1

SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

1.1 LINEAR ALGEBRAIC EQUATIONS

1. (a) Solving the last equation for x_4 yields $x_4 = 4/4 = 1$. We substitute this value of x_4 into the third equation and solve for x_3 to get

$$x_3 + 2(1) = -1 \qquad \Rightarrow \qquad x_3 = -3.$$

With these values of x_3, the second equation becomes

$$3x_2 + 2(-3) + (1) = 2 \qquad \Rightarrow \qquad 3x_2 = 7 \qquad \Rightarrow \qquad x_2 = \frac{7}{3}.$$

For x_1, we substitute the values of x_2 and x_4 into the first equation.

$$2x_1 + \left(\frac{7}{3}\right) - (1) = -2 \quad \Rightarrow \quad 2x_1 = -\frac{10}{3} \quad \Rightarrow \quad x_1 = -\frac{5}{3}.$$

Therefore, the answer is: $x_1 = -5/3$, $x_2 = 7/3$, $x_3 = -3$, and $x_4 = 1$.

Solutions Manual to Accompany Fundamentals of Matrix Analysis with Applications, First Edition. Edward Barry Saff and Arthur David Snider.

(b) Solving the first equation for x_1, we obtain $x_1 = 3$. We now substitute this value of x_1 into the second equation and solve for x_2.

$$-2(3) - 3x_2 = -12 \quad \Rightarrow \quad -3x_2 = -6 \quad \Rightarrow \quad x_2 = 2.$$

The third equation then becomes

$$(3) + (2) + x_3 = 5 \qquad \Rightarrow \qquad x_3 = 0.$$

Therefore, the answer is $x_1 = 3$, $x_2 = 2$, and $x_3 = 0$.

(c) From the third equation, we immediately get $x_4 = 1$. This value, when substituted into the first equation, yields

$$-x_3 + 2(1) = 1 \qquad \Rightarrow \qquad -x_3 = -1 \qquad \Rightarrow \qquad x_3 = 1.$$

From the fourth equation, we obtain

$$x_2 + 2(1) + 3(1) = 5 \qquad \Rightarrow \qquad x_2 = 0.$$

Finally, from the second equation, we conclude that

$$4x_1 + 2(0) + (1) = -3 \qquad \Rightarrow \qquad x_1 = -1.$$

So, the solution is $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 1$.

(d) The third equation implies that $x_1 = 1$. Then, the first equation says

$$2(1) + x_2 = 3 \qquad \Rightarrow \qquad x_2 = 1.$$

We now substitute these values of x_1 and x_2 into the second equation to get

$$3(1) + 2(1) + x_3 = 6 \qquad \Rightarrow \qquad x_3 = 1.$$

The solution to this problem is $x_1 = x_2 = x_3 = 1$.

3. To eliminate x_1 from the first and second equations, we subtract from the first equation the third equation multiplied by 3 and subtract from the second equation the third equation multiplied by 2, resp. Thus we get an equivalent system

$$\begin{aligned} -x_2 \quad &= -2 \\ -9x_2 - x_3 &= -16 \\ x_1 + 3x_2 + x_3 &= 3. \end{aligned}$$

From the first equation, we get $x_2 = 2$. Making the back substitutions into the second and third equations yields

$$\begin{aligned} &-9(2) - x_3 = -16 \quad \Rightarrow \quad -x_3 = 2 \quad \Rightarrow \quad x_3 = -2 \\ &x_1 + 3(2) + (-2) = 3 \quad \Rightarrow \quad x_1 = -1. \end{aligned}$$

The solution is $x_1 = -1$, $x_2 = 2$, and $x_3 = -2$.

5. We eliminate x_1 and x_4 from the first and the the last equations by subtracting from them the second equation. We also eliminate x_1 from the third equation by subtracting from it twice the second equation. The new system is

$$\begin{aligned} x_2 + x_3 \quad &= 1 \\ x_1 \quad + x_4 &= 0 \\ 2x_2 - x_3 - x_4 &= 6 \\ 2x_2 - x_3 \quad &= 0 \end{aligned}$$

Next, we eliminate x_2 from the third and fourth by subtracting from them the first equation multiplied by 2. This gives

$$\begin{aligned} x_2 + x_3 \quad &= 1 \\ x_1 \quad + x_4 &= 0 \\ -3x_3 - x_4 &= 4 \\ -3x_3 \quad &= -2 \end{aligned}$$

We now can go with the back substitution. The last equation gives $x_3 = 2/3$. With this value, we find x_1 and x_4 from the first and third equations, resp.

$$x_2 + \left(\frac{2}{3}\right) = 1 \quad \Rightarrow \quad x_2 = \frac{1}{3}$$
$$-3\left(\frac{2}{3}\right) - x_4 = 4 \quad \Rightarrow \quad -x_4 = 6 \quad \Rightarrow \quad x_4 = -6.$$

Finally, the second equation says $x_1 = -x_4 = 6$. so, the answer is $x_1 = 6, x_2 = 1/3, x_3 = 2/3$, and $x_4 = -6$.

7. To eliminate x_1 from the second equation, we multiply the first equation by $0.987/0.123$ and subtract the result from the second one. Thus, we get

$$\begin{aligned} 0.123x_1 \quad +0.456x_2 &= \quad 0.789 \\ -3.005x_2 &= -6.010 \end{aligned}$$

From the second equation, we get $x_2 = (-6.010)/(-3.005) = 2.000$. Substituting this value into the first equation, we get

$$\begin{aligned} 0.123x_1 + 0.456(2) = 0.789 \;&\Rightarrow\; 0.123x_1 = -0.123 \\ &\Rightarrow\; x_1 = -1.000 \end{aligned}$$

The solution, rounded to three decimal places, is $x_1 = -1.000$, $x_2 = 2.000$. The number of arithmetic operations required is 3 divisions, 3 multiplications, and 3 additions.

9. Following the notations in Problem 8, the coefficients in Problem 7 are

$$a = 0.123,\ b = 0.456,\ c = 0.789,$$
$$d = 0.987,\ e = 0.654,\ f = 0.321.$$

Applying the formulas given in Problem 8, we obtain

$$x = \frac{(0.789)(0.654) - (0.456)(0.321)}{(0.123)(0.654) - (0.456)(0.987)} = -1.000;$$
$$y = \frac{(0.123)(0.321) - (0.789)(0.987)}{(0.123)(0.654) - (0.456)(0.987)} = 2.000.$$

The number of arithmetic operations required is 2 divisions, 6 multiplications, and 6 additions.

11. To eliminate x_1 from the second equation, we subtract from it the first equation multiplied by $0.987/0.123$. Similarly, we eliminate x_1 from the third equation using the factor of $0.333/0.123$, and obtain

$$\begin{aligned} 0.123x_1 \quad +0.456x_2 \quad +0.789x_3 &= 0.111 \\ -3.005x_2 \quad -6.010x_3 &= -0.446 \\ -1.790x_2 \quad -2.913x_3 &= 0.587 \end{aligned}$$

We subtract from the third equation the second one multiplied by $(1.790/3.005)$:

$$\begin{aligned} 0.123x_1 \quad +0.456x_2 \quad +0.789x_3 &= 0.111 \\ -3.005x_2 \quad -6.010x_3 &= -0.446 \\ 0.667x_3 &= 0.853 \end{aligned}$$

Going from the third equation up, using the back substitution we find that

$$\begin{aligned} x_3 &= \frac{0.853}{0.667} = 1.279; \\ x_2 &= \frac{-0.446 + 6.010(1.279)}{-3.005} = -2.410; \\ x_1 &= \frac{0.111 - 0.456(-2.410) - 0.789(1.279)}{0.123} = 1.633. \end{aligned}$$

The number of arithmetic operations required is 6 divisions, 11 multiplications, and 11 additions.

13. (a) From the third equation we find that $x_3 = 5.16/1.42 \approx 3.6338$ (rounded to four decimal places). Substituting this value into the second equation, we get

$$x_2 = \frac{1.11 - 1.34(3.6338)}{2.73} \approx -1.3770$$

Finally, we use the values of x_2 and x_3 to find x_1 from the first equation.

$$x_1 = \frac{-4.22 - 7.29(-1.3770) + 3.21(3.6338)}{1.23} \approx 14.2137.$$

Rounding the results to two decimal places, the answer is

$$x_1 = 14.21,\ x_2 = -1.38,\ x_3 = 3.63.$$

The total number of arithmetic operations required is 3 divisions, 3 multiplications, and 3 additions.

(b) We start from the fourth equation to find x_4. Substituting its value into the third equation, we evaluate x_3, and so on. These computations give

$$\begin{aligned}
x_4 &= \frac{-1}{0.250} = -4.0000;\\
x_3 &= \frac{1 - 0.888(-4)}{0.999} = 4.5565;\\
x_2 &= \frac{-1 - 0.222(4.5565) - 0.333(-4)}{0.111} = -6.1220;\\
x_1 &= \frac{1 - 0.333(-6.1220) - 0.250(4.5565) - 0.200(-4)}{0.500}\\
&= 5.3990.
\end{aligned}$$

Rounding the results to three decimal places yields

$$x_1 = 5.399,\ x_2 = -6.122,\ x_3 = 4.557,\ x_4 = -4.000.$$

The number of arithmetic operations required is 4 divisions, 6 multiplications, and 6 additions.

15. No, the suggested procedure is inconsistent with the Gauss elimination rules. First of all, these rules require, on each step, the elimination of one of the variables from all but one equations. One can do some *preliminary steps* adding to an equation a multiple of another equation, but on each such step the *new system* must be considered.

In the problem, let's follow steps. Subtracting the second equation from the first, we get a *new system*:

$$\begin{aligned}
-x_1 + x_2 + x_3 &= 0\\
2x_1 + x_2 + x_3 &= 6\\
x_1 + x_2 + 3x_3 &= 6
\end{aligned}$$

Subtracting the third equation from the second, yields a *new system*:

$$\begin{aligned} -x_1 \quad +x_2 \quad +x_3 &= 0 \\ x_1 \qquad\quad -2x_3 &= 0 \\ x_1 \quad +x_2 \quad +3x_3 &= 6 \end{aligned}$$

Performing the last step suggested in the problem, we obtain

$$\begin{aligned} -x_1 \quad +x_2 \quad +x_3 &= 0 \\ x_1 \qquad\quad -2x_3 &= 0 \\ 2x_1 \qquad\quad +2x_3 &= 6 \end{aligned}$$

After these preliminary steps (that were not really necessary), we can now go with Gauss elimination procedure. Adding the second equation to the third yields

$$\begin{aligned} -x_1 \quad +x_2 \quad +x_3 &= 0 \\ x_1 \qquad\quad -2x_3 &= 0 \\ 3x_1 \qquad\qquad &= 6 \end{aligned}$$

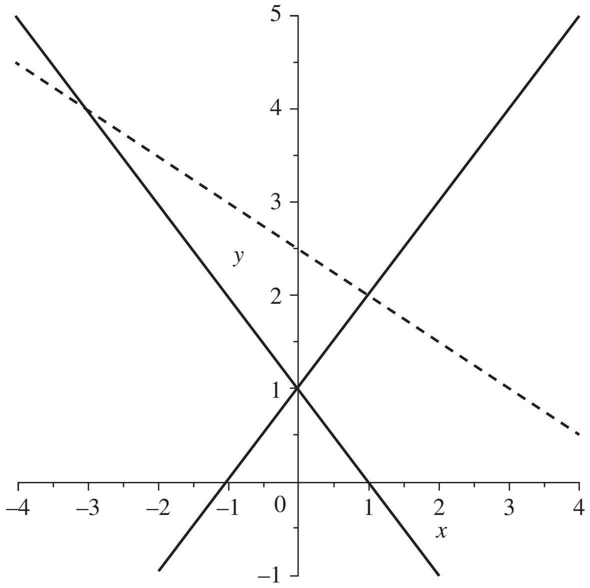
We can now use the back substitution method to solve the system.

$$x_1 = \frac{6}{3} = 2; \quad x_3 = \frac{x_1}{2} = 1; \quad x_2 = x_1 - x_3 = 1.$$

17. In the "derivation", a standard mistake was made: the equation $x^2 = (y-1)^2$ *does not* conclude that $x = y - 1$. The correct conclusion is $x = \pm(y-1)$. With the sign "+", we get the answer given; i.e. $x = 1$, $y = 2$. Choosing the sign "−", we get

$$-(y-1) + 2y = 5 \quad \Rightarrow \quad y = 4 \quad \Rightarrow \quad x = -(4-1) = -3.$$

The figure below indicates these *two* solutions – the points of intersection of the graphs of the equations.



19. The coordinates of both points must satisfy the equation $ax + by = c$. Thus, we have a system of two linear equations with three unknowns – a, b, and c:

$$\begin{aligned} ax_1 + by_1 &= c \\ ax_2 + by_2 &= c. \end{aligned}$$

Since there is always a line passing through two given points, this system is consistent. If the triple (a_0, b_0, c_0) is a solution, multiplying both equations by $k \neq 0$, we get an equivalent system

$$\begin{aligned} (ka_0)\, x_1 + (kb_0)\, y_1 &= (kc_0) \\ (ka_0)\, x_2 + (kb_0)\, y_2 &= (kc_0)\, . \end{aligned}$$

Thus, the triple (ka_0, kb_0, kc_0) also satisfies the conditions and, actually, determines the same line because

$$a_0x + b_0y = c_0 \quad \Leftrightarrow \quad (ka_0)\, x + (kb_0)\, y = (kc_0)\, .$$

Thus, the equation of a line is determined up to a non-zero constant multiple.

21. Since the points $(x_0, y_0), \ldots, (x_n, y_n)$ are on the graph $y = a_0 + \cdots + a_nx^n$, they must satisfy the equation. Substituting the coordinates of these points into the equation, yield the system of $(n+1)$ equations with $(n+1)$ unknowns – $a_0, \ldots, a_n$.

$$\begin{array}{llll} a_0 & +a_1x_0 & +\cdots+ & a_nx_0^n = y_0 \\ a_0 & +a_1x_1 & +\cdots+ & a_nx_1^n = y_1 \\ & \vdots & & \\ a_0 & +a_1x_n & +\cdots+ & a_nx_n^n = y_n \end{array}$$

23. Multiplying the second equation by $(1+i)$ we get

$$\begin{array}{ll} & (1+i)x_1 + (1+i)ix_2 + (1+i)(2-i)x_3 = (1+i)^2 \\ \Leftrightarrow & (1+i)x_1 + (i-1)x_2 + (3+i)x_3 = 2i. \end{array}$$

We eliminate now x_1 from the first equation by subtracting the above result from it (keeping the original second equation unchanged). Thus we get

$$\begin{array}{rrrl} & (3-i)x_2 & -(3+i)x_3 = & 0 \\ x_1 & +ix_2 & +(2-i)x_3 = & 1+i \\ & (-1+2i)x_2 & +(-2-3i)x_3 = & 0 \end{array}$$

To eliminate x_2 from the first equation, we multiply it by $(-1+2i)$ and then subtract from it the third equation multiplied by $(3-i)$. Thus, we get

$$\begin{array}{l} [-(3+i)(-1+2i) - (-2-3i)(3-i)]\, x_3 = 0 \\ \Rightarrow \qquad (9+7i)x_3 = 0, \end{array}$$

so that we have a new system

$$\begin{array}{rrrl} & & -(9+7i)x_3 = & 0 \\ x_1 & +ix_2 & +(2-i)x_3 = & 1+i \\ & (-1+2i)x_2 & +(-2-3i)x_3 = & 0 \end{array}$$

From the first equation, we find that $x_3 = 0$. Substituting this value into the third equation yields $x_2 = 0$. With zero values of x_2 and x_3, the second equation says that $x_1 = 1+i$. Therefore, the solution is

$$x_1 = 1+i,\ x_2 = 0,\ x_3 = 0.$$

25. If x and y are integers, then $6x$ and $4y$ are *even* numbers. Thus, their difference, $6x-4y$, is *even*, and so cannot be equal to an *odd* number 9.

27. The first equation is equivalent to

$$3(2x+4y) = 3(3) \qquad \Rightarrow \qquad 6x+12y = 9.$$

In the integers modulo 6, we have the sides as

$$(6x+12y) \mod 6 = 6(x+2y) \mod 6 = 0, \qquad 9 \mod 6 = 3.$$

Thus, the equation is *inconsistent* in integers modulo 6, and so is the system.

Concerning a solution in integers modulo 7, there are different ways to go. One of them is the following. First, we get a system that is equivalent to the given system by multiplying the first equation by 3 and the second equation by 2.

$$\begin{aligned} 6x \quad +12y &= 9 \\ 6x \quad +4y &= 6 \end{aligned}$$

Subtracting the second equation from the first one yields $8y = 3$. Rewriting this equation in integers modulo 7 (since $8 \mod 7 = 1$) as $y \mod 7 = 3$, the back substitution into the second equation in the original system yields

$$3x = 3-2y \qquad \Rightarrow \qquad (3x) \mod 7 = (3-2y) \mod 7 = 4 \mod 7.$$

Solving for x, we get $x \mod 7 = 6$. Therefore, in integers modulo 7, the solution to the given system is

$$x \mod 7 = 6, \; y \mod 7 = 3.$$

29. To simplify computations, we note that, for large n, the total number of each arithmetic operation can be approximated with a good *relative accuracy* by

$$\begin{aligned}
\text{Additions:} &\quad \frac{(n-1)n(2n+5)}{6} \approx \frac{n \cdot n \cdot (2n)}{6} = \frac{n^3}{3};\\
\text{Multiplications:} &\quad \frac{(n-1)n(2n+5)}{6} \approx \frac{n \cdot n \cdot (2n)}{6} = \frac{n^3}{3};\\
\text{Divisions:} &\quad \frac{n(n+1)}{2} \approx \frac{n^2}{2}.
\end{aligned}$$

Thus, the total number $N(n)$ of arithmetic operations can be approximated as

$$N(n) \approx \frac{n^3}{3} + \frac{n^3}{3} + \frac{n^2}{2} \approx \frac{2n^3}{3}.$$

For the "Thermal stress", $n = 10,000 = 10^4$. Therefore, using the performance of computers, we get that computers require approximately

$$\begin{aligned}
\text{Typical PC:} &\quad \frac{2\left(10^4\right)^3/3}{5 \times 10^9} = \frac{2 \times 10^{12}}{15 \times 10^9} \approx 133.33 \text{ (sec.)}\\
\text{Tianhe} - 2: &\quad \frac{2\left(10^4\right)^3/3}{3.38 \times 10^{16}} \approx \frac{10^{12}}{5.07 \times 10^{16}} \approx 1.97 \times 10^{-5} \text{ (sec.)}
\end{aligned}$$

to solve the system. Performing similar computations for other systems, we fill in the following table.

Model	***n***	***Typical PC***	***Tianhe-2***
Thermal Stress	10^4	133.333	1.97×10^{-5}
American Sign Language	10^5	1.33×10^5	0.020
Chemical Plant Modeling	3×10^5	3.59×10^6	0.532
Mechanics of Composite Materials	10^6	1.33×10^8	19.723
Electromagnetic Modeling	10^8	1.33×10^{14}	1.97×10^7
Computation Fluid Dynamics	10^9	1.33×10^{17}	1.97×10^{10}

1.2 MATRIX REPRESENTATION OF LINEAR SYSTEMS AND THE GAUSS-JORDAN ALGORITHM

1. We subtract the first row the second row multiplied by 2.

$$\left[\begin{array}{cc:c} 2 & 1 & 8 \\ 1 & -3 & -3 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{cc:c} 2 & 1 & 8 \\ 0 & -7 & -14 \end{array}\right]$$

Therefore, $x_2 = (-14)/(-7) = 2$, and the back substitution yields

$$2x_1 + (2) = 8 \qquad \Rightarrow \qquad x_1 = 3.$$

The solution is $x_1 = 3, x_2 = 2$.

3. We perform elementary row operations to reduce the given augmented coefficient matrix to upper triangular form.

$$\left[\begin{array}{ccc:cc} 4 & 2 & 2 & 8 & 0 \\ 1 & -1 & 1 & 4 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{array}\right] \quad \rho_2 \leftrightarrow \rho_1 \quad \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 4 & 2 \\ 4 & 2 & 2 & 8 & 0 \\ 3 & 2 & 1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{l} \rho_2 - 4\rho_1 \to \rho_2 \\ \rho_3 - 3\rho_1 \to \rho_3 \end{array} \quad \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 4 & 2 \\ 0 & 6 & -2 & -8 & -8 \\ 0 & 5 & -2 & -10 & -6 \end{array}\right]$$

$$6\rho_3 - 5\rho_2 \to \rho_3 \quad \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 4 & 2 \\ 0 & 6 & -2 & -8 & -8 \\ 0 & 0 & -2 & -20 & 4 \end{array}\right]$$

Evaluating x_3 from the third equation and making back substitutions we obtain the following solutions:

For the first system,

$$x_3 = \frac{-20}{-2} = 10, \; x_2 = \frac{-8 + 2(10)}{6} = 2, \; x_1 = 4 + (2) - (10) = -4.$$

For the second system,

$$x_3 = \frac{4}{-2} = -2,\ x_2 = \frac{-8+2(-2)}{6} = -2,$$
$$x_1 = 2 + (-2) - (-2) = 2.$$

5. For an $m \times n$ matrix **A**, let’s find a formula for the address of the element a_{ij}, $1 \le i \le m$, $1 \le j \le n$. This element is in the j’s column. Therefore, the number of preceding columns is $(j-1)$. Each of them contain m elements, and so the number of preceding elements is $m(j-1)$. The element a_{ij} is the i’s element in the j’s column, and so it has the number

$$\#\text{ of } a_{ij} = m(j-1) + i, \quad 1 \le i \le m, \quad 1 \le j \le n.$$

This is the answer to the part (c).

(a) We have $m = 7$ and $n = 8$. Using the formula derived for the part (c) yields

$$\#\text{ of } a_{1,7} = 7(7-1) + 1 = 43; \quad \#\text{ of } a_{7,1} = 7(1-1) + 7 = 7;$$
$$\#\text{ of } a_{5,5} = 7(5-1) + 5 = 33; \quad \#\text{ of } a_{7,8} = 7(8-1) + 7 = 56.$$

(d) Let’s first find the answer to this part, and then apply it in the part (b). From the formula in the part (c), we have

$$\frac{\#\text{ of } a_{ij} - i}{m} = j - 1 \quad \Leftrightarrow \quad j = \frac{\#\text{ of } a_{ij} - i + m}{m}.$$

Since j is a natural number and $1 \le i \le m$, we can eliminate the (unknown) i from this equation by writing

$$j = \left\lceil \frac{\#\text{ of } a_{ij}}{m} \right\rceil,$$

where $\lceil \cdot \rceil$ means the “ceiling function”, i.e., $\lceil x \rceil$ is the smallest integer that is not less than x. Once j is found, we can compute the index i using

$$i = \#\text{ of } a_{ij} - m(j-1).$$

(b) Here, $m = 5$, $n = 12$. With these values, the formulas in the part (d) give

$$\#4: \qquad j = \left\lceil \frac{4}{5} \right\rceil = 1, \quad i = 4 - 5(1-1) = 4 \;\Rightarrow\; a_{4,1};$$
$$\#20: \qquad j = \left\lceil \frac{20}{5} \right\rceil = 4, \quad i = 20 - 5(4-1) = 5 \;\Rightarrow\; a_{5,4};$$
$$\#50: \qquad j = \left\lceil \frac{50}{5} \right\rceil = 10, \quad i = 50 - 5(10-1) = 5 \;\Rightarrow\; a_{5,10}.$$

7. (a) Following the Gauss-Jordan elimination procedure, we proceed as follows.

$$\begin{bmatrix} 3 & 8 & 3 & \vdots & 7 \\ 2 & -3 & 1 & \vdots & -10 \\ 1 & 3 & 1 & \vdots & 3 \end{bmatrix}$$

$$(1/3)\rho_1 \to \rho_1 \quad \begin{bmatrix} 1 & 8/3 & 1 & \vdots & 7/3 \\ 2 & -3 & 1 & \vdots & -10 \\ 1 & 3 & 1 & \vdots & 3 \end{bmatrix}$$

$$\begin{array}{l} \rho_2 - 2\rho_1 \to \rho_2 \\ \rho_3 - \rho_1 \to \rho_3 \end{array} \quad \begin{bmatrix} 1 & 8/3 & 1 & \vdots & 7/3 \\ 0 & -25/3 & -1 & \vdots & -44/3 \\ 0 & 1/3 & 0 & \vdots & 2/3 \end{bmatrix}$$

$$(-3/25)\rho_2 \to \rho_2 \quad \begin{bmatrix} 1 & 8/3 & 1 & \vdots & 7/3 \\ 0 & 1 & 3/25 & \vdots & 44/25 \\ 0 & 1/3 & 0 & \vdots & 2/3 \end{bmatrix}$$

$$\begin{array}{l} \rho_1 - (8/3)\rho_2 \to \rho_1 \\ \rho_3 - (1/3)\rho_2 \to \rho_3 \end{array} \quad \begin{bmatrix} 1 & 0 & 17/25 & \vdots & -177/75 \\ 0 & 1 & 3/25 & \vdots & 44/25 \\ 0 & 0 & -1/25 & \vdots & 2/25 \end{bmatrix}$$

$$(-25)\rho_3 \to \rho_3 \quad \begin{bmatrix} 1 & 0 & 17/25 & \vdots & -177/75 \\ 0 & 1 & 3/25 & \vdots & 44/25 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$\begin{matrix} \rho_1 - (17/25)\rho_3 \to \rho_1 \\ \rho_2 - (3/25)\rho_3 \to \rho_2 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

Therefore, the answer is $x_1 = -1$, $x_2 = 2$, and $x_3 = -2$.

(b) This problem, actually, contains three systems of linear equations with the same coefficient matrix but different right-hand side columns. They are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, can proceed with the Gauss-Jordan elimination process for the coefficient matrix keeping a track of the right-hand sides on each step. For convenience, we first rearrange the equations.

$$\begin{bmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 3 & 2 & \vdots & 0 & 1 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \rho_3 \to \rho_1 \\ \rho_1 \to \rho_2 \\ \rho_2 \to \rho_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 3 & 2 & \vdots & 0 & 1 & 0 \end{bmatrix}$$

Then we proceed as follows:

$$\begin{matrix} \rho_2 - \rho_1 \to \rho_2 \\ \rho_3 - \rho_1 \to \rho_3 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 2 & 0 & \vdots & 1 & 0 & -1 \\ 0 & 3 & 1 & \vdots & 0 & 1 & -1 \end{bmatrix}$$

$$2\rho_3 - 3\rho_2 \to \rho_3 \quad \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 2 & 0 & \vdots & 1 & 0 & -1 \\ 0 & 0 & 2 & \vdots & -3 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c} (1/2)\rho_2 \to \rho_2 \\ (1/2)\rho_3 \to \rho_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right]$$

$$\rho_1 - \rho_3 \to \rho_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right]$$

Therefore, the solutions to the given systems, respectively, are

$$\begin{aligned} &x_1 = \frac{3}{2}, \quad x_2 = \frac{1}{2}, \quad x_3 = -\frac{3}{2}; \\ &x_1 = -1, \quad x_2 = 0, \quad x_3 = 1; \\ &x_1 = \frac{1}{2}, \quad x_2 = -\frac{1}{2}, \quad x_3 = \frac{1}{2}. \end{aligned}$$

9. Let us find out which approach, the standard one or the suggested modified one, is *more* efficient (meaning *less* time consuming). In other words, we have to compare the number of arithmetic operations, N_s and N_m, resp.

Intuitively, the modified approach should be more efficient:

In the standard way, given $1 \le j \le n$, to eliminate x_{ij}, $1 \le i \le n$, $i \neq j$, from the ith column, we have to perform arithmetics with entries on *all* rows ρ_i. In the modified procedure, in obtaining an upper triangular matrix, we deal only with the rows ρ_i, $i > j$. It is, approximately, twice less time consuming. Going then from the upper triangular coefficient matrix to a diagonal one, as described, essentially requires arithmetics performed *only* with the last column of the augmented matrix for the rows ρ_i, $1 \le i < j$. These arguments suggest that $N_m \le N_s$.

As an example, let us compute the number of arithmetic operations used in Problem 9(a), where we followed the standard Gauss-Jordan elimination process: $N_s = 27$. (Computations with obvious zero results are not counted.)

Using the modified way, we get $N_m = 20$. (Check it!)

Rigorous computations show that, for n large, the suggested modification of the Gauss-Jordan elimination procedure can save approximately 30% of the time spent for solving an $n \times n$ linear system vs the standard procedure.

1.3 THE COMPLETE GAUSS ELIMINATION ALGORITHM

1. To simplify computations, we rearrange the equations first.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 1 \\ 2 & 2 & -1 & 1 & \vdots & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 \\ 1 & 2 & -1 & 1 & \vdots & 0 \end{bmatrix} \begin{matrix} \rho_1 \to \rho_2 \\ \rho_2 \to \rho_4 \\ \rho_3 \to \rho_1 \\ \rho_4 \to \rho_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & \vdots & 0 \\ 1 & 1 & 1 & 1 & \vdots & 1 \\ 1 & 2 & -1 & 1 & \vdots & 0 \\ 2 & 2 & -1 & 1 & \vdots & 0 \end{bmatrix}$$

$$\begin{matrix} \rho_2 - \rho_1 \to \rho_2 \\ \rho_3 - \rho_1 \to \rho_3 \\ \rho_4 - 2\rho_1 \to \rho_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & 0 & \vdots & 1 \\ 0 & 2 & -1 & 0 & \vdots & 0 \\ 0 & 2 & -1 & -1 & \vdots & 0 \end{bmatrix}$$

$$\begin{matrix} \rho_4 - \rho_3 \to \rho_4 \\ \rho_3 - 2\rho_2 \to \rho_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & -3 & 0 & \vdots & -2 \\ 0 & 0 & 0 & -1 & \vdots & 0 \end{bmatrix}$$

From the third and fourth equations we find that

$$x_3 = \frac{-2}{-3} = \frac{2}{3}, \quad x_4 = \frac{0}{-1} = 0.$$

Substituting these values in the first two equations, we obtain

$$x_1 + (0) = 0 \Rightarrow x_1 = 0, \quad x_2 + \left(\frac{2}{3}\right) = 1 \Rightarrow x_2 = \frac{1}{3}.$$

Therefore, the solution is

$$x_1 = 0,\ x_2 = \frac{1}{3},\ x_3 = \frac{2}{3},\ x_4 = 0.$$

3. We follow the Gauss elimination procedure with back substitution.

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -4 & 11 \end{array}\right]$$

$$\begin{array}{l} \rho_2 - 2\rho_1 \to \rho_2 \\ \rho_3 + \rho_1 \to \rho_3 \end{array} \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & -4 & 11 \end{array}\right]$$

$$\begin{array}{l} \rho_4 + \rho_2 \to \rho_4 \\ -\rho_2 \to \rho_2 \end{array} \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -5 & 9 \end{array}\right]$$

$$\rho_2 - 3\rho_3 \to \rho_2 \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -5 & 9 \end{array}\right]$$

$$\rho_4 - \rho_2 \to \rho_4 \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 10 \end{array}\right]$$

The last equation is *inconsistent*, so is the original system.

5. Performing elementary row operations yields

$$\begin{bmatrix} 1 & 3 & 1 & \vdots & 2 \\ 3 & 4 & -1 & \vdots & 1 \\ 1 & -2 & -3 & \vdots & 1 \end{bmatrix} \begin{matrix} \rho_2 - 3\rho_1 \to \rho_2 \\ \rho_3 - \rho_1 \to \rho_3 \end{matrix} \begin{bmatrix} 1 & 3 & 1 & \vdots & 2 \\ 0 & -5 & -4 & \vdots & -5 \\ 0 & -5 & -4 & \vdots & -1 \end{bmatrix}$$

$$\rho_3 - \rho_2 \to \rho_3 \quad \begin{bmatrix} 1 & 3 & 1 & \vdots & 2 \\ 0 & -5 & -4 & \vdots & -5 \\ 0 & 0 & 0 & \vdots & 4 \end{bmatrix}$$

The system is *inconsistent* because of inconsistency of the third equation.

7. This augmented matrix defines three systems of linear equations in four unknowns with the same coefficient matrix **A** and right-hand side vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, where

$$A = \begin{bmatrix} 1 & -1 & -2 & 3 \\ 1 & 2 & 1 & -2 \\ 2 & 1 & -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

$$b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

We perform the Gauss elimination procedure with back substitution.

$$\begin{bmatrix} 1 & -1 & -2 & 3 & \vdots & 1 & 0 & 1 \\ 1 & 2 & 1 & -2 & \vdots & 1 & 0 & -1 \\ 2 & 1 & -1 & 1 & \vdots & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \rho_2 - \rho_1 \to \rho_2 \\ \rho_3 - 2\rho_1 \to \rho_3 \end{matrix} \begin{bmatrix} 1 & -1 & -2 & 3 & \vdots & 1 & 0 & 1 \\ 0 & 3 & 3 & -5 & \vdots & 0 & 0 & -2 \\ 0 & 3 & 3 & -5 & \vdots & 0 & 0 & -1 \end{bmatrix}$$

$$\rho_3 - \rho_2 \to \rho_3 \quad \left[\begin{array}{cccc|ccc} 1 & -1 & -2 & 3 & 1 & 0 & 1 \\ 0 & 3 & 3 & -5 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

In the first system, the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 1 \\ 0 & 3 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last equation is redundant. Taking $x_3 = s$ and $x_4 = t$ as free variables, we obtain from the second equation

$$x_2 = \frac{5x_4 - 3x_3}{3} = \frac{5t - 3s}{3}.$$

Back substitution of x_2, x_3, and x_4 into the first equation yields

$$x_1 = 1 + x_2 + 2x_3 - 3x_4 = 1 + \frac{5t - 3s}{3} + 2s - 3t = \frac{3 - 4t + 3s}{3}.$$

In the second system, we have

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 0 & 3 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Eliminating again the redundant third equation and letting $x_3 = s$ and $x_4 = t$ be free variables, we conclude that

$$x_2 = \frac{5x_4 - 3x_3}{3} = \frac{5t - 3s}{3};$$
$$x_1 = x_2 + 2x_3 - 3x_4 = \frac{5t - 3s}{3} + 2s - 3t = \frac{3s - 4t}{3}.$$

The third system,

$$\begin{bmatrix} 1 & -1 & -2 & 3 & \vdots & 1 \\ 0 & 3 & 3 & -5 & \vdots & -2 \\ 0 & 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

is inconsistent because of the last equation that says

$$(0)x_1 + (0)x_2 + (0)x_3 + (0)x_4 = 1,$$

which is wrong for any values of the variables involved.

9. We perform elementary row operations to reduce the given augmented coefficient matrix to upper trapezoidal form.

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & \vdots & 1 \\ 2 & -2 & 4 & 1 & 0 & \vdots & 5 \\ 3 & -3 & 6 & -1 & 1 & \vdots & -2 \end{bmatrix} \quad \begin{matrix} \rho_2 - 2\rho_1 \to \rho_2 \\ \rho_3 - 3\rho_1 \to \rho_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & 0 & \vdots & 3 \\ 0 & 0 & 0 & -1 & 1 & \vdots & -5 \end{bmatrix}$$

From the last two equations, we obtain

$$x_4 = 3, \quad -(3) + x_5 = -5 \qquad \Rightarrow \qquad x_5 = -2.$$

The first equation is a linear equation with three unknowns, x_1, x_2, and x_3. We can choose any two of them, whose values are *arbitrary*, and then solve the equation for the other variable. For example,

$$x_2 = s,\ x_3 = t, \quad \Rightarrow \quad x_1 - s + 2(t) = 1 \quad \Rightarrow \quad x_1 = 1 + s - 2t.$$

Thus, a solution to the given system can be written in the form

$$x_1 = 1 + s - 2t,\ x_2 = s,\ x_3 = t,\ x_4 = 3,\ x_5 = -2,$$

where s and t are *any* real numbers.

11. There are three systems of linear equations in this problem. All of them have the same *coefficient* matrix appeared in Example 3 but different right-hand sides vectors (columns). The first column is that given in Example 3 with the answer

$$x_1 = -\frac{2}{3} + 2s,\ x_2 = s,\ x_3 = \frac{1}{3} - 2s,\ x_4 = -\frac{1}{3},\ x_5 = \frac{1}{3}.$$

To solve the other two problems, all that we need is to follow the elementary row operations in Example 3 with the matrix representing the right-hand side columns of the remaining equations. Thus we get

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho_1 \leftrightarrow \rho_3 \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(-2/3)\rho_1 + \rho_2 \to \rho_2 \quad \begin{bmatrix} 1 & 0 \\ 1/3 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho_2 \leftrightarrow \rho_3 \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1/3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} \rho_2 + \rho_4 \to \rho_4 \\ -\rho_2 + \rho_5 \to \rho_5 \end{matrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1/3 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\begin{matrix} (1/2)\rho_3 + \rho_4 \to \rho_4 \\ (-1/2)\rho_3 + \rho_5 \to \rho_5 \end{matrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1/3 & 0 \\ 7/6 & 0 \\ -7/6 & 0 \end{bmatrix}$$

$$-\rho_4 + \rho_5 \to \rho_5 \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1/3 & 0 \\ 7/6 & 0 \\ -7/3 & 0 \end{bmatrix}$$

Thus, the new augmented matrix for the second and the third systems is

$$\begin{bmatrix} 3 & 6 & 6 & 0 & 0 & \vdots & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & \vdots & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & \vdots & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 3 & \vdots & 7/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & -7/3 & 0 \end{bmatrix}$$

The first system here is inconsistent because the fifth equation is:

$$(0)x_1 + (0)x_2 + (0)x_3 + (0)x_4 + (0)x_5 = 0 \neq -\frac{7}{3}.$$

In the second system, the fifth equation is redundant. From the fourth equation we find that $x_5 = 0$. The back substitution into the third equation gives $x_4 = 0$. Thus, the first two equations give the system

$$\begin{aligned} 3x_1 + 6x_2 + 6x_3 &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned}$$

We can choose x_2 to be a free parameter: $x_2 = s$. Then $x_3 = -2x_2 = -2s$, and the back substitution into the first equation gives us

$$x_1 = -2x_2 - 2x_3 = -2(s) - 2(-2s) = 2s.$$

We can write the solution as a column vector in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ -2s \\ 0 \\ 0 \end{bmatrix},$$

where s is any number.

13. Starting with the original matrix, we perform the following elementary row operations.

$$2\rho_1 + \rho_2 \to \rho_2 \qquad \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 9 & 6 & \vdots & 2+\alpha \\ 0 & 3 & 2 & \vdots & 2 \end{bmatrix}$$

$$\rho_3 - (1/3)\rho_2 \to \rho_3 \qquad \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 9 & 6 & \vdots & \alpha+2 \\ 0 & 0 & 0 & \vdots & (4-\alpha)/3 \end{bmatrix}$$

If $\alpha \neq 4$, then the last equation is inconsistent, and so the system is inconsistent.

Assume now that $\alpha = 4$. Then the last equation becomes redundant, and we come up with a system of two linear equations in three unknowns, whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 9 & 6 & \vdots & 6 \end{bmatrix}$$

Letting $x_3 = s$ be a free parameter, we obtain

$$x_2 = \frac{6-6s}{9} = \frac{2-2s}{3}; \; x_1 = 1 - x_2 - x_3$$
$$= 1 - \frac{2-2s}{3} - s = \frac{1-s}{3}.$$

Thus, the system has infinitely many solutions if $\alpha = 4$.

15. Performing

$$\begin{bmatrix} 2 & 1 & 2 & \vdots & 1 \\ 2 & 2 & \alpha & \vdots & 1 \\ 4 & 2 & 4 & \vdots & 1 \end{bmatrix} \qquad \rho_3 - 2\rho_1 \to \rho_3 \qquad \begin{bmatrix} 2 & 1 & 2 & \vdots & 1 \\ 2 & 2 & \alpha & \vdots & 1 \\ 0 & 0 & 0 & \vdots & -1 \end{bmatrix}$$

The new third equation does not involve α, and it is inconsistent. Therefore, the system has no solution for any α.

17. Since, in Example 3, the values of $x_4 = -1/3$ and $x_5 = 1/3$ are uniquely determined, their back substitution into the last augmented matrix yields

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 0 \\ 0 & 2 & 1 & \vdots & 1/3 \end{bmatrix} \quad \rho_1 - \rho_2 \to \rho_1 \quad \begin{bmatrix} 1 & 0 & 1 & \vdots & -1/3 \\ 0 & 2 & 1 & \vdots & 1/3 \end{bmatrix}$$

If $x_1 = s$ is a free parameter, then the first equation yields

$$x_3 = -\frac{1}{3} - x_1 = -\frac{1}{3} - s = -\frac{1+3s}{3}.$$

From the second equation, we then obtain

$$2x_2 = \frac{1}{3} - x_3 = \frac{1}{3} + \frac{1+3s}{3} \qquad \Rightarrow \qquad x_2 = \frac{2+3s}{6}.$$

Therefore, in terms of the free parameter x_1, the answer in Example 3 is

$$x_1 = s,\ x_2 = \frac{2+3s}{6},\ x_3 = -\frac{1+3s}{3},\ x_4 = -\frac{1}{3},\ x_5 = \frac{1}{3}.$$

19. (a) This is a system of three equations with four unknowns. The coefficient matrix is in upper trapezoidal form with non-zero entries on the main diagonal. Taking x_4 as a *free* parameter, we can find the value of x_3 from the last equation, and then use the back substitutions to find x_2 and x_1. Therefore, the system has infinitely many solutions.

(b) The last equation is redundant. Removing it from the augmented matrix, we obtain an upper triangular coefficient matrix whose pivot elements equal to 1. Therefore, the system has a unique solution.

(c) No solution because the last equation is inconsistent.

(d) The last equation is redundant. We can find x_5 from the fourth equation and, performing back substitution, x_4 from the third equation. Substituting these values into the first two equations, we will get a system of two equations with three unknowns, x_1, x_2, and x_3, whose augmented matrix is upper trapezoidal form with non-zero pivot elements in each row. Taking x_3 as a *free* parameter, we than can solve the system for x_2 and x_1 in terms of x_3. Therefore, the given system has infinitely many solutions.

1.4 ECHELON FORM AND RANK

1. This matrix is in a row echelon form (follows from the definition). The rank of this matrix is 1 because it has just one nonzero row.

3. This matrix is in a row echelon form. Since this matrix has *no* zero rows, its rank is 3.

5. This matrix is in a row echelon form. No zero row. Therefore, the rank equals the number of rows, i.e., 2.

7. This augmented matrix is in a row echelon form having

$$\text{rank}\,[\mathbf{A}|\mathbf{b}] = \text{rank}\,[\mathbf{A}] = 2.$$

Therefore, the system is consistent.

Since the number of unknowns (the number of columns) is $3 > 2$, we conclude that the system has infinitely many solutions.

9. This augmented matrix is in a row echelon with

$$\text{rank}\,[\mathbf{A}|\mathbf{b}] = \text{rank}\,[\mathbf{A}] = 1.$$

Thus, the system is consistent.

Since the number of unknowns (the number of columns) is $3 > 1$, we conclude that the system has infinitely many solutions.

11. This augmented matrix is in a row echelon;

$$\text{rank}\,[\mathbf{A}|\mathbf{b}] = \text{rank}\,[\mathbf{A}] = 4.$$

Thus, the system is consistent.

Since the number of unknowns (the number of columns), 4 , is the same as the rank, we conclude that the system has a unique solution.

13. (a) This homogeneous system has fewer equations (2) than unknowns (3). Thus, for any α, it has infinitely many solutions.

(b) The number of equations (3) in this homogeneous system equals the number of unknowns. Thus, we have to compute the rank of the coefficient matrix and find the value(s) of α, for which the rank is less than 3. In this and only in this case, the system will have nontrivial solutions.

$$\begin{bmatrix} 1 & 1 & 1 \\ \alpha & 4 & 3 \\ 4 & 3 & 2 \end{bmatrix} \quad \begin{matrix} \rho_2 - \alpha\rho_1 \to \rho_2 \\ \rho_3 - 4\rho_1 \to \rho_3 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4-\alpha & 3-\alpha \\ 0 & -1 & -2 \end{bmatrix}$$

$$\rho_2 \leftrightarrow \rho_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 4-\alpha & 3-\alpha \end{bmatrix}$$

$$\rho_3 + (4-\alpha)\rho_2 \to \rho_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \alpha - 5 \end{bmatrix}$$

The rank of this matrix is less than 3 (namely, equals to 2) if and only if

$$\alpha - 5 = 0 \quad \Leftrightarrow \quad \alpha = 5.$$

15. We have to verify that

$$\#\text{ columns } \mathbf{A}^{\text{ech}} = \text{rank}\left[\mathbf{A}^{\text{ech}}|\mathbf{b}\right] + \#\text{ free parameters.}$$

(a) The given coefficient matrix is in a row echelon form, whose rank, 3. equals the number of unknowns (number of columns in it). Thus, this system has a unique solution, and so the number of free parameters is 0. The equation to be verified becomes

$$3 \overset{?}{=} 3 + 0$$

with the obvious answer.

(b) The coefficient matrix is in a row echelon form. It has 6 columns (unknowns) and rank 3. x_6 is uniquely defined from the third equation: $x_6 = 2$. The back substitution into the first two equations results two equations with 5 unknowns:

$$\begin{bmatrix} 3 & 5 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 3 & 2 & \vdots & -10 \end{bmatrix}$$

In the second equation, we have 2 free parameters; for example, $x_4 = v$ and $x_5 = w$ give $x_3 = -10 - 3v - 2w$. These values, when substituted into the first equation, say that

$$3x_1 + 5x_2 = 0.$$

One of the variables, say x_2, can be taken as a third free parameter, $3u$. Then, the solution to the given system is

$$x_1 = -5u,\ x_2 = 3u,\ x_3 = -10 - 3v - 2w,$$
$$x_4 = v,\ x_5 = w,\ x_6 = 2.$$

The total number of free parameters, u, v, and w, is three. The question then is about whether or not the equality

$$6 \stackrel{?}{=} 3 + 3$$

holds, with the obvious positive answer.

(c) The coefficient matrix is in a row echelon form. It has 6 columns (unknowns) and rank 3. x_6 is uniquely defined from the third equation: $x_6 = 2$. The back substitution into the first two equations results two equations with 5 unknowns:

$$\begin{bmatrix} 3 & 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 1 & 0 & 0 & \vdots & -10 \end{bmatrix}$$

From the second equation, we have $x_3 = -10$, and the back substitution into the first equation gives $x_1 = 4$. Therefore, we get

$$x_1 = 4,\ x_3 = -10,\ x_6 = 2,$$

and three other variables (x_2, x_4, and x_5) can take any values – they are free parameters. Since $6 = 3{+}3$, the equality in question holds.

17. Let $\mathbf{x} = [x_j]$, $\mathbf{y} = [y_j]$, and $\mathbf{A} = [a_{ij}]$. Since $\mathbf{x}$ and $\mathbf{y}$ are solutions to the homogeneous system $[\mathbf{A}|\mathbf{0}]$, for any $1 \le i \le n$ we have

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \qquad \sum_{j=1}^{n} a_{ij}y_j = 0.$$

Summing these two equations, we get

$$\sum_{j=1}^{n} a_{ij}x_j \pm \sum_{j=1}^{n} a_{ij}y_j = \sum_{j=1}^{n} (a_{ij}x_j \pm a_{ij}y_j) = \sum_{j=1}^{n} a_{ij}(x_j \pm y_j) = 0.$$

These equations tell us that the n-by-1 column vectors $[x_j \pm y_j]$ are solutions to $[\mathbf{A}|\mathbf{0}]$. According to the addition/subtraction rules for vectors, these vectors are $\mathbf{x} \pm \mathbf{y}$.

19. Let $\mathbf{x} = [x_j]$ and $\mathbf{A} = [a_{ij}]$. Since $\mathbf{x}$ is a solution to the homogeneous system $[\mathbf{A}|\mathbf{0}]$, for any $1 \le i \le n$ we have

$$\sum_{j=1}^{n} a_{ij}x_j = 0.$$

Multiplying this equation by c yields

$$c\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} ca_{ij}x_j = \sum_{j=1}^{n} a_{ij}(cx_j) = 0.$$

This equations tell us that the n-by-1 column vector $[cx_j]$ is a solution to $[\mathbf{A}|\mathbf{0}]$. According to the scalar multiplication rule for vectors, this vector equals to $c\mathbf{x}$.

21. Let $\mathbf{x} = [x_j]$, $\mathbf{y} = [y_j]$, and $\mathbf{A} = [a_{ij}]$, $\mathbf{b} = [b_i]$. Since $\mathbf{x}$ and $\mathbf{y}$ are solutions to the system $[\mathbf{A}|\mathbf{b}]$, for any $1 \le i \le n$ we have

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad \sum_{j=1}^{n} a_{ij}y_j = b_i.$$

Subtracting these two equations, we get

$$\sum_{j=1}^{n} a_{ij}x_j - \sum_{j=1}^{n} a_{ij}y_j = b_i - b_i = 0.$$

On the other hand,

$$\sum_{j=1}^{n} a_{ij}x_j - \sum_{j=1}^{n} a_{ij}y_j = \sum_{j=1}^{n} (a_{ij}x_j - a_{ij}y_j) = \sum_{j=1}^{n} a_{ij}(x_j - y_j).$$

These equations tell us that the n-by-1 column vector $[x_j - y_j]$ is a solution to the homogeneous system $[\mathbf{A}|\mathbf{0}]$. According to the subtraction rule for vectors, this vector is $\mathbf{x} - \mathbf{y}$.

23. (a) We will verify the first and the third equation. Others are similar.

For the first equation, we use **KCL** saying that, in a circuit, the *algebraic* sum of all currents at any junction node is 0. At the left node, we see one ***in***coming current, I_2, and two ***out***going currents, I_1 and I_3. Thus, we have the equation

$$I_2 - I_1 - I_3 = 0,$$

which is equivalent to the give equation.

For the third equation, we use **KVL** saying that the combined voltage around *any* loop in a circuit is 0. In the loop 1, we have the following voltage inputs: the current I_3 passing through the resistant $R = 5(\Omega)$ and the voltage source $E = 5(\text{V})$. Taking into account the orientation of the loop, we get

$$-(5)I_3 + 5 = 0 \quad \Leftrightarrow \quad 5I_3 = 5.$$

(b) In the matrix form, the system is

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 5 & 0 & 0 & -2 & 0 \end{array}\right]$$

We perform elementary row operations to reduce this matrix to a row echelon form.

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 5 & 0 & 0 & -2 & 0 \end{array}\right]$$

$$\rho_6 - \rho_3 \to \rho_6 \quad \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -2 & -5 \end{array}\right]$$

$$\rho_6 + \rho_5 \to \rho_6 \quad \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last equation is inconsistent and, therefore, the system is inconsistent.

(c) Changing the 5V battery to 6V battery affects only the loop 1 (the third equation). Thus we have

$$\left[\begin{array}{cccccc:c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 5 & 0 & 0 & -2 & 0 \end{array}\right]$$

$$\rho_6 - \rho_3 + \rho_5 \to \rho_6 \qquad \left[\begin{array}{cccccc:c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last equation is redundant. The rank of the coefficient matrix is 5, the number of unknowns is 6. Therefore, there are infinitely many solutions with $6 - 5 = 1$ free parameter.

From the equations three, four, and five we find, resp.,

$$I_3 = \frac{6}{5}, \quad I_4 = \frac{1}{3}, \quad I_6 = 3.$$

Substituting these values into the first two equations, we obtain

$$\begin{array}{l} I_1 - I_2 = -6/5 \\ I_2 - I_5 = -8/3 \end{array} \qquad (\rho_1 + \rho_2 \to \rho_1) \qquad \begin{array}{l} I_1 - I_5 = -58/15 \\ I_2 - I_5 = -8/3 \end{array}$$

Taking $I_5 = t$ as a free parameter, we get the solution

$$I_1 = -\frac{58}{15} + t,\ I_2 = -\frac{8}{3} + t,\ I_3 = \frac{6}{5},\ I_4 = \frac{1}{3},\ I_5 = t,\ I_6 = 3.$$

(d) The parametrized solution shows that increasing I_5 by t units increases I_1 and I_2 by the same amount, but does not affect the currents I_3, I_4, and I_6.

25. (a) For the jth spring, according to the Hooke's Law, we have $F_j = k_j x_j$. Since the system is in equilibrium, we have

$$\begin{aligned} F_1 + F_2 + F_3 = 650 \quad &\Rightarrow \quad 5000x_1 + 10000x_2 + 20000x_3 = 650 \\ &\Rightarrow \quad x_1 + 2x_2 + 4x_3 = 0.13. \end{aligned}$$

The torque equilibrium gives another equation:

$$F_1 = F_3 \quad \Rightarrow \quad 5000x_1 = 20000x_3 \quad \Rightarrow \quad x_1 - 4x_3 = 0.$$

Finally, from the geometry considerations, we get

$$x_2 = \frac{x_1 + x_3}{2} \qquad \Rightarrow \qquad x_1 - 2x_2 + x_3 = 0.$$

We arrange these three equations in the order convenient for Gauss elimination.

$$\begin{array}{cccrr} x_1 & & -4x_3 & = & 0 \\ x_1 & -2x_2 & +x_3 & = & 0 \\ x_1 & +2x_2 & +4x_3 & = & 0.13 \end{array}$$

with the augmented matrix

$$\left[\begin{array}{rrr:r} 1 & 0 & -4 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & 4 & 0.13 \end{array}\right]$$

$$\begin{array}{l} \rho_2 - \rho_1 \to \rho_2 \\ \rho_3 - \rho_1 \to \rho_3 \end{array} \quad \left[\begin{array}{rrr:r} 1 & 0 & -4 & 0 \\ 0 & -2 & 5 & 0 \\ 0 & 2 & 8 & 0.13 \end{array}\right]$$

$$\rho_3 + \rho_2 \to \rho_3 \quad \left[\begin{array}{rrr:r} 1 & 0 & -4 & 0 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & 13 & 0.13 \end{array}\right]$$

Therefore, the solution is

$$x_3 = \frac{0.13}{13} = 0.01, \quad x_2 = \frac{5x_3}{2} = 0.025, \quad x_1 = 4x_3 = 0.04.$$

(b) The equilibrium conditions for forces say that

$$\begin{array}{rrrcr} F_1 & & -F_3 & = & 0 \\ F_1 & +F_2 & +F_3 & = & 650 \end{array} \quad \Leftrightarrow \quad \left[\begin{array}{rrr:r} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 650 \end{array}\right]$$

$$\rho_2 - \rho_1 \to \rho_2 \qquad \left[\begin{array}{rrr:r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 650 \end{array}\right]$$

The coefficient matrix has rank 2 but the number of columns (unknowns) is 3. Thus, the system has infinitely many solutions.

1.5 COMPUTATIONAL CONSIDERATIONS

1. This system has the augmented matrix

$$\left[\begin{array}{rr:r} 0.003 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

First, let us try to perform the Gauss elimination procedure with this matrix, i.e., without pivoting.

$$\left[\begin{array}{rr:r} 0.003 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

$$\frac{\rho_1}{0.003} \to \rho_1 \qquad \left[\begin{array}{rr:r} 1 & 1.971 \times 10^4 & 1.972 \times 10^4 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

$$\rho_2 - 5.291\rho_1 \to \rho_2 \qquad \left[\begin{array}{cc:c} 1 & 1.971\times10^4 & 1.972\times10^4 \\ 0 & -1.043\times10^5 & -1.039\times10^4 \end{array}\right]$$

$$\rho_2/(-1.043\times10^5) \to \rho_2 \qquad \left[\begin{array}{cc:c} 1 & 1.971\times10^4 & 1.972\times10^4 \\ 0 & 1 & 9.962\times10^{-2} \end{array}\right]$$

Therefore, the solution obtained is

$$x_2 = 9.962\times10^{-2}, \quad x_1 = 1.776\times10^4.$$

With pivoting, the augmented matrix is

$$\left[\begin{array}{cc:c} 5.291 & -6.130 & 46.78 \\ 0.003 & 59.14 & 59.17 \end{array}\right]$$

Performing the Gauss elimination yields

$$\left[\begin{array}{cc:c} 5.291 & -6.130 & 46.78 \\ 0.003 & 59.14 & 59.17 \end{array}\right]$$

$$\frac{\rho_1}{5.291} \to \rho_1 \qquad \left[\begin{array}{cc:c} 1 & -1.159 & 8.841 \\ 0.003 & 59.14 & 59.17 \end{array}\right]$$

$$\rho_2 - 0.003\rho_1 \to \rho_2 \qquad \left[\begin{array}{cc:c} 1 & -1.159 & 8.841 \\ 0 & 59.14 & 59.14 \end{array}\right]$$

Therefore,

$$x_2 = \frac{59.14}{59.14} = 1.000, \quad x_1 = 8.841 + 1.159(1.000) = 10.00.$$

The comparison of the answers obtained without and with pivoting indicate that the former is absolutely inaccurate (due to the machine specifications), and the latter gives the accurate answer.

3. The system satisfying the given conditions is

$$\begin{array}{rrcr} 0.004x_1 & +2.8x_2 & = & 252.8 \\ x_1 & +x_2 & = & 290 \end{array}$$

or, in the matrix form

$$\left[\begin{array}{rrcr} 0.004 & 2.8 & \vdots & 252.8 \\ 1 & 1 & \vdots & 290 \end{array}\right]$$

We reduce this matrix to a row echelon form keeping in mind that the machine retains only 3 significant digits after each computation.

$$\left[\begin{array}{rrcr} 0.004 & 2.8 & \vdots & 252.8 \\ 1 & 1 & \vdots & 290 \end{array}\right]$$

$$\rho_2 - \rho_1/0.004 \to \rho_2 \quad \left[\begin{array}{rrcr} 0.004 & 2.8 & \vdots & 252.8 \\ 0 & -699 & \vdots & -62900 \end{array}\right]$$

Therefore,

$$x_2 = \frac{-62900}{-699} = 89.9, \quad x_1 = \frac{252.8 - 2.8(89.9)}{0.004} = 270.$$

With pivoting, we deal with the augmented matrix

$$\left[\begin{array}{rrcr} 1 & 1 & \vdots & 290 \\ 0.004 & 2.8 & \vdots & 252.8 \end{array}\right]$$

$$\rho_2 - (0.004)\rho_1 \to \rho_2 \quad \left[\begin{array}{rrcr} 1 & 1 & \vdots & 290 \\ 0 & 2.79 & \vdots & 251 \end{array}\right]$$

Therefore,

$$x_2 = \frac{251}{2.79} = 89.9, \quad x_1 = 290 - 89.9 = 200.$$

5. (a) We start with the augmented matrix corresponding to the first two systems given in this problem.

$$\left[\begin{array}{rrrrr|r|r} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & s \end{array}\right]$$

and perform the following elementary row operations following the Gauss elimination procedure.

$$\left[\begin{array}{rrrrr|r|r} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & s \end{array}\right]$$

$$\begin{array}{l} \rho_2 + \rho_1 \to \rho_2 \\ \rho_3 + \rho_1 \to \rho_3 \\ \rho_4 + \rho_1 \to \rho_4 \\ \rho_5 + \rho_1 \to \rho_5 \end{array} \left[\begin{array}{rrrrr|r|r} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & -1 & 1 & 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 1 & 2 & 2 & 0 \\ 0 & -1 & -1 & -1 & 2 & 2 & s \end{array}\right]$$

$$\begin{array}{l} \rho_3 + \rho_2 \to \rho_3 \\ \rho_4 + \rho_2 \to \rho_4 \\ \rho_5 + \rho_2 \to \rho_5 \end{array} \left[\begin{array}{rrrrr|r|r} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 0 \\ 0 & 0 & -1 & 1 & 4 & 4 & 0 \\ 0 & 0 & -1 & -1 & 4 & 4 & s \end{array}\right]$$

$$\begin{array}{c} \rho_4 + \rho_3 \to \rho_4 \\ \rho_5 + \rho_3 \to \rho_5 \end{array} \left[\begin{array}{ccccc:c:c} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & 8 & 8 & 0 \\ 0 & 0 & 0 & -1 & 8 & 8 & s \end{array}\right]$$

$$\rho_5 + \rho_4 \to \rho_5 \left[\begin{array}{ccccc:c:c} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 16 & 16 & s \end{array}\right]$$

(b) The pattern is clear. Since the last column in the original augmented matrix is the sum of two preceding rows, the largest value in the third system in n-row case will occur in the position $(n, n+3)$ and equals to $2^{n-1} + s$.

(c) From the row echelon form obtained in part (a) (for $n = 5$) we conclude that, for the first system,

$$\begin{aligned} x_5 &= 16/16 = 1, \\ x_4 &= 8 - 8(1) = 0, \\ x_3 &= 4 - 4(1) = 0, \\ x_2 &= 2 - 2(1) = 0, \\ x_1 &= 1 - (1) = 0. \end{aligned}$$

For the second system,

$$\begin{aligned} x_5 &= s/16, \\ x_4 &= -8\,(x_5) = -s/2, \\ x_3 &= -4\,(x_5) = -s/4, \\ x_2 &= -2\,(x_5) = -s/8, \\ x_1 &= -x_5 = -s/16. \end{aligned}$$

For the third system, the value of x_4 will be the sum of values of this variable in the first two systems; namely, $x_4 = 0 + (-s/2) = -s/2$.