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RADIAL DEA MODELS

KAORU TONE
National Graduate Institute for Policy Studies, Tokyo, Japan

1.1 INTRODUCTION

Data envelopment analysis (DEA) models started from the seminal paper by Charnes, Cooper and Rhodes [1] (hereafter referred to as CCR). This opened up fertile territory for efficiency evaluation. This paper has been cited by more than 20 000 papers as of the publication date of this book. CCR extended Farrell's work [2] to models with multiple inputs and multiple outputs by utilizing linear programming technology and succeeded in establishing DEA as a powerful basis for efficiency analysis.

1.2 BASIC DATA

DEA compares the relative efficiency of a set of enterprises, called DMUs (decision-making units), which have common input and output factors. Let the numbers of DMUs, inputs and outputs be n, m and s, respectively. We denote input i and output r of DMU$_j$ by x_{ij} $(i=1,\ldots,m; j=1,\ldots,n)$ and y_{rj} $(r=1,\ldots,s; j=1,\ldots,n)$, respectively. The input and output vectors for DMU$_h$ $(h=1,\ldots,n)$ are defined as $\mathbf{x}_h=(x_{1h},\ldots,x_{mh})^T$ and $\mathbf{y}_h=(y_{1h},\ldots,y_{sh})^T$. The input and output matrices are defined

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as $\mathbf{X} = (x_{ij}) (i = 1, \ldots, m;\ j = 1, \ldots, n)$ and $\mathbf{Y} = (y_{rj}) (r = 1, \ldots, s;\ j = 1, \ldots, n)$. We assume $\mathbf{X} > \mathbf{0}$ and $\mathbf{Y} > \mathbf{0}$.[1] For the input, smaller is better, while for the output, larger is better. We evaluate DMUs by the ratio scale of output/input.

1.3 INPUT-ORIENTED CCR MODEL

Let the weights of the inputs and outputs be $\mathbf{v} = (v_1, \ldots, v_m) \geq \mathbf{0}$ and $\mathbf{u} = (u_1, \ldots, u_m) \geq \mathbf{0}$. The input-oriented CCR model evaluates the efficiency of a DMU $(\mathbf{x}_h, \mathbf{y}_h)(h = 1, \ldots, n)$ by solving the following fractional programming problem:

[Ratio form]

$$\max_{\mathbf{v},\mathbf{u}} \theta = \frac{u_1 y_{1h} + \cdots + u_s y_{sh}}{v_1 x_{1h} + \cdots + v_m x_{mh}} \tag{1.1}$$

$$\begin{aligned} \text{s.t.} \quad & \frac{u_1 y_{1j} + \cdots + u_s y_{sj}}{v_1 x_{1j} + \cdots + v_m x_{mj}} \leq 1 \quad (j = 1, \ldots, n) \\ & \mathbf{v} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0} \end{aligned} \tag{1.2}$$

This fractional program can be transformed into the following equivalent linear program:

[Multiplier form]

$$\max_{\mathbf{v},\mathbf{u}} \theta = u_1 y_{1h} + \cdots + u_s y_{sh} \tag{1.3}$$

$$\begin{aligned} \text{s.t.} \quad & v_1 x_{1h} + \cdots + v_m x_{mh} = 1 \\ & v_1 x_{1j} + \cdots + v_m x_{mj} - u_1 y_{1j} - \cdots - u_s y_{sj} \geq 0 \ (j = 1, \ldots, n) \\ & \mathbf{v} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0} \end{aligned} \tag{1.4}$$

The dual to the above LP can be described as follows:

[Envelopment form]

$$\min_{\boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+} \theta \tag{1.5}$$

$$\begin{aligned} \text{s.t.} \quad & x_{i1}\lambda_1 + \cdots + x_{in}\lambda_n - \theta x_{ih} \leq 0 \text{ or } x_{i1}\lambda_1 + \cdots + x_{in}\lambda_n - \theta x_{ih} + s_i^- = 0 \ (i = 1, \ldots, m) \\ & y_{r1}\lambda_1 + \cdots + y_{rn}\lambda_n \geq y_{rh} \text{ or } y_{r1}\lambda_1 + \cdots + y_{rn}\lambda_n - s_r^+ = y_{rh} \ (r = 1, \ldots, s) \\ & \lambda_j \geq 0 \ (\forall j),\ s_i^- \geq 0 \ (\forall i),\ s_r^+ \geq 0 \ (\forall r) \end{aligned} \tag{1.6}$$

[1] In some models, we can relax these assumptions.

$\boldsymbol{\lambda}$, $\mathbf{s}^-$ and $\mathbf{s}^+$ are the intensity, input-slack and output-slack vectors, respectively. This model aims at minimizing inputs while producing at least the given output level.

Let an optimal solution to [Envelopment form] be $\theta^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*}$ and $\mathbf{s}^{+*}$.

Definition 1.1 (CCR score)
The CCR score of DMU$_h$ is defined by θ^*.

Definition 1.2 (Strongly efficient)
DMU$_h$ is *strongly* CCR efficient if $\theta^* = 1$ and ($\mathbf{s}^{-*} = \mathbf{0}$ and $\mathbf{s}^{+*} = \mathbf{0}$) for all optimal solutions to [Envelopment form].

Definition 1.3 (Weakly efficient)
DMU$_h$ is *weakly* CCR efficient if $\theta^* = 1$ and ($\mathbf{s}^{-*} \neq \mathbf{0}$ or $\mathbf{s}^{+*} \neq \mathbf{0}$) for some optimal solutions to [Envelopment form].

Definition 1.4 (Inefficient)
DMU$_h$ is CCR inefficient if $\theta^* < 1$.

Definition 1.5 (Production possibility set)
From the data matrices $\mathbf{X}$ and $\mathbf{Y}$, we define the production possibility set P by

$$P = \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x} \geq \mathbf{X}\boldsymbol{\lambda}, \mathbf{y} \leq \mathbf{Y}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\} \tag{1.7}$$

Figure 1.1 shows a typical production possibility set in two dimensions for the single-input and single-output case. In this example, the possibility set is determined by B and the ray from the origin through B is the efficient frontier. DMU A is inefficient and its input-oriented score is $PQ/PA = 0.5$.

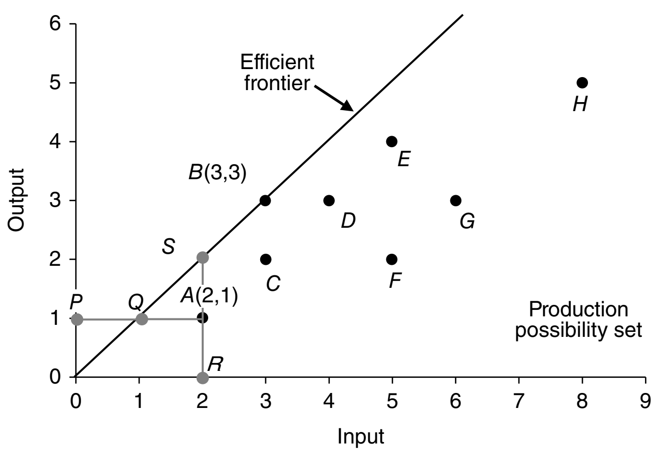

Figure 1.1 Production possibility set for the CCR model.

1.3.1 The CRS Model

This model is called the *constant-returns-to-scale* (CRS) model.

Definition 1.6 (Reference set)
For an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ to [Envelopment form], we define the reference set of DMU$_h$ by

$$E(h) = \left\{ i | \lambda_j^* > 0;\ j = 1, \ldots, n \right\} \tag{1.8}$$

The reference set is not always uniquely determined.

Definition 1.7 (CCR projection)
The CCR projection is defined as

$$\hat{\mathbf{x}}_h = \theta^* \mathbf{x}_h - \mathbf{s}^{-*},\ \hat{\mathbf{y}}_h = \mathbf{y}_h + \mathbf{s}^{+*} \tag{1.9}$$

Theorem 1.1 The projected $(\hat{\mathbf{x}}_h, \hat{\mathbf{y}}_h)$ is strongly CCR efficient.

1.4 THE INPUT-ORIENTED BCC MODEL

The envelopment form of the BCC (Banker–Charnes–Cooper) model [3] is defined as follows:

[Envelopment form of the BCC model]

$$\min_{\boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+} \theta \tag{1.10}$$

$$\begin{aligned} \text{s.t.}\ & x_{i1}\lambda_1 + \cdots + x_{in}\lambda_n - \theta x_{ih} \le 0 \text{ or } x_{i1}\lambda_1 + \cdots + x_{in}\lambda_n - \theta x_{ih} + s_i^- = 0\ (i = 1, \ldots, m) \\ & y_{r1}\lambda_1 + \cdots + y_{rn}\lambda_n \ge y_{rh} \text{ or } y_{r1}\lambda_1 + \cdots + y_{rn}\lambda_n - s_r^+ = y_{rh}\ (r = 1, \ldots, s) \\ & \lambda_1 + \cdots \lambda_n = 1 \\ & \lambda_j \ge 0\ (\forall j),\ s_i^- \ge 0\ (\forall i),\ s_r^+ \ge 0\ (\forall r) \end{aligned} \tag{1.11}$$

The multiplier form is as follows:

[Multiplier form]

$$\max_{\mathbf{v}, \mathbf{u}, u_0} \theta = u_1 y_{1h} + \ldots + u_s y_{sh} - u_0 \tag{1.12}$$

$$\begin{aligned} \text{s.t.}\ & v_1 x_{1h} + \cdots + v_m x_{mh} = 1 \\ & v_1 x_{1j} + \cdots + v_m x_{mj} - u_1 y_{1j} - \cdots - u_s y_{sj} + u_0 \ge 0\ (j = 1, \ldots, n) \\ & \mathbf{v} \ge \mathbf{0}, \mathbf{u} \ge \mathbf{0}, u_0 \text{ free in sign} \end{aligned} \tag{1.13}$$

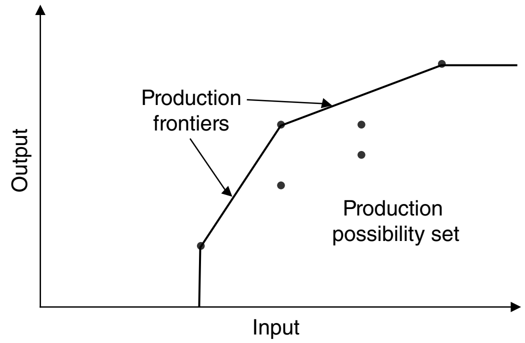

Figure 1.2 Production possibility set for the BCC model.

The equivalent BCC fractional program is obtained from the multiplier form as follows:

[Ratio form of the BCC model]

$$\max_{\mathbf{v},\mathbf{u},u_0} \theta = \frac{u_1 y_{1h} + \cdots + u_s y_{sh} - u_0}{v_1 x_{1h} + \cdots + v_m x_{mh}} \tag{1.14}$$

$$\begin{aligned} \text{s.t.} \quad & \frac{u_1 y_{1j} + \cdots + u_s y_{sj} - u_0}{v_1 x_{1j} + \cdots + v_m x_{mj}} \leq 1 \quad (j = 1, \ldots, n) \\ & \mathbf{v} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0}, u_0 \text{ free in sign} \end{aligned} \tag{1.15}$$

Figure 1.2 shows a typical production possibility set for the BCC model.

1.4.1 The VRS Model

This model is called the *variable-returns-to-scale* (VRS) model.

1.5 THE OUTPUT-ORIENTED MODEL

This model attempts to maximize the outputs while using no more than the observed amount of any input:

$$\eta^* = \max \eta \tag{1.16}$$

$$\begin{aligned} \text{s.t.} \quad & x_{i1}\lambda_1 + \cdots + x_{in}\lambda_n \leq x_{ih}0 \text{ or } x_{i1}\lambda_1 + \cdots + x_{in}\lambda_n + s_i^- = x_{ih} \ (i = 1, \ldots, m) \\ & y_{r1}\lambda_1 + \cdots + y_{rn}\lambda_n \geq \eta y_{rh} \text{ or } y_{r1}\lambda_1 + \cdots + y_{rn}\lambda_n - s_r^+ = \eta y_{rh} \ (r = 1, \ldots, s) \\ & \lambda_j \geq 0 \ (\forall j), \ s_i^- \geq 0 \ (\forall i), \ s_r^+ \geq 0 \ (\forall r) \end{aligned} \tag{1.17}$$

We define the output-oriented efficiency θ^* as the inverse of η^*:

$$\theta^* = 1/\eta^* \tag{1.18}$$

In Figure 1.1, DMU A has $\eta^* = RS/RA = 2$ and hence its output-oriented score is 0.5. In the CCR model, the input- and output-oriented scores are identical, whereas in the BCC model they are usually different.

1.6 ASSURANCE REGION METHOD

In the optimal weight (v_i^*, u_j^*) of a DEA model, we may see many zeros – showing that the DMU has a weakness in the corresponding items compared with other (efficient) DMUs. Large differences in weights from item to item may also be a concern. This leads to the assurance region method, which imposes constraints on the relative magnitudes of the weights for special items. For example, we may add a constraint on the ratio of weights for Input 1 and Input 2 as follows:

$$L_{12} \le v_2/v_1 \le U_{12} \tag{1.19}$$

where L_{12} and U_{12} are lower and upper bounds that the ratio v_2/v_1 may assume. See [4] for details.

1.7 THE ASSUMPTIONS BEHIND RADIAL MODELS

These models assume a proportional reduction of the inputs (such as $\theta^*\mathbf{x}_h$) and a proportional expansion of the outputs (such as $\eta^*\mathbf{y}_h$). In some instances, these assumptions are too restrictive. This has led to the development of non-radial models.

1.8 A SAMPLE RADIAL MODEL

We show an example of a radial model here. Table 1.1 represents 12 hospitals with two inputs, Doctor and Nurse, and two outputs, Outpatient and Inpatient, where (I) and (O) indicate input and output, respectively.

Table 1.2 reports scores for the hospital example, both input-oriented (CCR-I, BCC-I) and output-oriented (CCR-O, BCC-O), while Figure 1.3 shows a graphical comparison. The scores for CCR-I and CCR-O are identical.[2]

[2] Software for the CCR, BCC and other models is included in DEA-Solver Pro V13 (http://www.saitech-inc.com). See also Appendix A.

TABLE 1.1 A hospital example.

Hospital	(I) Doctor	(I) Nurse	(O) Outpatient	(O) Inpatient
A	20	151	100	90
B	19	131	150	50
C	25	160	100	55
D	27	168	180	72
E	25	158	80	66
F	55	255	150	60
G	33	235	170	70
H	31	206	130	60
I	30	244	110	60
J	50	290	250	100
K	53	306	230	110
L	38	284	150	90

TABLE 1.2 Efficiency scores obtained by radial models.

Hospital	CCR-I	CCR-O	BCC-I	BCC-O
A	1	1	1	1
B	1	1	1	1
C	0.6915	0.6915	0.8344	0.6916
D	1	1	1	1
E	0.7208	0.7208	0.8797	0.7332
F	0.5490	0.5490	0.5555	0.6524
G	0.7048	0.7048	0.7676	0.8693
H	0.6366	0.6366	0.6602	0.7253
I	0.5651	0.5651	0.6417	0.6809
J	0.8046	0.8046	1	1
K	0.7694	0.7694	1	1
L	0.6362	0.6362	0.7556	0.8919

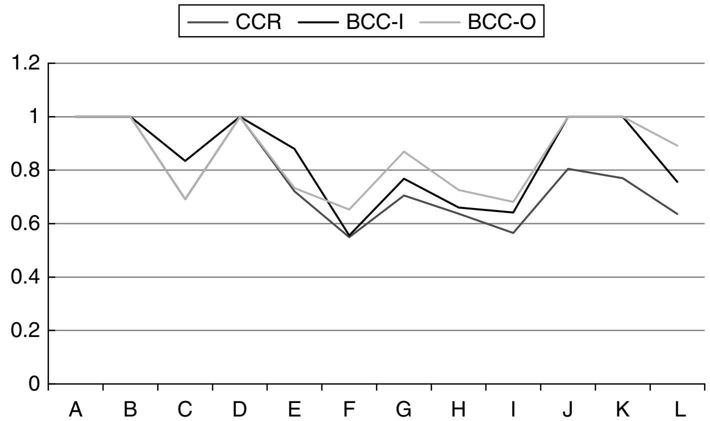

Figure 1.3 Comparison of scores.

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