# 1

# PRELIMINARIES OF FUZZY SET THEORY



This chapter presents the notations, definitions of fuzzy numbers (namely triangular, trapezoidal, Gaussian, double parametric form), type of differentiability, theorems/lemma related to fuzzy/fuzzy fractional differential equations, and fuzzy/interval arithmetic, which are relevant to the current investigation. Several excellent books related to this have also been written by different authors representing the scope and various aspects of fuzzy set theory such as in Zimmermann (2001), Jaulin et al. (2001), Ross (2004), Hanss (2005), Moore (1966), and Chakraverty (2014). These books also give an extensive review on fuzzy set theory and its applications, which may help the reader in understating the basic concepts of fuzzy set theory and its application.

**Definition 1.1 Interval** An interval $\tilde{x}$ is denoted by $[\underline{x}, \overline{x}]$ on the set of real numbers $R$ given by

$$\tilde{x} = [\underline{x}, \overline{x}] = \{x \in R : \underline{x} \le x \le \overline{x}\}. \tag{1.1}$$

We have only considered closed intervals throughout this thesis, although there exist various other types of intervals such as open and half-open intervals. $\underline{x}$ and $\overline{x}$ are known as the left and right end points, respectively, of the interval $\tilde{x}$ in Eq. (1.1).

*Fuzzy Arbitrary Order System: Fuzzy Fractional Differential Equations and Applications*, First Edition.
Snehashish Chakraverty, Smita Tapaswini, and Diptiranjan Behera.

Let us now consider two arbitrary intervals $\widetilde{x} = [\underline{x}, \overline{x}]$ and $\widetilde{y} = [\underline{y}, \overline{y}]$. These two intervals are said to be equal if they are in the same set. Mathematically, it only happens when corresponding end points are equal. Hence, one may write

$$\widetilde{x} = \widetilde{y} \quad \text{if and only if} \quad \underline{x} = \underline{y} \quad \text{and} \quad \overline{x} = \overline{y}. \tag{1.2}$$

For the given two arbitrary intervals $\widetilde{x} = [\underline{x}, \overline{x}]$ and $\widetilde{y} = [\underline{y}, \overline{y}]$, interval arithmetic operations such as addition (+), subtraction (−), multiplication (×), and division (/) are defined as follows:

$$\widetilde{x} + \widetilde{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \tag{1.3}$$

$$\widetilde{x} - \widetilde{y} = [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \tag{1.4}$$

$$\widetilde{x} \times \widetilde{y} = [\min S, \max S], \quad \text{where} \ S = \{\underline{x} \times \underline{y}, \underline{x} \times \overline{y}, \overline{x} \times \underline{y}, \overline{x} \times \overline{y}\}, \tag{1.5}$$

and

$$\widetilde{x} / \widetilde{y} = [\underline{x}, \overline{x}] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right] \quad \text{if} \ 0 \notin \widetilde{y}. \tag{1.6}$$

Now if $k$ is a real number and $\widetilde{x} = [\underline{x}, \overline{x}]$ is an interval, then the multiplication of them is given by

$$k\widetilde{x} = \begin{cases} [k\overline{x}, k\underline{x}], & k < 0, \\ [k\underline{x}, k\overline{x}], & k \geq 0. \end{cases} \tag{1.7}$$

**Definition 1.2 Fuzzy Number** A fuzzy number $\widetilde{U}$ is convex, normalized fuzzy set $\widetilde{U}$ of the real line $R$ such that

$$\{\mu_{\widetilde{U}}(x) : R \to [0, 1], \quad \forall x \in R\},$$

where, $\mu_{\widetilde{U}}$ is called the membership function of the fuzzy set, and it is piecewise continuous. There exists a variety of fuzzy numbers. But in this study, we have used only the triangular, trapezoidal, and Gaussian fuzzy numbers. So, we define these three fuzzy numbers as follows.

**Definition 1.3 Triangular Fuzzy Number (TFN)** A triangular fuzzy number (TFN) $\widetilde{U}$ is a convex, normalized fuzzy set $\widetilde{U}$ of the real line $R$ such that

1. There exists exactly one $x_0 \in R$ with $\mu_{\widetilde{U}}(x_0) = 1$ ($x_0$ is called the mean value of $\widetilde{U}$), where $\mu_{\widetilde{U}}$ is called the membership function of the fuzzy set.
2. $\mu_{\widetilde{U}}(x)$ is piecewise continuous.

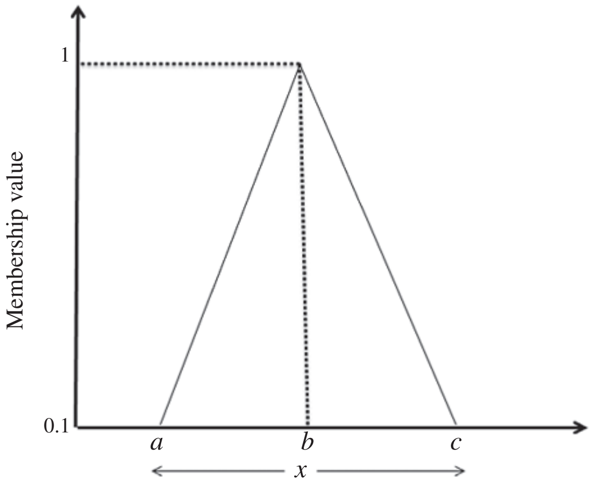


**Figure 1.1** Triangular fuzzy number

Let us consider an arbitrary TFN $\widetilde{U} = (a, b, c)$. The membership function $\mu_{\widetilde{U}}$ of $\widetilde{U}$ is defined as follows:

$$\mu_{\widetilde{U}}(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x \geq c. \end{cases}$$

The TFN $\widetilde{U} = (a, b, c)$ can be represented by an ordered pair of functions through $r$-cut approach, namely $[\underline{u}(r),\ \overline{u}(r)] = [(b-a)r + a, -(c-b)r + c]$ where, $r \in [0,\ 1]$ (Fig. 1.1).

**Definition 1.4 Trapezoidal Fuzzy Number (TrFN)** We consider an arbitrary trapezoidal fuzzy number (TrFN) $\widetilde{U} = (a, b, c, d)$. The membership function $\mu_{\widetilde{U}}$ of $\widetilde{U}$ is given as

$$\mu_{\widetilde{U}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{d-x}{d-c}, & c \leq x \leq d \\ 0, & x \geq d. \end{cases}$$

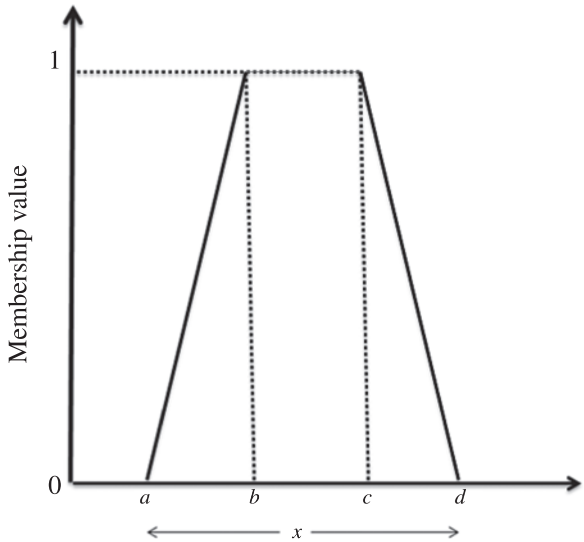


**Figure 1.2** Trapezoidal fuzzy number

The TrFN $\widetilde{U} = (a,\ b,\ c, d)$ can be represented with an ordered pair of functions through $r$-cut approach that is $[\underline{u}(r),\ \overline{u}(r)] = [(b-a)r + a,\ -(d-c)r + d]$ where, $r \in [0,\ 1]$ (Fig. 1.2).

**Definition 1.5 Gaussian Fuzzy Number (GFN)** Let us now define an arbitrary asymmetrical Gaussian fuzzy number, $\widetilde{U} = (\delta, \sigma_l, \sigma_r)$. The membership function $\mu_{\widetilde{U}}$ of $\widetilde{U}$ will be as follows:

$$\mu_U(x) = \begin{cases} \exp\left[\frac{-(x-\delta)^2}{2\sigma_l^2}\right] & \text{for } x \le \delta \\ \exp\left[\frac{-(x-\delta)^2}{2\sigma_r^2}\right] & \text{for } x \ge \delta \end{cases} \qquad \forall x \in R,$$

where, the modal value is denoted as $\delta$ and $\sigma_l, \sigma_r$ denote the left-hand and right-hand spreads (fuzziness), respectively, corresponding to the Gaussian distribution. For symmetric Gaussian fuzzy number, the left-hand and right-hand spreads are equal, that is, $\sigma_l = \sigma_r = \sigma$. So the symmetric Gaussian fuzzy number may be written as $\widetilde{U} = (\delta, \sigma, \sigma)$ and the corresponding membership function may be defined as $\mu_{\widetilde{U}}(x) = \exp\{-\beta(x-\delta)^2\}\ \ \forall x \in R$ where $\lambda = 1/2\sigma^2$. The symmetric Gaussian fuzzy number (Fig. 1.3) in parametric can be represented as

$$\widetilde{U} = [\underline{u}(r),\ \overline{u}(r)] = \left[\delta - \sqrt{-\frac{(\log_e r)}{\lambda}},\ \delta + \sqrt{-\frac{(\log_e r)}{\lambda}}\right], \quad \text{where } r \in [0,\ 1].$$

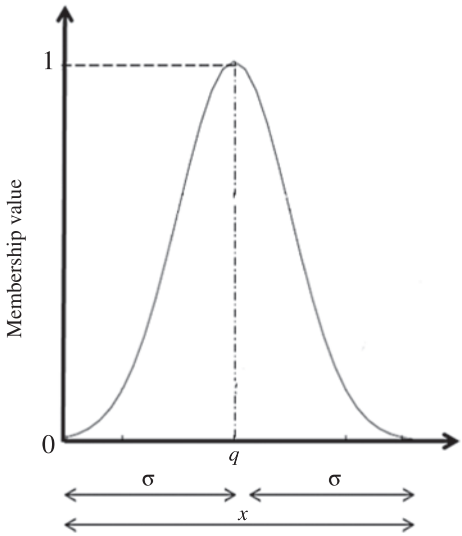


**Figure 1.3** Gaussian fuzzy number

For all the aforementioned type of fuzzy numbers, the lower and upper bounds of the fuzzy numbers satisfy the following requirements:

(i) $\underline{u}(r)$ is a bounded left-continuous nondecreasing function over $[0, 1]$;
(ii) $\overline{u}(r)$ is a bounded right-continuous nonincreasing function over $[0, 1]$;
(iii) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

**Definition 1.6 Double Parametric Form of Fuzzy Number** Using the $r$-cut approach as discussed in Definitions 1.2–1.5 for all the fuzzy numbers, we have $\widetilde{U} = [\underline{u}(r), \overline{u}(r)]$. Now one may write this as crisp number with double parametric form as $\widetilde{U}(r,\ \beta) = \beta(\overline{u}(r) - \underline{u}(r)) + \underline{u}(r)$ where $r$ and $\beta \in [0, 1]$. To obtain the lower and upper bounds of the solution in single parametric form, we may use $\beta = 0$ and 1, respectively. This may be represented as $\widetilde{U}(r,\ 0) = \underline{u}(r)$ and $\widetilde{U}(r,\ 1) = \overline{u}(r)$.

**Definition 1.7 Fuzzy Center** Fuzzy center of an arbitrary fuzzy number $\widetilde{u} = [\underline{u}(r), \overline{u}(r)]$ is defined as $\widetilde{u}^{c} = \frac{\underline{u}(r)+\overline{u}(r)}{2}$, for all $0 \leq r \leq 1$.

**Definition 1.8 Fuzzy Radius** Fuzzy radius of an arbitrary fuzzy number $\widetilde{u} = [\underline{u}(r), \overline{u}(r)]$ is defined as $\Delta\widetilde{u} = \frac{\overline{u}(r)-\underline{u}(r)}{2}$ for all $0 \leq r \leq 1$.

**Definition 1.9 Fuzzy Width** Fuzzy space or width of an arbitrary fuzzy number $\widetilde{u} = [\underline{u}(r), \overline{u}(r)]$ is defined as $|\underline{u}(r) - \overline{u}(r)|$, for all $0 \leq r \leq 1$.

**Definition 1.10 Fuzzy Arithmetic** For any two arbitrary fuzzy numbers $\widetilde{x} = [\underline{x}(r), \overline{x}(r)]$, $\widetilde{y} = [\underline{y}(r), \overline{y}(r)]$ and scalar $k$, the fuzzy arithmetic is similar to the interval arithmetic defined as follows:

(i) $\widetilde{x} = \widetilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$
(ii) $\widetilde{x} + \widetilde{y} = [\underline{x}(r) + \underline{y}(r),\ \overline{x}(r) + \overline{y}(r)]$
(iii) $\widetilde{x} \times \widetilde{y} = [\min(S), \max(S)]$ where

$$S = \{\underline{x}(r) \times \underline{y}(r), \underline{x}(r) \times \overline{y}(r), \overline{x}(r) \times \underline{y}(r), \overline{x}(r) \times \overline{y}(r)\}$$

(iv) $k\widetilde{x} = \begin{cases} [k\overline{x}(r), k\underline{x}(r)], & k < 0 \\ [k\underline{x}(r), k\overline{x}(r)], & k \geq 0 \end{cases}$
(v) $\frac{\widetilde{x}}{\widetilde{y}} = [\underline{x}(r), \overline{x}(r)] \times \left[\frac{1}{\overline{y}(r)}, \frac{1}{\underline{y}(r)}\right]$, where $0 \notin \widetilde{y}$, where $0 \notin \widetilde{y}$.

**Definition 1.11** Let $F : (a, b) \to R_F$ and $t_0 = (a, b)$ (Khastan et al., 2011; Chalco-Cano and Roman-Flores, 2008). $X$ is called differentiable at $t_0$, if there exists $F'(t_0) \in R_F$ such that

(i) for all $h > 0$ sufficiently close to 0, the Hukuhara difference $F(t_0 + h)\,\Theta\, F(t_0)$ and $F(t_0)\,\Theta\, F(t_0 - h)$ exists and (in metric $D$)

$$\lim_{h \to 0^+} \frac{F(t_0 + h)\Theta F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0)\Theta F(t_0 - h)}{h} = F'(t_0),$$

or

(ii) for all $h > 0$ sufficiently close to 0, the Hukuhara difference $F(t_0)\Theta F(t_0 + h)$ and $F(t_0 - h)\Theta F(t_0)$ exists and (in metric $D$)

$$\lim_{h \to 0^+} \frac{F(t_0)\Theta F(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{F(t_0 - h)\Theta F(t_0)}{-h} = F'(t_0)$$

Chalco-Cano and Roman-Flores (2008) used Definition 1.11 to obtain the following results.

**Theorem 1.1** Let $F : (a, b) \to R_F$ and denote $[F(t; r)] = [\underline{f}(t; r), \overline{f}(t; r)]$ for each $r \in [0,\ 1]$.

(i) If $F$ is differentiable of the first type (I), then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions, and we have $[F'(t; r)] = [\underline{f}'(t; r), \overline{f}'(t; r)]$.
(ii) If $F$ is differentiable of the second type (II), then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions, and we have $[F'(t; r)] = [\overline{f}'(t; r), \underline{f}'(t; r)]$.

**Proof** The proof of the theorem is given in Chalco-Cano and Roman-Flores (2008).

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