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Introduction: Background and Motivation

1.1 What This Book Is All About

The book introduces linear and nonlinear structural analysis through a combination of of mesh generation, solid mechanics and a new finite element methodology called ***c-type finite element method*** (Ray, 1999, 2003, 2004, 2005, 2007, 2008). The ultimate objective is to present the largest possible (curved) beam, plate and shell elements undergoing extremely large displacement and rotation, and to apply these to solve standard industrial problems. Any finite element method is only as strong as its weakest link. In other words, the book is not just about unification of mesh generation and finite element methodology but it strives to serve as a reference for budding researchers, engineers, analysts, upper division and graduate students and teachers by demonstrating what various interdisciplinary machinery has to be accurately harnessed to devise a solid and conducive theoretical framework upon which to build a robust, reliable and efficient numerical methodology for linear and nonlinear static and dynamic analysis of beams, plates and shells. As indicated, the principal goal of the book is to produce the largest possible arbitrary shaped elements (a) defined and restricted solely by the requirements of geometry, material, loading and support conditions, (b) avoiding computational problems such as locking in the conventional finite element methods and (c) presenting new, accurate and explicit expressions for resolution of the symmetry issue of the tangent operator for beams, plates and shells in areas of extreme nonlinearity. The 'mega-sized' elements may result in substantial cost saving and reduced bookkeeping for the subsequent finite element analysis, and a reduced engineering manpower requirement for the final quality assurance. For example, the explicit algebraic and symmetric expressions of the tangent operator, as presented in the book, are an absolute necessity for computational cost efficiency, especially in repetitive calculations that are commonly associated with nonlinear problems. It must be recognized that the requirements for numerical convergence should be purely adaptive and subservient to the main delineating factors already mentioned. However, this strategy of computer generation of mega-elements of arbitrary shape, as it turns out, takes its toll on the analyst. Firstly, only accurate theoretical formulation can be used for the underlying continuum or solid mechanics principles without unnecessary 'short-circuiting' by proliferation of ad hoc numerical manipulations. Secondly, it demands that the applicable finite element method be devised to successfully accept computer generated elements with arbitrarily distorted shapes, with edges (or faces) consisting of up to truly 3D curved boundaries (or surfaces)

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with natural twist and bend (e.g. for shell elements). Thirdly, for these 'hyperelements' with conformity, the finite element method must be able to accommodate effortlessly and naturally C^0, C^1 or C^2 inter-element continuity on demand.

1.2 A Brief Historical Perspective

Every meaningful structural analysis is an exercise in abstraction about a structural system in the real world, so just as with any other natural or man-made phenomenon, the viability and safety of the structural system is intricately associated with the methodology underpinning such abstraction. More specifically, for a structural system, abstractions lie in the geometric modeling of its material body, its relevant support conditions and its imposed loadings, and finally its material properties; we associate the sum total of these abstractions with a structural theory. Moreover, each of these abstractions defines the extent to which a particular structural theory can efficiently and logically predict and control the response of a structural system. To paraphrase Einstein's incisive comment to Heisenberg that led to the latter's discovery of the uncertainty principle, every theory, like a mirror or a horse's blinder, filters and determines what we can see of the real world. Naturally, lest we miss out on important real-world phenomena, as structural analysts, we have to critically evaluate structural theories, propounded in both the distant and the recent past, so that we can be successful in the ultimate goal of our exercise in abstraction, namely, the prediction and control of structural response to external stimuli. Translated into actual methodology for solid bodies, these abstractions reduce to two fundamentally complementary disciplines: solid mechanics and numerical analysis – each determining and harnessing the strength of the other.

1.2.1 Operational Mechanics

For hundreds of years, even before the digital age, the basic theoretical premises of linear and nonlinear solid mechanics involving the study of the deformation of a body, transmission of force through it and the characterization of its material properties, have been well established, and they were perfected during the twentieth century (Eringen, 1962; Green and Zerna, 1968; Ogden, 1997; Malvern, 1969). However, without the computational power of modern computers, the forms of the various equations in solid mechanics, while accurately describing real world problems, were not of much use for finding numerical solutions for material bodies of very complicated geometry and intricate support conditions, subjected to complex loading systems. Thus, before the computer era, solid mechanics had no unifying numerical methodology, and so structural problems were solved on a case-by-case basis using a variety of different analytical methods.

1.2.2 Conventional Finite Element Methods

With the coming of the digital era, numerical methods became much more dominant. Over 60 years ago, an energy-based methodology called the finite element method (Turner, *et al.*, 1956) made its triumphant entry into the realm of numerical structural analysis, its theory having been established earlier (Courant, 1943). Its chief advance was to choose triangular elements with a complete set of basis functions following Pascal's triangle, but soon after, the application of similar basis functions to quadrilateral elements proved to be a poor choice because arbitrarily curtailed and incomplete polynomials resulted in interpolation problems requiring various ad hoc numerical artifacts such as under integration and reduced integration, to rectify numerous locking issues that resulted. Both the conventional finite element methods – h-type with Lagrangian and

Hermite basis functions (Zienkiewicz and Taylor, 2000) and p-type with Legendre basis functions (Szabo and Babuska, 1991), – suffered from these ill-conceived ideas.

While the general theoretical formulation of the displacement finite element method of analysis is based on rigorous variational method, the practice of finite element method is another story – a harrowing experience because, in an ad-hoc manner, it tried in vain to tear apart the fundamental and inalienable dictum of the Rayleigh–Ritz–Galerkin method that says: "… method is a 'package deal', and neither requires nor permits the user to make independent decisions about different parts of the problem" (Strang and Fix, 1973, p. 33). Based on the most common construction mechanism of global basis or Ritz functions from appropriate mapping of the chosen elemental or local basis functions, the ultimate success or failure of a practiced or computer-implemented finite element method essentially depends on the choice of the local basis functions (also known as the elemental shape functions), the adequacy of the mapping functions and the evaluations of the integrals. It is no wonder then that for more than 30 years, the primary focus of all notable fundamental researches in the field of finite element analysis has been on devising the "best" shape functions and associated mapping functions. It is instructive to briefly review the interpolation failure and concomitant patchy, ad hoc remedies with the ill side effects and to present some extremely important conceptual problems that remained unsolved by the conventional finite element methods, insofar as interpolation is concerned.

h-type Methodology: Hermitian shape functions, C^1 and C^2, are used for situations where flexural strain exists such as in the case of beam. The nodes in these cases are what can be considered as multiple nodes; that is, apart from the values, the slopes and/or curvatures are also taken as degrees of freedom at a physical location. The simplest element in the family has two end nodes with two degrees of freedom each for a total of four degrees of freedom. A cubic Hermite is used for this. The inter-element continuity that can be imposed is that the slope or the first derivatives of the functions are continuous, that is, C^1 continuity. To speed up convergence where the solution is smooth or has curved flexural elements, one can use a quintic Hermite with curvature or bending moment continuity, that is, C^2 continuity. Note that because of the inclusion of the derivative components as degrees of freedom, the interpolation loses its barycentric nature or convexity. The isoparametric elements are obtained following the assumption and procedure that the functional representation of the deformational behavior is employed in representation of the element geometry. In other words, the displacement vector components and the geometry use the same shape functions. This, in turn, forces the conventional element to have as many internal nodes as necessary to make geometry description isoparametric to the displacement function. For example, for a quadratic interpolation function for a truss element, it becomes a three-node element with one internal node. For a conventional isoparametric element, the accuracy increases with the corresponding element size measured by h, the diameter of the smallest circle encompassing the element, and, hence, also identified as h-type elements. The conventional h-type elements in two dimensions are interpolated with polynomials of degree one, two and sometimes three. The main elements, that have been known and used for quite some time, can be categorized into two major groups: Lagrangian and serendipity. These are obtained by taking the tensor product of the one-dimensional Lagrangian elements. The mapping functions are linear. For quadratic elements, say, the Lagrangian tensor product element consists naturally of nine nodes – four corner nodes, four edge nodes and one interior node. For the serendipity element, in this case, the interior node is dropped in favor of incomplete interpolation. The higher-order h-type elements are obtained analogously and may be found in any of the standard texts already mentioned.

p-type Methodology: From later developments come the so-called hierarchical elements of increasing polynomial degree $\boldsymbol{p}$, and hence, identified as p-type elements. The main idea behind these elements is that the next improved element of higher degree is achieved by retaining the earlier expressions of all degrees less than the improved one. In other words, to obtain an element of degree $p, p \geq 2$, only higher-order additional terms are introduced to the shape functions of the element of $p - 1$ degree. Moreover, these additional shape functions must vanish at old nodes, then since the p^{th} derivative of an old $(p - 1)$ degree shape function always vanishes, this condition is chosen to insert a new node, and finally the new shape function is scaled so that it assumes the value of unity at this generalized displacement node. The combination is not barycentric or convex, and the degrees of freedom consist of both functions and their derivatives as in the mixed formulations. For p-type quadrilateral elements, there is more than one variant of the polynomial spaces. The shape functions are categorized in those spaces into three groups: nodal, side and interior modes. The basic idea behind the shape functions is the same as in one dimension, that is, choosing Legendre polynomials as the primary functions. However, in order to enforce the fundamental property of basis functions, namely that each assumes a value of unity at one node and vanishes at the rest of the nodes, blending functions similar to h-type elements are used.

Triangular Elements: The triangular elements, by contrast, have long been described in the barycentric coordinates. The root geometry has been described in three coordinates, suggesting a three-dimensional figure. However, a closer look will reveal that the root element is still planar, that is, a subset of two-dimensional space, $\mathbb{R}^2$, because only two of the coordinates are independent and all three are related; the barycentric coordinates are essentially the area coordinates in two dimensions. In other words, any point in two dimensions can be expressed as convex combinations of the vertex of a triangle containing the point. The coefficients of combinations are the barycentric coordinates of the point. As shown in this book, the three coordinates are proportional to the three smaller triangles generated by joining the point to each vertex of the triangle. All shape functions can be defined on the standard triangle in terms of these barycentric coordinates. For higher-degree h-type triangular elements, edge nodes (e.g. for degree 2) and inside nodes (e.g. for degree three) are included in the element. For the hierarchic p-type elements, for $p = 1, 2$, the shape functions for the triangular elements are exactly the same as those for the h-type elements. The mapping functions are chosen as linear for straight edge triangles. For large curved triangles, mapping functions are isoparametric for $p = 2$. For higher-order triangular elements, mapping functions are developed based on a blending method. For the purpose of critical evaluation to be introduced later, this concludes a brief presentation of the characteristics of the shape functions and the mapping functions for both families, namely the h-type and the p-type, of conventional finite elements.

1.2.2.1 Problems of the Conventional Finite Element Methods

We start by referring to the comment by MacNeal (1994) that the “days of pioneering” or the days of “heuristics, hunches and experimental data to guide . . . design choices” is over, and that we should take design of finite elements “ . . . less as an art and more as a science”. The noteworthy applied mathematicians such as Strang and Fix appeared to be too apologetic, maybe because the engineers beat the mathematicians and discovered the finite element method; they also appeared to have been temptingly permissive by announcing that the serious violations of the theory are mere “variational crimes”: “ . . . it [the rule] is broken every day, and for good reason” (MacNeal, 1994). No wonder then that MacNeal is forced to believe that “mathematical rigor falls well short of the goal of converting finite element design into science”. In any case, because of these conflicting signals, scores of books keep appearing that simply repeat the same old (“variational crime”-ridden) information relating to the shape functions. The main goal, therefore, will be first to identify the various problems that crept into the mainstream conventional finite element method. However, in order for us to recognize the violations of the conventional finite element method, we need to reiterate the main dictates of the variational theorems: geometry must be accurately interpolated with extreme care for curved boundaries; the trial functions must have adequate derivatives across the element boundaries required by the inter-element continuity requirement; essential boundary conditions must be honored; and the integrals must be computed accurately.

Power Series as Basis Functions: One look at the history of the development of finite elements will reveal that primarily the power series made up of monomials of Cartesian position coordinates constituted the basis functions. Naturally, the Pascal triangle had been the guiding post for determining the completeness of the polynomials made up of such monomials. But, innocuous and familiar as these are, the polynomials constructed from these monomials, in their indiscriminate use as basis functions, produced some very distressing phenomena.

Induced Anisotropy: Once such commitment to power series has been made – the notion of completeness of the polynomials making up the basis functions became important for both convergence and suitability of the shape functions. Depending on the displacement solution function, convergence of incomplete polynomial basis functions to the solution is not guaranteed. Moreover, the basis functions based on incomplete polynomials can easily induce anisotropy for triangular or tetrahedral elements in the description of, say, the stiffness matrix, that is, the definition has to depend on the orientation of the coordinate axes. Similar loss of symmetry and hence induced anisotropy may result for arbitrary quadrilateral or hexahedral elements.

Incomplete Strain States: Incomplete polynomials as basis functions fail to invoke all the internal strain states that may result due to all the possible imposed conditions. For example, a four-node linear conventional quadrilateral element can only represent two internal linear strain states as opposed to required six. In fact, this condition afflicts all serendipity and Lagrange elements in two or three dimensions. Absence of complete strain states gives rise to what is known as ***locking*** which we shall deal with in the chapter on linear application. Based on the preceding discussion, polynomial basis functions made up of monomials do not seem to be a good choice. In fact, p-type hierarchical elements dropped monomials in favor of Legendre polynomials and thus avoided these problems. However, completeness is not the only dominating issue in the choice of the basis functions and mapping functions. We now embark on one such important issue.

Inter-element Continuity: As indicated in this book about the theory of Rayleigh–Ritz–Galerkin or finite element method, the trial and test spaces are finite dimensional subspaces of the solution space. The accuracy of the solution increases with the corresponding increase in the dimensionality of the subspaces. Furthermore, the energy in the error is equal to the error in the energy, and the approximated strain energy corresponding to any finite dimensional approximation is always less than that of the actual strain energy associated with exact solution. This, of course, is easy to understand when one considers that the description of an infinite dimensional displacement function by a finite set of basis functions only imposes additional constraints on the system. The additional stiffness, in turn, reduces the displacement and hence the strain energy of the system. The strain energy increases with increase in the number of basis functions. For this to continue to happen, however, the basis functions must satisfy some continuity requirements. For example, for plane stress problems in two dimensions, the trial function, for the Rayleigh–Ritz–Galerkin method, must be such that its first derivative has finite energy, that is, that it must be at least continuous, so that the virtual work expression is meaningful. In general, the trial functions must have finite energy (in the energy norm) in their m^{th} derivatives when the system-governing differential equation is of order $2m$. In finite element analysis, the geometry is reconstructed as an assemblage of finite elements. The above condition, then, becomes that the basis functions must be of class C^{m-1} across the element boundary. For plane stress analysis, $m = 1$, so the trial functions must be at least C^0 across the boundary, whereas for plate analysis, $m = 2$ and the corresponding continuity requirement is C^1, that is, the first derivative of the displacement function must be continuous. Any element that violates this requirement is known as the ***non-conforming*** element (Taylor *et al.*, 1976); otherwise, it is ***conforming***. In general, to establish, say, C^0 continuity at any edge between any two elements, the shape functions for a node connected to the edge must be identical for the two elements, and the shape functions for the rest of the nodes must vanish at the edge. For example, it can easily be shown that the constant strain, three-node triangle is conforming, and so are triangular or tetrahedral elements with edge nodes, as long as the edges are straight and all edges have same number of nodes. However, for the general four-node quadrilateral element, a simple coordinate transformation by rotation will show that the element is non-conforming. The only quadrilateral element that is conforming is the four-node rectangular element. All other general quadrilateral elements with edge nodes on the straight edges are non-conforming. Finally, if the edges of the elements, triangular or quadrilateral, are curved, the elements are non-conforming. All these elements made up of basis functions by incomplete power series of position coordinates lack the barycentric or convexity property of interpolation, and thus are devoid of invariance under affine – for example, rotational – transformations. In fact, as will be shown later, the problem associated with the general curved conventional elements of today's finite element method is one of interpolation that translates into lack of inter-element continuity in general. Historically speaking, however, these earlier problematic h-type elements found their partial savior in what is known as parametric mapping and rigid body rotation.

Parametric Mapping and Rigid Body Rotation: In the theoretical formulation of the isoparametric finite element method, it is assumed that the elemental basis functions are expressed in a root element, and the displacement functions are then mapped isoparametrically with the geometry. For the case where the geometry is straight and can be expressed by a linear root element, the displacement functions could be of higher order, needing conventionally edge or interior nodes; here, the mapping of the root element is known as ***subparametric***. In the reverse situation of curved geometry with simple strain states, the number of basis functions necessary to completely describe the geometry may be higher than that needed for displacement description; this mapping goes by the name of ***superparametric***. For superparametric elements, the representation of rigid body rotation and constant strain states is affected adversely. The history of construction of conventional finite elements with variable nodes appeared to have failed to realize one fundamental mathematical relationship: *the conversion in description of non-parametric functions to parametric description automatically and inescapably fixes all the internal nodes*. The displacement functions or the geometry coordinates, to start with, are non-parametric functions. To generate ***isoparametry*** between them, both needed to be first transformed into parametric entities. This then, by the principle just stated, fixes the intermediate nodes, be it just the edge nodes for one dimension, or the edge, face or internal nodes in their higher dimensional counterpart. Any discussion on distortion of the intermediate nodes is irrelevant and futile. In fact, if complete isoparametry is established, guaranteeing a barycentric or convex combination of the basis functions of the root element, the ensuing finite element, irrespective of any elemental distortion, will be conforming and complete. Short of guaranteed completeness of the polynomials, we would like to have a device by which various conventional elements can be measured for their acceptance. We now engage in discussing one such tool.

Patch Test: In earlier days, when terms were being added or dropped for designing new finite elements in ad hoc fashion, a standard numerical test was necessary for evaluating their worth. One such test is the patch test. The idea behind the patch test is simple. A chunk of elements representing a system is plucked out and subjected some known boundary conditions for which the internal solutions are known. It is expected, then, for acceptance of the elements, that the corresponding computed internal quantities should be the same as those known expectations. In other words, if a linear black box (the element) is correct, the output and input must correspond. To make calculations easy, the boundary is taken simple as well, such as square or cube for two and three dimensions, respectively. The constant strain patch test is considered standard for two- dimensional in-plane load conditions, because as the elements reduce sufficiently in size, the state of the strain (or derivative of the displacement) becomes constant in the limit. So, for convergence, the element should be able to represent such a state of strain. For plate bending, this strain becomes the bending curvature. In any case, linear displacement-equivalent forces are applied on the external nodes and solved for internal node displacements. The derivatives of this solution must match the known constant strains. There are several versions of patch tests. A patch test was initially conceived as a numerical experimentation for element validation. For smooth solution problems, the patch test was later shown mathematically to be a necessary and sufficient test for convergence of h-type elements. It appears quite clear to the author that the patch test is, in the main, an interpolation check. However, it serves to detect other failures such as integration or equilibrium failures. Because of the conformity (satisfying equilibrium requirements) and completeness (satisfying interpolation requirements) of the conventional linear isoparametric elements having only corner nodes, generation of constant strain or linear displacement state is extremely easy. Similarly, the conventional variable-node higher-order isoparametric elasticity elements with straight edges or faces with evenly spaced nodes can pass a higher-order patch test. As rectangular elements with evenly spaced edge nodes, the serendipity elements fail the patch test while its counterpart full-noded Lagrangian elements do not. Because of incomplete quadratic terms, isoparametric three- or four-noded bending elements cannot pass the constant curvature or quadratic displacement patch test. For curved elements, triangular (and tetrahedral) or quadrilateral (and hexahedral) with variable nodes arbitrarily placed cannot pass the quadratic displacement or constant patch test as necessary for plate and shell elements. As will be shown in the subsequent chapters, this is clearly an interpolation failure associated with elements with variable nodes under a curved situation. Finally, of course, the linear elements fail to pass the constant curvature patch test. In other words, these elements should not be used to model, say, plate or shell systems.

Locking Problems: The discussion on the patch test has been included merely to show the innate interpolation problems that have been afflicting conventional finite elements. More specifically, these translated into other serious numerical maladies known as ***locking***. We will therefore now look at the phenomenon of locking only in conventional rectangular elements. Other elements such as triangular elements experience the same locking phenomena, so we will dscribe the remedial methods that have been tried, but not altogether successfully. The futility crept in because none of these so-called tricks ever tried to correct the innate interpolation problems. In fact, these remedies, which include reduced and selective integration, incompatible modes and so on, do not appear to be numerically robust in the classical sense of consistency, convergence and stability. As a result, we are burdened with the additional problem of spurious modes introduced by reduced or selective integration. Simple solutions based on exact interpolation theory that allow the reader to appreciate the strength of the new c-type finite element method are presented in Chapter 9 about linear applications.

Shear and Membrane Locking and Shape Dependence: The major debacle of interpolation failure is locking, that is, elements showing unacceptably high stiffness, almost bordering on rigidity, in certain displacement states. In other words, the low order or variable node elements are failing to respond flexibly enough for or failing to recognize as it should, certain imposed conditions. Moreover, it was recognized that the severity of this phenomenon of locking depends on the shape of the element. Finally, the shape may be boiled down to a specific parameter that allegedly perpetrates such locking.

Now let us briefly examine the conventional state-of-the-art remedies.

Reduced/Selective Integration, Bubble Mode and Drilling Freedom: The linear four-node element failed to interpolate the quadratic term, and thus experienced dilatation locking.

Likewise, inability to handle quadratic term gave rise to spurious shear strain. This, in turn, contributed to the shear locking. As we already know, the numerical integration to evaluate, say, the stiffness matrix, is performed by selecting Gauss or other quadrature points. Thus, an immediate remedy to shear locking was offered by ***selective under-integration*** (Zienkiewicz *et al.*, 1971) by evaluating the integral involving shear at only one Gauss point, which effectively wiped out the shear contribution to the stiffness matrix. The problem with this remedy is that the reduced integration, is obviously less accurate. More importantly, it creates undesirable ***spurious modes***, that is, artificial singularities or eigenmodes induced by possible reduction in the rank of the stiffness matrix. Another ad hoc remedy was to include quadratic interpolation terms: so-called ***bubble modes*** (Wilson, 1973), without introducing any additional node, to handle both shear and dilatation locking. This, of course, renders the element non-conforming as two nodes cannot possibly represent a quadratic term. In line with the addition of modes of flexibility just described, modes known as ***drilling freedoms*** have been devised to resolve locking problems. Drilling freedoms (Allman, 1984) are nodal rotational degrees of freedom about an axis perpendicular to the plane of the element. These in-plane rotational degrees of freedom, in improving membrane performance, may create problems for bending performance of curved shell elements, where these degrees of freedom are already coupled with the bending modes. But, in general, most of these patchy remedies that worked on regular rectangular shaped element, fail to eradicate locking when applied to general h-type isoparametric curved elements of arbitrary shape. Moreover, these varieties of so-called remedial elements for specific displacement states increase the number of elements available commercially to such an extent that an ordinary user can be easily confused as to their use. It is extremely important that the element library be simple and user-friendly.

Plate and Shell Elements: For our purpose, we need only to note that for arbitrary shaped plate and curved shell elements, the interpolation problems and proposed solutions are similar to those already described. The remedies that have been offered for offsetting interpolation failure are not robust and are highly dependent on the element shapes. For example, trapezoidal plate or shell elements can still create problems of transverse shear locking.

1.2.3 Essential Mesh Generation: Curves and Surfaces

At around the same time as the introduction of the finite element method to structural analysis, a different numerical revolution, unnoticed by the finite element analysts, was underway primarily in the car industry (Coons and Herzog, 1967; Bezier, 1972; de Casteljau, 1986; Gordon and Riesenfeld 1974a,b), to improve upon the production of automotives with arbitrarily curved bodies: namely, **computer-aided geometric design** (CAGD). Of all the new methods of CAGD, **solid modeling** (Mantyla, 1988; Hoffmann, 1989) and **boundary representation** (B-Rep) methods (Barnhill, 1974; Faux and Pratt, 1979; Bohm, 1981, 1984; Foley and van Dam, 1982; Yamaguchi, 1988; Farin, 1992; Piegl and Tillman, 1995) took the leading role. Solid modeling is based on using geometrically simple elements as building blocks with set-theoretic operations of union, intersection, complementarity and so on, but without any information on inter-element continuity. On the other hand, B-Rep methods are all about building the geometry from simple shapes, from curves to surfaces to solids, with information and control over the inter-element boundaries. The foundations of these methods are based on complete Bernstein polynomials (Davis, 1975) as shape factors in Bezier control vector form and Schoenberg B-spline (Schoenberg, 1946) curves in de Boor–Cox (de Boor, 1972; Cox, 1972) control forms. The most important aspect of this revolution, in so far as it could relate to possible application to finite element analysis, is that B-splines are constructed geometrically as opposed to algebraically (Prenter, 1975) and thus provide the necessary local support for each finite element; in the ultimate analysis, these polynomials are complete for interpolation and approximation with local support.

1.3 Symbiotic Structural Analysis

To solve nonlinear structural problems such as beams, plates and shells under large displacement and rotation, it is not enough to engage the first part, namely, the mesh generation technique, and, the second part, namely, the c-type finite element method, free of ‘variational crimes’ that lead to locking, ad hoc numerical artefacts, and so on, without reformatting the fundamental axioms of solid mechanics to corresponding computational forms. In other words, the transformed governing equations of solid mechanics applied to beams, plates and shells form the third essential part of the system for possible production of largest finite elements. The reason for using such a transformation is primarily that finite rotations, unlike displacements of Euclidean space, traverse a configuration space that is a Riemannian curve space necessitating ***covariant derivatives*** as opposed to ***Gateaux derivatives***; this book tries to draw together these three components in developing the largest possible nonlinear beam, plate and shell static and dynamic finite elements.

1.4 Linear Curved Beams and Arches

The study of curved beams in finite element development is extremely important for anticipating problems for a given proposed method, when extended to the more general case of curved elements, namely, shells. In the linear regime, for many years, Timoshenko- or Mindlin-type curved beam and arch elements reportedly suffered from what is commonly known as shear and membrane locking in the thin regime (Ashwell & Sabir, 1971; Ashwell *et al.*, 1971). This, has triggered, for thirty years, a range of interpolationally inadequate approaches with limited success in terms of accuracy, rate of convergence and general applicability. Unfortunately, there are textbooks (e.g. Cook, 1994; Dow, 1999), that constantly refer to these methods as incurable observations, and this is misleading for the aspiring analysts. Nonetheless, we list a few such attempts with their associated pitfalls: the application of reduced integration of shear and membrane energy producing spurious modes of deformations in thin limits (Stolarski and Belytschko, 1982; Panadian, 1989); the truncation of so-called field-inconsistent strains resulting in low convergence rate (Babu and Prathap, 1986); the application of mixed polynomial-trigonometric displacement field restricting its general applicability to non-circular curved beams (Cantin & Clough, 1968; Ashwell, 1971); the utilization of higher-order quintic-quintic independent displacement fields (Dawe, 1974); the problem-specific penalty relaxation method (Tessler & Spiridigliozzi, 1986); the application of so-called quasi-conforming technique (Shi and Voyiadjis, 1991); and the circular curve-specific application of quadratic curvature interpolation (Lee and Sin, 1994). Finally, there is a method in which so-called material finite elements (MFE) are generated by analysis applicable only to linear circularly curved beams resulting in polynomial distributions for the field variables that are consistent with the equilibrium equations (Raveendranath *et al.*, 1999). The elements are lock-free with good predictability for field variables but unfortunately the shear strain distribution is restricted to being constant, resulting in only approximately constant shear force over each element. All these methods mentioned are either numerically ineffective or problem-specific. Also, there continues to be an unnecessary general consensus that to be locking free, a consistent formulation must have the order of interpolation for tangential displacement one order higher than that of normal displacement (Meck, 1980). For formulation in the linear regime of truly three-dimensional curved beams, we refer to the original works of Love (1944, 1972). For warping with seven degrees of freedom – three translations, three rotations and one twist, we refer to Washizu (1964).

1.5 Geometrically Nonlinear Curved Beams and Arches

One main focus of this book, as already indicated, is to demonstrate the versatility, simplicity and accuracy of the c-type method as applied to three-dimensional, beams and arches. The beams may be shear-deformable and extensible, straight or truly three-dimensional space curved with natural twist and bend, and finally geometrically nonlinear with extreme rotations and displacements without warping. The three-dimensional curved beam problem in the geometrically nonlinear regime – that is, for finite deformations – was first reported by Reissner (1973) and then in (1981) with an improved formulation. However, the analysis is restricted to second-order approximations on the rotation parameter and presents no numerically explicit expressions for stiffness, and so on. The work of Argyris (1982) is, in an unduly long and confusing sense (Spring, 1986), an introduction to theoretical kinematics for the mechanics community. For formulation in the extreme geometrically nonlinear regime, see the work of Simo (1985). In the analytical formulation, the possibility of a naturally twisted and bent situation is suggested. However, the numerical examples presented in the referenced work modeled curved beams as an assemblage of straight elements, resulting in a larger than necessary number of elements as will be demonstrated later. Simo and Vu-Quoc (1986) succeeded in enriching this method by considering finite rotation as an algebraic structure, namely, a ***Lie group***. Unfortunately, this paper missed the well-known role of covariant derivatives in parameters that travel on a differentiable manifold, that is, generalized multi-dimensional surfaces. However, this otherwise outstanding paper was taken as a standard by several other authors to justify their conclusions about the symmetry property of the tangent stiffness matrix. For example, Crisfield (1990), following a co-rotational formulation, wrongly asserted that even for a conservative system, the tangent stiffness matrix is unsymmetrical everywhere except at an equilibrium state. In view of this, we refer later to a simple lemma from Kreyszig (1959) to bring out the essence of the matter: in summary, the Gateaux derivatives are only appropriate on linear vector spaces, not on manifolds, for linearization and symmetry of the virtual functional, because, unlike ‘flat-spaces’ (e.g. Euclidean) where vectors are ‘free’ vectors, the Riemannian and differentiable manifolds accommodate only ‘bound’ vectors. An explicit proof for the multi-dimensional surface was eventually reported by Simo (1992). The tangent spaces are linear vector spaces, thus restriction of the rotation parameterization to elements of these spaces can avoid issues relating to covariant derivatives and allow the use of the familiar Gateaux derivatives needed for linearization. Following this idea, a series of papers appeared, starting with Cardona and Geradin (1988), where the symmetry of the geometric stiffness was correctly assumed, but without any proof. Similarly, in a related paper (Ibrahimbegovic, 1995), because of a less than instructive explicit expression of the same, the symmetry could only be verified but not proved. Another paper (Gruttmann *et al.*, 2000) presented a proof but because of the Cartesian nature of the analysis, the expression for the geometric stiffness remained rather convoluted. Of course, covariant derivatives are always needed for a curvilinear formulation, say, in the development of the equilibrium equations as we will show. We attend to this problem by choosing all the parameters as vector fields measured in curvilinear coordinates for various reasons. The first is rooted in our intention to unite the geometric mesh generation and the subsequent finite element analysis. Thus, the information – Bezier geometric controls, curvature and twist in Frenet frames – produced by a mesh generation provides the starting point for our analysis. Secondly, applications of the covariant derivatives for the stress resultant and stress couples in deriving the equilibrium equations and subsequent virtual work principle result in a direct proof and an elegant yet explicit expression for both the material and the geometric stiffness. Thirdly, the elements with only six degrees of freedom per node can be directly incorporated into the standard direct stiffness based finite element packages. Finally, in the numerical examples presented in the literature, almost invariably, curved beams are modeled as an assemblage of straight elements resulting in more than the necessary number of elements. We treat curved beams as mega-elements with truly three-dimensional attributes of natural bend and twist, and thus use a minimum number of elements for analysis, as will be demonstrated. The book presents the nonlinear beam theory for both quasi-static and dynamic loading conditions. For the quasi-static problems a Newton-type iteration scheme has been employed, similar to (Crisfield, 1981), to solve various benchmark problems; no attempt has been made in this book to apply the theory to dynamic problems.

1.6 Geometrically Nonlinear Plates and Shells

Another important focus of the book, lies in a demonstration of the versatility, simplicity and accuracy of the c-type method as applied to shells. The shells may be shear-deformable and extensible, and geometrically cylindrical, spherical or truly arbitrary surfaces, and finally geometrically nonlinear with extreme rotations and displacements and subjected to quasi-static or dynamic loading. Computationally speaking, the methodologies for modeling and solution of shell structures can be generally categorized into four groups: (1) the earliest finite element approach consisted of facet-type plate elements with loss of slope continuity for Love–Kirchhoff-type shells that need -continuity, and that are plagued by the shear locking problem when dealing with Mindlin–Reissner-type shells with shear strains; (2) three-dimensional solid (brick) elements are used as shell elements based on 3D elasticity equation with a large number of nodes and degrees of freedoms. For thinner shells, these elements broke down with erratic behavior and locking problem and ill-conditioning; (3) A finite element modeling known as degenerative shell elements from three-dimensional shell brick elements based on Mindlin–Reissner-type shell theory with reduced integration; and (4) shell elements modeled after various old classical shell theories (Wempner, 1989; Ahmad *et al.*, 1970; Zienkiewicz *et al.*, 1971; Pawsey and Clough, 1971).

For theoretical developments, we can divide the various efforts into two main groups: (1) in the ***direct approach*** the shell is taken at the very start as a 2D object and all the laws are developed for this, including the constitutive laws; Cosserat shells belong to this category (Naghdi, 1972); (2) the ***derived approach***, where various suitable constraints are applied to the three-dimensional shell-like body. If the main assumption is that the normals to the surface remain normal during deformation and inextensible, this results in what is known as the Love–Kirchhoff theory of shells with only three degrees of freedom at each point; otherwise, if normals just remain straight and inextensible, we have the Mindlin–Reissner model of shells with five degrees of freedom. Finally, inclusion of extensibility accommodates the full set of six degrees of freedom and goes by the name of Timoshenko-type shell theory (Pietraszkiewicz, 1979; Simo *et al.*, 1990). All these approaches suffer from adhocism in either their theoretical underpinning or their numerical evaluation. In this book, a modified method is developed following Reissner (1960) and Libai & Simmonds (1998). In this concept, we start from the exact three-dimensional balance laws and derive the variational functional exactly; any approximation is contained where it should naturally belong, that is the constitutive theory, which is at best empirical. The variational functional is linearized for numerical evaluation by a Newton-type iterative method by recognizing that the rotation tensor travels on a curve space and hence the absolute necessity of the covariant derivative of curve spaces over the Gateaux derivative of so-called 'flat' Euclidean space; for the corresponding constitutive theory, we follow Chroscielewski *et al.*, (1992).

1.7 Symmetry of the Tangent Operator: Nonlinear Beams and Shells

In nonlinear computational mechanics the prevalent Newton-type methods require linearization of the virtual functional for subsequent finite element formulation. The ad hoc practice of symmetrizing the tangent operator away from equilibrium, based on an application of Gateaux derivatives (a simple generalization of partial derivatives) in the linearization of parameters on a Riemannian surface, resulted in the claim that the geometric stiffness is non-symmetric away from equilibrium even for conservative systems. From a computational point of view, even for feasibility of producing a convergent solution without shear locking, and so on with *truly curved and large* beam or shell element, it is imperative that the concomitant theoretical formulation be derived in the most accurate and cost-effective sense. For beam and shell formulations in the geometrically nonlinear regimes, this brings us to the discussion of the following topics that show the essential motivation of our book.

Symmetry of the Geometric Stiffness – Conservative Systems: The three-dimensional curved beam formulation with a rotating or spinning coordinate system, extended in shell theory by Simmonds and Danielson (1970), in the geometric nonlinear regime (i.e. for finite deformations) was first reported by Reissner (1973) and subsequently again in (1981), with an improved formulation. However, the analysis is restricted to second-order approximations on the rotation parameter and presents no numerically explicit expressions for the residual and stiffness needed for a finite element development. Simo and Vu-Quoc (1986) enriched this by consideration of finite rotation as an algebraic structure, namely, a Lie group. Unfortunately, their paper overlooked the well-known role of the covariant (absolute) derivative in consistent linearization with respect to parameters that travel on a smooth manifold, that is, generalized multi-dimensional surfaces. To quote Simo and Vu-Quoc (1986 p. 80): "we show ... the global geometric stiffness arising from the (consistently) linearized weak form is non-symmetric, *even for conservative loading*, at a non equilibrium configuration ... Argyris and coworkers [5–8] point out that this lack of symmetry *inevitably* arises at the element (local) level" (emphasis added). The mistake is rooted in the application of the Gateaux derivative (a generalized partial derivative applicable only to Euclidean or "flat" surfaces) to define and compute the Hessian from the virtual work functional defined on Riemannian or "curved" surfaces. As a result, to overcome the inevitable dilemma, they introduced an ad hoc procedure that has since become known as the ***symmetrization of the geometric stiffness***. Simo recovered from this adhocism in a subsequent paper (1992) published four years later. However, Simo's original assertion was taken as a standard by several other authors to justify their mistaken conclusions about the symmetry property of the tangent stiffness operator. For example, Crisfield stated, (p. 148, Crisfield, 1990): "The tangent matrix is ... non-symmetric. ... *for the conservative problems* analyzed, symmetry is *recovered* as an equilibrium state is reached" (emphasis added). Based on this, the paper goes off at a tangent to recommend future research: "a study of the geometric stiffness matrix in order to discover which terms may be reasonably neglected without serious cost to the performance" (Crisfield, 1990 p. 148).

Symmetry of the Geometric Stiffness – Non-conservative Systems: For general three-dimensional situations, it will be recognized that, as indicated by Ziegler (1977), a moment loading (i.e. either distributed or end moments) makes the system non-conservative and so the resulting geometric stiffness is non-symmetric. But, as we will show, an identification of the terms responsible for the asymmetry suggests that the non-symmetrical structure may be computationally rather weak. Thus, in the absence of moment loading or where the end boundary conditions are purely kinematic with no distributed body moments, the system is conservative, and thus the geometric stiffness is definitely symmetric as discussed in the previous section. For two-dimensional or planar systems with planar deformations, the reduction of the 3D expression of the geometric stiffness to the 2D situation will reveal that all moment-related terms, including the reactive and externally applied distributed or end moments, vanish identically, leaving the geometric stiffness unconditionally symmetric.

Theoretical motivation: One important result of our theoretical motivation, *is that we show that the symmetry of the tangent operator is* ***always*** *guaranteed for a conservative system, in equilibrium or away from the equilibrium, if, instead of the Gateaux derivative, the concept of the covariant derivative is applied in the definition of consistent linearization*. Note that in any repetitive numerical solution (as in nonlinear problems): (a) the computational saving in working with a symmetric matrix, as opposed to a non-symmetric matrix, could be considerable and thus, always optimal, (b) an exact explicit definition, as opposed to an implicit one such as a derivative form or an approximate one, is always advantageous computationally. Thus, for a finite element formulation in the nonlinear regime with very large elements, it is not enough to prove that the tangent stiffness is symmetric for a conservative system. It is also imperative that the tangent stiffness present itself in its most simple and computationally effective form. In this sense, all papers published in the literature to date are not optimally suitable. Simo and Vu-Quoc (1986) and other similar papers presented the tangent stiffness in differential form. Cardona and Geradin (1988), where the symmetry of the geometric stiffness was correctly assumed, had to settle for an approximate expression of the tangent stiffness (on p. 2428, the comment after equation (146)). Similarly, in a related paper (Ibrahimbegovic, 1995), because of the disordered nature of the derived expression, the symmetry could not be proved. Gruttmann *et al.* (2000) presented a proof, but because of the Cartesian nature of the analysis, the expression for the geometric stiffness remained somewhat convoluted.

Computational motivation: As a principal result of our computational motivation, we present in this book a crucial ***central lemma*** (Section 10.1) *involving the virtual rotation and incremental rotation of a beam that avoids awkward derivation* [which forced Cardona and Geradin (1988) to settle for an approximation] *of second derivatives of tensors on a Riemannian surface. With the help of this lemma, along with other results, we present an accurate expression for the tangent stiffness in an optimal, cost-effective operational form*. In this book, for several reasons, we choose all the parameters to be vector or tensor fields measured in curvilinear coordinates. This is first rooted in our intention of uniting the geometric mesh generation and the subsequent finite element analysis. Thus, the information, namely, Bezier geometric controls, curvature and twist in Frenet frames, produced by a mesh generation provides the starting point of our analysis. Secondly, applications of the covariant derivatives for the stress resultant and stress couples in deriving the equilibrium equations and subsequent linearization of the virtual work functional result in a direct proof of symmetry and an elegant yet explicit expression for both the material and the geometric part of the stiffness. Thirdly, the elements with only six degrees of freedom per node can be directly incorporated into a standard displacement-based finite element system. In conclusion, with the goal of developing a truly three-dimensional, geometrically nonlinear curved finite element, a geometrically exact curvilinear, nonlinear formulation for the truly three-dimensional curved beams/arches and shells is presented in the book. The symmetry question of the geometric stiffness matrix is fully treated and shows that the tangent operator is symmetric for a conservative system. The moment loading is discussed thoroughly by identification of specific terms of non-symmetry. The two-dimensional specialization establishes the unconditional symmetry of the operator. Of utmost importance, considering the requirement of repetitive generation of the tangent operator, is the presentation of a closed form expression for the associated geometric stiffness in its most simple, direct and explicit form compared to those available in the published literature.

Finally, a few words about the chronological history is in order. The original work by the author of geometrically applying Bernstein–Bezier and B-spline bases to beams was initiated in 1995, and a paper (Ray, 1999) contained the c-type method, noting the importance of choosing appropriate basis functions for a finite element method. Note that about 13 years before any paper containing a similar finite element method (the iso-geometric method) was published, a single conference paper (Reus, 1992) was found to exist through an extensive literature search; the present author, having been completely unaware of the paper, developed the c-type method independently.

1.8 Road Map of the Book

Based on the discussions of this chapter on background and motivations, the book is designed to accomplish its goals through a logical sequence of chapters, the content of each of the chapters is as follows:

Chapter 1: this introductory chapter.

Chapter 2: presents some essential mathematical entities from real analysis such as sets, group, algebra and functions and continuity, normed spaces, the Sobolev space, the Vainberg principle, and Gram–Schmidt orthogonalization.

Chapter 3: introduces tensors in general, and then particularizes to second-order tensors; it identifies tensors both as linear functionals and dyadics or polyadics; it presents various properties and the differential and integral calculus of tensors so that all structural equations can be written in compact absolute notation devoid of cumbersome indicial description except where absolutely necessary.

Chapter 4: discusses in detail the representation and properties of rotational tensors necessary for analysis of structures with finite rotational responses; it covers both the theoretical and computational details pertaining to rotation tensor, and thus introduces the mathematical object called the quaternion.

Chapter 5: deals with the theory and computation of real curves in anticipation of its application to beams; in particular, it presents curves in the form of Bernstein polynomials and Bezier controls. It then includes B-spline representation geometrically by composite Bezier curves for local support.

Chapter 6: deals with the theory and computation of real surfaces in anticipation of its application to plates and shells; more specifically, it presents surfaces in the form of Bernstein polynomials and Bezier controls. It extends further to include B-spline representation geometrically by composite Bezier surfaces for local support.

Chapter 7: presents briefly the essential elements of nonlinear solid mechanics: deformation (strain), transmission of force (stress), balance laws and the constitutive theory of hyperelastic materials.

Chapter 8: introduces finite element theory as an energy method, and it gives a presentation of the new c-type finite element method.

Chapter 9: applies c-type finite element method to linear structural problems involving rods, straight and curved beams, plane stress and plane strain elements; it also provides solutions for various locking problems.

Chapter 10: develops the theory for nonlinear beams subjected to quasi-static and dynamic loading, resulting in extremely large displacement and rotation responses, and applies the c-type finite element method to solving numerical problems with extremely large beam elements.

Chapter 11: develops the theory for nonlinear plates and shells subjected to quasi-static and dynamic loading resulting in extremely large displacement and rotation responses and applies c-type finite element method to solve numerical problems with extremely large plate and shell elements.

References

Ahmad *et al.* (1970) Analysis of thick and thin shell structures by curved finite elements, *International Journal for Numerical Methods in Engineering* **2**(3), 419–451

Allman, D.J. (1984) A compatible triangular element including vertex rotations for plane elasticity analysis. *Computers & Structures*, **19**, 1–8.

Argyris, J.H. (1982) An excursion into large rotations. *Computer Methods in Applied Mechanics and Engineering*, **32**, 85–155.

Ashwell, D.G. and Sabir, A.B. (1971) Limitations of certain curved finite elements when applied to arches. *International Journal of Mechanical Sciences*, **13**, 133–139.

Ashwell, D.G. *et al*. (1971) Further study the application of curved finite element to circular arches. *International Journal of Mechanical Sciences*, **13**, 507–517.

Babu, C.R. and Prathap, G. (1986) A linear thick curved beam element. *International Journal for Numerical Methods in Engineering*, **23**, 1313–1328.

Barnhill, R.E. (1974) *Smooth Interpolation Over Triangles, in Computer Aided Geometric Design*, Academic Press.

Bathe, K.J. and Bolourchi, S. (1979) Large displacement analysis of three-dimensional beam structures. *International Journal for Numerical Methods in Engineering*, **14**, 961–986.

Bezier, P.E. (1972) Numerical control in automobile design and manufacture of curved surfaces, in *Curved Surfaces in Engineering*, IPC Science and Technology Press, London.

Bohm, W. (1981) Generating the Bezier points of B-spline curves and surfaces. *Computer Aided Design*, **13**(6), 365–366

Bohm, W. *et al.* (1984) A survey of curve and surface methods in CAGD. *Computer-Aided Geometric Design*, **I**, 1–60.

Cantin, G. and Clough, R.W. (1968) A curved cylindrical shell finite element. *AIAA Journal*, **6**, 1057–1062.

Cardona, A. and Geradin, M. (1988) A beam finite element non-linear theory with finite rotations. *International Journal for Numerical Methods in Engineering*, **26**, 2403–2438.

Chroscielewski, J. *et al.* (1992) Genuinely resultant shell finite elements accounting for geometric and material non-linearity. *International Journal for Numerical Methods in Engineering*, **35**, 63–94.

Cook, R.D. *et al.* (1994) *Concepts and Applications of Finite Element Analysis*, 3rd edn, John Wiley & Sons, Inc., NY, pp. 343–350.

Coons, S.A. and Herzog, B. (1967) *Surfaces for computer-aided aircraft design*. Paper 67-895, AIAA.

Courant, R. (1943) Variational methods for the solution of problems of equilibrium and vibration. *Bulletin of the American Mathematical Society*, **49**, 1–23.

Cox, M.G. (1972) The Numerical Evaluation of B-splines. *Journal of the Institute of Mathematics and its Application*, **10**, 134–149.

Crisfield, M.A. (1981) A fast incremental/iterative solution procedure that handles 'snap-through'. *Computers & Structures*, **13**, 55–62.

Crisfield, M.A. (1990) A consistent co-rotational formulation for non-linear, three-dimensional, beam-elements. *Computer Methods in Applied Mechanics and Engineering*, **81**, 131–150.

Davis, P.J. (1975) *Interpolation & Approximation*, Dover Publications, Inc., NY.

Dawe, D.J. (1974) Numerical studies using circular arch finite elements. *Computers & Structures*, **4**, 729–740.

De Boor, C. (1972) On calculating with B-splines. *Journal of Approximation Theory*, **6**, 50–62.

de Casteljau, P. (1986) *Shape Mathematics and CAD*, Kogan Page, London.

Dow, J.O. (1999) *A Unified Approach to the Finite Element Method and Error Analysis Procedures*, Academic Press.

Eringen, A.C. (1962) *Nonlinear Theory of Continuous Media*, McGraw-Hill, Inc.

Farin, G. (1992) *Curves and Surfaces for Computer Aided Geometric Design - A Practical Guide*, 3rd edn, Academic Press.

Faux, L. and Pratt, M. (1979) *Computational Geometry for Design & Manufacture*, Ellis Horwood.

Foley, J. and van Dam, A. (1982) *Fundamentals of Interactive Computer Graphics*, Addison-Wesley.

Gordon, W.J. and Riesenfeld, R.F. (1974a) Bernstein-Bezier methods for the computer-aided design of free-form curves and surfaces. *Journal of the Association for Computing Machinery*, **21**(2), 293–310.

Gordon, W.J. and Riesenfeld, R.F. (1974b) B-spline curves and surfaces, in *Computer Aided Geometric Design*, Academic Press, NY.

Green, A.E. and Zerna, W. (1968) *Theoretical Elasticity*, 2nd edn, Oxford University Press.

Gruttmann, F. *et al.* (2000) Theory and numerics of three-dimensional beams with elastoplastic material behavior. *International Journal for Numerical Methods in Engineering*, **48**, 1675–1702.

Hoffmann, C. (1989) *Geometric & Solid Modeling – An Introduction*, Morgan Kaufmann Publishers, Inc., CA.

Huddleston, J.V. (1968) Finite deflections and snap-through of high circular arches. *Journal of Applied Mechanics*, **35**(4), 763–769.

Ibrahimbegovic, A. *et al.* (1995) Computational aspects of vector-like parametrization of three-dimensional finite rotations. *International Journal for Numerical Methods in Engineering*, **38**, 3653–3673.

Kreyszig, E. (1959) *Differential Geometry*, University of Toronto Press.

Lang, S. (1968) *Linear Algebra*, 3rd Printing, Addison-Wesley Publishing Co., Inc.

Lee, P.G. and Sin, H.C. (1994) Locking-free curved beam element based on curvature. *International Journal for Numerical Methods in Engineering*, **37**, 989–1007.

Libai, A. and Simmonds, J.G. (1998) *Nonlinear Theory of Elastic Shells*, 2nd edn, Cambridge University Press.

Love, A.E.H. (1944) *The Mathematical Theory of Elasticity*, Dover, New York.

MacNeal, R.H. (1994) *Finite Elements: Their Design and Performance*, Mercel Dekker, Inc.

Malvern, L.E. (1969) *Introduction to the Mechanics of Continuous Medium*, Prentice Hall, Inc., NJ.

Mantyla, M. (1988) *An Introduction to Solid Modeling*, Computer Science Press, Helsinki.

Meck, H.R. (1980) An accurate polynomial displacement function for finite ring elements. *Computers & Structures*, **11**, 265–269.

Naghdi, P.M. (1972) The theory of plates and shells, in *Handbuch der Physik Band VIa/2*, Springer-Verlag, pp. 425–640.

Ogden, R.W. (1997) *Non-Linear Elastic Deformations*, Dover Publications, Inc., NY.

Palazotto, A.N. and Dennis, S.T. (1992) *Nonlinear analysis of shell structures*. AIAA Education series, Washington, DC, pp. 205–206.

Panadian, N. *et al.* (1989) Studies on performance of curved beam finite elements for analysis of thin arches. *Computers & Structures*, **31**, 997–1002.

Pawsey, S.E. and Clough, RW. (1971) Improved numerical integration of thick shell finite elements. *International Journal for Numerical Methods in Engineering*, **3**, 545–586.

Piegl, L. and Tiller, W. (1995) *The NURBS Book*, Springer-Verlag, Berlin.

Pietraszkiewicz, W. (1979) *Finite Rotations and Lagrangian Description in the Nonlinear Theory of Shells*, PWN, Warszawa=Poznan.

Prathap, G. (1985) The curved beam/deep arch finite ring element revisited. *International Journal for Numerical Methods in Engineering*, **21**, 389–407.

Prathap, G. and Shashirekha, B.R. (1993) Variationally correct assumed strain field for the simple Curved beam Element. *International Journal for Numerical Methods in Engineering*, **37**, 989–1007.

Prenter, P.M. (1975) *Splines and Variational Methods*, John Wiley & Sons.

Raveendranath, P. *et al.* (1999) A two-noded locking-free shear flexible curved beam element. *International Journal for Numerical Methods in Engineering*, **44**, 265–280.

Ray, D. (1999) Beam flexure and plane stress (unpublished)

Ray, D. (2003) C-type method of unified CAMG & FEA. Part I: Beam and arch mega-elements – 3D linear and 2D non-linear. *International Journal for Numerical Methods in Engineering*, **58**, 1297–1320.

Ray, D. (2004) c-Type Method of Unified CAMG & FEA: Part II: Beam & Arch Mega-Elements – True 3D Curved, and Geometrically Nonlinear: An Exact Total & Incremental Curvilinear Formulation. (unpublished)

Ray, D. (2004) c-Type Method of Unified CAMG & FEA: Geometrically Nonlinear Curved Beams, Plates & Shells, Indian Association for Computational Mechanics. International Congress on Computational Mechanics and Simulation. (ICCMS-04), December, IIT-Kanpur.

Ray, D. (2004) Part III: On the Symmetry of the Tangent Operator of Nonlinear Beams: An Exact Curvilinear Formulation, (unpublished)

Ray, D. (2007) Optimal Nonlinear Beam, Plate & Shell Elements & c-type FEM. 9th U.S. Congress on Computational Mechanics, San Francisco.

Ray, D. (2008) Largest Geometrically Exact Nonlinear Beam, Plate & Shell Elements & c-type FEM. Proceedings of 6th International Conference, IASS-IACM 2008 – 6th ICCSSS, Cornell Univ., NY.

Reddy, J.N. and Liu, C.F. (1985) A higher order shear deformation theory of laminated elastic shells. *International Journal of Engineering Science*, **23**(3), 319–330

Reissner, E. (1960) On some problems in shell theory. Proceedings of First Symposium Naval Structural Mech., pp. 74–114.

Reissner, E. (1972) On one-dimensional finite strain beam theory: The plane problem. *Journal of Applied Mathematics and Physics*, **23**, 795–804.

Reissner, E. (1973) On one-dimensional large displacement finite strain beam theory. *Studies in Applied Mathematics*, **52**, 87–95.

Reissner, E. (1981) On finite deformations of space-curved beams. *Journal of Applied Mathematics and Physics*, **32**, 734–744.

Reus, J. (1992) Mechanical Deformations of Hyperpatch Solids. Second. International Conference On Computer Graphics and Visual Technology, December, pp. 147–158.

Schoenberg, I.J. (1946) Contributions to the problem of approximation of equidistant data by analytic functions. *Quarterly Applied Mathematics*, **4**, 45–99.

Shi, G. and Voyiadjis, G.Z. (1991) Simple and efficient shear flexible Two-node arch/beam and four-node cylindrical shell/plate elements. *International Journal for Numerical Methods in Engineering*, **31**, 759–776.

Simmonds, J.G. and Danielson, D.A. (1970) Nonlinear shell theory with finite rotation vector. *Proceedings of Koninklijke Nederlandse Akademie, Wetenschappen, B*, **73**, 460–478.

Simo, JC. (1985) A finite strain beam formulation. The three-dimensional dynamic problem. *Part I. Computer Methods in Applied Mechanics and Engineering*, **49**, 55–70.

Simo, J.C. and Vu-Quoc, L. (1986) A three-dimensional finite strain rod model. Part II: Computational Aspects. *Computer Methods in Applied Mechanics and Engineering*, **58**(1), 79–116.

Simo, J.C. *et al.* (1990) On a stress resultant geometrically exact shell model. Part III: Computational aspects of the nonlinear theory. *Computer Methods in Applied Mechanics and Engineering*, **79**, 21–70.

Simo, J.C. (1992) The (symmetric) Hessian for geometrically nonlinear models in solid mechanics: Intrinsic definition and geometric interpretation. *Computer Methods in Applied Mechanics and Engineering*, **58**(1), 189–200.

Spring, KW. (1986) Euler parameters and the use of quaternion algebra in the manipulation of finite rotations: a review. *Mechanism and Machine Theory*, **21**(5), 365–373.

Stolarski, H. and Belytschko, T. (1982) Membrane locking and reduced integration for curved elements. *Journal of Applied Mechanics*, **49**, 172–178

Strang, G. and Fix, G. (1973) *An Analysis of the Finite Element Method*, Prentice Hall.

Szabo, B. and Babuska, I. (1991) *Finite Element Analysis*, John Wiley & Sons, Inc.

Taylor, RL, *et al.* (1976) A nonconforming element for stress analysis. *International Journal for Numerical Methods in Engineering*, **10**, 1211–1219.

Taylor, R.L. and Ray, D. (2005) Rods – Some Developments for Large Displacements. 5th International Conference on Computation of Shell and Spatial Structures, Salzburg, Austria.

Tessler, A. and Spiridigliozzi, L. (1986) Curved beam elements with penalty relaxation. *International Journal for Numerical Methods in Engineering*, **23**, 2245–2262.

Timoshenko, S and Goodier, J, (1951) *Theory of Elasticity*, 2nd Edition, McGraw-Hill, Inc.

Turner, M.J., Clough, R.W., Martin, H.C., and Topp, LJ. (1956) Stiffness & deflection analysis of complex structures. *Journal of Aerospace Science*, **23**, 805–823.

Vainberg, M.M. (1964) *Variational Methods for the Study of Nonlinear Operators*, Holden-Day. Inc.

Washizu, K. (1964) Some considerations on a naturally curved and twisted slender beam. *Journal of Mathematics and Physics*, **43**(2), 111–116.

Wempner, G. (1989) Mechanics and finite elements of shells. *Applied Mechanics Reviews*, **42**, 129–142.

Wilson, E. *et al.* (1973) Incompatible displacement models, in *Numerical and Computer Methods in Structural Mechanics* (eds S.T. Fenves *et al.*), Academic Press, pp. 43–57.

Yamaguchi, F. (1988), *Curves and Surfaces in Computer Aided Geometric Design*, Springer-Verlag, Berlin.

Ziegler, H. (1977) *Principles of Structural Stability*, 2nd edn, Berkhauser, Basel.

Zienkiewicz, O.C, Taylor, R.L., and Too, J.M. (1971) Reduced integration techniques in general analysis of plates and shells. *International Journal for Numerical Methods in Engineering*, **3**, 275–290.

Zienkiewicz, O.C. and Taylor, R.L. (2000) *The Finite Element Method - Basic Formulattion and Fundamentals*, Butterworth.