CHAPTER ONE

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Introduction

In this introductory chapter we formulate several problems that illustrate basic ideas that reoccur frequently in this book.

In Section 1.1 we discuss two mathematical models, one from physics and one from population biology. Each mathematical model is a differential equation—an equation involving the rate of change of a variable with respect to time. Using these models as examples, we introduce some basic terminology, explore the notion of a solution of a differential equation, and end with an overview of the art and craft of mathematical modeling.

It is not always possible to find analytic, closed-form solutions of a differential equation. In Section 1.2 we look at two graphical methods for studying the qualitative behavior of solutions: phase lines and direction fields. Although we will learn how to sketch direction fields by hand, we will use the computer to draw them.

Sections 1.1 and 1.2 give us a glimpse of two of the three major methods of studying differential equations, the **analytical** method and the **geometric** method, respectively. We defer study of the third major method—**numerical**—to Chapter 8. However, you may study the first three sections of Chapter 8 immediately after Chapter 1.

In Section 1.3 we present some important definitions and commonly used terminology in conjunction with different ways of classifying differential equations. Classification schemes provide organizational structure for the book and help give you perspective on the subject of differential equations.

1.1 Mathematical Models and Solutions

Many of the principles, or laws, underlying the behavior of the natural world are statements, or relations, involving rates in which one variable, say, y, changes with respect to another variable, t, for example. Most often, these relations take the form of equations containing y and certain of the derivatives $y', y'', \ldots, y^{(n)}$ of y with respect to t. The resulting equations are then referred to as **differential equations**. Some examples of differential equations that will be studied in detail later on in the text, are:

$y' = r\left(1 - \frac{y}{K}\right)y,$ an equation for population dynamics,

$my'' + \gamma y' + ky = 0,$ the equation for a damped spring-mass system, and

$\theta'' + \frac{g}{l}\sin(\theta) = 0,$ the pendulum equation.

The subject of differential equations was motivated by problems in mechanics, elasticity, astronomy, and geometry during the latter part of the 17th century. Inventions (or discoveries) in theory, methods, and notation evolved concurrently with innovations in calculus. Since their early historical origins, the number and variety of problems to which differential equations are applied have grown substantially. Today, scientists and engineers use differential equations to study problems in all fields of science and engineering, as well as in several of the business and social sciences. Some representative problems from these fields are shown below.

Applications of Differential Equations

- airplane and ship design
- earthquake detection and prediction
- controlling the flight of ships and rockets
- modeling the dynamic behavior of nerve cells
- describing the behavior of economic systems
- heat transfer
- wave propagation
- weather forecasting
- designing medical imaging technologies
- determining the price of financial derivatives
- forecasting and managing the harvesting of fish populations
- designing optimal vaccination policies to prevent the spread of disease

The common thread that links these applications is that they all deal with systems that evolve in time. Differential equations is the mathematical apparatus that we use to study such systems.

We often refer to a differential equation that describes some physical process as a **mathematical model** of the process; many such models are discussed throughout this book. In this section we construct a model from physics and a model from population biology. Each model results in an equation that can be solved by using an integration technique from calculus. These examples suggest that even simple differential equations can provide useful models of important physical systems.

Heat Transfer: Newton's Law of Cooling

EXAMPLE 1

If a material object is hotter or colder than the surrounding environment, its temperature will approach the temperature of the environment. If the object is warmer than the environment, its temperature will decrease. If the object is cooler than the environment, its temperature will increase. Sir Isaac Newton postulated that the rate of change of the temperature of the object is negatively proportional to the difference between its temperature and the temperature of the surroundings (the **ambient temperature**). This principle is referred to as **Newton's law of cooling**.

Suppose we let $u(t)$ denote the temperature of the object at time t, and let T be the ambient temperature (see Figure 1.1.1). Then du/dt is the rate at which the temperature of the object changes. From Newton, we know that du/dt is proportional to $-(u - T)$. Introducing a positive constant of proportionality k called the **transmission coefficient**, we then get the differential equation

$$\frac{du}{dt} = -k(u - T), \qquad \text{or} \qquad u' = -k(u - T). \tag{1}$$

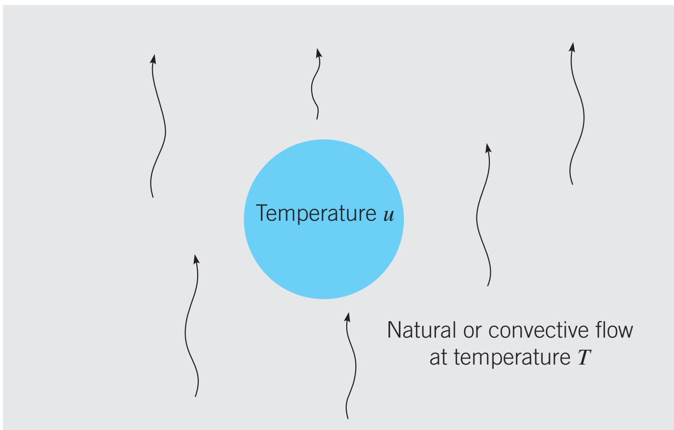

FIGURE 1.1.1 Newton's Law of Cooling: The time rate of change of u, du/dt, is negatively proportional to $u - T$: $du/dt \propto -(u - T)$.

Note that the minus sign on the right side of Eq. (1) causes du/dt to be negative if $u(t) > T$, while du/dt is positive if $u(t) < T$. The transmission coefficient measures the rate of heat exchange between the object and its surroundings. If k is large, the rate of heat exchange is rapid. If k is small, the rate of heat exchange is slow. This would be the case, for example, if the object was surrounded by thick insulating material.

The temperatures u and T are measured in either degrees Fahrenheit (°F) or degrees Celsius (°C). Time is usually measured in units that are convenient for expressing time intervals over which significant changes in u occur, such as minutes, hours, or days. Since the left side of Eq. (1) has units of temperature per unit time, k must have the units of $(\text{time})^{-1}$.

Newton's law of cooling is applicable to situations in which the temperature of the object is approximately uniform at all times. This is the case for small objects that conduct heat easily, or containers filled with a fluid that is well mixed. Thus, we expect the model to be reasonably accurate in predicting the temperature of a small copper sphere, a well-stirred cup of coffee, or a house in which the air is continuously circulated, but the model would not be very accurate for predicting the temperature of a roast in an oven.

Terminology

Let us assume that the ambient temperature T in Eq. (1) is a constant, say, $T = T_0$, so that Eq. (1) becomes

$$u' = -k(u - T_0). \tag{2}$$

In Section 1.2 we consider an example in which T depends on t. Common mathematical terminology for the quantities that appear in this equation are:

time	t	is an **independent variable**,
temperature	u	is a **dependent variable** because it depends on t,
	k and T_0	are **parameters** in the model.

The equation is an **ordinary differential equation** because it has one, and only one, independent variable. Consequently, the derivative in Eq. (2) is an ordinary derivative. It is a **first order** equation because the highest order derivative that appears in the equation is the first derivative. The dependency of u on t implies that u is, in fact, a function of t, say, $u = \phi(t)$. Thus when we write Eq. (2), three questions may, after a bit of reflection, come to mind:

1. "Is there actually a function $u = \phi(t)$, with derivative $u' = d\phi/dt$, that makes Eq. (2) a true statement for each time t?" If such a function exists, it is called a solution of the differential equation.
2. "If the differential equation does have a solution, how can we find it?"
3. "What can we do with this solution, once we have found it?"

In addition to methods used to derive mathematical models, answers to these types of questions are the main subjects of inquiry in this book.

Solutions and Integral Curves

By a **solution** of Eq. (2), we mean a differentiable function $u = \phi(t)$ that satisfies the equation. One solution of Eq. (2) is $u = T_0$, since Eq. (2) reduces to the identity $0 = 0$ when T_0 is substituted for u in the equation. In other words, "It works when we put it into the equation." The constant solution $u = T_0$ is referred to as an **equilibrium solution** of Eq. (2). Although simple, equilibrium solutions usually play an important role in understanding the behavior of other solutions. In Section 1.2 we will consider them in a more general setting.

If we assume that $u \neq T_0$, we can discover other solutions of Eq. (2) by first rewriting it in the form

$$\frac{du/dt}{u - T_0} = -k. \tag{3}$$

By the chain rule the left side of Eq. (3) is the derivative of $\ln |u - T_0|$ with respect to t, so we have

$$\frac{d}{dt} \ln |u - T_0| = -k. \tag{4}$$

Then, by integrating both sides of Eq. (4), we obtain

$$\ln|u - T_0| = -kt + C, \tag{5}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of Eq. (5), we find that

$$|u - T_0| = e^{-kt+C} = e^C e^{-kt}, \tag{6}$$

or

$$u - T_0 = \pm e^C e^{-kt}. \tag{7}$$

Thus

$$u = T_0 + ce^{-kt} \tag{8}$$

is a solution of Eq. (2), where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that if we allow c to take the value zero, then the constant solution $u = T_0$ is also contained in the expression (8). The expression (8) contains all possible solutions of Eq. (2) and is called the **general solution** of the equation.

Given a differential equation, the usual problem is to find solutions of the equation. However, it is also important to be able to determine whether a particular function is a solution of the equation. Thus, if we were simply asked to verify that u in Eq. (8) is a solution of Eq. (2), then we would need to substitute $T_0 + ce^{-kt}$ for u in Eq. (2) and show that the equation reduces to an identity, as we now demonstrate.

EXAMPLE 2

Verify by substitution that $u = T_0 + ce^{-kt}$, where c is an arbitrary real number, is a solution of Eq. (2),

$$u' = -k(u - T_0), \tag{9}$$

on the interval $-\infty < t < \infty$.

Substituting $\phi(t) = T_0 + ce^{-kt}$ for u in the left side of the equation gives $\phi'(t) = -kce^{-kt}$ while substituting $\phi(t)$ for u into the right side yields $-k(T_0 + ce^{-kt} - T_0) = -kce^{-kt}$. Thus, upon substitution, Eq. (2) reduces to the identity

$$\underbrace{-kce^{-kt}}_{\phi'(t)} = \underbrace{-kce^{-kt}}_{-k(\phi(t)-T_0)}, \qquad -\infty < t < \infty,$$

for each real number c and each value of the parameter k.

Integral Curves. The geometrical representation of the general solution (8) is an infinite family of curves in the tu-plane called **integral curves**. Each integral curve is associated with a particular value of c; it is the graph of the solution corresponding to that value of c.

Although we can sketch, by hand, qualitatively correct integral curves described by Eq. (8), we will assign numerical values to k and T_0, and then use a computer to plot the graph of Eq. (8) for some different values of c. Setting $k = 1.5$ day^{-1} and $T_0 = 60°$F in Eq. (2) and Eq. (8) gives us

$$\frac{du}{dt} = -1.5(u - 60), \tag{10}$$

with the corresponding general solution

$$u = 60 + ce^{-1.5t}. \tag{11}$$

In Figure 1.1.2 we show several integral curves of Eq. (10) obtained by plotting the graph of the function in Eq. (11) for different values of c. Note that all solutions approach the equilibrium solution $u = 60$ as $t \to \infty$.

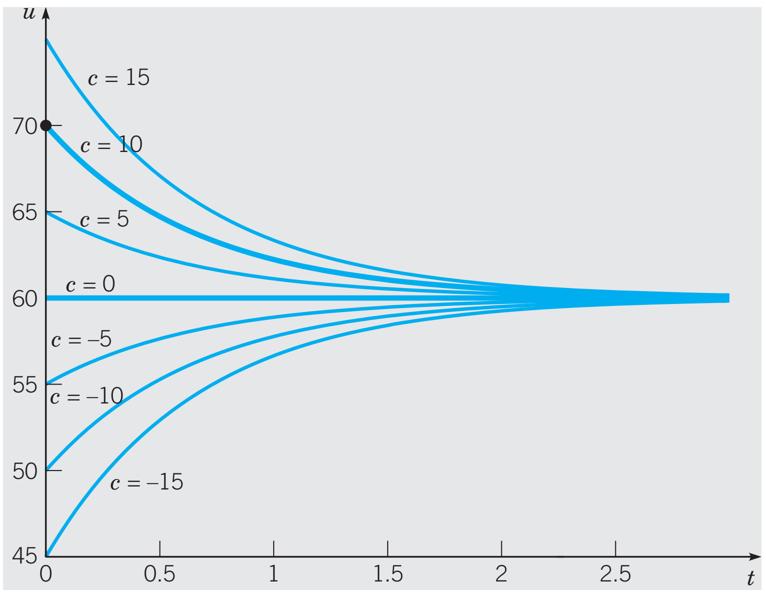

FIGURE 1.1.2 Integral curves of $u' = -1.5(u - 60)$. The curve corresponding to $c = 10$ in Eq. (11) is the graph of $u = 60 + 10e^{-1.5t}$, the solution satisfying the initial condition $u(0) = 70$. The curve corresponding to $c = 0$ in Eq. (11) is the graph of the equilibrium solution $u = 60$, which satisfies the initial condition $u(0) = 60$.

Initial Value Problems

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this by specifying a point that must lie on the graph of the solution. For example, to determine the constant c in Eq. (11), we could require that the temperature have a given value at a certain time, such as the value 70 at time $t = 0$. In other words, the graph of the solution must pass through the point (0, 70). Symbolically, we can express this condition as

$$u(0) = 70. \tag{12}$$

Then, substituting $t = 0$ and $u = 70$ into Eq. (11), we obtain

$$70 = 60 + c.$$

Hence $c = 10$, and by inserting this value in Eq. (11), we obtain the desired solution, namely,

$$u = 60 + 10e^{-1.5t}. \tag{13}$$

The graph of the solution (13) is the thick curve, labeled by $c = 10$, in Figure 1.1.2. The additional condition (12) that we used to determine c is an example of an **initial condition**.

The differential equation (10) together with the initial condition (12) form an **initial value problem**.

Note that the solution of Eq. (10) subject to the initial condition $u(0) = 60$ is the equilibrium solution $u = 60$, the thick curve labeled by $c = 0$ in Figure 1.1.2.

Population Biology

Next we consider a problem in population biology. To help control the field mouse population in his orchards, in an economical and ecofriendly way, a fruit farmer installs nesting boxes for barn owls, predators for whom mice are a natural food supply. In the absence of predators we assume that the rate of change of the mouse population is proportional to the current population; for example, if the population doubles, then the number of births per unit time also doubles. This assumption is not a well-established physical law (such as the laws of thermodynamics, which underlie Newton's law of cooling in Example 1), but it is a common initial hypothesis[1] in a study of population growth. If we denote time by t and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{14}$$

where the proportionality factor r is called the **rate constant** or **growth rate**.

As a simple model for the effect of the owl population on the mouse population, let us assume that the owls consume the mice at a constant predation rate a. By modifying Eq. (14) to take this into account, we obtain the equation

$$\frac{dp}{dt} = rp - a, \tag{15}$$

where both r and a are positive. Thus the rate of change of the mouse population, dp/dt, is the net effect of the growth term rp and the predation term $-a$. Depending on the values of p, r, and a, the value of dp/dt may be of either sign.

EXAMPLE 3

Suppose that the growth rate for the field mice is 0.5/month and that the owls kill 15 mice per day. Determine appropriate values for the parameters in Eq. (15), find the general solution of the resulting equation, and graph several solutions, including any equilibrium solutions.

We naturally assume that p is the number of individuals in the mouse population at time t. We can choose our units for time to be whatever seems most convenient; the two obvious possibilities are days or months. If we choose to measure time in months, then the growth term is $0.5p$ and the predation term is $-(15 \text{ mice/day}) \cdot (30 \text{ days/month}) = -450$ mice/month, assuming an average month of 30 days. Thus Eq. (15) becomes

$$\frac{dp}{dt} = 0.5p - 450, \tag{16}$$

where each term has the units of mice/month.

By following the same steps that led to the general solution of Eq. (2), we find that the general solution of Eq. (16) is

$$p = 900 + ce^{t/2}, \tag{17}$$

where c is again a constant of integration.

[1] A somewhat better model of population growth is discussed in Section 2.5.

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Integral curves for Eq. (16) are shown in Figure 1.1.3. For sufficiently large values of p it can be seen from the figure, or directly from Eq. (16) itself, that dp/dt is positive, so that solutions increase. On the other hand, for small values of p the opposite is the case. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in Eq. (16) and then solving for p, we find the equilibrium solution $p = 900$ for which the growth term and the predation term in Eq. (16) are exactly balanced. This corresponds to the choice $c = 0$ in the general solution (17).

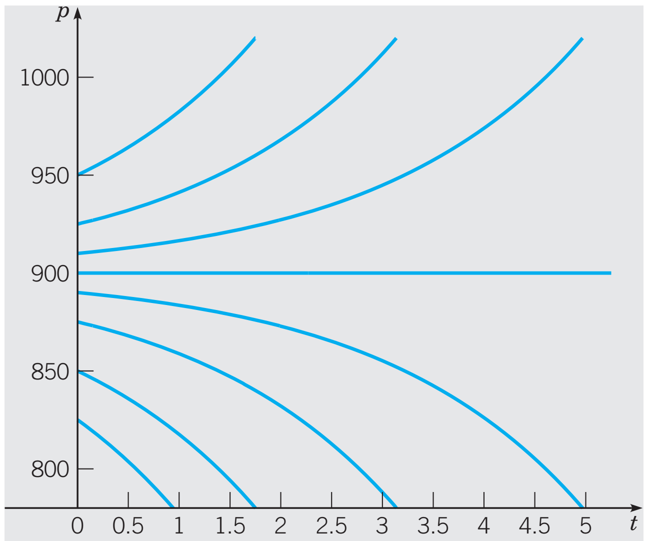


FIGURE 1.1.3 Integral curves, including the equilibrium solution $p = 900$, for $p' = 0.5p - 450$.

Solutions of the more general equation (15), in which the growth rate and the predation rate are unspecified, behave very much like those of Eq. (16). The equilibrium solution of Eq. (15) is $p = a/r$. Solutions above the equilibrium solution increase, while those below it decrease.

Constructing Mathematical Models

Mathematical modeling is the craft, and art, of using mathematics to describe and understand real-world phenomena. A viable mathematical model can be used to test ideas, make predictions, and aid in design and control problems that are associated with the phenomena. For instance, in Example 1, we constructed the differential equation

$$\frac{du}{dt} = -k(u - T) \tag{18}$$

to model heat exchange between an object and its surroundings. Recall that $u(t)$ is the time-dependent variable representing the temperature of the object and T is the temperature of the surroundings. If the value of u is known at time $t = 0$, and the values of the parameters T and k are known, solutions of this differential equation tell us what the temperature of the object will be for times $t > 0$.

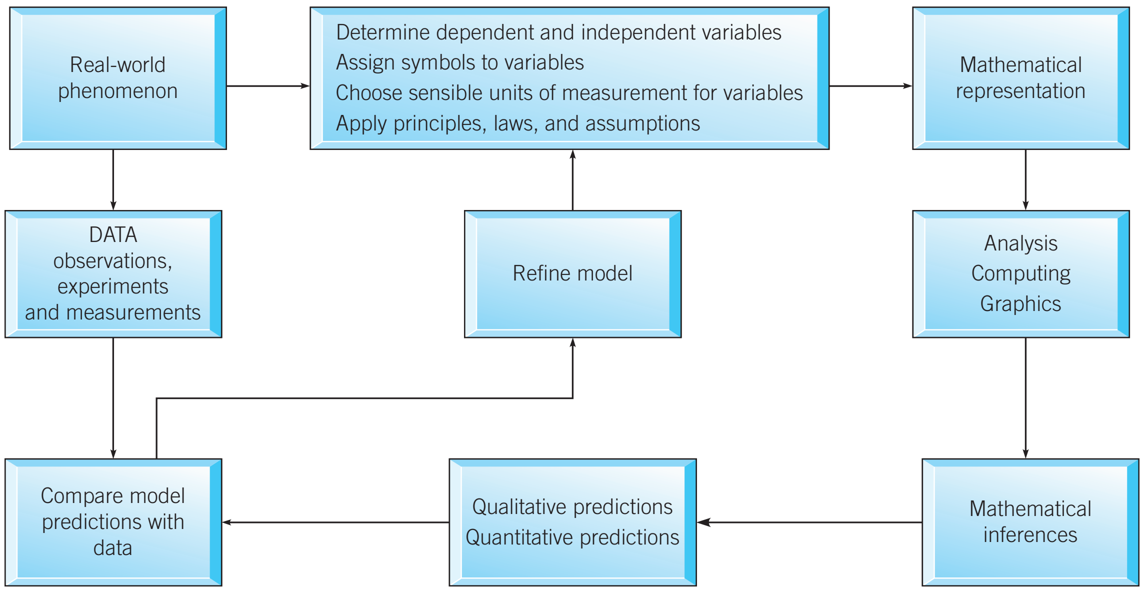


FIGURE 1.1.4 A diagram of the modeling process.

The steps used to arrive at Eq. (18) are typical of the steps used to construct any mathematical model. It is, therefore, worthwhile to illustrate the general process by a system flow diagram, as in Figure 1.1.4.

In the Problems for this section, and for many other sections of this textbook, we ask you to construct differential equation models of various real-world phenomena. In constructing mathematical models, you will find that each problem is different. Although the modeling process, in broad outline, is well represented by the above diagram, it is not a skill that can be reduced to the observance of a set of prescribed rules. Successful modeling usually requires that the modeler be intimate with the field in which the problem originates. However experience has shown that the very act of attempting to construct a mathematical model forces the modeler to ask the most cogent questions about the phenomenon being investigated:

1. What is the purpose of the model?
2. What aspects of the phenomenon are most important for the intended uses of the model?
3. What can we measure or observe?
4. What are the relevant variables; what is their relationship to the measurements?
5. Are there well-established principles (such as physical laws, or economic laws) to guide us in formulating the model?
6. In terms of the variables, how do we mathematically represent the interaction of various components of the phenomenon?
7. What simplifying assumptions can we make?
8. Do conclusions and predictions of the model agree with experiment and observations?
9. What additional experiments are suggested by the model?
10. What are limitations of the model?

For many applied mathematicians, engineers, and scientists, mathematical modeling is akin to poetry—an art form and creative act employing language that adheres to form and conventions. Likewise, there are rules (e.g., physical laws) that the mathematical modeler must follow, yet he or she has access to a myriad of mathematical tools (the language) for describing the phenomenon under investigation. History abounds with the names of scientists, mathematicians, and engineers, driven by the desire to understand nature and advance technology, who have engaged in the practice of mathematical modeling: Newton, Euler, von Kármán, Verhulst, Maxwell, Rayleigh, Navier, Stokes, Heaviside, Einstein, Schrödinger, and so on. Their contributions have literally changed the world. Nowadays, mathematical modeling is carried out in universities, government agencies and laboratories, business and industrial concerns, policy think tanks, and institutes dedicated to research and education. For many practitioners of mathematical modeling, it is, in a sense, their *raison d'être*.

PROBLEMS

1. Newton's Law of Cooling. A cup of hot coffee has a temperature of 200°F when freshly poured, and is left in a room at 70°F. One minute later the coffee has cooled to 190°F.

(a) Assume that Newton's law of cooling applies. Write down an initial value problem that models the temperature of the coffee.
(b) Determine when the coffee reaches a temperature of 170°F.

2. Blood plasma is stored at 40°F. Before it can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 minutes (min) for the plasma to warm to 90°F. Assume Newton's law of cooling applies. How long will it take the plasma to warm to 90°F if the oven temperature is set at 100°F?

3. At 11:09 p.m. a forensics expert arrives at a crime scene where a dead body has just been found. Immediately, she takes the temperature of the body and finds it to be 80°F. She also notes that the programmable thermostat shows that the room has been kept at a constant 68°F for the past 3 days. After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5°F. This last temperature reading was taken exactly one hour after the first one. The next day the investigating detective asks the forensic expert, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6°F) prior to death, what does she tell the detective?

4. Population Problems. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.

(a) Find the rate constant r if the population doubles in 30 days.
(b) Find r if the population doubles in N days.

5. The field mouse population in Example 3 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.
(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
(c) Find the initial population p_0 if the population is to become extinct in 1 year.

6. Radioactive Decay. Experiments show that a radioisotope decays at a rate negatively proportional to the amount of the isotope present.

(a) Use the following variables and parameters to write down and solve an initial value problem for the process of radioactive decay: t = time; $a(t)$ = amount of the radioisotope present at time t; a_0 = initial amount of radioisotope; r = decay rate, where $r > 0$.
(b) The **half-life**, $T_{1/2}$, of a radioisotope is the amount of time it takes for a quantity of the radioactive material to decay to one-half of its original amount. Find an expression for $T_{1/2}$ in terms of the decay rate r.

7. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

(a) If 100 milligrams (mg) of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.
(b) Find an expression for the amount of thorium-234 present at any time t.
(c) Find the time required for the thorium-234 to decay to one-half its original amount.

8. Classical Mechanics. The differential equation for the velocity v of an object of mass m, restricted to vertical motion and subject only to the forces of gravity and air resistance, is

$$m\frac{dv}{dt} = -mg - \gamma v. \qquad \text{(i)}$$

In Eq. (i) we assume that the drag force, $-\gamma v$ where $\gamma > 0$ is a drag coefficient, is proportional to the velocity.

Acceleration due to gravity is denoted by g. Assume that the upward direction is positive.

(a) Show that the solution of Eq. (i) subject to the initial condition $v(0) = v_0$ is

$$v = \left(v_0 + \frac{mg}{\gamma}\right) e^{-\gamma t/m} - \frac{mg}{\gamma}.$$

(b) Sketch some integral curves, including the equilibrium solution, for Eq. (i). Explain the physical significance of the equilibrium solution.

(c) If a ball is initially thrown in the upward direction so that $v_0 > 0$, show that it reaches its maximum height when

$$t = t_{\max} = \frac{m}{\gamma} \ln\left(1 + \frac{\gamma v_0}{mg}\right).$$

(d) The terminal velocity of a baseball dropped from a high tower is measured to be 33 m/s. If the mass of the baseball is 145 grams (g) and $g = 9.8$ m/s^2, what is the value of γ?

(e) Using the values for m, g, and γ in part (d), what would be the maximum height attained for a baseball thrown upward with an initial velocity $v_0 = 30$ m/s from a height of 2 m above the ground?

9. For small, slowly falling objects, the assumption made in Eq. (i) of Problem 8 that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

(a) Write a differential equation for the velocity of a falling object of mass m if the drag force is proportional to the square of the velocity. Assume that the upward direction is positive.

(b) Determine the limiting velocity after a long time.

(c) If $m = 0.025$ kilograms (kg), find the drag coefficient so that the limiting velocity is -35 m/s.

Mixing Problems. Many physical systems can be cast in the form of a mixing tank problem. Consider a tank containing a solution—a mixture of solute and solvent–such as salt dissolved in water. Assume that the solution at concentration $c_i(t)$ flows into the tank at a volume flow rate $r_i(t)$ and is simultaneously pumped out at the volume flow rate $r_o(t)$. If the solution in the tank is well mixed, then the concentration of the outflow is $Q(t)/V(t)$, where $Q(t)$ is the amount of solute at time t and $V(t)$ is the volume of solution in the tank. The differential equation that models the changing amount of solute in the tank is based on the principle of conservation of mass,

$$\underbrace{\frac{dQ}{dt}}_{\text{rate of change of } Q(t)} = \underbrace{c_i(t) r_i(t)}_{\text{rate in}} - \underbrace{\{Q(t)/V(t)\}\, r_o(t)}_{\text{rate out}}, \qquad \text{(i)}$$

where $V(t)$ also satisfies a mass conservation equation,

$$\frac{dV}{dt} = r_i(t) - r_o(t). \qquad \text{(ii)}$$

If the tank initially contains an amount of solute Q_0 in a volume of solution, V_0, then initial conditions for Eqs. (i) and (ii) are $Q(0) = Q_0$ and $V(0) = V_0$, respectively.

10. A tank initially contains 200 liters (L) of pure water. A solution containing 1 g/L enters the tank at a rate of 4 L/min, and the well-stirred solution leaves the tank at a rate of 5 L/min. Write initial value problems for the amount of salt in the tank and the amount of brine in the tank, at any time t.

11. A tank contains 100 gallons (gal) of water and 50 ounces (oz) of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture flows out at the same rate. Write an initial value problem for the amount of salt in the tank at any time t.

12. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.

(b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

13. Pharmacokinetics. A simple model for the concentration $C(t)$ of a drug administered to a patient is based on the assumption that the rate of decrease of $C(t)$ is negatively proportional to the amount present in the system,

$$\frac{dC}{dt} = -kC,$$

where k is a rate constant that depends on the drug and its value can be found experimentally.

(a) Suppose that a dose administered at time $t = 0$ is rapidly distributed throughout the body, resulting in an initial concentration C_0 of the drug in the patient. Find $C(t)$, assuming the initial condition $C(0) = C_0$.

(b) Consider the case where doses of C_0 of the drug are given at equal time intervals T, that is, doses of C_0 are administered at times $t = 0, T, 2T, \ldots$. Denote by C_n the concentration immediately after the nth dose. Find an expression for the concentration C_2 immediately after the second dose.

(c) Find an expression for the concentration C_n immediately after the nth dose. What is $\lim_{n\to\infty} C_n$?

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106*, no. 2 (1999), pp. 127–135.

14. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

(b) How much of the drug is present in the bloodstream after a long time?

Continuously Compounded Interest. The amount of money $P(t)$ in an interest bearing account in which the principal is compounded continuously at a rate r per annum and in which money is continuously added, or subtracted, at a rate of k dollars per annum satisfies the differential equation

$$\frac{dP}{dt} = rP + k. \quad \text{(i)}$$

The case $k < 0$ corresponds to paying off a loan, while $k > 0$ corresponds to accumulating wealth by the process of regular contributions to an interest bearing savings account.

15. Show that the solution to Eq. (i), subject to the initial condition $P(0) = P_0$, is

$$P = \left(P_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r}. \quad \text{(ii)}$$

Use Eq. (ii) in Problem 15 to solve Problems 16 and 17.

16. According to the International Institute of Social History (Amsterdam), the amount of money used to purchase Manhattan Island in 1626 is valued at \$1,050 in terms of today's dollars. If that amount were instead invested in an account that pays 4% per annum with continuous compounding, what would be the value of the investment in 2020? Compare with the case that interest is paid at 6% per annum.

17. How long will it take to pay off a student loan of \$20,000 if the interest paid on the principal is 5% and the student pays \$200 per month. What is the total amount of money repaid by the student?

18. Derive Eq. (ii) in Problem 15 from the discrete approximation to the change in the principal that occurs during the time interval $[t, t + \Delta t]$,

$$P(t + \Delta t) \cong P(t) + (r\Delta t)P(t) + k\Delta t,$$

assuming that $P(t)$ is continuously differentiable on $t \geq 0$. [*Hint:* Substitute $P(t + \Delta t) = P(t) + P'(t)\Delta t + (1/2)P''(\hat{t})(\Delta t)^2$), where $t < \hat{t} < t + \Delta t$, simplify, divide by Δt, and let $\Delta t \to 0$.]

Miscellaneous Modeling Problems

19. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

20. Archimedes's *principle of buoyancy* states that an object submerged in a fluid is buoyed up by a force equal to the weight of the fluid displaced. An experimental, spherically shaped sonobuoy of radius 1/2 m with a mass m kg is dropped into the ocean with a velocity of 10 m/s when it hits the water. The sonobuoy experiences a drag force due to the water equal to one-half its velocity. Write down a differential equation describing the motion of the sonobuoy. Find values of m for which the sonobuoy will sink and calculate the corresponding terminal sink velocity of the sonobuoy. The density of seawater is $\rho_0 = 1.025$ kg/L.

1.2 Qualitative Methods: Phase Lines and Direction Fields

In Section 1.1 we were able to find solutions of the differential equations

$$\frac{du}{dt} = -k(u - T_0) \quad \text{and} \quad \frac{dp}{dt} = rp - k \quad (1)$$

by using a simple integration technique. Do not assume that this is always possible. Finding closed-form analytic solutions of differential equations can be difficult or impossible. Fortunately, it is possible to obtain information about the qualitative behavior of solutions by using elementary ideas from calculus and graphical methods; we consider two such methods in this section—phase line diagrams and direction fields.

Qualitative behavior refers to general properties of the differential equation and its solutions such as existence of equilibrium points, behavior of solutions near equilibrium points, and long-time behavior of solutions.[1] Qualitative analysis is important to the mathematical modeler because it can provide insight into even a very complicated model without having to find an exact solution or an approximation to an exact solution. It can show, often with only a small amount of effort, whether the equations are a plausible model of the phenomenon being studied. If not, what changes need to be made in the equations?

Autonomous Equations: Equilibrium Solutions and the Phase Line

A first order **autonomous** differential equation is an equation of the form

$$\frac{dy}{dt} = f(y). \tag{2}$$

The distinguishing feature of an autonomous equation is that the independent variable, in this case t, does not appear on the right side of the equation. For instance, the two equations appearing in (1) are autonomous. Other examples of autonomous equations are

$$p' = rp(1 - p/K), \qquad x' = \sin x, \qquad \text{and} \qquad y' = \sqrt{k^2/y - 1},$$

where r, K, and k are constants. However, the equations

$$u' + ku = kT_0 + kA\sin\omega t, \qquad x' = \sin(tx), \qquad \text{and} \qquad y' = -y + t$$

are not autonomous because the independent variable t does appear on the right side of each equation.

Equilibrium Solutions. The first step in a qualitative analysis of Eq. (2) is to find constant solutions of the equation. If $y = \phi(t) = c$ is a constant solution of Eq. (2), then $dy/dt = 0$. Therefore any constant solution must satisfy the algebraic equation

$$f(y) = 0. \tag{3}$$

These solutions are called **equilibrium solutions** of Eq. (2) because they correspond to no change or variation in the value of y as t increases or decreases. Equilibrium solutions are also referred to as **critical points**, **fixed points**, or **stationary points** of Eq. (2).

Equilibrium solutions, although simple, are usually important for understanding the behavior of other solutions of the differential equation. To obtain information about other solutions, we draw the graph of $f(y)$ versus y. Figure 1.2.1 shows a generic plot of $f(y)$, where the equilibrium points are $y = a$, b, and c. It is convenient to think of the variable y as the position of a particle whose motion along the horizontal axis is governed by Eq. (2). The corresponding velocity of the particle, dy/dt, is prescribed by Eq. (2).

At points where the velocity of the particle $dy/dt = f(y) > 0$, so that y is an increasing function of t, the particle moves to the right. This is indicated in Figure 1.2.1 by placing on the y-axis arrows that point to the right in the intervals $y < a$ and $b < y < c$ where $f(y) > 0$. At points where the velocity of the particle $dy/dt = f(y) < 0$, so that y is a decreasing function of t, the particle moves to the left. This is indicated in Figure 1.2.1 by placing on the y-axis arrows that point to the left in the intervals $a < y < b$ and $y > c$, where $f(y) < 0$.

[1] In addition, the qualitative properties of differential equations include results about existence and uniqueness of solutions, intervals of existence, and dependence of solutions on parameters and initial conditions. These issues will be addressed in Sections 2.4 and 2.5.

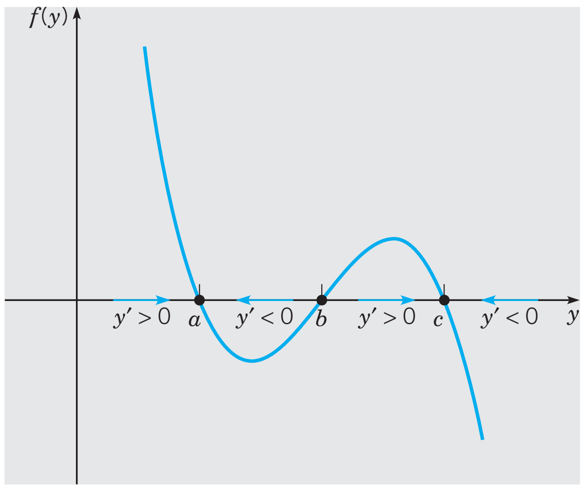


FIGURE 1.2.1 A generic graph of the right side of Eq. (2). The arrows on the y-axis indicate the direction in which y is changing [given by the sign of $y' = f(y)$] for each possible value of y. At the equilibrium points $y = a$, b, and c, $dy/dt = 0$.

The particle is stationary at the equilibrium points $y = a, b$, and c since $dy/dt = 0$ at each of those points.

The horizontal line in Figure 1.2.1 is referred to as the **phase line**, or the **one-dimensional phase portrait** of Eq. (2). The information contained in the phase line can be used to sketch the qualitatively correct integral curves of Eq. (2) by drawing it vertically just to the left of the ty-plane, as shown in Figure 1.2.2. We first draw the equilibrium solutions $y = a, b$ and c; then we draw a representative sampling of other curves that are increasing when $y < a$ and $b < y < c$ and decreasing when $a < y < b$ and $y > c$, as shown in Figure 1.2.2*b*.

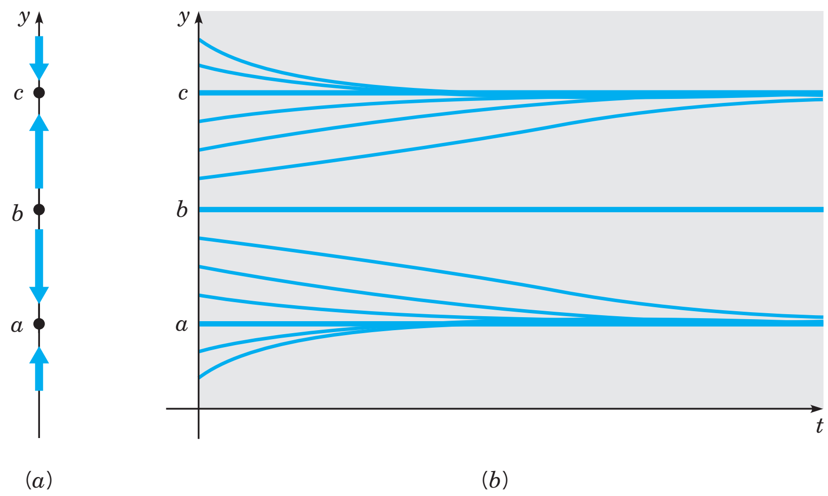


FIGURE 1.2.2 (*a*) The phase line. (*b*) Plots of y versus t.

Stability of Equilibrium Points. In the drawings of the phase line notice that arrows drawn on either side of the equilibrium point $y = a$ point toward $y = a$. Consequently, solution curves in Figure 1.2.2*b* that start sufficiently close to $y = a$ approach $y = a$ as $t \to \infty$. Similarly, arrows drawn on either side of $y = c$ in Figures 1.2.1 and 1.2.2*a* point toward $y = c$. It follows that solution curves that start sufficiently close to $y = c$ approach $y = c$ as $t \to \infty$, as shown in Figure 1.2.2*b*. The equilibrium points $y = a$ and $y = c$ are said to be **asymptotically stable**. On the other hand, arrows in the phase line that lie on either side of the equilibrium point $y = b$ point away from $y = b$. Correspondingly, solution curves that start near $y = b$ move away from $y = b$ as t increases. The equilibrium point $y = b$ is said to be **unstable**.

To facilitate our understanding of asymptotically stable and unstable equilibrium points, it is again useful to think of y as the position of a particle whose dynamics are governed by Eq. (2). A particle, perturbed slightly via some disturbance, from an asymptotically stable equilibrium point, will move back toward that point. However, a particle situated at an unstable equilibrium point, subjected to any disturbance, will move away from that point. *All real-world systems are subject to disturbances, most of which are unaccounted for in a mathematical model. Therefore, systems residing at unstable equilibrium points are not likely to be observed in the real world.*

EXAMPLE 1

Draw phase line diagrams for Eq. (2) of Section 1.1,

$$\frac{du}{dt} = -k(u - T_0), \qquad \text{where } k > 0, \tag{4}$$

and use it to discuss the behavior of all solutions as $t \to \infty$. Compare behaviors for two different values of k, $0 < k_1 < k_2$.

As shown in Figure 1.2.3*a*, the graph of $f(u) = -k_1(u - T_0)$ versus u is a straight line with slope $-k_1 < 0$ that intersects the phase line at $u = T_0$, the only equilibrium solution of Eq. (4). Since $u' > 0$ if $u < T_0$ and $u' < 0$ if $u > T_0$, all arrows on the phase line point toward $u = T_0$, which is therefore asymptotically stable. Consequently, any solution $u = \phi(t)$ of Eq. (4) satisfies

$$\lim_{t\to\infty} \phi(t) = T_0.$$

Equation (4) and Figure 1.2.3*a* also show that the absolute value of the instantaneous rate of heat exchange (as measured by $|u'|$) is an increasing function of the difference between the temperature of the object and the temperature of the surroundings,

$$|u'| = k_1|u - T_0|.$$

Thus the slope of any solution curve will be steeper at points far away from T_0 compared to points that are close to T_0. Furthermore the slope will approach zero as $|u - T_0| \to 0$. Solution curves consistent with these observations are shown in Figure 1.2.3*b*.

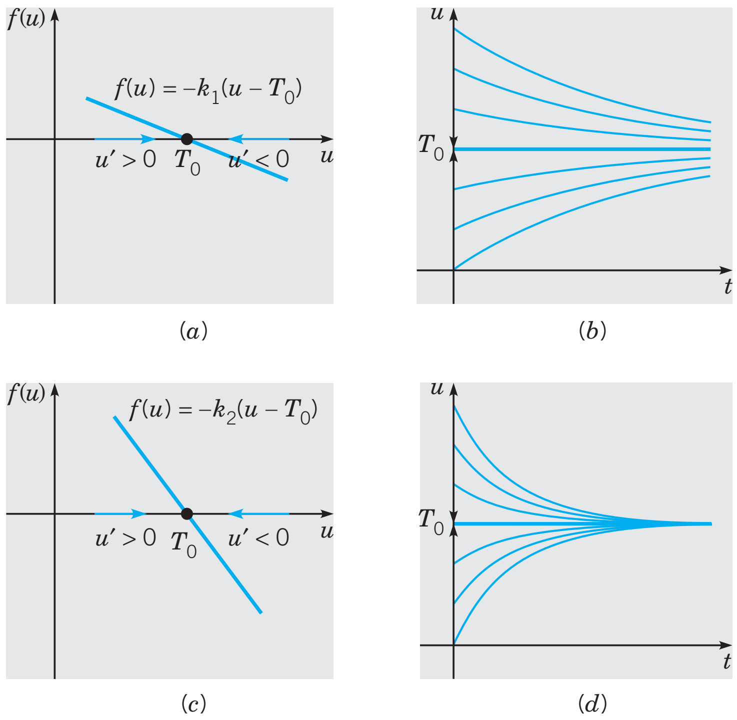


FIGURE 1.2.3 (*a*) and (*c*) Phase lines for $du/dt = -k(u - T_0)$, $k = k_1$ and k_2, where $k_1 < k_2$. The heavy blue arrows on the u-axis indicate the direction in which u is changing [given by the sign of $u'(t)$] for each possible value of u. For a given temperature difference $u - T_0$, the instantaneous rate of heat exchange depends on the slope $-k$ of the line. The parameter k is called the transmission coefficient. (*b*) and (*d*) Corresponding solutions of $du/dt = -k(u - T_0)$, where the phase line information in (*a*) and (*c*) is overlaid on the vertical axes. The rate of approach to equilibrium is governed by k. If k is small, the rate of heat exchange is slow. If k is large, the rate of heat exchange is rapid.

EXAMPLE 2

Draw a phase line diagram for the mouse population growth model, Eq. (15) of Section 1.1,

$$\frac{dp}{dt} = rp - a, \qquad \text{where } r, a > 0, \tag{5}$$

and use it to describe the behavior of all solutions as $t \to \infty$. Discuss implications of the model for the fruit farmer.

The only equilibrium solution of Eq. (5) is $p = a/r$. A plot of $f(p) = rp - a$ versus p in Figure 1.2.4*a* illustrates that $p' < 0$ when $p < a/r$, and $p' > 0$ when $p > a/r$. Thus the arrows on the p-axis point away from the equilibrium solution, which is unstable. Corresponding solution curves are shown in Figure 1.2.4*b*; note that the phase line diagram is overlaid on the p-axis.

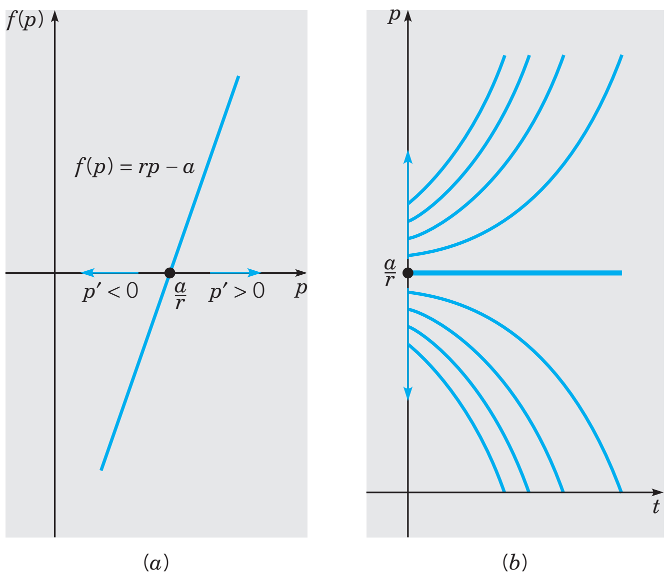


FIGURE 1.2.4 (a) The phase line for Eq. (5), $dp/dt = rp - a$, where $r, a > 0$. The slope r of the line corresponds to the growth rate of the mouse population. The direction of the arrows on the p-axis shows that the equilibrium solution $p = a/r$ is unstable. (b) Integral curves for Eq. (5).

Since the equilibrium solution is unstable, as time passes, an observer may see a mouse population either much larger or much smaller than the equilibrium population, but the equilibrium solution itself will not, in practice, be observed. Without the possible benefits of a more accurate and complex population model,[2] one inference that the fruit farmer might draw is that if he wants to control the mouse population, then he must install enough nesting boxes for the owls, thereby increasing the harvest rate a, to ensure that the mouse population $p(t)$ is always less than a/r. Thus a/r is a threshold value that should never be exceeded by $p(t)$ if the control strategy is to succeed.

This model also suggests a number of questions that the fruit farmer may wish to pursue, perhaps with assistance from a biologist who is knowledgeable about life cycles and habitats of field mice and owls:

- What is the growth rate of a field mouse population when there is an abundant food supply?
- How many mice per day does a barn owl consume?
- How do we estimate the size of the mouse population?
- Should we model the owl population?
- What will be a sustainable owl population if the mouse population drops to an economically acceptable level?

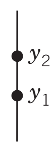
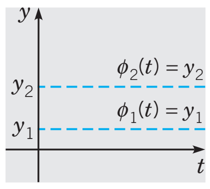
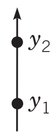
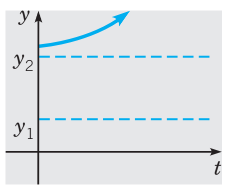
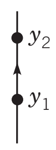
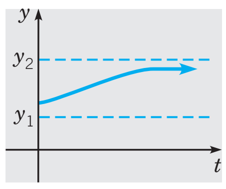
In each of the above examples, equilibrium solutions are important for understanding how other solutions of the given differential equation behave. An equilibrium solution may be thought of as a solution that serves as a reference to other, often nearby, solutions. An

[2] More elaborate population models appear in Sections 2.5 and 7.4.

asymptotically stable equilibrium solution is often referred to as an **attractor** or **sink**, since nearby solutions approach it as $t \to \infty$. On the other hand, an unstable equilibrium solution is referred to as a **repeller** or **source**.

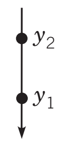
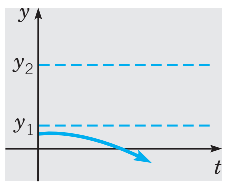
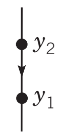
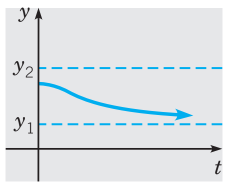
The main steps for creating the phase line and a rough sketch of solution curves for a first-order autonomous differential equation are summarized in Table 1.2.1.

TABLE 1.2.1 Procedure for drawing phase lines and sketching solution curves for an autonomous equation.

	Illustration	
Step	Phase Line	Solution Curves
1. Find the equilibrium solutions of $dy/dt = f(y)$.	Solve $f(y) = 0$.	
2. Sketch the equilibrium solutions. These partition the phase line and ty-plane into disjoint regions.	Plot equilibrium solutions as points along a vertical line in increasing order as you move upward along the line. For instance, if $y_1 < y_2$ are equilibrium solutions, the phase line looks like 	Plot equilibrium solutions as dashed horizontal lines in the ty-plane. For instance, if $0 < y_1 < y_2$ are equilibrium solutions, the ty-plane looks like 
3. In each region, assess the sign of $f(y)$.		
(a) If $f(y) > 0$, then the solution curves passing through points in that region are increasing for all t, and either: (i) $\lim_{t \to \infty} y(t) = \infty$ if there is no larger equilibrium solution.	Affix arrowheads appropriately in each region. 	Sketch a representative solution curve in each region. 
(ii) $\lim_{t \to \infty} y(t) = y_2$ if y_2 is the next larger equilibrium solution.		

(continued)

TABLE 1.2.1 Procedure for drawing phase lines and sketching solution curves for an autonomous equation. *(continued)*

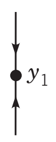
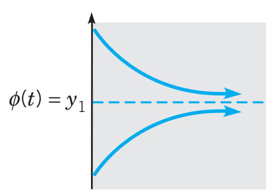
Step	Illustration: Phase Line	Illustration: Solution Curves
(b) If $f(y) < 0$, then the solution curves passing through points in that region are decreasing for all t, and either:	Affix arrowheads appropriately in each region.	Sketch a representative solution curve in each region.
(i) $\lim_{t\to\infty} y(t) = -\infty$ if there is no smaller equilibrium solution.		
(ii) $\lim_{t\to\infty} y(t) = y_1$ if y_1 is the next smaller equilibrium solution.		

Classification of Equilibrium Solutions

There are four possible arrow patterns that can encase a given equilibrium point of Eq. (2). The behavior of the solution curves "nearby" is different for each arrow pattern, resulting in different classifications of the corresponding equilibrium points. Suppose y_1 is an equilibrium point of Eq. (2). We illustrate the four possibilities in Table 1.2.2.

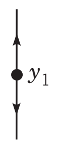
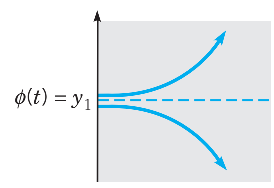
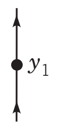
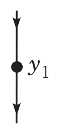
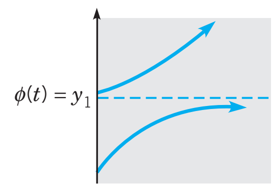
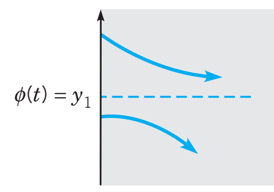
Remark. We use the same classification for the equilibrium *solution curve* $y = \phi(t) = y_1$ as we do for the equilibrium *point* y_1, with the only change being that the word "point" is replaced by "solution" in each case in Table 1.2.2.

TABLE 1.2.2 Classification of equilibrium points of (2).

Phase Line	Sample Solution Curves	Verbal Interpretation	Classification y_1 is a(n) …
		Solution curves passing through points whose y-values close to y_1 on either side tend toward y_1 asymptotically as $t \to \infty$.	**asymptotically stable equilibrium point**

(continued)

TABLE 1.2.2 Classification of equilibrium points of (2). *(continued)*

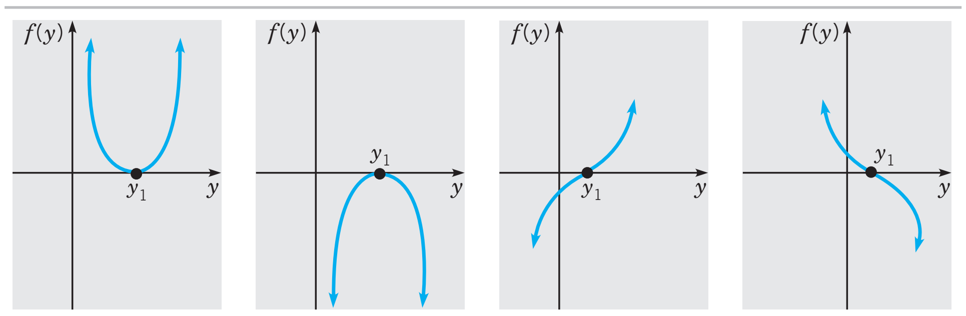
Phase Line	Sample Solution Curves	Verbal Interpretation	Classification y_1 is a(n) ...
		Solution curves passing through points whose y-values close to y_1 on either side tend away from y_1 as $t \to \infty$.	**unstable equilibrium point**
 	 	Solution curves tend away from y_1 if they pass through points whose y-values are close to y_1 on one side, but they tend toward y_1 asymptotically as $t \to \infty$ if they pass through points whose y-values are close to y_1 on the opposite side.	**semistable equilibrium point**

Linearization About an Equilibrium Point

Since the classification of an equilibrium point y_1 depends only on the behavior *near* y_1, we can extract its classification from certain features of the graph of f near y_1. Assume that f is differentiable in a vicinity of y_1 and suppose that $f'(y_1) < 0$. Then the graph of f in this vicinity resembles its tangent line, which has a negative slope. So, the graph of f is decreasing in this vicinity and hence the continuity of f implies that

- $f(y)$ is positive when $y < y_1$ and y is close by y_1.
- $f(y)$ is negative when $y > y_1$ and y is close by y_1.

Thus, the phase line must look like the one for an asymptotically stable equilibrium point. Similar reasoning shows that if $f'(y_1) > 0$, then y_1 must be an unstable equilibrium point. If $f'(y_1) = 0$, classifying y_1 from this information alone is impossible because the graph of f can exhibit one of several different situations near y_1, including the following:



We summarize this discussion as the following theorem.

THEOREM 1.2.1

Linearization About an Equilibrium Point. Let y_1 be an equilibrium point of Eq. (2) and assume that f has a continuous derivative in a vicinity of y_1.

i. If $f'(y_1) < 0$, then y_1 is an asymptotically stable equilibrium point.
ii. If $f'(y_1) > 0$, then y_1 is an unstable equilibrium point.
iii. If $f'(y_1) = 0$, then more information is needed to classify y_1.

Solutions and Direction Fields for $y' = f(t, y)$

Phase line diagrams allow us to infer qualitative properties of solutions of autonomous equations, that is, equations of the form $y' = f(y)$. More generally, the right side of a first order equation can depend on both the dependent and independent variables. The standard, or **normal form**, for a first order differential equation is

$$\frac{dy}{dt} = f(t, y). \tag{6}$$

Here f is a given function of the two variables t and y, sometimes referred to as the **rate function**. If the independent variable t appears explicitly in the rate function, then the equation is said to be **nonautonomous**.

A **solution** of Eq. (6) is a differentiable function $y = \phi(t)$ that satisfies the equation. This means that if we substitute $\phi(t)$ into the equation in place of the dependent variable y, the resulting equation

$$\phi'(t) = f(t, \phi(t)) \tag{7}$$

must be true for all t in the interval where $\phi(t)$ is defined. Equation (7) may be read as "at each point $(t, \phi(t))$ the slope $\phi'(t)$ of the line tangent[3] to the integral curve must be equal to $f(t, \phi(t))$" (see Figure 1.2.5).

It is not necessary to have a solution of Eq. (6) to draw direction field vectors. If a solution passes through the point (t, y), then the slope of the direction vector at that point is given by $f(t, y)$. Thus a direction field for equations of the form (6) can be constructed by evaluating f at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation.

Direction Fields for Autonomous Equations. Since the right side of an autonomous equation $y' = f(y)$ does not depend on t, slopes of direction field vectors for autonomous equations can vary only in the vertical direction of the ty-plane. Thus the slope of each direction field vector on a horizontal line $y = \alpha$, where α is a constant, will be $f(\alpha)$, as we now illustrate.

[3]Recall from calculus that the direction vector for the tangent line at $\mathbf{r}(t) = t\,\mathbf{i} + \phi(t)\,\mathbf{j}$ is $\mathbf{r}'(t) = 1\,\mathbf{i} + \phi'(t)\,\mathbf{j}$, which has slope $\phi'(t)$. Here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the horizontal and vertical directions, respectively, of the xy-plane.

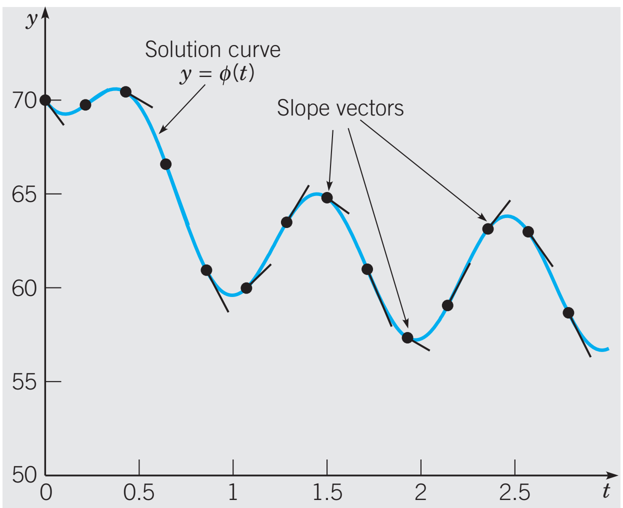


FIGURE 1.2.5 The path taken by an integral curve of a differential equation $y' = f(t, y)$ is determined by the slope vectors generated by $f(t, y)$ at each point on the path.

EXAMPLE 3

Draw a direction field for Eq. (4):

$$\frac{du}{dt} = -k(u - T_0).$$

Our task is simplified slightly if we assign numerical values to k and T_0, but the procedure is the same regardless of which values we choose. If we let $k = 1.5$ and $T_0 = 60$, then Eq. (4) becomes

$$du/dt = -1.5(u - 60) \tag{4a}$$

Suppose that we choose a value for u. Then, by evaluating the right side of Eq. (4a), we can find the corresponding value of du/dt. For instance, if $u = 70$, then $du/dt = -15$. This means that the slope of a solution $u = \phi(t)$ has the value -15 at any point where $u = 70$. We can display this information graphically in the tu-plane by drawing short line segments with slope -15 at several points on the line $u = 70$. Similarly, if $u = 50$, then $du/dt = 15$, so we draw line segments with slope 15 at several points on the line $u = 50$. We obtain Figure 1.2.6 by proceeding in the same way with other values of u. Figure 1.2.6 is an example of what is called a **direction field** or sometimes a **slope field**.

The importance of Figure 1.2.6 is that each line segment is a tangent line to the graph of a solution of Eq. (4a). Consequently, by looking at the direction field, we can visualize how solutions of Eq. (4a) vary with time. On a printed copy of a direction field we can even sketch (approximately) graphs of solutions by drawing curves that are always tangent to line segments in the direction field. Thus the general geometric behavior of the integral curves can be inferred from the direction field in Figure 1.2.6.

This approach can be applied equally well to the more general Eq. (4), where the parameters k and T_0 are unspecified positive numbers. The conclusions are essentially the same. The equilibrium solution of Eq. (4) is $u = T_0$. Solutions below the equilibrium solution increase with time, those above it decrease with time, and all other solutions approach the equilibrium solution as t becomes large.

The connection between integral curves and direction fields is an important concept for understanding how the right side of a differential equation, such as $u' = -k(u - T_0)$,

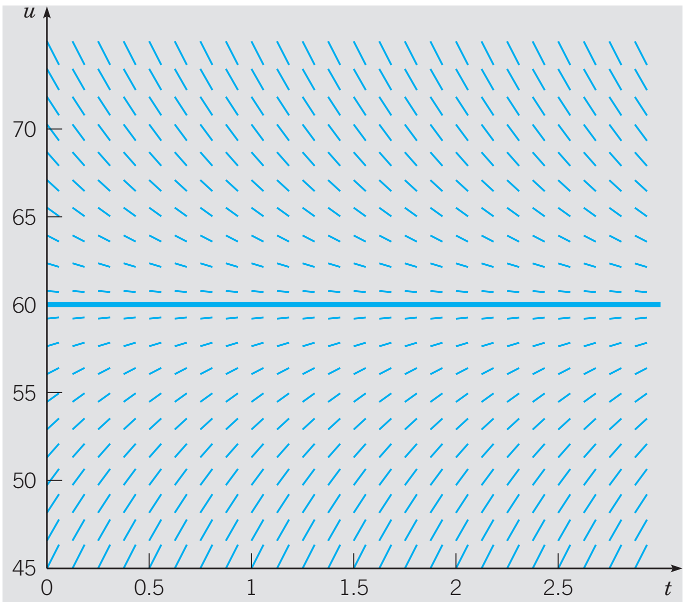


FIGURE 1.2.6 Direction field and equilibrium solution $u = 60$ for $u' = -1.5(u - 60)$.

determines the behavior of solutions and gives rise to the integral curves. However, using modern software packages, it is just as easy to plot the graphs of numerical approximations to solutions as it is to draw direction fields. We will frequently do this because the behavior of solutions of a first order equation is usually made most clear by overlaying the direction field with a representative set of integral curves, as shown in Figure 1.2.7. Such a sampling of integral curves facilitates visualization of the many other integral curves determined by the direction field generated by the right side of the differential equation.

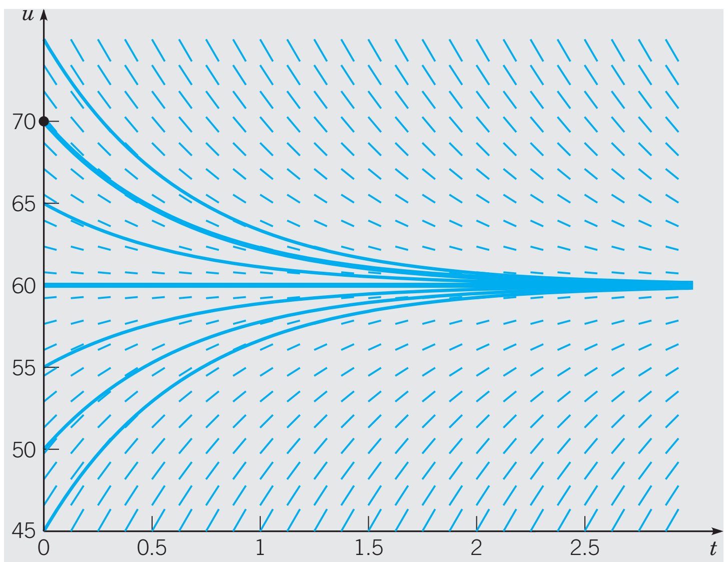


FIGURE 1.2.7 Direction field for $u' = -1.5(u - 60)$ overlaid with the integral curves shown in Figure 1.1.2.

If in Eq. (6) both the dependent variable y and the independent variable t appear explicitly on the right side of the equation, then the slopes of the direction field vectors will vary with both t and y. To illustrate, we consider the following extension of Example 1 in Section 1.1, an application of Newton's law of cooling to the heating and cooling of a building subject to periodic diurnal variation in the external air temperature.

EXAMPLE 4

Heating and Cooling of a Building

Consider a building, thought of as a partly insulated box, that is subject to external temperature fluctuations. Construct a model that describes the temperature fluctuations inside the building.

Let $u(t)$ and $T(t)$ be the internal and external temperatures, respectively, at time t. Assuming that the air inside and outside the enclosure is well mixed, we use Newton's law of cooling, just as we did in Example 1, Section 1.1, to get the differential equation

$$\frac{du}{dt} = -k\big[u - T(t)\big]. \tag{8}$$

However we now allow for the external temperature to vary with time. If we assume that the temperature of the external air is described by

$$T(t) = T_0 + A \sin \omega t, \tag{9}$$

then Eq. (8) can be written as

$$\frac{du}{dt} + ku = kT_0 + kA \sin \omega t. \tag{10}$$

In Problem 30 we ask you to verify that

$$u = T_0 + \frac{kA}{k^2 + \omega^2}(k \sin \omega t - \omega \cos \omega t) + ce^{-kt} \tag{11}$$

is a solution of Eq. (10), where c is an arbitrary real constant. In Chapter 2 we present a systematic general method for solving a class of first order equations of which Eq. (10) is a member.

To construct a direction field and integral curves for Eq. (10), we suppose that

$$k = 1.5(\text{day})^{-1}, \quad T_0 = 60°\text{F}, \quad A = 15°\text{F}, \quad \text{and} \quad \omega = 2\pi.$$

Thus t is measured in days and

$$T(t) = 60 + 15\sin(2\pi t) \tag{12}$$

corresponds to a daily variation of 15°F above and below a mean temperature of 60°F. Inserting these values for the parameters into Eqs. (10) and (11) gives

$$\frac{du}{dt} + 1.5u = 90 + 22.5 \sin 2\pi t \tag{13}$$

with a corresponding general solution

$$u = 60 + \frac{22.5}{2.25 + \pi^2}(1.5 \sin 2\pi t - 2\pi \cos 2\pi t) + ce^{-1.5t}. \tag{14}$$

Figure 1.2.8 shows a direction field and several integral curves for Eq. (13) along with a graph of the exterior temperature $T(t)$. The behavior of solutions is a bit more complicated than those shown in Figure 1.2.7 because the right side of Eq. (13) depends on the independent variable t as well as the dependent variable u. Figure 1.2.8 shows that after approximately two days, all solutions begin to exhibit similar behavior. From the general solution (14), it is evident that for large t,

$$u(t) \approx U(t) = 60 + \frac{22.5}{2.25 + 4\pi^2}\,(1.5\sin 2\pi t - 2\pi\cos 2\pi t)\,, \tag{15}$$

since $ce^{-1.5t} \to 0$ as $t \to \infty$. The function $U(t)$ in expression (15) is referred to as the **steady-state solution** of Eq. (13). Using trigonometric identities, we can write $U(t)$ in the form (see Problem 31)

$$U(t) = 60 + \frac{22.5}{\sqrt{2.25 + 4\pi^2}}\sin(2\pi t - \delta) \approx 60 + 3.4831\sin(2\pi t - 1.33645), \tag{16}$$

where $\delta = \cos^{-1}(1.5/\sqrt{2.25 + 4\pi^2})$. Comparing $U(t)$ with $T(t)$, we see that for large t the air temperature within the building varies sinusoidally at the same frequency as the external air temperature, but with a time lag of $t_{\text{lag}} = 1.33645/(2\pi) = 0.2127$ days and an amplitude of only 3.4831°F about a mean temperature of 60°F. Does the qualitative behavior of the steady state solution agree with what you expect based on physical reasoning and experience?

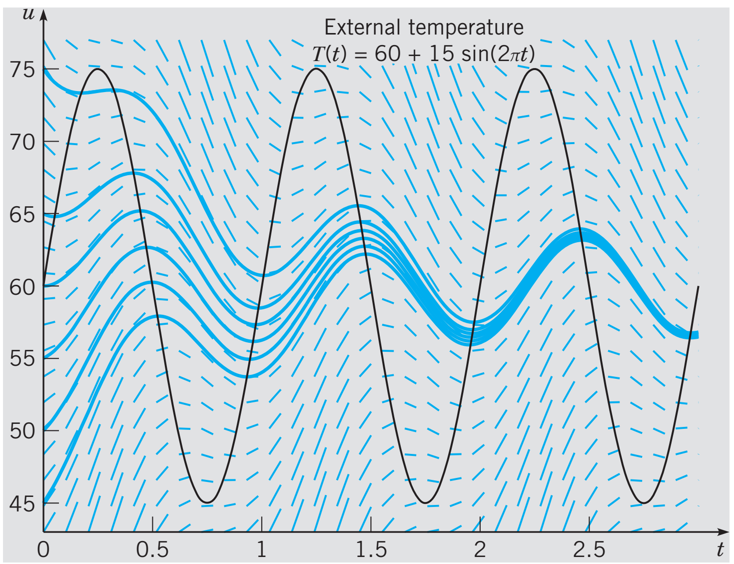


FIGURE 1.2.8 Direction field and integral curves for $u' + 1.5u = T(t)$. The variation in external temperature is described by $T(t) = 60 + 15\sin 2\pi t$.

PROBLEMS

Phase Line Diagrams. Problems 1 through 7 involve equations of the form $dy/dt = f(y)$. In each problem, sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.

1. $dy/dt = y(y-1)(y-2), \quad y_0 \ge 0$

2. $dy/dt = e^y - 1, \quad -\infty < y_0 < \infty$

3. $dy/dt = e^{-y} - 1, \quad -\infty < y_0 < \infty$

4. $dy/dt = -2(\arctan y)/(1 + y^2), \quad -\infty < y_0 < \infty$

5. $dy/dt = y^2(y+1)(y-3), \qquad -\infty < y_0 < \infty$

6. $dy/dt = ay + by^2, \qquad a > 0, \quad b > 0, \quad y_0 \geq 0$

7. $dy/dt = ay + by^2, \qquad a > 0, \quad b > 0, \quad -\infty < y_0 < \infty$

Problems 8 through 13 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.

8. $dy/dt = -k(y-1)^2, \qquad k > 0, \quad -\infty < y_0 < \infty$

9. $dy/dt = y^2(y^2-1), \qquad -\infty < y_0 < \infty$

10. $dy/dt = y(1-y^2), \qquad -\infty < y_0 < \infty$

11. $dy/dt = ay - b\sqrt{y}, \qquad a > 0, \quad b > 0, \quad y_0 \geq 0$

12. $dy/dt = y^2(4-y^2), \qquad -\infty < y_0 < \infty$

13. $dy/dt = y^2(1-y)^2, \qquad -\infty < y_0 < \infty$

Direction Fields. In each of Problems 14 through 19 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

14. $y' = 3 - 2y$

15. $y' = 2y - 3$

16. $y' = 3 + 2y$

17. $y' = -1 - 2y$

18. $y' = 1 + 2y$

19. $y' = y + 2$

In each of Problems 20 through 23 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than the solutions shown in Figures 1.2.6 and 1.2.7.

20. $y' = y(4-y)$

21. $y' = -y(5-y)$

22. $y' = y^2$

23. $y' = y(y-2)^2$

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.2.9 through 1.2.14. In each of Problems 24 through 29 identify the differential equation that corresponds to the given direction field.

(a) $y' = 2y - 1$
(b) $y' = 2 + y$
(c) $y' = y - 2$
(d) $y' = y(y+3)$
(e) $y' = y(y-3)$
(f) $y' = 1 + 2y$
(g) $y' = -2 - y$
(h) $y' = y(3-y)$
(i) $y' = 1 - 2y$
(j) $y' = 2 - y$

24. The direction field of Figure 1.2.9.

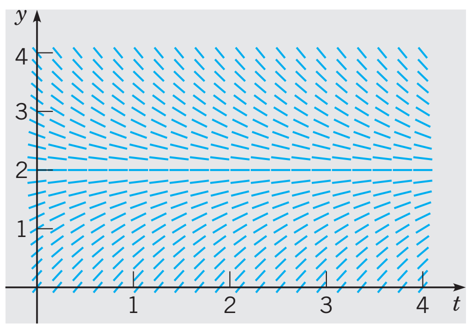


FIGURE 1.2.9 Direction field for Problem 24.

25. The direction field of Figure 1.2.10.

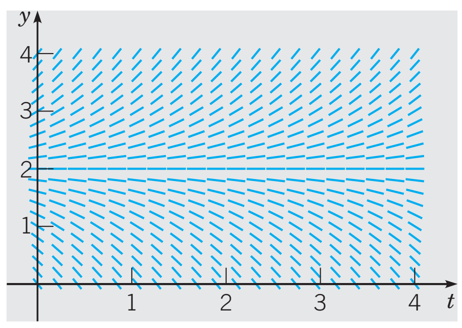


FIGURE 1.2.10 Direction field for Problem 25.

26. The direction field of Figure 1.2.11.

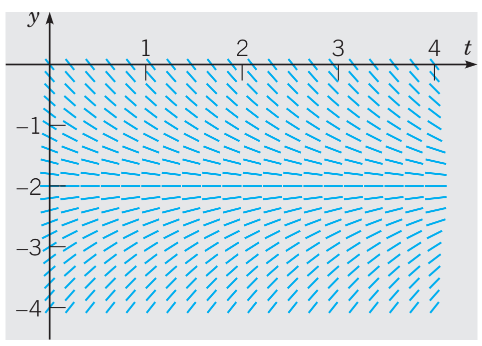


FIGURE 1.2.11 Direction field for Problem 26.

27. The direction field of Figure 1.2.12.

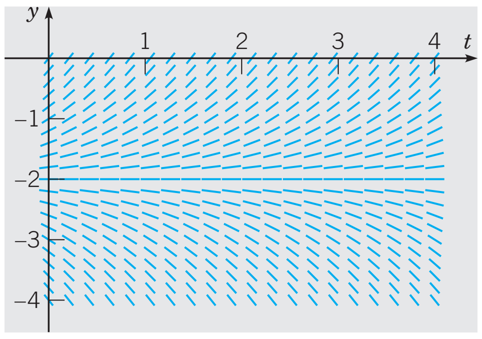


FIGURE 1.2.12 Direction field for Problem 27.

28. The direction field of Figure 1.2.13.

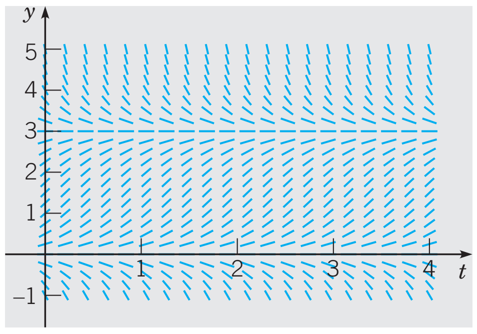


FIGURE 1.2.13 Direction field for Problem 28.

29. The direction field of Figure 1.2.14.

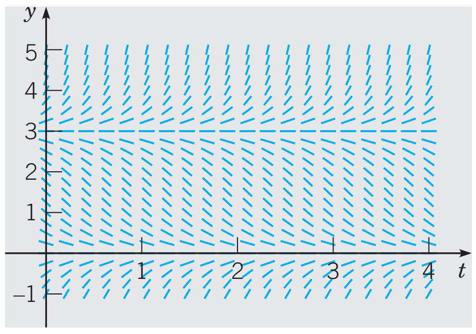


FIGURE 1.2.14 Direction field for Problem 29.

30. Verify that the function in Eq. (11) is a solution of Eq. (10).

31. Show that $A\sin\omega t + B\cos\omega t = R\sin(\omega t - \delta)$, where $R = \sqrt{A^2 + B^2}$ and δ is the angle defined by $R\cos\delta = A$ and $R\sin\delta = -B$.

Applications.

32. If in the exponential model for population growth, $dy/dt = ry$, the constant growth rate r is replaced by a growth rate $r(1 - y/K)$ that decreases linearly as the size of the population increases, we obtain the logistic model for population growth,

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right), \qquad \text{(i)}$$

in which K is referred to as the *carrying capacity* of the population. Sketch the graph of $f(y)$, find the critical points, and determine whether each is asymptotically stable or unstable.

33. An equation that is frequently used to model the population growth of cancer cells in a tumor is the Gompertz equation

$$\frac{dy}{dt} = ry\ln(K/y),$$

where r and K are positive constants.

(a) Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.

(b) For each y in $0 < y \le K$, show that dy/dt, as given by the Gompertz equation, is never less than dy/dt, as given by the logistic equation, Eq. (i) in Problem 32.

34. In addition to the Gompertz equation (see Problem 33), another equation used to model the growth of cancerous tumors is the Bertalanffy equation

$$\frac{dV}{dt} = aV^{2/3} - bV,$$

where a and b are positive constants. This model assumes that the tumor grows at a rate proportional to surface area, while the loss of tumor mass due to cell death is proportional to the volume of the tumor. Sketch the graph of $f(V)$ versus V, find the critical points, and determine whether each is asymptotically stable or unstable.

35. A chemical of fixed concentration c_i flows into a continuously stirred tank reactor at a constant volume flow rate r_i and flows out at the same rate. While in the reactor, the chemical undergoes a simple reaction in which it disappears at a rate proportional to the concentration:

$$\frac{dc}{dt} = \frac{r_i}{V}c_i - r_i\frac{c}{V} - kc, \qquad \text{(i)}$$

where V is the volume of the reactor and k is the rate of reaction.

(a) Use the dimensionless variables

$$C = \frac{c}{c_i}, \qquad \tau = \frac{t}{V/r_i}$$

to express Eq. (i) in dimensionless form

$$\frac{dC}{d\tau} = 1 - C - \alpha C, \qquad \text{(ii)}$$

where

$$\alpha = \frac{kV}{r_i}.$$

(b) Determine the equilibrium solution of Eq. (ii), draw a phase line diagram, and determine whether the equilibrium solution is asymptotically stable or unstable. Then sketch a phase portrait with a representative set of solution curves.

36. A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3},$$

where α is the coefficient of evaporation.

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

(c) Find a condition that must be satisfied if the pond is not to overflow.

37. The Solow model of economic growth (ignoring the effects of capital stock depreciation) is

$$k' = \sigma f(k) - (n+g)k, \qquad \text{(i)}$$

where k is *capital stock per unit of effective labor*, $f(k)$ is *GDP per unit of effective labor*, and σ, $0 < \sigma < 1$, is the fraction of gross domestic product (GDP) devoted to investment. The parameters n and g, growth rates of labor L and technology A, respectively, appear in the equations

$$\frac{dL}{dt} = nL, \qquad \frac{dA}{dt} = gA.$$

The product $A(t)L(t)$ is referred to as **effective labor** and the **output** Y of the economy is given by $Y = ALf(k)$. Assume that the production function $f(k)$ satisfies the following conditions:

(i) $f(0) = 0, \quad f(k) > 0$ for $k > 0$,
(ii) $f'(k) > 0, \quad f''(k) < 0$ for $k > 0$,
(iii) $\lim_{k\to 0} f'(k) = \infty, \quad \lim_{k\to\infty} f'(k) = 0$.

For example, the function $f(k) = ck^{\alpha}$ where $0 < \alpha < 1$ satisfies conditions (i), (ii), and (iii).

(a) Draw a phase line diagram of Eq. (i) by sketching the graphs of *actual investment* $\sigma f(k)$ and *break-even investment* $(n+g)k$ on the same set of coordinate axes. Show that Eq. (i) has an asymptotically stable equilibrium solution k^*.

(b) When $k = k^*$, we say the Solow economy is on its *balanced growth path*. Show that when $k = k^*$, the output of the economy grows at the combined growth rates of labor and technology,

$$\frac{dY}{dt} = (n+g)Y.$$

1.3 Definitions, Classification, and Terminology

In Sections 1.1 and 1.2 we gave examples to introduce you to a number of important topics in the context of first order differential equations: *mathematical modeling, solutions, integral curves, initial value problems, phase line diagrams, direction fields, equilibrium points,* and *concepts of stability*. Prior to embarking on an in-depth study of first order equations, we briefly step back and give you a broader view of differential equations by presenting a few important definitions, introducing some commonly used terminology, and discussing different ways that differential equations are classified. This background information will enhance your understanding of the subject in the following ways:

- It will provide you with an organizational framework for the subject;
- it will acquaint you with some of the language used to discuss the subject in a sensible manner;
- it will give you perspective on the subject as a whole.

We begin with a definition of a differential equation.

DEFINITION 1.3.1

Differential Equation. An equation that contains derivatives of one or more unknown functions with respect to one or more independent variables is said to be a **differential equation**.

The a priori unknown functions referred to in Definition 1.3.1 are dependent variables. When we write down a differential equation, such as Eq. (2) in Section 1.1,

$$\frac{du}{dt} = -k(u - T_0),$$

the unknown function u is considered to be a function of t, and is therefore a dependent variable. Thus Definition 1.3.1 may be alternatively expressed as "an equation that contains derivatives of one or more dependent variables with respect to one or more independent variables is said to be a **differential equation**."

Definition 1.3.1 underlies the following classifications based on (i) the number of independent variables, (ii) the number of unknown functions, and (iii) the highest order derivatives that appear in the equations.

Ordinary and Partial Differential Equations

We make a distinction between differential equations in which there is only one independent variable and differential equations in which there are two or more independent variables.

If the unknown function (or functions) depend on a single independent variable, then the only derivatives that appear in the equation are ordinary derivatives. In this case the differential equation is said to be an **ordinary differential equation** (**ODE**). For example, all of the differential equations that appear in Section 1.1 are ODEs.

If the unknown function (or functions) depend on more than one independent variable, and partial derivatives appear in the equation, then the differential equation is said to be a **partial differential equation** (**PDE**). Examples of PDEs are the three archetypal equations of mathematical physics shown in Table 1.3.1.

TABLE 1.3.1 Three archetypal PDEs of mathematical physics.

heat equation,	$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$	independent variables t and x.	(1)
wave equation,	$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2},$	independent variables t and x.	(2)
Laplace's equation,	$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$	independent variables x and y.	(3)

In Eq. (1) $u(x, t)$ is the temperature of a metal rod at position x at time t; in Eq. (2) $y(x, t)$ is the vertical displacement from equilibrium of a horizontal vibrating string at position x at time t; in Eq. (3) $V(x, y)$ is the electric potential at the point (x, y) in a metal plate with a prescribed distribution of electric charge around the boundary. All of these equations have counterparts with $n \geq 3$ independent variables.

Systems of Differential Equations

Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient and is referred to as a *scalar equation*. All of the differential equations that appear in Section 1.1 are scalar ODEs. However, if there are two or more unknown functions, then a system of equations is required. Systems arise whenever there are two or more components that interact in some manner.[1] For example, the Lotka–Volterra, or predator–prey equations are important in ecological modeling. They have the form

$$\begin{aligned} dx/dt &= \quad ax - \alpha xy \\ dy/dt &= -cy + \gamma xy. \end{aligned} \tag{4}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. These equations provide an example of what is often one of the main problems confronting the mathematical modeler: "In terms of the variables, how do we mathematically represent the interaction of various components of the phenomenon?" The system (4) arises from the following assumptions. The prey are assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation; this exponential growth is represented in the first equation of the system (4) by the term ax. The rate of predation upon the prey is assumed to be proportional to the product of the predator and prey populations; this is represented above by $-\alpha xy$. If either x or y is zero, then there is no predation. In the second equation γxy represents the growth of the predator population. A different constant is used since the rate at which the predator population grows is not necessarily equal to the rate at which it consumes the prey. The term $-cy$ represents the loss rate of the predators due to either natural death or emigration; it leads to an exponential decay in the absence of prey. The constants a, α, c, and γ are based on empirical observations and depend on the particular species being studied. Systems of equations are discussed in Chapters 3, 6, and 7; in particular, the Lotka–Volterra equations are examined in Section 7.4. In some areas of applications, it is not unusual to encounter very large systems containing hundreds, or even thousands of equations.

Order

The **order** of a differential equation is the order of the highest derivative, ordinary or partial, that appears in the equation. The equations in Section 1.1 are all first order ODEs, while each of Eqs. (1), (2), and (3) is a second order PDE. The equation

$$ay'' + by' + cy = f(t), \tag{5}$$

where a, b, and c are given constants, and f is a given function, is a second order ODE. Equation (5) is a useful model of physical systems, for example, the motion of a mass attached to a spring, or the current in an electric circuit; we will consider it in detail in Chapter 4. More generally, the equation

$$F[t, u(t), u'(t), \dots, u^{(n)}(t)] = 0 \tag{6}$$

is an ODE of the nth order. Equation (6) expresses a relation between the independent variable t and the values of the function u and its first n derivatives $u'(t), u''(t), \dots, u^{(n)}(t)$. It is convenient and customary to write y for $u(t)$ with $y', \dots, y^{(n)}$ standing for $u'(t), u''(t), \dots, u^{(n)}(t)$. Thus Eq. (6) is written as

$$F\left[t, y, y', \dots, y^{(n)}\right] = 0. \tag{7}$$

[1]We will frequently use the word *system* to refer to (i) a real-world group or combination of interrelated, interdependent, or interacting elements forming a collective entity, and (ii) a system of equations that model that entity. Although closely related and often identified with one another, they are not the same.

For example,

$$y''' + 2e^t y'' + yy' = t^4 \tag{8}$$

is a third order equation for $y = u(t)$. Occasionally, other letters will be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative. Thus we assume Eq. (7) can be written as

$$y^{(n)} = f(t, y, y'', \ldots, y^{(n-1)}). \tag{9}$$

We study only equations of the form (9), although in the process of solving them, we often find it convenient to rewrite them in other forms.

Linear and Nonlinear Equations

DEFINITION 1.3.2

Linear Differential Equation. An nth order ordinary differential equation $F(t, y, y', \ldots, y^{(n)}) = 0$ is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).^2 \tag{10}$$

The functions $a_0, a_1, \ldots, a_n$, called the **coefficients** of the equation, can depend at most on the independent variable t. Equation (10) is said to be **homogeneous** if the term $g(t)$ is zero for all t. Otherwise, the equation is **nonhomogeneous**.

Important special cases of Eq. (10) are first order linear equations, $a_0(t)y' + a_1(t)y = g(t)$, the subject of Section 2.2, and second order linear equations,

$$a_0(t)y'' + a_1(t)y' + a_2(t)y = g(t),$$

which we take up in Chapter 4.

An ODE that is not of the form (10) is a **nonlinear equation**. The distinction between a linear ODE and a nonlinear ODE hinges only on how the dependent variable y and its derivatives $y', y'', \ldots, y^{(n)}$ appear in the equation: for an equation to be linear, they can appear in no other way except as designated by the form (10).

Common reasons that an ODE is nonlinear are that there are terms in the equation in which the dependent variable y or any of its derivatives

(i) are arguments of a nonlinear function, for example, terms such as $\sin y$, e^{-y}, or $\sqrt{1+y^2}$,
(ii) appear as products, or are raised to a power other than 1, such as y^2 and yy'.

These statements also apply to equations in which there are two or more dependent variables, that is, to systems of differential equations. The presence of such terms often makes it easy to determine that an equation is nonlinear by observation. For example, Eq. (8) is nonlinear because of the term yy'. Each equation in system (4) is nonlinear because of the terms that involve the product xy of the dependent variables.

[2] Comparing Eq. (10) with Eq. (7), we see that an nth order ODE is linear only if

$$F\left[t, y, y', \ldots, y^{(n)}\right] = a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y - g(t). \tag{11}$$

If this is the case, F is said to be a **linear** function of the variables $y, y', y'', \ldots, y^{(n)}$.

To show that a given equation is linear, you need only match its coefficients with Eq. (10) of appropriate order, as we show in the following example.

EXAMPLE 1

Show that

$$x^3y''' + 3x^2y'' + 4y = \sin(\ln x) \tag{12}$$

is a linear differential equation and state whether the equation is homogeneous or nonhomogeneous.

If, in Eq. (10), the independent variable is chosen to be x instead of t, and we set $n = 3$, $a_0(x) = x^3$, $a_1(x) = 3x^2$, $a_2(x) = 0$, $a_3(x) = 4$, and $g(x) = \sin(\ln x)$, we see that Eq. (10) reduces to $x^3y''' + 3x^2y'' + 4y = \sin(\ln x)$. Since $g(x)$ is not the zero function, the equation is nonhomogeneous.

Solutions

In Example 2, Section 1.1, we showed that directly substituting

$$u = T_0 + ce^{-kt}, \quad c \text{ an arbitrary constant,} \tag{13}$$

into the differential equation

$$\frac{du}{dt} = -k(u - T_0) \tag{14}$$

results in the identity

$$-kce - kt = -kce - kt, \quad -\infty < t < \infty,$$

and therefore the function in Eq. (13) is a solution of Eq. (14). The following definition generalizes this notion of a solution to nth order differential equations.

DEFINITION 1.3.3

Solution of a Differential Equation. A **solution** of the ordinary differential equation (9) on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \dots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f\left[t, \phi(t), \phi'(t), \dots, \phi^{(n-1)}(t)\right] \tag{15}$$

for every t in $\alpha < t < \beta$.

Thus, to determine if a given function is a solution of a differential equation, we substitute the function into the equation. If, upon substitution, the differential equation reduces to an identity, then the function is a solution. Otherwise, the function is not a solution.

EXAMPLE 2

Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are solutions of

$$y'' + y = 0 \tag{16}$$

on the interval $-\infty < t < \infty$.

Since $y_1'(t) = -\sin t$ and $y_1''(t) = -\cos t$, substituting $y_1(t)$ into Eq. (16) yields

$$\underbrace{-\cos t}_{y_1''(t)} + \underbrace{\cos t}_{y_1(t)} = 0$$

for all t. Therefore $y_1(t)$ is a solution of Eq. (16) on $-\infty < t < \infty$. Similarly, substituting $y_2(t)$ into Eq. (16) gives

$$\underbrace{-\sin t}_{y_2''(t)} + \underbrace{\sin t}_{y_2(t)} = 0$$

for all t, so $y_2(t)$ is also a solution of Eq. (16) on $-\infty < t < \infty$.

EXAMPLE 3

Show that $y(x) = c_1x^2 + c_2x^2 \ln x + \frac{1}{4}\ln x + \frac{1}{4}$, where c_1 and c_2 are arbitrary constants, is a solution of

$$x^2y'' - 3xy' + 4y = \ln x \tag{17}$$

on the interval $0 < x < \infty$.

Subsituting $y(x)$ into the left side of Eq. (17) yields

$$x^2\underbrace{\left(2c_1 + c_2(2\ln x + 3) - \frac{1}{4x^2}\right)}_{y''(x)} - 3x\underbrace{\left(2c_1x + c_2(2x\ln x + 3) + \frac{1}{4x}\right)}_{y'(x)} + 4\underbrace{\left(c_1x^2 + c_2x^2\ln x + \frac{1}{4}\ln x + \frac{1}{4}\right)}_{y(x)}$$

$$= c_1(2x^2 - 6x^2 + 4x^2) + c_2(2x^2\ln x + 3x^2 - 6x^2\ln x - 3x^2 + 4x^2\ln x) - \frac{1}{4} - \frac{3}{4} + \ln x + 1$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \ln x.$$

Thus $y(x)$ satisfies Eq. (17) for all $0 < x < \infty$).

Initial Value Problems

Recall that in Section 1.1 we found solutions of certain equations by a process of direct integration. For instance, we found that the equation

$$\frac{du}{dt} = -k(u - T_0) \tag{18}$$

has the solution

$$u = T_0 + ce^{-kt}, \tag{19}$$

where c is an arbitrary constant. Each value of c corresponds to an integral curve in the tu-plane. If we want the solution that satisfies the condition

$$u(t_0) = u_0, \tag{20}$$

that is, the integral curve in the tu-plane that passes through the point (t_0, u_0), we substitute Eq. (19) into Eq. (20) to get

$$T_0 + ce^{-kt_0} = u_0. \tag{21}$$

Solving Eq. (21) for c gives

$$c = e^{kt_0}(u_0 - T_0), \tag{22}$$

and then replacing c in Eq. (19) by the right side of Eq. (22), we get

$$u = T_0 + (u_0 - T_0)e^{-k(t-t_0)}, \tag{23}$$

the solution of the initial value problem consisting of Eq. (18) and Eq. (20).

Let us now consider the simple second order equation

$$\frac{d^2y}{dt^2} = 0. \tag{24}$$

Integrating Eq. (24) twice results in two undetermined constants:

$$y(t) = c_1 t + c_2. \tag{25}$$

Values for c_1 and c_2 may be determined, for example, by requiring two initial conditions

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_1. \tag{26}$$

Substituting the solution (25) into Eqs. (26) gives

$$c_1 t_0 + c_2 = y_0 \quad \text{and} \quad c_1 = y_1.$$

Since $c_1 = y_1$, it follows that $c_2 = y_0 - t_0 y_1$. The solution of the initial value problem consisting of Eqs. (24) and (26) is therefore the straight line in the ty-plane with slope y_1 that passes through the point(t_0, y_0),

$$y = y_1(t - t_0) + y_0.$$

In general, solving an nth order ordinary differential equation results in n constants of integration $c_1, c_2, \dots, c_n$. In applications, these constants of integration are determined by a set of auxiliary constraints, called **initial conditions**, on the solutions.

DEFINITION 1.3.4

Initial Value Problem. An **initial value problem** for an nth order differential equation

$$y^{(n)} = f(t, y, y'', \dots, y^{(n-1)}) \tag{27}$$

on an interval I consists of Eq. (27) together with n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots, \quad y^{(n-1)}(t_0) = y_{n-1} \tag{28}$$

prescribed at a point $t_0 \in I$, where $y_0, y_1, \dots, y_{n-1}$ are given constants.

Thus $y = \phi(t)$ is a **solution of the initial value problem** (27), (28) on I if, in addition to satisfying Eq. (27) on I,

$$\phi(t_0) = y_0, \quad \phi'(t_0) = y_1, \quad \dots, \quad \phi^{(n-1)}(t_0) = y_{n-1}.$$

EXAMPLE 4

Show that $\phi(t) = 2\cos t - 3\sin t$ is a solution of the initial value problem

$$y'' + y = 0, \qquad y(0) = 2, \quad y'(0) = -3, \tag{29}$$

on the interval $-\infty < t < \infty$.

Just as in Example 2, we substitute $\phi(t)$ into $y'' + y = 0$ to find that

$$\underbrace{-2\cos t + 3\sin t}_{\phi''(t)} + \underbrace{2\cos t - 3\sin t}_{\phi(t)} = 0$$

for all t. Therefore $\phi(t)$ is a solution of $y'' + y = 0$ on $-\infty < t < \infty$. Next we must check to see if the initial conditions specified in the initial value problem (29) are satisfied. Since

$$\phi(0) = 2\cos 0 - 3\sin 0 = 2$$

and

$$\phi'(0) = -2\sin 0 - 3\cos 0 = -3,$$

we conclude that $\phi(t)$ is a solution of the initial value problem (29).

PROBLEMS

In each of Problems 1 through 6, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$

2. $(1 + y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

4. $\dfrac{dy}{dt} + ty^2 = 0$

5. $\dfrac{d^2y}{dt^2} + \sin(t + y) = \sin t$

6. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$

Show that Eq. (10) can be matched to each equation in Problems 7 through 12 by a suitable choice of n, coefficients $a_0, a_1, \ldots, a_n$, and function g. In each case, state whether the equation is homogeneous or nonhomogeneous.

7. $\dfrac{dQ}{dt} = -\left(\dfrac{1}{1+t}\right)Q + 2\sin t$

8. $\dfrac{d^2y}{dt^2} = ty$

9. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = \ln x, \quad x > 0$

10. $\dfrac{d}{dx}\left[(1 - x^2)\dfrac{d}{dx}P_n\right] + n(n + 1)P_n = 0, \quad n$ constant

11. $\dfrac{d^4y}{dt^4} + (\cos t)\dfrac{d^2y}{dt^2} + y = e^{-t}\sin t$

12. $\dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0, \quad \lambda$ constant

In each of Problems 13 through 20, verify that each given function is a solution of the differential equation.

13. $y'' - y = 0; \quad y_1(t) = e^t, \quad y_2(t) = \cosh t$

14. $y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t$

15. $ty' - y = t^2; \quad y = 3t + t^2$

16. $y'''' + 4y''' + 3y = t; \quad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3$

17. $2t^2y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}$

18. $t^2y'' + 5ty' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2}\ln t$

19. $y'' + y = \sec t, \quad 0 < t < \pi/2; \quad y = (\cos t)\ln\cos t + t\sin t$

20. $y' - 2ty = 1; \quad y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 21 through 24, determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

21. $y' + 2y = 0$

22. $y'' - y = 0$

23. $y'' + y' - 6y = 0$

24. $y''' - 3y'' + 2y' = 0$

In each of Problems 25 and 26, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

25. $t^2y'' + 4ty' + 2y = 0$

26. $t^2y'' - 4ty' + 4y = 0$

In Problems 27 through 31, verify that $y(t)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(t)$ satisfies the given initial condition.

27. $y' + 2y = 0; \quad y(t) = Ce^{-2t}, \quad y(0) = 1$

28. $y' + (\sin t)y = 0; \quad y(t) = Ce^{\cos t}, \quad y(\pi) = 1$

29. $y' + (2/t)y = (\cos t)/t^2$; $y(t) = (\sin t)/t^2 + C/t^2$, $y(1) = \frac{1}{2}$

30. $ty' + (t+1)y = t$; $y(t) = (1 - 1/t) + Ce^{-t}/t$, $y(\ln 2) = 1$

31. $2y' + ty = 2$; $y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + Ce^{-t^2/4}$, $y(0) = 1$

32. Verify that the function $\phi(t) = c_1 e^{-t} + c_2 e^{-2t}$ is a solution of the linear equation

$$y'' + 3y' + 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied:

(a) $y(0) = -1$, $y'(0) = 4$
(b) $y(0) = 2$, $y'(0) = 0$

33. Verify that the function $\phi(t) = c_1 e^t + c_2 te^t$ is a solution of the linear equation

$$y'' - 2y' + y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied:

(a) $y(0) = 3$, $y'(0) = 1$
(b) $y(0) = 1$, $y'(0) = -4$

34. Verify that the function $\phi(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$ is a solution of the linear equation

$$y'' + 2y' + 5y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied:

(a) $y(0) = 1$, $y'(0) = 1$
(b) $y(0) = 2$, $y'(0) = 5$