CHAPTER **1**

The Heston Model for European Options

Abstract

Here, we present the European call price under the Heston model. We first present the model and then illustrate that the call price in the Heston model can be expressed as the sum of two terms that each contains an in-the-money probability but obtained under a separate measure, a result demonstrated by Bakshi and Madan (2000). We then show how to incorporate a continuous dividend yield and how to compute the price of a European put, and demonstrate that the numerical integration can be speed up by consolidating the two numerical integrals into a single integral. Finally, we derive the Black-Scholes model as a special case of the Heston model.

CIR process, European call, characteristic function, dividend yield, put-call parity, Black-Scholes

In this chapter, we present the European call price under the Heston model. We first present the model and then illustrate that the call price in the Heston model can be expressed as the sum of two terms that each contains an in-the-money probability, but obtained under a separate measure, a result demonstrated by Bakshi and Madan (2000). We then show how to incorporate a continuous dividend yield and how to compute the price of a European put, and demonstrate that the numerical integration can be speeded up by consolidating the two numerical integrals into a single integral. Finally, we derive the Black-Scholes model as a special case of the Heston model.

MODEL DYNAMICS

The Heston model assumes that the underlying stock price, S_t, follows a Black-Scholes–type stochastic process, but with a stochastic variance, v_t, that follows a Cox, Ingersoll, and Ross (1985) process. Hence, the Heston model is represented by the bivariate system of stochastic differential equations (SDEs),

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t}
\end{aligned}, \tag{1.1}
$$

where $E^{\mathbb{P}}\left[dW_{1,t}dW_{2,t}\right] = \rho dt$. We will sometimes drop the time index and write $S = S_t$, $v = v_t$, $W_1 = W_{1,t}$ and $W_2 = W_{2,t}$ for notational convenience. The parameters of the model are

μ the drift of the process for the stock;

$\kappa > 0$ the mean reversion speed for the variance;

$\theta > 0$ the mean reversion level for the variance;

$\sigma > 0$ the volatility of the variance;

$v_0 > 0$ the initial (time zero) level of the variance;

$\rho \in [-1, 1]$ the correlation between the two Brownian motion W_1 and W_2; and

λ the volatility risk parameter (discussed below).

We will see in Chapter 2 that these parameters affect the distribution of the terminal stock price in a manner that is intuitive. Some authors refer to v_0 as an unobserved initial state variable rather than a parameter. Because volatility cannot be observed, only estimated, and because v_0 represents this state variable at time zero, this characterization is sensible. For the purposes of estimation, however, many authors treat v_0 as a parameter like any other. Parameter estimation is covered in Chapter 6.

The stock price and variance follow the processes in Equation (1.1) under the historical measure $\mathbb{P}$, also called the physical measure. For pricing purposes, however, we need the processes for (S_t, v_t) under the risk-neutral measure $\mathbb{Q}$. The risk-neutral process is

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{v_t} S_t d\tilde{W}_{1,t} \\
dv_t &= \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t} d\tilde{W}_{2,t}
\end{aligned}
, \tag{1.2}
$$

where $E^{\mathbb{Q}}\left[d\tilde{W}_{1,t}d\tilde{W}_{2,t}\right] = \rho dt$. The risk-neutral parameters of the variance process are where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$.

Note that when $\lambda = 0$, we have $\kappa^* = \kappa$ and $\theta^* = \theta$ so that the parameters under the physical and risk-neutral measures are the same. Throughout this book, we set $\lambda = 0$, but this is not always needed. Indeed, λ is embedded in the risk-neutral parameters κ^* and θ^*. Hence, when we estimate the risk-neutral parameters to price options, we do not need to estimate λ. Estimation of λ is the subject of its own research, such as that by Bollerslev et al. (2011). For notational simplicity, throughout this book we will drop the asterisk on the parameters and the tilde on the Brownian motion when it is obvious that we are dealing with the risk-neutral measure.

For details on how the risk-neutral process is constructed, see Rouah (2013).

THE HESTON EUROPEAN CALL PRICE

In this section, we show that the call price in the Heston model can be expressed in a manner that resembles the call price in the Black-Scholes model, which we present in Equation (1.16). Authors sometimes refer to this characterization of the call price as "Black-Scholes–like" or "à la Black-Scholes." The time-t price of a European call on a non-dividend-paying stock with spot price S_t, when the strike is K and the time

to maturity is $\tau = T - t$, is the discounted expected value of the payoff under the risk-neutral measure $\mathbb{Q}$

$$
\begin{aligned}
C(K) &= e^{-r\tau} E^{\mathbb{Q}}[(S_T - K)^+] \\
&= S_t P_1 - K e^{-r\tau} P_2.
\end{aligned}
\tag{1.3}
$$

The last line of (1.3) is the "Black-Scholes–like" call price formula, with P_1 replacing $\Phi(d_1)$, and P_2 replacing $\Phi(d_2)$ in the Black-Scholes call price (1.16).

By introducing two measures, $\mathbb{Q}$ and $\mathbb{Q}^S$, the European call price of Equation (1.3) can be written

$$
C(K) = S_t \mathbb{Q}^S(S_T > K) - K e^{-r\tau} \mathbb{Q}(S_T > K).
\tag{1.4}
$$

The measure $\mathbb{Q}$ uses the bond B_t as the numeraire, while the measure $\mathbb{Q}^S$ uses the stock price S_t. Bakshi and Madan (2000) present a derivation of the call price expressed as (1.4), but under a general setup. As shown in their paper, the change of measure that leads to (1.4) is valid for a wide range of models, including the Black-Scholes and Heston models. We will see later in this chapter that when S_T follows the lognormal distribution specified in the Black-Scholes model, then $\mathbb{Q}^S(S_T > K) = \Phi(d_1)$ and $\mathbb{Q}(S_T > K) = \Phi(d_2)$. Hence, the characteristic function approach to pricing options, pioneered by Heston (1993), applies to the Black-Scholes model also.

In the Heston model, it can be shown that P_1 and P_2 in (1.3) can be written

$$
P_j = \Pr_j(\ln S_T > \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right] d\phi,
\tag{1.5}
$$

where $i = \sqrt{-1}$ is the imaginary unit, $\Pr_1 = \mathbb{Q}^S$, $\Pr_2 = \mathbb{Q}$, and $f_j(\phi; x, v)$ are the characteristic functions for the logarithm of the terminal stock price $x_T = \ln S_T$. Heston (1993) postulates that these characteristic functions are of the log linear form,

$$
f_j(\phi; x_t, v_t) = \exp\left(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t\right),
\tag{1.6}
$$

where C_j and D_j are constant coefficients and $\tau = T - t$ is the time to maturity.

The coefficients can be shown to be

$$
D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)
\tag{1.7}
$$

and

$$
C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right],
\tag{1.8}
$$

where $a = \kappa\theta$,

$$
d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)},
$$
$$
g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j},
$$

and with $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $b_1 = \kappa + \lambda - \rho\sigma$, and $b_2 = \kappa + \lambda$. See Rouah (2013) for a complete derivation of the coefficients (1.7) and (1.8).

We use two functions to implement the Heston call price in VBA using the trapezoidal rule, HestonProb and HestonPriceTrapz. The first function calculates the characteristic functions and returns the real part of the integrand. The functions allow to price calls or puts, and allow for a dividend yield, as explained in the following section. The following code snippet is taken from the Excel file Ch1Trapz.xls. To conserve space, parts of the functions have been omitted.

```
Function HestonProb(phi As Double,...) As Double
' C = (r-q)*phi*tau*i + a/sigma^2*((b - rho*sigma*i*phi + d)...
bigC = cAdd(Term3, term4)
' D = (b - rho*sigma*i*phi + d)/sigma^2 * ((1-exp(d*tau))...
bigD = cProd(Term5, Term6)
' The characteristic function.
' f = cExp(C + D*v0 + phi*x*i)
f = cExp(cAdd(cAdd(bigC, dv0), phixi))
' Return the real part of the integrand.
' HestonProb = real(Exp(-i * phi * Log(K)) * f / i / phi)
philogKi = cProd(Complex(-Log(K) * phi, 0), i)
E = cDiv(e2, cProd(i, Complex(phi, 0)))
HestonProb = cReal(E)
End Function
```

The second function calculates the price of a European call $C(K)$, or European put $P(K)$, by put-call parity in Equation (1.13). The function calls the HestonProb function at every point of the integration grid and uses the trapezoidal rule for integration when all the integration points have been calculated.

```
Function HestonPriceTrapz(PutCall As String,...) As Double
' Integration increment and integration grid
dphi = (Uphi - Lphi) / (N - 1)
phi(1) = Lphi
For j = 2 To N
    phi(j) = phi(j - 1) + dphi
Next j
' Weights for trapezoidal rule
W(1) = 0.5 * dphi
W(N) = 0.5 * dphi
For j = 2 To N - 1
    W(j) = dphi
Next j
' Build the integrands for P1 and P2;
```

```
For j = 1 To N
    int1(j) = HestonProb(phi(j), ..., 1) * W(j)
    int2(j) = HestonProb(phi(j), ..., 2) * W(j)
Next j
' The integrals and the probabilities
I1 = Sum(int1)
I2 = Sum(int2)
P1 = 0.5 + 1 / pi * I1
P2 = 0.5 + 1 / pi * I2
' The call price and put price by put call parity
HestonC = S * Exp(-q * T) * P1 - K * Exp(-r * T) * P2
HestonP = HestonC - S * Exp(-q * T) + K * Exp(-r * T)
End Function
```

Pricing European calls and puts is straightforward. For example, the price of a six-month European put with strike $K = 100$ on a dividend-paying stock with spot price $S = 100$ and yield $q = 0.02$, when the risk-free rate is $r = 0.03$ and using the parameters $\kappa = 5$, $\sigma = 0.5$, $\rho = -0.8$, $\theta = v_0 = 0.05$, and $\lambda = 0$, along with the integration grid $\phi \in [1e^{-8}, 100]$ with 500 points is 5.7589. The price of the call with identical features is 6.2527. This is illustrated in Figure 1.1, which is a screen shot of the Excel file Ch1Trapz.xls.

If there is no dividend yield so that $q = 0$, then as expected, the put price decreases to 5.3789, and the call price increases to 6.8677.

	B	C		E	F
	Heston Price Using the Trapezoidal Integration Rule				
	Option Features			**Option Flavor**	**Heston Price**
	Spot Price (S)	100.0000		Call (C)	6.2527
	Strike Price (K)	100.0000		Put (P)	5.7589
	Time to Maturity (τ)	0.5000			
	Risk free rate (r)	0.0300		**Integration Grid**	
	Dividend (q)	0.0200		Lower Limit	1.00E-08
				Upper Limit	100.00
	Heston Parameters			Number of abscissas	500
	kappa (κ)	5.00			
	theta (θ)	0.05			
	sigma (σ)	0.50			
	v0	0.05			
	rho (ρ)	-0.80			
	lambda (λ)	0.00			

FIGURE 1.1 Heston Price Using the Trapezoidal Rule

The Excel files Ch1GLa.xls and Ch1GLe_GLo.xls implement the Heston price using Gauss-Laguerre integration, and Gauss-Legendre and Gauss-Lobatto integration, respectively, each using 32 integration points. These rules will be covered in Chapter 5. The pricing functions in these files are nearly identical to those for the trapezoidal rule, except that the integration points and weights are hard-coded outside the function. The functions also allow for the "Little Trap" formulation of the characteristic function of Albrecher et al. (2007), which we discuss in the following chapter.

The VBA function HestonProb from the Excel file Ch1GLa.xls illustrates this point.

```
Function HestonProb(...,trap as integer) As Double
If trap = 0 Then
    ' Original Heston formulation of the c.f.
    bigG = cDiv(Num2, Den2)
    bigC = cAdd(Term3, term4)
    bigD = cProd(Term5, Term6)
 ElseIf trap = 1 Then
    ' Little Trap formulation of the c.f.
    bigG = cDiv(Num2, Den2)
    bigC = cAdd(Term3, term4)
    bigD = cProd(Term5, Term6)
End If
' The characteristic function.
f = cExp(cAdd(cAdd(bigC, Dv0), phixi))
' Return the real part of the integrand.
HestonProb = cReal(E)
End Function
```

The function HestonPriceGaussLaguerre uses 32-point Gauss-Laguerre integration to produce the call or put price. Note that the integration points (x) and weights (W) are passed as arguments to the function.

```
Function HestonPriceGaussLaguerre(..., x, W)
' Build the integrands for P1 and P2;
For j = 1 To N
    phi = x(j)
    weight = W(j)
    int1(j) = HestonProb(phi, ...) * weight
    int2(j) = HestonProb(phi, ...) * weight
Next j
End Function
```

	A	B	C	D	E	F	G	H
1								
2		Heston Price Using 32-Point Gauss-Laguerre Integration						
3								
4		Option Features			Option Flavor	Heston Price		
5		Spot Price (S)	100.0000		Call (C)	6.2524		
6		Strike Price (K)	100.0000		Put (P)	5.7586		
7		Time to Maturity (τ)	0.5000					
8		Risk free rate (r)	0.0300		Abscissas (x) and Weights (w)			
9		Dividend (q)	0.0200		32-point Gauss-Laguerre Numerical Integration			
10					Number	Abscissas	Weights	
11					1	0.04449	0.11419	
12		Heston Parameters			2	0.23453	0.26607	
13		kappa (κ)	5.00		3	0.57688	0.41879	
14		theta (θ)	0.05		4	1.07245	0.57253	
15		sigma (σ)	0.50		5	1.72241	0.72765	
16		v0	0.05		6	2.52834	0.88454	
17		rho (ρ)	-0.80		7	3.49221	1.04362	
18		lambda (λ)	0.00		8	4.61646	1.20535	
19		Trap (0=Heston, 1=Albrecher)	1		9	5.90396	1.37022	
20					10	7.35813	1.53878	
21					11	8.98295	1.71162	
22					12	10.78300	1.88948	
23					13	12.76370	2.07298	
24					14	14.93136	2.26328	
25					15	17.29141	2.46221	
26					16	19.85856	2.66004	

FIGURE 1.2 Heston Price Using Gauss-Laguerre Integration

Figure 1.2 shows that the Heston prices obtained using 32-point Gauss-Laguerre integration are very close to those obtained with the trapezoidal rule in Figure 1.1.

Some applications require VBA code for the Heston characteristic function. The HestonProb function can be modified to return the characteristic function itself, instead of the integrand. In certain instances, the integrand for P_j,

$$\text{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right], \tag{1.9}$$

is well behaved in that it poses no difficulties in numerical integration. This corresponds to an integrand that does not oscillate much, that dampens quickly so that a large upper limit in the numerical integration is not required, and that does not contain portions that are excessively steep. In other instances, the integrand is not well behaved, and numerical integration loses precision. To illustrate, in the Excel file Ch1IntegrandPlot.xls, we plot the first integrand ($j = 1$) in Equation (1.9). This plot appears in Figure 1.3. The integrand has a discontinuity at $\phi = 0$, but this does not show up in the figure.

The plot indicates an integrand that has a fair amount of oscillation, especially at short maturities, and that is steep near the origin. In Chapter 2, we investigate other problems that can arise with the Heston integrand.
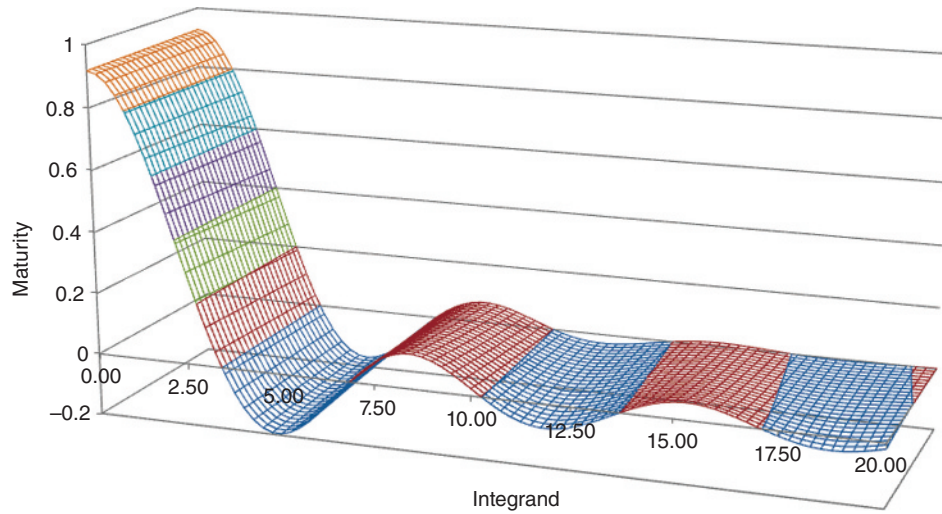
FIGURE 1.3   Heston Integrand and Maturity

DIVIDEND YIELD AND THE PUT PRICE

It is straightforward to include dividends into the model if it can be assumed that the dividend payment is a continuous yield, q. In that case, r is replaced by $r - q$ in Equation for the stock price process

$$dS_t = (r - q)S_t dt + \sqrt{v_t} S_t d\tilde{W}_{1,t}. \tag{1.10}$$

The solution for C_j becomes

$$C_j = (r - q)i\phi\tau + \frac{\kappa\theta}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right]. \tag{1.11}$$

To obtain the price $P(K)$ of a European put, first obtain the price $C(K)$ of a European call, using a slight modification of Equation (1.3) to include the term $e^{-q\tau}$ for the dividend yield, as explained by Whaley (2006),

$$C(K) = S_t e^{-q\tau} P_1 - K e^{-r\tau} P_2. \tag{1.12}$$

The put price is found by put-call parity

$$P(K) = C(K) + K e^{-r\tau} - S_t e^{-q\tau}. \tag{1.13}$$

CONSOLIDATING THE INTEGRALS

It is possible to regroup the integrals for the probabilities P_1 and P_2 into a single integral, which will speed up the numerical integration required in the call price calculation. Substituting the expressions for P_j into the call price in (1.12) and rearranging produces

$$
\begin{aligned}
C(K) = {} & \frac{1}{2}S_t e^{-q\tau} - \frac{1}{2}Ke^{-r\tau} \\
& + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K}}{i\phi}\left(S_t e^{-q\tau}f_1(\phi;x,v) - Ke^{-r\tau}f_2(\phi;x,v)\right)\right]d\phi.
\end{aligned}
\tag{1.14}
$$

The advantage of this consolidation is that only a single numerical integration is required instead of two, so the computation time will be reduced by almost one-half. The put price can be obtained by using put-call parity, with the call price calculated using (1.14).

The integrand of the consolidated form is in the function HestonProbConsol in the Excel file Ch1ConsolCF.xls.

```
Function HestonProbConsol(phi As Double...) As Double
' Return the real part of the integrand.
expphilogK = cExp(cProd(cProd(im, Complex(phi, 0)) ...
HestonProbConsol = cReal(cProd(Term1, Term2))
End Function
```

This function is then fed into the HestonPriceConsol function, which calculates the call price in accordance with Equation (1.14). The function uses the trapezoidal rule for numerical integration.

```
Function HestonPriceConsol(PutCall As String...) As Double
' Build the consolidated integrand
For j = 1 To N
    inte(j) = HestonProbConsol(phi(j),...) * W(j)
Next j
integral = Sum(inte)
' The call price and put price by put call parity
HestonC = 0.5 * S * Exp(-q * T) - 0.5 * K * Exp(-r * T)  ...
HestonP = HestonC - S * Exp(-q * T) + K * Exp(-r * T)
' Output the option price
```

```
If PutCall = "C" Then
    HestonPriceConsol = HestonC
Else
    HestonPriceConsol = HestonP
End If
End Function
```

The consolidated form produces exactly the same prices for the call and the put, but with roughly one-half of the computation time.

BLACK-SCHOLES AS A SPECIAL CASE

With a little manipulation, it is straightforward to show that the Black-Scholes model is nested inside the Heston model. The Black-Scholes model assumes the following dynamics for the underlying price S_t under the risk-neutral measure $\mathbb{Q}$

$$dS_t = rS_t + \sigma_{BS}S_t d\tilde{W}_t. \tag{1.15}$$

It is shown in many textbooks, such as that by Hull (2011) or Chriss (1996), that (1.15) can be solved for the spot price S_t. The result is that, at time t, the natural logarithm of the stock price at expiry $\ln S_T$ is distributed as a normal random variable with mean $\ln S_t + (r - \frac{1}{2}\sigma_{BS}^2)\tau$ and variance $\sigma_{BS}^2\tau$, where $\tau = T - t$ is the time to expiry. Consequently, the Black-Scholes call price is given by

$$C_{BS}(K) = S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \tag{1.16}$$

with

$$
\begin{aligned}
d_1 &= \frac{\ln(S_t/K) + (r + \sigma_{BS}^2/2)\tau}{\sigma_{BS}\sqrt{\tau}}, \\
d_2 &= \frac{\ln(S_t/K) + (r - \sigma_{BS}^2/2)\tau}{\sigma_{BS}\sqrt{\tau}} = d_1 - \sigma_{BS}\sqrt{\tau},
\end{aligned}
\tag{1.17}
$$

where $\Phi(x)$ is the standard normal cumulative distribution function. The volatility σ_{BS} is assumed to be constant.

If we set $\sigma = 0$, the volatility of variance parameter in the Heston model, then the Brownian component of the variance process in Equation (1.1) drops out. This will produce volatility that is time varying but deterministic. If we further set $\theta = v_0$, then this will produce volatility that is constant. Hence, setting $\sigma = 0$ and $\theta = v_0$ in the Heston model leads us to expect the same price as that produced by the Black-Scholes model, with $\sigma_{BS} = \sqrt{v_0}$ as the Black-Scholes implied volatility.

To implement the Black-Scholes model as a special case of the Heston model, we cannot simply substitute $\sigma = 0$ into the pricing functions, because that will lead to division by zero in the expressions for $C_j(\tau, \phi)$ and $D_j(\tau, \phi)$. Instead, we must use

$$D_j(\tau, \phi) = \frac{\left(u_j i\phi - \frac{1}{2}\phi^2\right)\left(1 - e^{-b_j\tau}\right)}{b_j} \tag{1.18}$$

and

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a\left(u_j i\phi - \frac{1}{2}\phi^2\right)}{b_j}\left[\tau - \frac{\left(1 - e^{-b_j\tau}\right)}{b_j}\right]. \tag{1.19}$$

See Rouah (2013) for a detailed derivation.

The function HestonProbZeroSigma is used to implement the Black-Scholes model as a special case of the Heston model (when $\sigma = 0$). This function is in the Excel file Ch1BSHeston.xls. To conserve space, only the crucial portions of the function are presented.

```
Function HestonProbZeroSigma(phi As Double,...) As Double
' D = (u*i*phi - phi^2/2)*(1-exp(-b*tau))/b;
bigD = cDiv(cProd(Term1, Term2), Term3)
'C = (r-q)*i*phi*tau + a*(u*i*phi-0.5*phi^2)/b ...
bigC = cAdd(Term1, Term2)
' The characteristic function.
' f = exp(C + D*theta + i*phi*x)
f = cExp(cAdd(cAdd(bigC, Dtheta), iphix))
' Return the real part of the integrand.
HestonProbZeroSigma = cReal(cDiv(Num1, Den1))
End Function
```

The function HestonPriceZeroSigma uses the trapezoidal rule to obtain the price when $\sigma = 0$. Again, only the relevant parts of the code are presented.

```
Function HestonPriceZeroSigma(...) As Double
' Integrands for probabilities P1 and P2
For j = 1 To N
    P1_int(j) = HestonProbZeroSigma(phi(j), ..., 1) * W(j)
    P2_int(j) = HestonProbZeroSigma(phi(j), ..., 2) * W(j)
Next j
End Function
```

With the settings $\tau = 0.5$, $S = K = 100$, $q = 0.02$, $r = 0.03$, $\kappa = 5$, $v_0 = \theta = 0.05$, and $\lambda = 0$, the Heston model and Black-Scholes model with $\sigma_{BS} = \sqrt{v_0}$ each return 6.4730 for the price of the call and 5.9792 for the price of the put.

CONCLUSION

The Heston model has become the most popular stochastic volatility model for pricing equity options. This is in part due to the fact that the call price in the model is available in closed form. Some authors refer to the call price as being in "semiclosed" form because of the numerical integration required to obtain P_1 and P_2. But the Black-Scholes model also requires numerical integration, to obtain $\Phi(d_1)$ and $\Phi(d_2)$. In this sense, the Heston model produces call prices that are no less closed than those produced by the Black-Scholes model. The difference is that programming languages often have built-in routines for calculating the standard normal cumulative distribution function, $\Phi(\cdot)$ (usually by employing a polynomial approximation), whereas the Heston probabilities are not built in and must be obtained using numerical integration. In the next chapter, we investigate some of the problems that can arise in numerical integration when the integrand

$$\text{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right]$$

is not well behaved. We encountered an example of such an integrand in Figure 1.3.