}^{q} a_i \boldsymbol{w}_i$ is an element in $\aleph(T)$. We can therefore write

$$\boldsymbol{u} - \sum_{i=1}^{q} a_i \boldsymbol{w}_i = \sum_{i=1}^{p} b_i \boldsymbol{u}_i, \tag{1.5}$$

for some scalars $b_1, b_2, \ldots, b_p$. From (1.5) it follows that $\boldsymbol{u}$ can be written as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p$; $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_q$, which proves (2).

From (1) and (2) we conclude that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p$; $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_q$ form a basis for U. Hence, $n = p + q$. ■

REFERENCE

Anderson, V. L. and Kempthorne, O. (1954). A model for the study of quantitative inheritance. *Genetics*, 39, 883–898.

EXERCISES

1.1 For $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -13 \\ -1 \\ 2 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$,

show the following:

(a) $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent, and find a linear relationship among them.

(b) $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_4$ are linearly independent, and find the linear combination of them that equals $(a, b, c)'$.

1.2 Let U and V be two vector spaces over R. The *Cartesian product* $U \times V$ is defined as the set of all ordered pairs $(\boldsymbol{u}, \boldsymbol{v})$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are elements in U and V, respectively. The sum of two elements, $(\boldsymbol{u}_1, \boldsymbol{v}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{v}_2)$ in $U \times V$ is defined as $(\boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{v}_1 + \boldsymbol{v}_2)$, and if α is a scalar, then $\alpha(\boldsymbol{u}, \boldsymbol{v})$ is defined as $(\alpha\boldsymbol{u}, \alpha\boldsymbol{v})$, where $(\boldsymbol{u}, \boldsymbol{v})$ is an element in $U \times V$. Show that $U \times V$ is a vector space.

1.3 Let $\boldsymbol{V}$ be a vector space spanned by the vectors,

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \; \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}, \text{ and } \boldsymbol{v}_4 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

(a) Show that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent.

(b) Show that $\boldsymbol{V}$ has dimension 2.

1.4 Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ be a basis for a vector space $\boldsymbol{U}$. Show that if $\boldsymbol{u}$ is any element in $\boldsymbol{U}$, then there exists a unique set of scalars, $a_1, a_2, \dots, a_n$, such that $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$, which proves the assertion in Note 3.

1.5 Let $\boldsymbol{U}$ be a vector subspace of $\boldsymbol{V}$, $\boldsymbol{U} \neq \boldsymbol{V}$. Show that $\dim(\boldsymbol{U}) < \dim(\boldsymbol{V})$.

1.6 Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m$ be elements in a vector space $\boldsymbol{U}$. The collection of all linear combinations of the form $\sum_{i=1}^{m} a_i \boldsymbol{u}_i$, where $a_1, a_2, \dots, a_m$ are scalars, is called the linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m$ and is denoted by $L(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m)$. Show that $L(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m)$ is a vector subspace of $\boldsymbol{U}$.

1.7 Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be vector spaces and let T be a linear transformation from $\boldsymbol{U}$ into $\boldsymbol{V}$.

1. Show that $\aleph(T)$, the null space of T, is a vector subspace of $\boldsymbol{U}$.

2. Show that $\Re(T)$, the range of T, is a vector subspace of $\boldsymbol{V}$.

1.8 Consider a vector subspace of R^4 consisting of all $\boldsymbol{x} = (x_1, x_2, x_3, x_4)'$ such that $x_1 + 3x_2 = 0$ and $2x_3 - 7x_4 = 0$. What is the dimension of this vector subspace?

1.9 Suppose that T is a linear transformation from R^3 onto R (the image of R^3 under T is all of R) given by $T(x_1, x_2, x_3) = 3x_1 - 4x_2 + 9x_3$. What is the dimension of its null space?

1.10 Let T be a linear transformation from the vector space U into the vector space V. Show that T is one-to-one if and only if whenever $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ are linearly independent in U, then $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \dots, T(\boldsymbol{u}_n)$ are linearly independent in V.

1.11 Let the functions x, e^x be defined on the closed interval [0, 1]. Show that these functions are linearly independent.