

## **PART I**

### **THE EMERGENCE AND BREAKDOWN OF COMPLEXITY**

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# FEATURES OF COMPLEXITY

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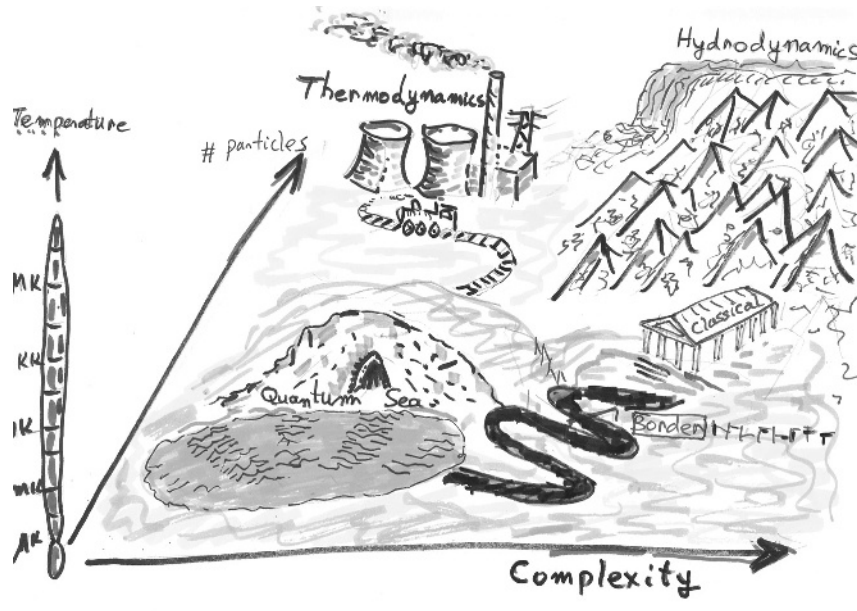
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## I. INTRODUCTION

A complex dynamical system is associated with a description which requires a number of variables comparable to the number of particles. If we accept that quantum mechanics is the basic theory of matter, we are faced with the dilemma of the emergence of dynamical complexity. One of the main pillars of quantum mechanics is the superposition principle. As a result, the theory is completely linear. Dynamical complexity is typically associated with nonlinear phenomena. Complexity can be quantified as the ratio between the number of variables required to describe the dynamics of the system to the number of degrees of freedom. Chaotic dynamics is such an example where this ratio is close to one [1]. Classical mechanics is generically nonlinear and therefore chaotic dynamics emerges.



**Figure 1.** The relation between complexity, the number of particles and temperature in the physical world. Complexity is measured by the ratio of the number of variables required to describe the dynamics of a system compared to the number of degrees of freedom.

Typical classical systems of even a few degrees of freedom can become extremely complex. The complexity can be associated with positive Kolmogorov entropy [2]. In contrast, quantum mechanics is regular. Strictly, closed quantum systems have zero Kolmogorov entropy [3]. How do these two fundamental theories which address dynamical phenomena have such striking differences? The issue of the emergence of classical mechanics from quantum theory is therefore a non-resolved issue despite many years of study (cf. Figure 1).

Thermodynamics is a rule-based theory with a very small number of variables. The theory of chaos has been invoked to explain the emergence of simplicity from the underlying complex classical dynamics. Chaotic dynamics leads to rapid loss of the ability to keep track of the systems trajectory. As a result, a coarse grain picture of self-averaging reduces the number of variables. Following this viewpoint, complexity is created in the singular transition between quantum and classical dynamics. When full chaos dominates, thermodynamics takes over.

*Quantum thermodynamics* is devoted to the study of thermodynamical processes within the context of quantum dynamics. This leads to an alternative direct route linking quantum mechanics and thermodynamics. This link avoids the indirect route to the theory through classical mechanics. The study is based on the

thermodynamic tradition of learning by example. In this context, it is necessary to establish quantum analogues of heat engines. These studies unravel the intimate connection between the laws of thermodynamics and their quantum origin [4–27]. The key point is that thermodynamical phenomena can be identified at the level of an individual small quantum device [28].

## II. THE EMERGENCE OF CLASSICAL DYNAMICS FROM THE UNDERLYING QUANTUM LAWS

### A. Insight from Quantum Control Theory

Quantum Control focuses on guiding quantum systems from initial states to targets governed by time-dependent external fields [29, 30]. Two interlinked theoretical problems dominate quantum control: the first is the existence of a solution and the second is how to find the control field. Controllability addresses the issue of the conditions on the quantum system which enable control. The typical control targets are state-to-state transformations or optimising a pre-specified observable. A more demanding task is implementing a unitary transformation on a subgroup of states. Such an implementation is the prerequisite for quantum information processing. The unitary transformation connects the initial wavefunction which encodes the computation input to the final wave function which encodes the computation output. Finding a control field for this task can be termed the quantum compiler problem. The existence of a solution for a unitary transformation is assured by the theorem of complete controllability [31–33]. In short, a system is completely controllable if the combined Hamiltonians of the control and system span a compact Lie algebra. Moreover complete controllability implies that all possible state-to-state transformations are guaranteed.

Finding a control field that implements the task is a complex inversion problem. Given the target unitary transformation  $\hat{U}(T)$  at final time  $T$  what is the control field that generates it?

$$i\hbar \frac{d}{dt} \hat{U} = \hat{H}(\epsilon) \hat{U}, \quad (1)$$

where  $\hat{U}(0) = \hat{I}$ , and  $\epsilon(t)$  is the control field. The methods developed to solve the inversion problem could be classified as global, such as Optimal Control Theory (OCT) [34–36], or local, for example, Local Control [37–39]. OCT casts the inversion task into an optimisation problem which is subsequently solved by an iterative approach. The number of iterations required to converge to a high fidelity solution is a measure of the complexity of the inversion.

How difficult is it to solve for the quantum compiler for a specific unitary transformation? This task scales at least factorially with the size of the transformation.

The rational is based on the simultaneous task of generating  $N - 1$  state-to-state transformations which constitute the eigenfunctions of the target unitary transformation. To set the relative phase of these transformations, a field that drives a superposition state to the final target time has to be found. All these individual control fields have to be orthogonal to all the other transformations, thus the scaling becomes  $N!$  more difficult than finding the field that generates an individual state-to-state transformation [40]. This scaling fits the notion that the general quantum compiler computation problem has to be hard in the class of NP problems [41]. If it would be an easy task, a unitary transformation could solve in one step all algorithmic problems.

The typical control Hamiltonian can be divided into an uncontrolled part  $\hat{\mathbf{H}}_0$ , the drift Hamiltonian, and a control Hamiltonian composed from an operator sub-algebra:

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + \sum_j \alpha_j(t) \hat{\mathbf{A}}_j, \quad (2)$$

where  $\alpha_j(t)$  is the control field for the operator  $\hat{\mathbf{A}}_j$  and the set of operators  $\{\hat{\mathbf{A}}\}$  form a closed small Lie sub-algebra. This model includes molecular systems controlled by a dipole coupling to the electromagnetic field. Complete controllability requires that the commutators of  $\hat{\mathbf{A}}_j$  and  $\hat{\mathbf{H}}_0$  span the complete algebra  $\mathbf{U}(N)$  where  $N = n^2 - 1$  and  $n$  is the size of the Hilbert space of the system. If  $\hat{\mathbf{H}}_0$  is part of the control algebra, the system is not completely controllable, that is, there are state-to-state transitions which cannot be accomplished. In the typical quantum control scenarios the size of the control sub-algebra is constant but the size of control space increases. For example, in coherent control of molecules by a light field, the three components of the dipole operator compose the control algebra. These operators are sufficient to completely control a vast number of degrees of freedom of the molecule.

Are systems still completely controllable in the more messy and complex real world? This task is associated with the control of an open quantum system where the controlled system is in contact with the environment. The theorems of controllability do not cover open quantum systems which remains an open problem. Coherent control which is based on interfering pathways is typically degraded by environmental noise or decoherence. Significant effort has been devoted to overcome this issue, mostly in the context of implementing gates for quantum computers. The remedy which is known as dynamical decoupling employs very fast control fields to reset the system on track [42–44].

We argue that there is a fundamental flaw in these remedies. Although the noise from the environment can be suppressed, the fast controls introduce a new source of noise originating from the controllers. The controller which generates the control field has to be fast in the timescale of the controlled system. This means

that the noise introduced by the controller can be modeled as a delta correlated Gaussian noise. For the control algebra of Eq. (2) we obtain:

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] + \sum_j \alpha_j(t)[\hat{A}_j, \hat{\rho}] + \sum_j \xi_j(t)[\hat{A}_j, [\hat{A}_j, \hat{\rho}]], \quad (3)$$

where  $\xi$  is determined by the noise in the controls [45].

Equation (3) can have a different interpretation. It describes a system subject to simultaneous weak quantum measurement of the set of operators  $\{\hat{A}\}$ . Quantum measurement causes the collapse of the system to an eigenstate of the measured operator. A weak measurement is a small step in this direction. It achieves only a small amount of information on the system and induces only partial collapse. Equation (3) describes a continuous process of a series of infinitely weak quantum measurements in time. A weak quantum measurement can be applied simultaneously to a set of non-commuting operators  $\{\hat{A}\}$ . In this case, the system collapses to a generalised coherent state associated with this control sub-algebra [46]. Generalized coherent states are states that have minimum uncertainty with respect to the operators of the algebra, that is, they are classical states.

When the size of the quantum system increases keeping the same control algebra, the noise from the control dominates. As a result, the system will collapse to a classical-like state. Superpositions of generalised coherent states will collapse to mixtures. The rate of collapse is proportional to the distance between the states in the superposition. This means that cat states which are superpositions of macroscopically distinguished state have a very short lifetime. The consequence is that their generation by coherent control becomes impossible.

For large quantum systems, noise on the control means that complete state-to-state control is lost [45]. This implies also that complete unitary control becomes impossible. Does this imply that quantum computers cannot be scaled up? The present analysis is sufficient only for a control model where the number of controls is restricted while the size of Hilbert space is increased. This applies to molecular quantum computers controlled by Nuclear Magnetic Resonance (NMR) or by light fields generated by pulse shaping techniques. The general problem of complete controllability of unitary operators subject to control noise where the number of controls is increased with the system size is still open.

Additional studies with the control Hamiltonian of Eq. (2) have shown that state-to-state control tasks from one generalised coherent state to another are relatively easy. A control field found for a small system size can be employed when the number of states increases with small adjustments. These can be classified as classical control tasks. For example, we could find a field that translates in space a coherent state in a nonlinear Morse oscillator. When we decreased  $\hbar$  and increased the number of states maintaining the same field, a high fidelity solution of the control task was obtained. This simple picture completely changed when

the control task was to generate a superposition of generalised coherent states from a single initial state [47,48]. In this case, the control field was not invariant to an increase in system size. We can conclude by stating that classical control tasks are simply robust and scalable while quantum control tasks are delicate requiring a high algorithmic complexity to generate. A noisy environment may be the cause of the emergence of classical-like dynamics from the underlying quantum foundation.

### III. THE EMERGENCE OF THERMODYNAMICAL PHENOMENA

Thermodynamics developed as a phenomenological theory, with the fundamental postulates based on experimental evidence. The theory was initiated by an analysis of a heat engine by Carnot [49]. The well-established part of the theory concerns quasistatic macroscopic processes near thermal equilibrium. Quantum theory, on the other hand, addresses the dynamical behaviour of systems at atomic and smaller length scales. The two disciplines rely upon different sets of axioms. However, one of the first developments, namely Planck's law, which led to the basics of quantum theory, was achieved thanks to consistency with thermodynamics. Einstein, following the ideas of Planck on blackbody radiation, quantised the electromagnetic field [50].

Cars, refrigerators, air conditioners, lasers, and power plants are all examples of heat engines. We are so accustomed to these devices that we take their operation for granted. Rarely a second thought is devoted to the unifying features governing their performance. Practically, all such devices operate far from the ideal maximum efficiency conditions set by Carnot [49]. To maximise the power output, efficiency is sacrificed. This tradeoff between efficiency and power is the focus of "finite-time thermodynamics". The field was initiated by the seminal paper by Curzon and Ahlborn [51]. From everyday experience, the irreversible phenomena that limit the optimal performance of engines [52] can be identified as losses due to friction, heat leaks and heat transport. Is there a unifying fundamental explanation for these losses? Is it possible to trace the origin of these phenomena to quantum mechanics?

Gedanken heat engines are an integral part of thermodynamical theory. Carnot in 1824 set the stage by analyzing an ideal engine [49]. Carnot's analysis preceded the systematic formulation that led to the first and second laws of thermodynamics. Amazingly, thermodynamics was able to keep its independent status despite the development of parallel theories dealing with the same subject matter. Quantum mechanics overlaps thermodynamics in that it describes the state of matter. But in addition, quantum mechanics includes a comprehensive description of dynamics. This suggests that quantum mechanics can generate a concrete interpretation of the word *dynamics* in thermodynamics leading to a fundamental basis for finite-time thermodynamics [5, 6, 53–56].



The following questions come to mind:

- How do the laws of thermodynamics emerge from quantum mechanics?
- What are the quantum origins of irreversible phenomena involving friction and heat transport?
- What is the relation between the quasistatic thermodynamical process and the quantum adiabatic theorem?

Heat engines can be roughly classified as reciprocating cycles such as the Otto or Carnot cycle or continuous resembling turbines. Each class has its advantages in connecting to quantum theory. We will demonstrate this connection with the quantum version of the Otto cycle for a reciprocating model and the quantum tricycle as the generic model of a continuous heat engine.

### A. The Quantum Otto Cycle

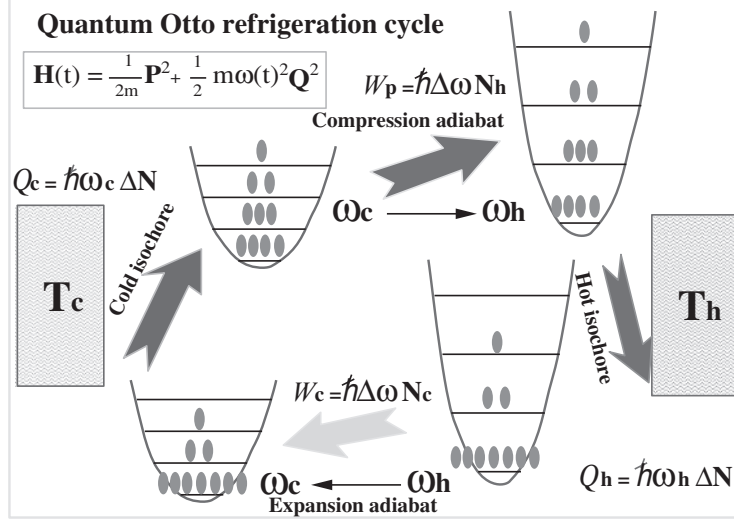
Nicolaus August Otto invented a reciprocating four-stroke engine in 1861 [57]. The basic components of the engine include hot and cold reservoirs, a working medium, and a mechanical output device. The cycle of the engine is defined by four branches (cf. Figure 2):

1. The hot *isochore*: heat is transferred from the hot bath to the working medium without change in the external control.
2. The power *adiabat*: the working medium expands by changing the external control producing work while isolated from the hot and cold reservoirs.
3. The cold *isochore*: heat is transferred from the working medium to the cold bath without control change.
4. The compression *adiabat*: the working medium is compressed by changing the external control consuming power while isolated from the hot and cold reservoirs.

The external control could be a change of volume. In the quantum version, the control is a change of the frequency of the confining potential of a trap [56] or the external magnetic field [58] in a magnetisation/demagnetisation device. The efficiency  $\eta$  of the cycle is limited to  $\eta_o \leq 1 - \frac{\omega_c}{\omega_h}$  where  $\omega_{h/c}$  is the frequency at the hot and cold extremes. As expected, the Otto efficiency is always smaller than the efficiency of the Carnot cycle  $\eta_o \leq \eta_c = 1 - \frac{T_c}{T_h}$ .

### B. Quantum Dynamics of the Working Medium

The quantum analogue of the Otto cycle requires a dynamical description of the working medium, the power output, and the heat transport mechanism. The



**Figure 2.** The quantum Otto refrigeration cycle. The cycle can operate at the level of a single quantum harmonic oscillator shuffling heat from a cold to hot reservoir while adjusting its frequency. The performance characteristics are equivalent to macroscopic Otto refrigerators.

dynamics on the state  $\hat{\rho}$  during the adiabatic branches is unitary and is the solution of the Liouville von Neumann equation [59]:

$$\frac{d}{dt} \hat{\rho}(t) = -\frac{i}{\hbar} [\hat{H}(t), \hat{\rho}(t)], \quad (4)$$

where  $\hat{H} = \hat{H}_0 + \hat{H}_c(\omega(t))$  is time-dependent during the evolution. Notice that generically  $[\hat{H}(t), \hat{H}(t')] \neq 0$ , since the drift Hamiltonian  $\hat{H}_0$  does not commute with the control  $\hat{H}_c$ .

The dynamics on the hot and cold *isochores* is an equilibration process of the working medium with a bath at temperature  $T_h$  or  $T_c$ . This is the dynamics of an open quantum system where the working medium is described explicitly and the influence of the bath implicitly:

$$\frac{d}{dt} \hat{\rho}(t) = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \mathcal{L}_D(\hat{\rho}), \quad (5)$$

where  $\mathcal{L}_D$  is the dissipative superoperator responsible for driving the working medium to thermal equilibrium, while the Hamiltonian  $\hat{H} = \hat{H}(\omega)$  is static. The equilibration is not complete since only a finite time  $\tau_h$  or  $\tau_c$  is allocated to the hot or cold *isochores*. The dissipative superoperator  $\mathcal{L}_D$  is cast into the semigroup form [60].

To summarize, the quantum model of the Otto cycle contains equations of motion for each of the branches. It differs from the thermodynamical model in that a finite time period is allocated to each of these branches. Solving these equations for different operating conditions allows to obtain the quantum thermodynamical observables. Can a simple thermodynamical picture emerge even for driven systems far from equilibrium dynamics?

### C. Quantum Thermodynamics

Thermodynamics is notorious in its ability to describe a process employing an extremely small number of variables. For a heat engine, the energy  $E$  and the entropy  $S$  seem obvious choices. A minimal set of quantum expectations  $\langle \hat{\mathbf{X}}_n \rangle$  constitutes the analogue description where  $\langle \hat{\mathbf{X}}_n \rangle = \text{Tr}\{\hat{\mathbf{X}}_n \hat{\rho}\}$ . The dynamics of this set is generated by the Heisenberg equation of motion:

$$\frac{d}{dt} \hat{\mathbf{X}} = \frac{\partial \hat{\mathbf{X}}}{\partial t} + \frac{i}{\hbar} [\hat{\mathbf{H}}, \hat{\mathbf{X}}] + \mathcal{L}_D^*(\hat{\mathbf{X}}), \quad (6)$$

where the first term addresses an explicitly time-dependent set of operators,  $\hat{\mathbf{X}}(t)$ .

The energy expectation  $E$  is obtained when  $\hat{\mathbf{X}} = \hat{\mathbf{H}}$ , that is,  $E = \langle \hat{\mathbf{H}} \rangle$ . The quantum analogue of the first law of thermodynamics [5, 61]:  $dE = d\mathcal{W} + d\mathcal{Q}$  is obtained by inserting  $\hat{\mathbf{H}}$  into Eq. (6):

$$\frac{d}{dt} \hat{\mathbf{E}} = \dot{\mathcal{W}} + \dot{\mathcal{Q}} = \left\langle \frac{\partial \hat{\mathbf{H}}}{\partial t} \right\rangle + \langle \mathcal{L}_D^*(\hat{\mathbf{H}}) \rangle. \quad (7)$$

The power is identified as  $\mathcal{P} = \dot{\mathcal{W}} = \langle \frac{\partial \hat{\mathbf{H}}}{\partial t} \rangle$ . The heat exchange rate becomes  $\dot{\mathcal{Q}} = \langle \mathcal{L}_D^*(\hat{\mathbf{H}}) \rangle$ . The Otto cycle contains the simplification that power is produced or consumed only on the *adiabats* and heat transfer takes place only on the *isochores*.

The thermodynamic state of the system is fully determined by the thermodynamical variables. Statistical thermodynamics adds the prescription that the state is determined by the maximum entropy condition subject to the constraints set by the thermodynamical observables [62–64].

The state of the working medium of all power-producing engines is not in thermal equilibrium. In order to generalize the canonical form, additional variables are required to define the state of the system. The maximum entropy state subject to this set of observables  $\langle \hat{\mathbf{X}}_j \rangle = \text{tr}\{\hat{\mathbf{X}}_j \hat{\rho}\}$  becomes:

$$\hat{\rho} = \frac{1}{Z} \exp \left( \sum_j \beta_j \hat{\mathbf{X}}_j \right), \quad (8)$$

where  $\beta_j$  are Lagrange multipliers. The generalized canonical form of Eq. (8) is meaningful only if the state can be cast in the canonical form during the complete cycle of the engine leading to  $\beta = \beta(t)$ . This requirement is called canonical invariance [65]. A necessary condition for canonical invariance is that the set of operators  $\hat{\mathbf{X}}$  in Eq. (8) is closed under the dynamics generated by the equation of motion. By knowing the initial values of the observables  $\langle \hat{\mathbf{X}}_j \rangle(0)$ , the Heisenberg equations of motion for the cycle can be solved. This leads to the values of the thermodynamical observables at any time  $\langle \hat{\mathbf{X}}_j \rangle(t)$ . If this condition is also sufficient for canonical invariance, then the state of the system can be reconstructed from a small number of quantum thermodynamical observables  $\langle \hat{\mathbf{X}}_j \rangle(t)$ .

The condition for canonical invariance on the unitary part of the evolution taking place on the *adiabats* is as follows: if the Hamiltonian is a linear combination of the operators in the set  $\hat{\mathbf{H}}(t) = \sum_m h_m \hat{\mathbf{X}}_m$  ( $h_m(t)$  are expansion coefficients), and the set forms a closed Lie algebra  $[\hat{\mathbf{X}}_j, \hat{\mathbf{X}}_k] = \sum_l C_l^{jk} \hat{\mathbf{X}}_l$  (where  $C_l^{jk}$  is the structure factor of the Lie algebra), then the set  $\hat{\mathbf{X}}$  is closed under the evolution [66]. In addition, canonical invariance prevails [67].

As an example for the Otto cycle with a working medium composed from a harmonic oscillator:

$$\hat{\mathbf{H}} = \frac{\hat{\mathbf{P}}^2}{2m} + \frac{m\omega(t)^2}{2} \hat{\mathbf{Q}}^2,$$

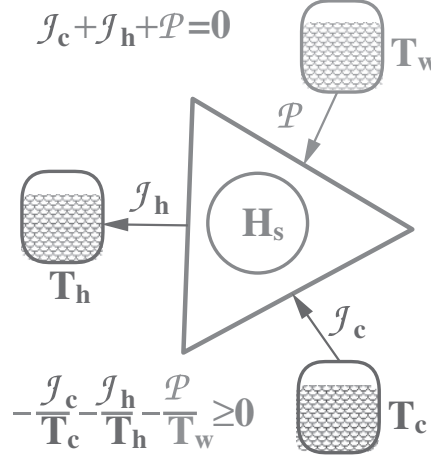
the set of the operators  $\hat{\mathbf{P}}^2$ ,  $\hat{\mathbf{Q}}^2$ , and  $\hat{\mathbf{D}} = (\hat{\mathbf{Q}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{Q}})$  form a closed Lie algebra. The Hamiltonian can be decomposed into the two first operators of the set  $\hat{\mathbf{P}}^2$  and  $\hat{\mathbf{Q}}^2$ . Therefore canonical invariance will result on the adiabatic branches. On the *isochores*, this set is also closed to the operation of  $\mathcal{L}_D$ . This means that the conditions of canonical invariance are fulfilled for this case [56]. Only two additional variables to the energy are able to completely describe the system even for conditions which are very far from thermal equilibrium.

The significance of canonical invariance is that all thermodynamical quantities become functions of a very limited set of quantum observables  $\langle \hat{\mathbf{X}}_j \rangle$ . The choice of operators  $\hat{\mathbf{X}}_j$  should reflect the most characteristic thermodynamical variables. This is an example of simplicity emerging from complexity.

#### D. The Quantum Tricycle

The minimum requirement for a continuous quantum thermodynamical device is a system connected simultaneously to three reservoirs [68]. These baths are termed hot, cold, and work reservoir, as described in Figure 3. A crucial point is that this device is nonlinear, combining three currents. A linear device cannot function as a heat engine or a refrigerator [69].

A quantum description requires a representation of the dynamics working medium and the three heat reservoirs. A reduced description is employed in which



**Figure 3.** The quantum tricycle: a quantum heat pump designed by the Hamiltonian  $\hat{\mathbf{H}}_s$  coupled to a work reservoir with temperature  $T_w$ , a hot reservoir with temperature  $T_h$  and a cold reservoir with temperature  $T_c$ . The heat and work currents are indicated. In steady state  $J_h + J_c + P = 0$ .

the dynamics of the working medium is described by the Heisenberg equation for the operator  $\hat{\mathbf{O}}$  for open systems [60, 70]:

$$\frac{d}{dt}\hat{\mathbf{O}} = \frac{i}{\hbar}[\hat{\mathbf{H}}_s, \hat{\mathbf{O}}] + \frac{\partial \hat{\mathbf{O}}}{\partial t} + \mathcal{L}_h(\hat{\mathbf{O}}) + \mathcal{L}_c(\hat{\mathbf{O}}) + \mathcal{L}_w(\hat{\mathbf{O}}), \quad (9)$$

where  $\hat{\mathbf{H}}_s$  is the system Hamiltonian and  $\mathcal{L}_g$  are the dissipative completely positive superoperators for each bath ( $g = h, c, w$ ). A minimal Hamiltonian describing the essence of the quantum refrigerator is composed of three interacting oscillators:

$$\begin{aligned} \hat{\mathbf{H}}_s &= \hat{\mathbf{H}}_0 + \hat{\mathbf{H}}_{int} \\ \hat{\mathbf{H}}_0 &= \hbar\omega_h \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}} + \hbar\omega_c \hat{\mathbf{b}}^\dagger \hat{\mathbf{b}} + \hbar\omega_w \hat{\mathbf{c}}^\dagger \hat{\mathbf{c}} \\ \hat{\mathbf{H}}_{int} &= \hbar\omega_{int} (\hat{\mathbf{a}}^\dagger \hat{\mathbf{b}} \hat{\mathbf{c}} + \hat{\mathbf{a}} \hat{\mathbf{b}}^\dagger \hat{\mathbf{c}}^\dagger). \end{aligned} \quad (10)$$

$\hat{\mathbf{H}}_{int}$  represents an annihilation of excitations on the work and cold bath simultaneous with creating an excitation in the hot bath.  $\hat{\mathbf{H}}_{int}$  is nonlinear in contrast to the linear  $\hat{\mathbf{H}}_0$ . In an open quantum system, the superoperators  $\mathcal{L}_g$  represent a thermodynamic isothermal partition allowing heat flow from the bath to the system. Such a partition is equivalent to the weak coupling limit between the system and bath where  $\hat{\rho} = \hat{\rho}_s \otimes \hat{\rho}_B$  at all times [6]. The superoperators  $\mathcal{L}_g$  are derived from the Hamiltonian:

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_s + \hat{\mathbf{H}}_h + \hat{\mathbf{H}}_c + \hat{\mathbf{H}}_w + \hat{\mathbf{H}}_{sh} + \hat{\mathbf{H}}_{sc} + \hat{\mathbf{H}}_{sw}, \quad (11)$$

where  $\hat{\mathbf{H}}_g$  are bath Hamiltonians and  $\hat{\mathbf{H}}_{sg}$  represent system bath coupling. Each of the oscillators is linearly coupled to a heat reservoir, for example, for the hot

bath:  $\hat{\mathbf{H}}_{sh} = \lambda_{sh}(\hat{\mathbf{a}}\hat{\mathbf{A}}_h^\dagger + \hat{\mathbf{a}}^\dagger\hat{\mathbf{A}}_h)$ . Each reservoir individually equilibrates the working medium to thermal equilibrium with the reservoir temperature. In general, the derivation of a thermodynamically consistent master equation is technically very difficult [71]. Typical problems are approximations that violate the laws of thermodynamics. We therefore require that the master equations fulfil the thermodynamical laws. Under steady state conditions of operation they become:

$$\begin{aligned} J_h + J_c + \mathcal{P} &= 0, \\ -\frac{J_h}{T_h} - \frac{J_c}{T_c} - \frac{\mathcal{P}}{T_w} &\geq 0, \end{aligned} \quad (12)$$

where  $J_k = \langle \mathcal{L}_k(\hat{\mathbf{H}}) \rangle$ . The first equality represents conservation of energy (first law) [72, 73], and the second inequality represents positive entropy production in the universe  $\Sigma_u \geq 0$  (second law). For refrigeration,  $T_w \geq T_h \geq T_c$ . From the second law, the scaling exponent  $\alpha \geq 1$  [7].

### E. The Third Law of Thermodynamics

There exist two seemingly independent formulations of the third law of thermodynamics, both originally stated by Nernst [74–76]. The first is a purely static (equilibrium) one, also known as the Nernst heat theorem, and can simply be phrased as follows:

- The entropy of any pure substance in thermodynamic equilibrium approaches zero as the temperature approaches zero.

The second is a dynamical one, known as the unattainability principle:

- It is impossible by any procedure, no matter how idealized, to reduce any assembly to absolute zero temperature in a finite number of operations.

Different studies investigating the relation between the two formulations have led to different answers regarding which of these formulations implies the other, or if neither does.

A more concrete version of the dynamical third law can be expressed as follows:

- No refrigerator can cool a system to absolute zero temperature at finite time.

This formulation enables us to quantify the third law, that is, evaluating the characteristic exponent  $\chi$  of the cooling process:

$$\frac{d}{dt}T \propto -T^\xi, \quad (13)$$

for  $T \rightarrow 0$ . Namely, for  $\xi \geq 1$  the system is cooled to zero temperature at finite time. The cold bath is modelled either by a system of harmonic oscillators (bosonic bath) or the ideal gas at low density, including the possible Bose–Einstein condensation

effect. To check under what conditions the third law is valid, we consider a finite cold bath with the heat capacity  $Cv(T_c)$  cooled down by the refrigerator with the optimised time-dependent parameter. The equation which describes the cooling process reads:

$$Cv(T_c) \frac{d}{dt} T_c = -\mathcal{J}_c \geq 0. \quad (14)$$

The third law would be violated if the solution  $T_c(t)$  reached zero at finite time. The heat current has the universal structure when  $T_c \rightarrow 0$ :

$$\mathcal{J}_c = \hbar \omega_c \lambda(\omega_c), \quad (15)$$

where  $\hbar \omega_c$  is the energy quant of transport and  $\lambda(\omega_c)$  the heat conductance rate. The optimal cooling rate is obtained when  $\omega_c \propto T_c$  [77, 78]. The ration of  $\lambda(T_c)/Cv(T_c) \propto T_c^{\frac{1}{2}}$  when  $T_c \rightarrow 0$  for a Bose/Fermi gas cold bath. For a phonon bath,  $\lambda(\omega_c)$  depends on the spectral density. An Ohmic bath has the ratio  $\lambda(T_c)/Cv(T_c) \propto 1$  which could violate the third law. This is consistent with the observation that for Ohmic spectral density, the system and bath have no ground state.

The analysis of the examples of both the discrete and continuous quantum heat engines shows that a thermodynamical description is valid at the level of a single quantum device. Consistency with thermodynamics has always proven to be correct. Apparent violations of the thermodynamical laws could always be attributed to faulty analysis [27]. Although there have been challenges to this rule [79], we still stand by our statement.

#### IV. PERSPECTIVE

The issue of complexity in dynamical systems is still not resolved. We tried to present a unifying framework based on Kolmogorov's idea of algorithmic complexity. We examined the control task which leads to quantum computing: generating a unitary transformation. If this task was easy, many complex algorithmic problems could be solved in parallel. This indicates that the problem of the quantum compiler is complex. Similar approaches could quantify the complexity of other quantum dynamical encounters, where much work should be done. Once noise is introduced, for example, in the control fields of coherent control, classical-like phenomena emerge. In addition, this noise on the controls can make the quantum compiler task unscalable. Similar external noise will also reduce quantum dynamics into classical-like localised states, thus pointing to a route of the emergence of classical phenomena from quantum mechanics.

Finite-time thermodynamics emerges naturally from quantum dynamics of open systems. Even a small quantum engine, due to inherent statistical fluctuations,

behaves like a thermodynamical device. A necessary condition for any quantum device is a nonlinear character combining at least three energy currents. This is the definition of the quantum thermodynamical tricycle. Quantum network composed from tricycles will display complex dynamics due to this nonlinearity. In a simple analysis of the quantum tricycle, we could study the emergence of the third law of thermodynamics.

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## DISCUSSION

### Session: IA

#### Speaker: Ronnie Kosloff

**Gregoire Nicolis said:** Your definition of complexity follows closely the ideas of algorithmic information theory developed by Kolmogorov and Chaitin. Now, according to this theory, the most complex objects that one may encounter are the random sequences. This is unlikely to describe adequately complexity as we observe it in nature where behind an apparent randomness one finds correlations and selection rules, reflecting the action of some underlying deterministic evolution laws [1].

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**Ronnie Kosloff answered:** The ideas of Kolmogorov and complexity have developed significantly and a whole new field in computer science emerged [1].

The ideas of algorithmic complexity had a great influence on the field of quantum computing. Nevertheless, they have not penetrated yet to the issue of dynamical complexity. I view this as one of the outstanding problems which will benefit both fields.

1. A. Sanjeev and B. Barak, *Computational Complexity: A Modern Approach* (Cambridge, ISBN 978-0-521-42426-4, 2007).