1

Elements of Mathematical Calculation

This chapter is an introduction presenting the elements of mathematical calculation that will be used in the book.

1.1 Vectors: Vector Operations

A *vector* (denoted by **a**) is defined by its numerical magnitude or modulus $|\mathbf{a}|$, by the direction Δ, and by sense. The vector is represented (Fig. 1.1) by an orientated segment of straight line.

The sum of two vectors **a**, **b** is the vector **c** (Fig. 1.2) represented by the diagonal of the parallelogram constructed on the two vectors; it reads

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{1.1}$$

The unit vector **u** of the vector **a** (or of the direction Δ) is defined by the relation

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{1.2}$$

If one denotes by **i**, **j**, **k** the unit vectors of the axes of dextrorsum orthogonal reference system *Oxyz*, and by a_x, a_y, a_z the projections of vector **a** onto the axes, then one may write the analytical expression

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \tag{1.3}$$

Dynamics of the Rigid Solid with General Constraints by a Multibody Approach, First Edition.
Nicolae Pandrea and Nicolae-Doru Stănescu.

Companion website: www.wiley.com/go/pandrea

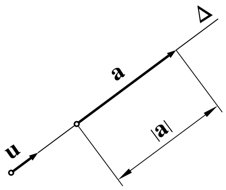

Figure 1.1 Representation of a vector.

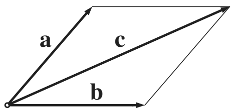

Figure 1.2 The sum of two vectors.

The scalar (dot) product of two vectors is defined by the expression

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\alpha, \tag{1.4}$$

where α is the angle between the two vectors.

We obtain the equalities

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0,\ \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1 \tag{1.5}$$

and, consequently, one deduces the analytical expressions

$$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z, \tag{1.6}$$

$$|\mathbf{a}|=\sqrt{a_x^2+a_y^2+a_z^2},\ |\mathbf{b}|=\sqrt{b_x^2+b_y^2+b_z^2}, \tag{1.7}$$

$$\cos\alpha=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}. \tag{1.8}$$

The vector (cross) product of two vectors, denoted by **c**,

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}, \tag{1.9}$$

is the vector perpendicular onto the plan of the vectors **a** and **b**, while the sense is given by the rule of the right screw when the vector **a** rotates over the vector **b** (making the smallest angle); the modulus has the expression

$$|\mathbf{c}|=|\mathbf{a}||\mathbf{b}|\sin\alpha, \tag{1.10}$$

α being the smallest angle between the vectors **a** and **b**.

One obtains the equalities

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}, \tag{1.11}$$

and the analytical expression

$$\mathbf{a}\times\mathbf{b}=\left(a_yb_z-a_zb_y\right)\mathbf{i}+\left(a_zb_x-a_xb_z\right)\mathbf{j}+\left(a_xb_y-a_yb_x\right)\mathbf{k}. \tag{1.12}$$

The mixed product of three vectors, defined by the relation $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and denoted by (**a**, **b**, **c**), leads to the successive equalities

$$(\mathbf{a},\mathbf{b},\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.13}$$

The mixed product (**a**, **b**, **c**) is equal to the volume with sign of the parallelepiped constructed having the three vectors as edges (Fig. 1.3). It is equal to zero if and only if the three vectors are coplanar.

The double vector product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ satisfies the equality

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{1.14}$$

The reciprocal vectors of the (non-coplanar) vectors **a**, **b**, **c** are defined by the expressions

$$\mathbf{a}^*=\frac{\mathbf{b}\times\mathbf{c}}{(\mathbf{a},\mathbf{b},\mathbf{c})},\ \mathbf{b}^*=\frac{\mathbf{c}\times\mathbf{a}}{(\mathbf{a},\mathbf{b},\mathbf{c})},\ \mathbf{c}^*=\frac{\mathbf{a}\times\mathbf{b}}{(\mathbf{a},\mathbf{b},\mathbf{c})}, \tag{1.15}$$

and satisfy the equality

$$(\mathbf{a}^*,\mathbf{b}^*,\mathbf{c}^*)=\frac{1}{(\mathbf{a},\mathbf{b},\mathbf{c})}. \tag{1.16}$$

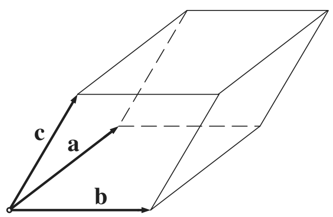

Figure 1.3 The geometric interpretation of the mixed product of three vectors.

An arbitrary vector **v** may be written in the form

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{a}^*)\mathbf{a} + (\mathbf{v}\cdot\mathbf{b}^*)\mathbf{b} + (\mathbf{v}\cdot\mathbf{c}^*)\mathbf{c}, \tag{1.17}$$

or as

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{a})\mathbf{a}^* + (\mathbf{v}\cdot\mathbf{b})\mathbf{b}^* + (\mathbf{v}\cdot\mathbf{c})\mathbf{c}^*. \tag{1.18}$$

1.2 Real Rectangular Matrix

By *real rectangular matrix* we understand a table with m rows and n columns ($m \neq n$)

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \tag{1.19}$$

where the *elements* a_{ij} are real numbers.

Sometimes, we use the abridged notation

$$[\mathbf{A}] = (a_{ij}) \text{ or } [\mathbf{A}] = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}. \tag{1.20}$$

The multiplication between a matrix and a scalar $\lambda \in \mathbb{R}$ is defined by the relation

$$\lambda[\mathbf{A}] = (\lambda a_{ij}), \tag{1.21}$$

while *the sum of two matrices of the same type* (with the same number of rows and the same number of columns) is defined by

$$[\mathbf{A}] + [\mathbf{B}] = (a_{ij} + b_{ij}). \tag{1.22}$$

The zero matrix or the null matrix is the matrix denoted by **[0]**, which has all its elements equal to zero.

The zero matrix verifies the relations

$$[\mathbf{A}] + [\mathbf{0}] = [\mathbf{0}] + [\mathbf{A}] = [\mathbf{A}]. \tag{1.23}$$

The transpose matrix $[\mathbf{A}]^{\mathrm{T}}$ is the matrix obtained transforming the rows of the matrix **[A]** into columns, that is

$$[\mathbf{A}]^{\mathrm{T}} = \left(a_{ji}\right). \tag{1.24}$$

The transposing operation has the following properties

$$\left[[\mathbf{A}]^{\mathrm{T}}\right]^{\mathrm{T}} = [\mathbf{A}], [[\mathbf{A}] + [\mathbf{B}]]^{\mathrm{T}} = [\mathbf{A}]^{\mathrm{T}} + [\mathbf{B}]^{\mathrm{T}}, \tag{1.25}$$

where we assumed that the sum can be performed.

The matrix with one column bears the name *column matrix* or *column vector* and it is denoted by $\{\mathbf{A}\}$, that is

$$\{\mathbf{A}\} = [a_{11} \quad a_{21} \quad \ldots \quad a_{m1}]^{\mathrm{T}}, \tag{1.26}$$

while the matrix with one row is called *row matrix* or *row vector* and is denoted as

$$[\mathbf{A}] = [a_{11} \quad a_{12} \quad \ldots \quad a_{1n}], \tag{1.27}$$

or

$$[\mathbf{A}] = \{\mathbf{A}\}^{\mathrm{T}}, \tag{1.28}$$

where

$$\{\mathbf{A}\} = [a_{11} \quad a_{12} \quad \ldots \quad a_{1n}]^{\mathrm{T}}. \tag{1.29}$$

If the matrix $[\mathbf{A}]$ has m rows and n columns, and the matrix $[\mathbf{B}]$ has n rows and p columns, *then* the two matrices *can be multiplied* and the result is a matrix $[\mathbf{C}]$ with m rows and p columns

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}], \tag{1.30}$$

where the elements c_{ij}, $1 \le i \le m$, $1 \le j \le p$, of the matrix $[\mathbf{C}]$ satisfy the equality

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \tag{1.31}$$

that is, the elements of the product matrix are obtained by multiplying the rows of matrix $[\mathbf{A}]$ by the columns of matrix $[\mathbf{B}]$.

The transpose of the product matrix is given by the relation

$$[[\mathbf{A}][\mathbf{B}]]^{\mathrm{T}} = [\mathbf{B}]^{\mathrm{T}}[\mathbf{A}]^{\mathrm{T}}. \tag{1.32}$$

In some cases, there may exist *matrices of matrices* and the multiplication is performed as in the following example

$$\begin{bmatrix} [\mathbf{A}_1] & [\mathbf{A}_2] \\ [\mathbf{A}_3] & [\mathbf{A}_4] \\ [\mathbf{A}_5] & [\mathbf{A}_6] \end{bmatrix} \begin{bmatrix} [\mathbf{B}_1] & [\mathbf{B}_2] \\ [\mathbf{B}_3] & [\mathbf{B}_4] \end{bmatrix} = \begin{bmatrix} [\mathbf{A}_1][\mathbf{B}_1]+[\mathbf{A}_2][\mathbf{B}_3] & [\mathbf{A}_1][\mathbf{B}_2]+[\mathbf{A}_2][\mathbf{B}_4] \\ [\mathbf{A}_3][\mathbf{B}_1]+[\mathbf{A}_4][\mathbf{B}_3] & [\mathbf{A}_3][\mathbf{B}_2]+[\mathbf{A}_4][\mathbf{B}_4] \\ [\mathbf{A}_5][\mathbf{B}_1]+[\mathbf{A}_6][\mathbf{B}_3] & [\mathbf{A}_5][\mathbf{B}_2]+[\mathbf{A}_6][\mathbf{B}_4] \end{bmatrix}, \tag{1.33}$$

where we assumed that the operations of multiplication and addition of matrices can be performed for each separate case.

1.3 Square Matrix

The matrix [**A**] is a *square matrix* if the number of rows is equal to the number of columns; hence

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \tag{1.34}$$

where the number n is *the dimension* or *the order of the matrix.*

The determinant associated to the matrix [**A**] is denoted by det[**A**].

If $[\mathbf{A}_{ij}]$ is the matrix obtained from the matrix [**A**] by the suppression of the row i and the column j, then *the algebraic complement* a_{ij}^* is given by the expression

$$a_{ij}^* = (-1)^{i+j}\det\left[\mathbf{A}_{ij}\right], 1 \le i, j \le n, \tag{1.35}$$

and the following relation holds true

$$\sum_{k=1}^{n} a_{ik}a_{jk}^* = \sum_{k=1}^{n} a_{kj}a_{ki}^* = \begin{cases} 0 & \text{for } i \neq j \\ \det[\mathbf{A}] & \text{for } i = j \end{cases}. \tag{1.36}$$

The determinants of the matrices satisfy the equalities

$$\det[\mathbf{A}] = \det[\mathbf{A}]^{\mathrm{T}}, \tag{1.37}$$

$$\det[[\mathbf{A}][\mathbf{B}]] = \det[\mathbf{A}]\cdot\det[\mathbf{B}], \tag{1.38}$$

where we assumed that the matrices [**A**] and [**B**] have the same order.

In general, *the multiplication of matrices* is not commutative,

$$[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}], \tag{1.39}$$

but it is associative and distributive, that is

$$[\mathbf{A}][[\mathbf{B}][\mathbf{C}]] = [[\mathbf{A}][\mathbf{B}]][\mathbf{C}] = [\mathbf{A}][\mathbf{B}][\mathbf{C}], \tag{1.40}$$

$$[\mathbf{A}][[\mathbf{B}] + [\mathbf{C}]] = [\mathbf{A}][\mathbf{B}] + [\mathbf{A}][\mathbf{C}], \tag{1.41}$$

where the matrices [**A**], [**B**] and [**C**] have the same order.

The trace of a matrix, denoted by Tr[**A**] is equal to the sum of the elements situated on the principal diagonal

$$\mathrm{Tr}[\mathbf{A}] = \sum_{i=1}^{n} a_{ii}. \tag{1.42}$$

The diagonal matrix is the matrix with all the elements equal to zero, except *some elements* situated on the principal diagonal.

The unity matrix, generally denoted by [**I**], is *the diagonal matrix* that has all the elements of the principal diagonal equal to unity,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{1.43}$$

The unity matrix verifies the relations

$$[\mathbf{A}][\mathbf{I}] = [\mathbf{I}][\mathbf{A}] = [\mathbf{A}]. \tag{1.44}$$

The adjunct matrix $\mathbf{A}^*$ is defined by the relation

$$[\mathbf{A}^*] = \left(a_{ij}^*\right). \tag{1.45}$$

The matrix [**A**] is called *singular* if $\det[\mathbf{A}] = 0$; it is called a *non-singular* one if $\det[\mathbf{A}] \neq 0$.

The non-singular matrices [**A**] admit *inverse* matrices $[\mathbf{A}]^{-1}$; the inverse matrices fulfill the conditions

$$[\mathbf{A}]^{-1} = \frac{1}{\det[\mathbf{A}]}[\mathbf{A}^*], \tag{1.46}$$

$$[\mathbf{A}][\mathbf{A}]^{-1} = [\mathbf{A}]^{-1}[\mathbf{A}] = [\mathbf{I}], \tag{1.47}$$

$$\left[[\mathbf{A}]^{\mathrm{T}}\right]^{-1} = \left[[\mathbf{A}]^{-1}\right]^{\mathrm{T}}. \tag{1.48}$$

The matrix [**A**] is called *symmetric* if

$$[\mathbf{A}] = [\mathbf{A}]^{\mathrm{T}}; \tag{1.49}$$

it is called *anti-symmetric* or *skew* if

$$[\mathbf{A}] = -[\mathbf{A}]^{\mathrm{T}}. \tag{1.50}$$

The matrix [**A**] is called *orthogonal* if it fulfills the condition

$$[\mathbf{A}][\mathbf{A}]^{\mathrm{T}} = [\mathbf{I}]. \tag{1.51}$$

The orthogonal matrix [**A**] satisfies the equalities

$$[\mathbf{A}]^{\mathrm{T}} = [\mathbf{A}]^{-1}, \det[\mathbf{A}] = \pm 1. \tag{1.52}$$

The equation of nth degree

$$\det[\lambda[\mathbf{I}] - [\mathbf{A}]] = 0 \tag{1.53}$$

is *the characteristic equation* of the matrix [**A**]; its roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called *the eigenvalues of the matrix* [**A**].

The vectors $\left\{\mathbf{v}^{(m_i)}\right\}$ which are obtained from the equality

$$[\mathbf{A}]\left\{\mathbf{v}^{(m_i)}\right\} = \lambda_m\left\{\mathbf{v}^{(m_i)}\right\}, 1 \le m \le k, \tag{1.54}$$

are called *eigenvectors* and, if the matrix [**A**] is a symmetric one, then its eigenvectors are orthogonal

$$\left\{\mathbf{v}^{(r)}\right\}^{\mathrm{T}}\left\{\mathbf{v}^{(s)}\right\} = 0, \text{ if } s \neq r. \tag{1.55}$$

Using the notation

$$b_j = \mathrm{Tr}\left[[\mathbf{A}]^j\right], \tag{1.56}$$

one obtains the characteristic equation

$$\sum_{j=0}^{n} c_{n-j}\lambda^{j} = 0, \tag{1.57}$$

where the coefficients c_j are given by the iterative relations

$$c_0 = 0,\ c_j = -\frac{1}{j}\sum_{k=0}^{j-1} c_k b_{j-k}. \tag{1.58}$$

Observation 1.3.1.

i. The eigenvalues of the matrix [**A**] of order n can be real or complex, distinct or not.
ii. One or more eigenvectors correspond to an eigenvalue λ_m, depending on the order of multiplicity for that eigenvalue.
iii. No matter if the eigenvalue is real or not, keeping into account that the matrix [**A**] has real components, the eigenvectors associated to that eigenvalue are matrices with n rows and one column, with real elements.

Observation 1.3.2. Let us consider that the matrix [**A**] is a square one, of order 3.

i. If the eigenvalues are real and distinct $\lambda_i \in \mathbb{R}$, $\lambda_i \neq \lambda_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$, then the eigenvalues are obtained by solving three matrix equations of the form

$$[\mathbf{A}]\{\mathbf{v}_i\} = \lambda_i\{\mathbf{v}_i\}, i = 1, 2, 3. \tag{1.59}$$

ii. If the eigenvalues are real, but two of them are equal, $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$, $\lambda_1 = \lambda_2$, $\lambda_3 \neq \lambda_1$, then the eigenvalues result by solving the matrix equations

$$[\mathbf{A}]\{\mathbf{v}_1\} = \lambda_1\{\mathbf{v}_1\}, ([\mathbf{A}] - \lambda_1[\mathbf{I}])\{\mathbf{v}_2\} = \{\mathbf{v}_1\}, [\mathbf{A}]\{\mathbf{v}_3\} = \lambda_3\{\mathbf{v}_3\}. \tag{1.60}$$

iii. If the eigenvalues are real and equal, $\lambda_i = \lambda$, $i = 1, 2, 3$, then the eigenvector are obtained by solving the matrix equations

$$[\mathbf{A}]\{\mathbf{v}_1\} = \lambda\{\mathbf{v}_1\}, ([\mathbf{A}] - \lambda[\mathbf{I}])\{\mathbf{v}_2\} = \{\mathbf{v}_1\}, ([\mathbf{A}] - \lambda[\mathbf{I}])\{\mathbf{v}_3\} = \{\mathbf{v}_2\}. \tag{1.61}$$

iv. If the eigenvalues are one real, $\lambda_1 \in \mathbb{R}$, and two complex conjugate, $\lambda_2 = \alpha + \mathrm{i}\beta$, $\lambda_3 = \alpha - \mathrm{i}\beta$, $\alpha, \beta \in \mathbb{R}$, $\mathrm{i}^2 = -1$, then the eigenvectors result by solving the matrix equations

$$[\mathbf{A}](\{\mathbf{v}_2\} + \mathrm{i}\{\mathbf{v}_3\}) = (\alpha + \mathrm{i}\beta)(\{\mathbf{v}_2\} + \mathrm{i}\{\mathbf{v}_3\}); \tag{1.62}$$

1.4 Skew Matrix of Third Order

Starting from the relation of definition (1.49), it results that a third order skew matrix may be written in the form

$$[\mathbf{B}] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{1.63}$$

One associates to the skew matrix [**B**] *the column matrix (vector)*

$$\{\mathbf{b}\} = [b_1 \;\; b_2 \;\; b_3]^{\mathrm{T}} \tag{1.64}$$

and the vector

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}. \tag{1.65}$$

It results the equality

$$[\mathbf{B}]\{\mathbf{b}\} = \{\mathbf{0}\}. \tag{1.66}$$

Being given the skew matrices [**A**], [**B**], and the eigenvectors associated to these matrices, then the vector equality

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{1.67}$$

may be put in the matrix expression

$$[\mathbf{A}]\{\mathbf{b}\} = -[\mathbf{B}]\{\mathbf{a}\}. \tag{1.68}$$

For *the skew matrix* [**B**] one may write the following relations (obtained by elementary calculation)

$$\det[\mathbf{B}] = 0, \tag{1.69}$$

$$[\mathbf{B}]^2 = -\left(b_1^2 + b_2^2 + b_3^2\right)[\mathbf{I}] + \{\mathbf{b}\}\{\mathbf{b}\}^{\mathrm{T}}, \tag{1.70}$$

$$[\mathbf{B}]^3 = -\left(b_1^2 + b_2^2 + b_3^2\right)[\mathbf{B}]. \tag{1.71}$$

For the *skew* matrices [**A**], [**B**] and the associated vectors **a**, **b**, denoting the vector product by **c**, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, and by [**C**] the associated skew matrix, one obtains the relations

$$[\mathbf{A}][\mathbf{B}] = -(a_1b_1 + a_2b_2 + a_3b_3)[\mathbf{I}] + \{\mathbf{b}\}\{\mathbf{a}\}^{\mathrm{T}}, \tag{1.72}$$

$$[\mathbf{B}][\mathbf{A}] = -(a_1b_1 + a_2b_2 + a_3b_3)[\mathbf{I}] + \{\mathbf{a}\}\{\mathbf{b}\}^{\mathrm{T}}, \tag{1.73}$$

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}] - [\mathbf{B}][\mathbf{A}] = \{\mathbf{b}\}\{\mathbf{a}\}^{\mathrm{T}} - \{\mathbf{a}\}\{\mathbf{b}\}^{\mathrm{T}}, \tag{1.74}$$

$$\begin{gathered}[\mathbf{C}]^2 = (a_1b_1 + a_2b_2 + a_3b_3)\left[\{\mathbf{a}\}\{\mathbf{b}\}^{\mathrm{T}} - \{\mathbf{b}\}\{\mathbf{a}\}^{\mathrm{T}}\right] - \\ \left(b_1^2 + b_2^2 + b_3^2\right)\{\mathbf{a}\}\{\mathbf{a}\}^{\mathrm{T}} - \left(a_1^2 + a_2^2 + a_3^2\right)\{\mathbf{b}\}\{\mathbf{b}\}^{\mathrm{T}}.\end{gathered} \tag{1.75}$$

If the matrix [**A**] is an *arbitrary* third order one, and the matrices [**B**], [**C**] are *skew* ones, then the matrix

$$[\mathbf{D}] = [\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}] \tag{1.76}$$

is a *skew* matrix, and the associated column matrices {**b**}, {**c**}, {**d**} satisfy the equalities

$$\{\mathbf{d}\} = [\mathbf{A}^*]\{\mathbf{b}\}, \tag{1.77}$$

$$[\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}]\{\mathbf{c}\} = -[\mathbf{C}][\mathbf{A}^*]\{\mathbf{b}\}, \tag{1.78}$$

where $[\mathbf{A}^*]$ is the adjunct matrix of the matrix [**A**].

When the matrix [**A**] is orthogonal, one obtains the equalities

$$\{\mathbf{d}\} = [\mathbf{A}]^{\mathrm{T}}\{\mathbf{b}\}, [\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}]\{\mathbf{c}\} = -[\mathbf{C}][\mathbf{A}]^{\mathrm{T}}\{\mathbf{b}\}. \tag{1.79}$$

More general, if the matrix [**A**] has k rows and 3 columns, then it results that the kth order square matrix

$$[\mathbf{D}] = [\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}] \tag{1.80}$$

is a skew matrix; moreover, it results that if $k = 1$, then the matrix [**D**] is the zero matrix with only one element.

Sometimes, in the analytical calculations, it is useful to use the skew matrices associated to the unit vectors **i**, **j**, **k**,

$$[\mathbf{U}_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, [\mathbf{U}_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, [\mathbf{U}_3] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.81}$$

and the column matrices

$$\{\mathbf{u}_1\}=\begin{bmatrix}1\\0\\0\end{bmatrix},\{\mathbf{u}_2\}=\begin{bmatrix}0\\1\\0\end{bmatrix},\{\mathbf{u}_3\}=\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{1.82}$$

respectively.

One obtains the expressions

$$[\mathbf{B}]=\sum_{i=1}^{3}b_i[\mathbf{U}_i],\{\mathbf{b}\}=\sum_{i=1}^{3}b_i\{\mathbf{u}_i\}, \tag{1.83}$$

$$[\mathbf{U}_i]\left[\mathbf{U}_j\right][\mathbf{U}_i]=[\mathbf{0}],(\forall)\ i\neq j, \tag{1.84}$$

$$[\mathbf{U}_1]=[\mathbf{U}_2][\mathbf{U}_3]-[\mathbf{U}_3][\mathbf{U}_2] \tag{1.85}$$

and the analogous,

$$[\mathbf{U}_1][\mathbf{U}_2][\mathbf{U}_3]+[\mathbf{U}_3][\mathbf{U}_2][\mathbf{U}_1]=[\mathbf{0}]. \tag{1.86}$$

and the analogous.

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