

# 1 *Linear Equations and Mathematical Concepts*

<b>1.1 Solving Linear Equations</b>	<b>2</b>
<i>Example 1.1.1 Solving a Linear Equation</i>	3
<i>Example 1.1.2 Solving for y</i>	4
<i>Example 1.1.3 Simple Interest</i>	4
<i>Example 1.1.4 Investment</i>	4
<i>Example 1.1.5 Gasoline Prices</i>	5
<i>Example 1.1.6 Breaking a Habit</i>	5
<b>1.2 Equations of Lines and Their Graphs</b>	<b>7</b>
<i>Example 1.2.1 Ordered Pair Solutions</i>	8
<i>Example 1.2.2 Intercepts and Graph of a Line</i>	9
<i>Example 1.2.3 Slope-Intercept Form</i>	10
<i>Example 1.2.4 Point-Slope Form</i>	11
<i>Example 1.2.5 Temperature Conversion</i>	11
<i>Example 1.2.6 Salvage Value</i>	12
<i>Example 1.2.7 Parallel or Perpendicular Lines</i>	12
<b>1.3 Solving Systems of Linear Equations</b>	<b>15</b>
<i>Example 1.3.1 Graphical Solutions to a System of Linear Equations</i>	16
<i>Example 1.3.2 Substitution – Consistent System</i>	17
<i>Example 1.3.3 Substitution–Inconsistent System</i>	17
<i>Example 1.3.4 Elimination Method – Consistent System</i>	18
<i>Example 1.3.5 Market Equilibrium</i>	18
<b>1.4 The Numbers <math>\pi</math> and <math>e</math></b>	<b>21</b>
The Number $\pi$	21
<i>Example 1.4.1 A Biblical <math>\pi</math></i>	22
The Number $e$	22
<b>1.5 Exponential and Logarithmic Functions</b>	<b>24</b>
Exponential Functions	24
<i>Example 1.5.1 Using Laws of Exponents</i>	24
<i>Example 1.5.2 Simplifying Exponential Equations</i>	25
<i>Example 1.5.3 Simplifying Exponential Exponents</i>	26
Logarithmic Functions	26
<i>Example 1.5.4 Exponential and Logarithmic Forms</i>	27
<i>Example 1.5.5 Simplifying Logarithms</i>	28
<i>Example 1.5.6 Simplifying Logarithmic Expressions</i>	28
<i>Example 1.5.7 Base 10 Exponents</i>	29
<i>Example 1.5.8 Base <math>e</math> Exponents</i>	30
<b>1.6 Variation</b>	<b>32</b>
<i>Example 1.6.1 Summer Wages</i>	33
<i>Example 1.6.2 Hooke’s Law</i>	33

## 2 LINEAR EQUATIONS AND MATHEMATICAL CONCEPTS

<i>Example 1.6.3 Geometric Similarity</i>	34
<i>Example 1.6.4 Volume of a Sphere</i>	34
<i>Example 1.6.5 Seawater Density</i>	35
<i>Example 1.6.6 Illumination</i>	35
<i>Example 1.6.7 Beam Strength</i>	36
<b>1.7 Unit Conversions and Dimensional Analysis</b>	<b>38</b>
<i>Example 1.7.1 Dimensional Correctness</i>	38
<i>Example 1.7.2 Speed Conversion</i>	39
<i>Example 1.7.3 Time Conversion</i>	39
<i>Example 1.7.4 Atoms of Nitrogen</i>	40
<i>Example 1.7.5 Atoms of Hydrogen</i>	40
<b>Historical Notes and Comments</b>	<b>41</b>

### 1.1 SOLVING LINEAR EQUATIONS

Mathematical descriptions, often as **algebraic expressions**, usually consist of alphanumeric characters and special symbols.

Physicists describe the distance,  $s$ , that an object falls under gravity in time,  $t$ , by  $s = (1/2)gt^2$ . Here, the letters  $s$  and  $t$  are **variables** since their values may change, while,  $g$ , the acceleration of gravity is considered constant. While any letters can represent variables, typically the later letters of the alphabet are customary. The use of  $x$  and  $y$  is generic. Sometimes, it is convenient to use a letter that is descriptive of the variable, as  $t$  for time.

Earlier letters of the alphabet are customary for fixed values or **constants**. However, exceptions are widespread. The equal sign, a special symbol, is used to form an **equation**. An equation equates algebraic expressions. Numerical values for variables that preserve equality are called **solutions** to the equations.

For example,  $5x + 1 = 11$  is an equation in a single variable,  $x$ . It is a **conditional** equation since it is only true when  $x = 2$ . Equations that hold for all values of the variable are called **identities**. For example,  $(x + 1)^2 = x^2 + 2x + 1$  is an identity. By solving an equation, values of the variables that satisfy the equation are determined.

An equation in which only the first powers of variables appear is a **linear equation**. Every linear equation in a single variable can be solved using some or all of these properties:

**Substitution** – Substituting one expression for an equivalent one does not alter the original equation. For example,  $2(x - 3) + 3(x - 1) = 21$  is equivalent to  $2x - 6 + 3x - 3 = 21$  or  $5x - 9 = 21$ .

**Addition** – Adding (or subtracting) a quantity to each side of an equation leaves it unchanged. For example,  $5x - 9 = 21$  is equivalent to  $5x - 9 + 9 = 21 + 9$  or  $5x = 30$ .

**Multiplication** – Multiplying (or dividing) each side of an equation by a nonzero quantity leaves it unchanged. For example,  $5x = 30$  is equivalent to  $(5x)(1/5) = (30)(1/5)$  or  $x = 6$ .

***To Solve Single Variable Linear Equations***

1. Resolve fractions.
2. Remove grouping symbols.
3. Use addition and/or subtraction to move variable terms to one side of the equation.
4. Divide the equation by the variable coefficient.
5. Verify the solution in the original equation as a check.

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***Example 1.1.1 Solving a Linear Equation***

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Solve  $(3x/2) - 8 = (2/3)(x - 2)$ .

***Solution:***

To remove fractions, multiply both sides of the equation by 6, the least common denominator of 2 and 3. The revised equation becomes

$$9x - 48 = 4(x - 2).$$

Next, remove grouping symbols to yield

$$9x - 48 = 4x - 8.$$

Now, subtract  $4x$  and add  $48$  to both sides to yield

$$9x - 4x - 48 + 48 = 4x - 4x - 8 + 48 \text{ or } 5x = 40.$$

Finally, divide both sides by 5 (the coefficient of  $x$ ) to attain  $x = 8$ . The result,  $x = 8$  is checked by substitution in the original equation:

$$(3(8)/2) - 8 = (2/3)(8 - 2)$$

$$12 - 8 = (2/3)(6)$$

$$4 = 4 \text{ checks!}$$

The solution  $x = 8$  is correct!

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Equations often contain more than one variable. To solve linear equations in several variables simply bring the variable of interest to one side. Proceed as for a single variable, considering the other variables as constants for the moment.

**Example 1.1.2 Solving for y**

Solve for y:  $5x + 4y = 20$ .

**Solution:**

Move terms with y to one side of the equation and any remaining terms to the opposite side. Here,  $4y = 20 - 5x$ . Next, divide both sides by 4 to yield  $y = 5 - (5/4)x$ .

**Example 1.1.3 Simple Interest**

“Interest equals principal times rate times time” expresses the well-known simple interest formula,  $I = prt$ . Solve for time t.

**Solution:**

Clearly,  $I = (pr)t$  and  $pr$  becomes the coefficient of t. Dividing by  $pr$  gives  $t = I/pr$ .

Mathematics is often called “the language of science” or “the universal language.” To study phenomena or situations of interest, mathematical expressions and equations are used to create a **mathematical model**. Extracting information from the mathematical model provides solutions and insights. These suggestions may aid in modeling skills.

**To Solve Word Problems**

1. Read the problem carefully.
2. Identify the quantity of interest and possibly useful formulas.
3. A diagram may help.
4. Assign symbols to variables and other unknown quantities.
5. Translate words into an equation(s) using symbols for variables and unknowns.
6. Solve for the quantity of interest.
7. Check the solution and whether the proper question has been answered.

**Example 1.1.4 Investment**

Ms. Brown invests \$5000 to yield 1% annual interest. What will she earn in 1 year?

**Solution:**

Here, the principal (original investment) is \$5000. The interest rate is 0.01 (expressed as a decimal) and the time is 1 year.



Using the simple interest formula,  $I = prt$ , Ms. Brown's interest is

$$I = (\$5000)(0.01)(1) = \$50$$

After 1 year, her capital becomes  $p + prt = \$5000 + \$50 = \$5050$ .

### Example 1.1.5 Gasoline Prices

The June, 2014, East Coast regular grade gasoline average price (including tax) was about \$3.64 per gallon. The comparable West Coast average was about \$4.00 per gallon.

- What was the average regular grade gasoline price on the East Coast for 12 gallons of fuel?
- What was the average regular grade gasoline price on the West Coast for 25 gallons of fuel?

**Solution:**

- On average, on the East Coast 12 gallons cost  $(12)(3.64) = \$43.68$ .
- On average, on the West Coast 25 gallons cost  $(25)(\$4.00) = \$100.00$ .

◆ The famous yesteryear comedy team of Bud Abbott and Lou Costello used arithmetic shenanigans as the basis for many of their routines. The duo are probably best known for their “Who’s on first” baseball routine. Google Ivars Peterson’s “Math Trek” for some fun!

### Example 1.1.6 Breaking a Habit

One theory for breaking an adverse habit (smoking, snacking, childish behavior, etc.) is to delay successive gratifications. Suppose a wait time of  $w$  hours before gratifying a desire. Next, an increment of  $v$  hours to  $w + v$  hours to gratification. On the next occasion, the wait time is  $w + 2v$ , and so on. Determine the wait time before gratification for the  $n^{\text{th}}$  time.

**Solution:**

The first wait occurs at time  $w$ , the next  $v$  hours later so that the  $n^{\text{th}}$  time is

$$w + (n - 1)v, \quad n = 1, 2, \dots$$

**EXERCISES 1.1**

*In Exercises 1–6 determine whether the equation is an identity, a conditional equation, or a contradiction.*

- |                                   |                            |
|-----------------------------------|----------------------------|
| 1. $3x + 1 = 4x - 5$              | 4. $4x + 3(x + 2) = x + 6$ |
| 2. $2(x + 1) = x + x + 2$         | 5. $4(x + 3) = 2(2x + 5)$  |
| 3. $5(x + 1) + 2(x - 1) = 7x + 6$ | 6. $3x + 7 = 2x + 4$       |

*In Exercises 7–27 solve for the variable.*

- |                                   |  |
|-----------------------------------|--|
| 7. $5x - 3 = 17$                  | 18. $3x + 5(x - 2) = 2(x + 7)$                   |
| 8. $3x + 2 = 2x + 7$              | 19. $3s - 4 = 2s + 6$                            |
| 9. $2x = 4x - 10$                 | 20. $5(z - 3) + 3(z + 1) = 12$                   |
| 10. $x/3 = 10$                    | 21. $7t + 2 = 4t + 11$                           |
| 11. $4x - 5 = 6x - 7$             | 22. $(1/3)x + (1/2)x = 5$                        |
| 12. $5x + (1/3) = 7$              | 23. $4(x + 1) + 2(x - 3) = 7(x - 1)$             |
| 13. $0.6x = 30$                   | 24. $1/3 = (3/5)x - (1/2)$                       |
| 14. $3x/5 - 1 = 2 - (1/5)(x - 5)$ | 25. $\frac{x + 8}{2x - 5} = 2$                   |
| 15. $2/3 = (4/5)x - (1/3)$        | 26. $\frac{3x - 1}{7} = x - 3$                   |
| 16. $4(x - 3) = 2(x - 1)$         | 27. $8 - \{4[x - (3x - 4) - x] + 4\} = 3(x + 2)$ |
| 17. $5(x - 4) = 2x + 3(x - 7)$    |  |

*In Exercises 28–35 solve for the variable indicated.*

28. Solve:  $5x - 2y + 18 = 0$  for  $y$ .
29. Solve:  $6x - 3y = 9$  for  $x$ .
30. Solve:  $y = mx + b$  for  $x$ .
31. Solve:  $3x + 5y = 15$  for  $y$ .
32. Solve:  $A = p + prt$  for  $p$ .
33. Solve:  $V = LWH$  for  $W$ .
34. Solve:  $C = 2\pi r$  for  $r$ .
35. Solve:  $Z = \frac{x - \mu}{\sigma}$  for  $x$ .
36. The sum of three consecutive positive integers is 81. Determine the largest integer.
37. Sally purchased a used car for \$1300 and paid \$300 down. If the car is to be paid off in five equal monthly installments, what are her monthly payments?

38. A man's suit, marked down 20%, sold for \$120. What was the original price?
39. If the marginal propensity to consume  $m = 0.75$  and consumption,  $C$ , is 11 million units when disposable income is \$2 million, find the "consumption function." (Hint: use  $C = mx + b$ , where  $b$  is a constant).
40. An extension to a fire station costs \$100,000. The annual maintenance cost increases with the number of fire engines housed by \$2500 each. If \$115,000 has been allocated for the first year, how many additional fire engines can be housed?
41. The speed of light is much greater than the speed of sound, so lightning is seen before the sound of thunder is heard. An observer's distance from the flash can be calculated from the time between the sight of lightning and the sound of thunder. The distance,  $d$  (in miles), from the storm can be modeled as  $d = 4.5t$ , where time,  $t$ , is in seconds.
- How far is a storm if thunder is heard 2 seconds after the lightning is seen?
  - If a storm is 18 miles away, how long before the sound of thunder is heard?
42. A worker has 40 hours to produce two types of items, A and B. Each unit of A requires 3 hours to complete and each item of B, 2 hours. After completing eight units of B, the remaining time was spent on units of A. How many units of A were produced?
43. An employee's share of Social Security Payroll Tax was 6.2% in 2003 on the first \$87,000 of earnings. This amount was matched by the employer. Develop a linear model for an employee's Social Security Payroll Tax.
44. An employee works 37.5 hours at a \$10 hourly wage. Federal tax deductions are 6.2% to Social Security, 1.45% to Medicare Part A, and 15% for income tax. What is the after tax take-home pay?
45. The body surface area (BSA) and weight (Wt) in infants and children weighing between 3 and 30 kilograms has been reported to follow the linear relationship

$$\text{BSA} = 1321 + 0.3433\text{Wt} \quad (\text{BSA is in square centimeters and Wt is in grams})$$

- Determine the BSA for a child weighing 20 kilograms.
- If a child's BSA is 10,325 square centimeters, estimate its weight in kilograms. Current, J.D., "A Linear Equation for Estimating the Body Surface Area in Infants and Children.", The Internet Journal of Anesthesiology 1998; 2(2).

## 1.2 EQUATIONS OF LINES AND THEIR GRAPHS

Mathematical models express features of interest. In the managerial, social, and natural sciences and engineering, linear equations are often used to relate quantities of interest. Therefore, a thorough understanding of linear equations is important.

The standard form of a linear equation is  $ax + by = c$ , where  $a$ ,  $b$ , and  $c$  are real-valued constants. It is characterized by the first power of the exponents.

**Standard Form of a Linear Equation**

$$ax + by = c$$

$a$ ,  $b$ , and  $c$  are real numbered constants,  $a$  and  $b$  not both zero.

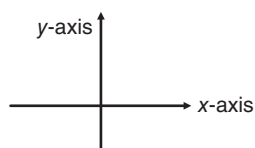
**Example 1.2.1**    *Ordered Pair Solutions*

Do the ordered pairs  $(3, 5)$  and  $(1, 7)$  satisfy the linear equation  $2x + y = 9$ ?

**Solution:**

An ordered pair satisfies an equation if equality is preserved. Substituting the point  $(3, 5)$  yields  $2(3) + 5 \neq 9$ . The ordered pair  $(3, 5)$  is not a solution to the equation. For  $(1, 7)$ , the substitution yields  $2(1) + 7 = 9$ . This is true.

A **graph** is a pictorial representation of a function. It consists of points that satisfy the function. **Cartesian coordinates** are used to represent the relative positions of points in a plane or in space. In a plane, a point  $P$  is specified by the coordinates or ordered pair  $(x, y)$  representing its distance from two perpendicular intersecting straight lines, called the  $x$ -axis and the  $y$ -axis, respectively (see figure).



To determine the  **$x$ -intercept** of a line (its intersection with the  $x$ -axis), set  $y = 0$  and solve for  $x$ . Likewise, for the  **$y$ -intercept** set  $x = 0$  and solve for  $y$ . Two distinct points uniquely determine a line in a graph of a linear equation. An additional point can be a check as the three points must be **collinear**, that is, lie on the same line. The coordinate axes may be differently scaled.

Cartesian coordinates are so named to honor the mathematician **René Descartes** (Historical Notes).

◆ In Navajo religion, medicine men heal by “balancing of forces.” Sand paintings in healing ceremonies use reflection symmetry to show paired forces.

Sometimes “fourfold” symmetry is found. Fourfold symmetry is reflection both horizontally and vertically as in the four quadrants of a Cartesian system. Such symmetry arises in many Native American cultures as an organizing principle. For example, prayers are to “the four winds” and teepees are made with four base poles, each placed in one of the four compass directions.



**Example 1.2.2** *Intercepts and Graph of a Line*

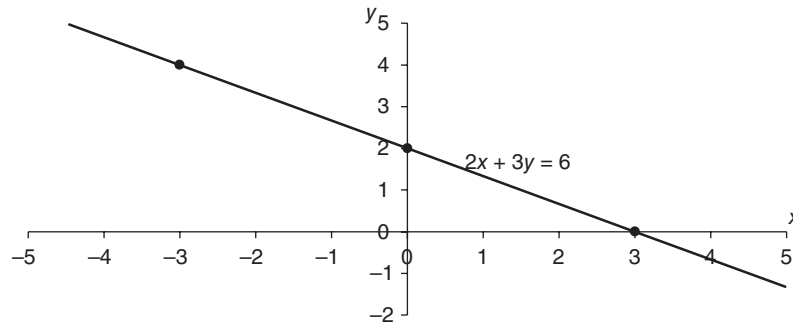
Find the  $x$ - and  $y$ -intercepts of the line  $2x + 3y = 6$  and graph its equation.

**Solution:**

When  $x = 0$ ,  $3y = 6$ ; the  $y$ -intercept is  $y = 2$ . When  $y = 0$ ,  $2x = 6$ ; the  $x$ -intercept is  $x = 3$ .

The two intercepts,  $(3, 0)$  and  $(0, 2)$ , being two points of  $2x + 3y = 6$ , uniquely determine the line.

As a check, arbitrarily choose a value for  $x$ , say  $x = -3$ . Then,  $2(-3) + 3y = 6$  or  $3y = 12$ , so  $y = 4$ . Therefore,  $(-3, 4)$  is another point on the line. Note the three points on the graph.



When either  $a$  or  $b$  in the standard equation  $ax + by = c$  is zero, it reduces to a single value for the remaining variable. So, if  $y = 0$ ,  $ax = c$  and  $x = c/a$ , a vertical line. If  $x = 0$ ,  $by = c$  and  $y = c/b$ , a horizontal line.

**Vertical and Horizontal Lines**

For  $ax + by = c$

The graph of  $x = c/a$ , ( $b = 0$ ), is a vertical line.

The graph of  $y = c/b$ , ( $a = 0$ ), is a horizontal line.

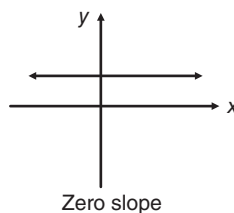
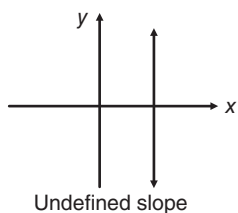
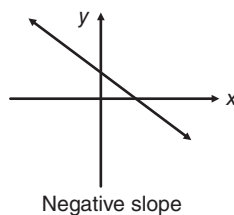
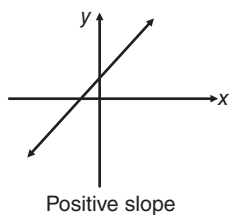
It is often useful to express the equations of lines in other (and equivalent) formats. The **slope**,  $m$ , of a line can be described in several ways, for example, as “the rise divided by the run” or as “the change in  $y$ , denoted by  $\Delta y$ , divided by the change in  $x$ ,  $\Delta x$ .” A line with a positive slope increases (rises) from left to right ( $/$ ), while a line with negative slope decreases (falls) ( $\backslash$ ).



### *Slope*

$$\text{Slope} = m = \frac{\text{Rise}}{\text{Run}} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The **slope-intercept form** of a line is  $y = mx + b$ , where  $m$  is the slope and  $b$  its  $y$ -intercept. A horizontal line has a slope of zero. A vertical line has an infinite (undefined) slope, as there is no change in  $x$  for any change in  $y$ .



### *Slope-Intercept Form of a Linear Equation*

$$y = mx + b$$

where  $m$  is the slope and  $b$  the  $y$ -intercept.

A linear equation in standard form ( $ax + by = c$ ) is written in slope-intercept form by solving for  $y$ .

#### *Example 1.2.3    Slope-Intercept Form*

Write  $2x + 3y = 6$  in slope-intercept form and identify the slope and  $y$ -intercept.

**Solution:**

Solving for  $y$ ,  $y = (-2/3)x + 2$ . By inspection, the slope is  $-2/3$  ("line falls") and the  $y$ -intercept is  $(0, 2)$ , in agreement with the previous example.



A linear equation can also be written in a **point-slope form**:  $y - y_1 = m(x - x_1)$ . Here,  $(x_1, y_1)$  is a point on the line and  $m$  is the slope.

***Point-Slope Form of a Linear Equation***

$$y - y_1 = m(x - x_1)$$

where  $m$  is the slope and  $(x_1, y_1)$  a point on the line.

**Example 1.2.4**     *Point-Slope Form*

Find the equation of a line passing through  $(2, 4)$  and  $(5, 13)$  in point-slope form.

**Solution:**

First, the slope  $m = \frac{13 - 4}{5 - 2} = \frac{9}{3} = 3$ . Now, using  $(2, 4)$  in the point-slope form, we have  $y - 4 = 3(x - 2)$ . (Using the point  $(5, 13)$  yields the equivalent  $y - 13 = 3(x - 5)$ .) For the slope-intercept form, solving for  $y$  yields  $y = 3x - 2$ . The standard form is  $3x - y = 2$ .

Incidentally, as noted earlier, while the generic symbols  $x$  and  $y$  are most common, other letters are used to denote variables. Earlier  $I = prt$  was used to express accrued interest. Economists use  $q$  and  $p$  for quantity and price, respectively, and scientists often use  $F$  and  $C$  for Fahrenheit and Celsius temperatures or  $p$  and  $v$  for gas pressure and volume, and so on.

**Example 1.2.5**     *Temperature Conversion*

Water boils at 212 degrees Fahrenheit or 100 degrees Celsius and freezes at 32 degrees Fahrenheit or 0 degrees Celsius. Find a linear relation between Celsius and Fahrenheit temperatures.

**Solution:**

Let the Celsius temperature be the input variable, denoted by  $C$ . The Fahrenheit temperature is the output variable,  $F$ , in a linear relation. The ordered pairs are  $(100, 212)$  and  $(0, 32)$ . For the slope-intercept form,  $m = \frac{32 - 212}{0 - 100} = \frac{-180}{-100} = \frac{9}{5}$ . The second ordered pair provides the vertical  $y$ -intercept by inspection. Therefore,  $F = (9/5)C + 32$  is the widely used relation to easily convert Celsius to Fahrenheit temperatures. Similarly, there is a relation to connect Fahrenheit to Celsius temperatures.

A common mathematical model for depreciation (equipment, buildings, etc.) relates current value “ $y$ ” (dollars) to age “ $x$ ” (years). Straight Line Depreciation (SLD) is a common choice. In an SLD model, annual depreciation is the same each year of useful life. Any remaining value is the “salvage value.”

**Example 1.2.6 Salvage Value**

Equipment value (dollars) at time  $t$  (years) is  $V(t) = -10,000t + 80,000$  and its useful life expectancy is 6 years. Determine the original value, salvage value, and annual depreciation.

**Solution:**

The original value, at  $t = 0$ , was the salvage value, at  $t = 6$  years, the end of useful life, is  $\$20,000 = (-10,000(6) + 80,000)$ . Finally, the slope, the annual depreciation is  $\$10,000$ .

Note that parallel lines have the same slope. Two lines are perpendicular if their slopes are negative reciprocals.

**Parallel and Perpendicular Lines**

Two lines are parallel if their slopes are equal

$$(m_1 = m_2)$$

Two lines are perpendicular if their slopes are negative reciprocals

$$(m_1 = -1/m_2)$$

**Example 1.2.7 Parallel or Perpendicular Lines**

Are these pairs of lines parallel, perpendicular, or neither?

- a)  $3x - y = 1$  and  $y = (1/3)x - 4$
- b)  $y = 2x + 3$  and  $y = (-1/2)x + 5$
- c)  $y = 7x + 1$  and  $y = 7x + 3$

**Solution:**

- a) Two slopes are required. By inspection, the slope of the second line is  $1/3$ . The line  $3x - y = 1$  in slope-intercept form is  $y = 3x - 1$  so the slope is 3. Since the slopes are neither equal nor negative reciprocals, the lines must intersect.
- b) The slopes are 2 and  $(-1/2)$ . Since these are negative reciprocals, the two lines are perpendicular.
- c) The lines have the same slope (with different intercepts), so they are parallel.

◆ “What Makes an Equation Beautiful,” the title of an article in *The New York Times* of October 24, 2004, is not likely to excite everyone’s interest, especially for the linear equations you studied in this chapter. Fair enough!

However, if linear equations are fairly new to you, be assured that they are a building block for more advanced – and more interesting equations.

Some physicists were recently asked: “Which equations are the greatest?” According to the article, some were nominated for the breadth of knowledge they capture, for their historical importance, and for reshaping our perception of the universe.

### EXERCISES 1.2

1. Find the  $x$ - and  $y$ -intercepts for the following:

- |                   |                  |
|-------------------|------------------|
| a) $5x - 3y = 15$ | d) $9x - y = 18$ |
| b) $y = 4x - 5$   | e) $x = 4$       |
| c) $2x + 3y = 24$ | f) $y = -2$      |

2. Find slopes and  $y$ -intercepts for the following:

- |                      |                   |
|----------------------|-------------------|
| a) $y = (2/3)x + 8$  | d) $6y = 4x + 3$  |
| b) $3x + 4y = 12$    | e) $5x = 2y + 10$ |
| c) $2x - 3y - 6 = 0$ | f) $y = 7$        |

3. Find the slopes of lines defined by these points:

- |                       |                             |
|-----------------------|-----------------------------|
| a) (3, 6) and (-1, 4) | d) (2, 3) and (2, 7)        |
| b) (1, 6) and (2, 11) | e) (2, 6) and (5, 6)        |
| c) (6, 3) and (12, 7) | f) (5/3, 2/3) and (10/3, 1) |

4. Find equations for the lines

- with a slope of 4 passing through (1, 7);
- passing through (2, 7) and (5, 13);
- with undefined slope passing through (2, 5/2);
- with  $x$ -intercept 6 and  $y$ -intercept -2;
- with slope 5 and passing through (0, -7);
- passing through (4, 9) and (7, 18).

5. Plot graphs of

- |                 |                   |
|-----------------|-------------------|
| a) $y = 2x - 5$ | c) $3x + 5y = 15$ |
| b) $x = 4$      | d) $2x + 7y = 14$ |

14 LINEAR EQUATIONS AND MATHEMATICAL CONCEPTS

6. Plot graphs of

a)  $2x - 3y = 6$

c)  $y = (-2/3)x + 2$

b)  $y = -3$

d)  $y = 4x - 7$

7. Are the pairs of lines parallel, perpendicular or neither?

a)  $y = (5/3)x + 2$  and  $5x - 3y = 10$

b)  $6x + 2y = 4$  and  $y = (1/3)x + 1$

c)  $2x - 3y = 6$  and  $4x - 6y = 15$

d)  $y = 5x - 4$  and  $3x - y = 4$

e)  $y = 5$  and  $x = 3$

8. Find equations for the lines

a) through  $(2, 3)$  and parallel to  $y = 5x - 1$ .

b) through  $(1, 4)$  and perpendicular to  $2x + 3y = 6$ .

c) through  $(5, 7)$  and perpendicular to  $x = 6$ .

d) through  $(4, 1)$  and parallel to  $x = 1$ .

e) through  $(2, 3)$  and parallel to  $2y = 5x + 4$ .

9. Can a linear equation not have an  $x$ -intercept? Have more than one  $x$ -intercept? Have no  $y$ -intercept? Have more than one  $y$ -intercept?

10. Find a formula to convert Fahrenheit to Celsius temperatures. Hint: use the freezing and boiling temperatures of water.

11. A new machine costs \$75,000 and has a salvage value of \$21,000 after 9 years. Find a linear equation to model its SLD.

12. A new car costs \$28,000 and after 5 years it has a trade-in value of \$3000. Find a linear equation to model its SLD.

13. A car requires 7 gallons of gasoline to travel 245 miles and 12 gallons to travel 420 miles. Determine a linear relationship to express miles traveled as a function of the gasoline usage.

14. A piece of office equipment was purchased for \$50,000 and after 10 years had a salvage value of \$5000. Express its depreciation by a linear equation.

15. A firm pays \$1100 monthly rent on a building (a fixed cost). Each unit of product costs \$5 (a variable cost of operation). Form a linear model for the total monthly cost to produce  $x$  items.

16. A skateboard sells for \$24. Determine the revenue function for sales of skateboards.

17. A car rental company charges \$50 per day for a medium-sized car and 30 cents per mile of travel.

a) Find the cost equation for renting a medium-sized car for a single day.

b) How many miles can be traveled in a day on a budget of \$110?

18. At the ocean's surface, water pressure equals an air pressure of about 15 pounds per square inch. Below the surface, water pressure increases by 4.43 pounds per square inch for each 10 foot descent.
  - a) Express water pressure as a function of ocean depth.
  - b) At what depth is water pressure 80 pounds per square inch?
19. A department store priced a man's suit at \$84 that wholesaled at \$70. Also, a woman's dress priced at \$48 wholesaled at \$40. If the markup policy is linear and is reflected in the prices of these items, relate the retail price,  $R$ , to the store's cost,  $C$ .
20. The demand for a certain product is linearly related to its price. The product was priced at \$1.50 and 40 items were sold. When it was priced at \$6, only 22 items were sold. Determine a linear relationship between the price,  $x$ , and the sales,  $y$ .

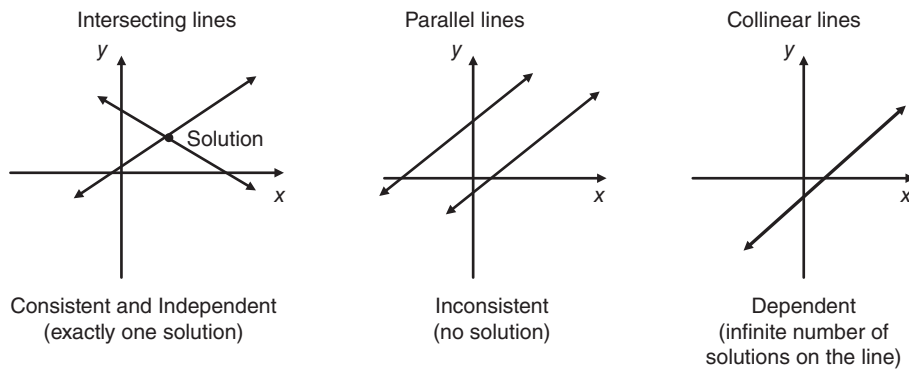
### 1.3 SOLVING SYSTEMS OF LINEAR EQUATIONS

To solve systems of two or more linear equations is to find common (intersection) points among them. The following methods are available:

- (1) Graphing      (2) Substitution      (3) Elimination (addition)

To solve a system of linear equations in two variables graphically, they are graphed on common axes. If there is a common solution, it is at their intersection.

The graph of a system of two linear equations in two variables has one of these properties: they intersect at a point, are parallel, or are collinear. These three possibilities are illustrated in the following figures.



Whether a system of two linear equations is **consistent**, **inconsistent**, or **dependent** is apparent from the slope-intercept form  $y = mx + b$ .

- When the slopes differ, the system has a unique solution.
- When the slopes are equal and the  $y$ -intercepts differ, the lines are parallel. Such systems are inconsistent (have no solution).

- When the slopes and  $y$ -intercepts are equal, the lines are collinear (identical). There are an infinity of solutions that satisfy both equations.

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**Example 1.3.1**     *Graphical Solutions to a System of Linear Equations*

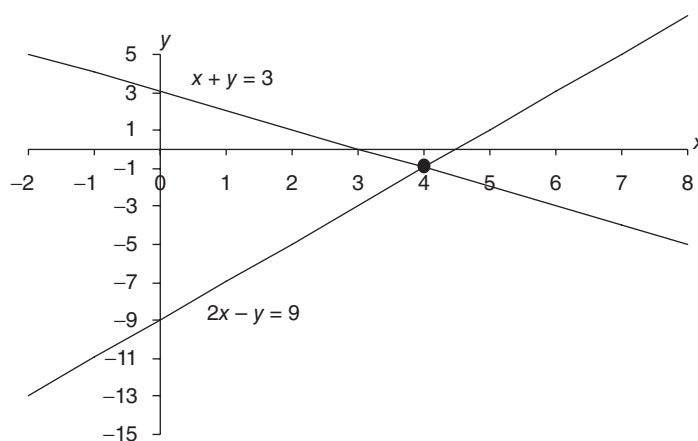

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Solve this system graphically:

$$x + y = 3$$

$$2x - y = 9$$
**Solution:**

As seen from the graph, the lines intersect and the solution to this system is at  $(4, -1)$ .



Graphing is not usually practical because of the need for precision in plotting.

An algebraic method of **substitution** can be used to solve systems of equations. In this method, one equation is solved for a variable and substituted in the other equation. It can be helpful to minimize calculations by thoughtful choice of substitution.

***Solving by Substitution***

1. Solve for a variable in either equation as convenient. (For example, choose a variable with an integer coefficient, preferably  $\pm 1$ , if possible.)
2. Substitute the chosen variable in the remaining equation. The result is an equation in one variable.
3. Solve the equation in one variable. (This variable cannot be the variable from Step 1.)
4. Substitute the value determined in Step 3 into the equation from Step 1 to determine the value of the remaining variable.
5. Check your solution in each of the original equations.



**Example 1.3.2**    *Substitution – Consistent System*

Solve the following system by substitution:  $2x + y = 9$   
 $3x + 5y = 17$

**Solution:**

First, observe coefficients. Since the coefficient of  $y$  in the first equation is unity, it is chosen to solve for  $y$  and to substitute into the second equation. Solving for  $y$  in the first equation yields  $y = 9 - 2x$ . Substituting this into the second equation gives  $3x + 5(9 - 2x) = 3x + 45 - 10x = 17$ . Simplifying,  $7x = 28$  and  $x = 4$ . Since  $y = 9 - 2x$ , we conclude that  $y = 1$ . The ordered pair solution is, therefore,  $(4, 1)$ . This solution satisfies both equations.

In the preceding example, we solved for  $y$  in the first equation. One could have solved for  $y$  in the second equation or for  $x$  in either equation to start the process. In solving the system, we found a single solution. The system was consistent and independent. A dependent system leads to an identity. An inconsistent system always yields a false condition.

**Example 1.3.3**    *Substitution–Inconsistent System*

Solve by substitution:  $4x - 2y = 6$   
 $2x - y = 1$

**Solution:**

It is preferable to solve for  $y$  in the second equation:  $y = 2x - 1$ . Substituting into the first equation yields  $4x - 2(2x - 1) = 4x - 4x + 2 = 6$ . The result “ $2 = 6$ ” is false, and the system is inconsistent and lacks a solution.

A third method to algebraically solve a system of equations is the **addition** or **elimination method**. Here, one seeks to multiply the equations by factors so that when they are added one of the variables is eliminated.

**Solving by Elimination (Addition)**

1. Write each equation in standard form  $Ax + By = C$ .
2. If necessary, multiply one or both equations by a constant, so their sum contains only one variable (the other variable having been eliminated).
3. Solve for the remaining variable in the resulting equation.
4. Substitute the value from Step 3 into either of the original equations to find the value of the remaining variable.
5. Check your solution in the original system.

**Example 1.3.4 Elimination Method – Consistent System**

Solve by elimination:

$$\begin{aligned} 3x - y &= 8 \\ x + 2y &= 5 \end{aligned}$$

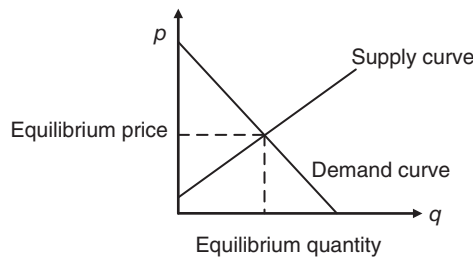
**Solution:**

Sometimes it is easier to eliminate a variable whose coefficients have opposite signs. Here,  $y$  is the preferred variable for elimination. This is accomplished by multiplying the first equation by 2, as

$$\begin{aligned} 6x - 2y &= 16 \\ x + 2y &= 5 \end{aligned}$$

Now, adding the two equations yields  $7x = 21$  or  $x = 3$ . Using  $x = 3$ , we determine that  $y = 1$  to give  $(3, 1)$  as the ordered pair solution to the system.

Economists know that consumer demand for a commodity is related to its price. According to the *Law of Demand*, the quantity demanded increases as price decreases. Just as a consumer's willingness to buy is related to price, a manufacturer's willingness to supply goods is related to the realized price. In the *Law of Supply*, the quantity supplied increases as price increases. *Market equilibrium* occurs when the demand equals supply. The following figure illustrates the intersection when supply and demand are equal.

**Example 1.3.5 Market Equilibrium**

Determine the equilibrium quantity,  $q$ , and price,  $p$ , when the demand function for a commodity is  $p = -5q + 20$  and the supply function by  $p = 3q + 4$ .

**Solution:**

Using substitution to solve the system of equations yields  $-5q + 20 = 3q + 4$ .  
So,  $8q = 16$  or  $q = 2$  and  $p = 10$ .

*Total cost* consists of two components, a fixed cost and a variable cost. A *fixed cost* as, say, monthly rent, is independent of the “level of production.” *Variable cost* changes with

production level. *Revenue* equals the price of an item multiplied by its demand. *Profit* is the difference between revenue and total cost. Solving the equations for total cost and for revenue simultaneously, their intersection yields the *break-even* point, where profit is zero.

◆ In Keynesian economic theory, consumption,  $C$  (in dollars), is the linear relation  $C = mx + b$ , where  $x$  is disposable income in dollars;  $m$ , marginal propensity to consume; and  $b$ , a scaling constant.

*Marginal cost (MC)* is the cost associated with one additional unit of production. When cost is a linear function, MC is the slope. When revenue is linear, its slope is the *marginal revenue (MR)*. That is, MR is the revenue from an additional unit of sales.

### EXERCISES 1.3

- Which ordered pairs are solutions to the system  $2x + y = 7$  and  $x + y = 5$ ?
  - (3, 1)
  - (2, 3)
  - (4, -1)
- Which ordered pairs are solutions to the system  $2x + 2y = 4$  and  $x + y = 2$ ?
  - (2, 0)
  - (0, 2)
  - (1, 1)
- Write each equation in slope-intercept form. Determine by inspection whether the system is consistent, inconsistent, or dependent.
 

a) $3y = -x + 8$ $x + y = 6$	b) $x + 2y = 7$ $2x = -4y + 14$	c) $3x + 2y = 7$ $y = (-3/2)x + 5$
---------------------------------	------------------------------------	---------------------------------------
- Determine the solution graphically:
 

a) $x + y = 4$ $2x + 2y = 6$	b) $x + 2y = 5$ $2x + y = 4$	c) $y = (1/3)x - 2$ $x - 3y = 6$
---------------------------------	---------------------------------	-------------------------------------
- Determine the solution graphically:
 

a) $x + 2y = 5$ $3x + y = 5$	b) $2x + 3y = 12$ $2x - y = 4$	c) $y = (1/4)x + 1$ $2x - 3y = 2$
---------------------------------	-----------------------------------	--------------------------------------
- Use the substitution method to solve:
 

a) $y = 5$ $2x + y = 7$	b) $x - 1 = 0$ $x + 3y = 4$	c) $2x + 3y = 8$ $x + 2y = 5$
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20 LINEAR EQUATIONS AND MATHEMATICAL CONCEPTS

7. Use the substitution method to solve:

a)  $x = 3$   
 $x + 3y = 9$

b)  $y - 2 = 0$   
 $x + 3y = 9$

c)  $x + y = 5$   
 $y = -x + 3$

8. Use the elimination (addition) method to solve:

a)  $x + y = 2$   
 $x - y = 4$

b)  $2x + 3y = 6$   
 $4x - y = 5$

c)  $2x + y = 5$   
 $2y = -4x + 10$

d)  $y = -x + 4$   
 $2x + 2y = 7$

9. Use the elimination (addition) method to solve:

a)  $-x + 2y = 5$   
 $x + y = 4$

b)  $4x + 3y = 35$   
 $2x - y = 5$

c)  $x + 4y = 13$   
 $2y = -4x + 10$

d)  $y = x + 4$   
 $2x + 3y = 12$

10. A real estate agent receives a weekly salary plus a sales commission. If the agent earns \$550 in a week when a \$50,000 home is the only sale and \$850 in a week when a \$150,000 home is the only sale, calculate the weekly salary and percentage commission.

11. An elementary school plans a fund-raiser. Students are to sell boxes of cookies for \$4 each and boxes of candies for \$5 each. The students sold 2400 items for \$10,500. How many boxes of each were sold?

12. A chemist requires 500 milliliters of a 10% solution. Solutions of 5% and 25% are available. How many milliliters of each solution should be mixed?

13. Octane is a measure of a gasoline's ability to resist "knock" or "pinging." Gasoline pumps offer three unleaded octane grades: regular (87), mid-grade (89), and premium (93). How many gallons of regular should be mixed with premium to have 100 gallons of 92% octane?

14. A student works two part-time jobs and earned \$185 by working a total of 25 hours at jobs that pay \$7 and \$8 per hour. How many hours did the student work at each job?

15. Given the demand and supply functions

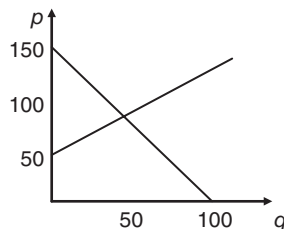
$$\text{Demand } p = -2q + 320$$

$$\text{Supply } p = 8q + 20, \text{ where } p \text{ is the price and } q \text{ the quantity.}$$

- Sketch the supply and demand curves on a coordinate system.
- Label market equilibrium on your sketch.
- Solve algebraically for the coordinates of market equilibrium.

16. A vendor pays \$250 monthly rent for product assembly space. Assembled items cost \$10 and are sold for \$20 each. Determine the revenue function, total cost function, profit function, and the break-even point.

17. On the graph



- Label both supply and demand lines.
- Label the market equilibrium point and estimate its coordinates.

18. Find the market equilibrium for these supply and demand functions

$$\text{Demand : } p = -0.5q + 54$$

$$\text{Supply : } p = 1.2q + 37$$

#### 1.4 THE NUMBERS $\pi$ AND $e$

“There are numbers and there are numbers!” to coin a cliché. Integers are familiar to you along with fractions and decimals to comprise **rational numbers**. The square roots, cube roots, and so on of numbers that are not perfect squares, cubes, and so on are called **irrational numbers** and **complex numbers**. They arise as the roots of linear, quadratic, and other equations known as **polynomials**.

However, there are other numbers in nature that are not the roots of polynomial equations – they are known as **transcendental numbers**. Every transcendental number is irrational. Prominent among these amazing transcendental numbers are  $\pi$  and  $e$ . Both make appearances in the text. While we briefly introduce them here, an Internet search yields much fascinating information about them.

◆ The 1998 movie “Pi” features a mathematician obsessed with finding numerical patterns in life. A list of films in which mathematics seems to play some role may be found at <http://world.std.com/~reinhold/mathmovies.html>.

##### The Number $\pi$

The Greek letter “pi,” written  $\pi = 3.14159 \dots$ , is likely already familiar to you. The circumference of a circle,  $C$ , of radius  $r$  and diameter  $d$ , is  $2\pi r$  or  $\pi d$ . Therefore,  $\pi = C/2r$  or  $C/d$ .

The ancients sought to construct a square, having the same area as a given circle, using only a compass and a straight edge. Known as “squaring the circle,” it was one of the most baffling challenges in geometry. Centuries of trials concluded that the area of any circle can never exactly equal the area of a square. For the area of a circle of radius  $r$ ,  $\pi r^2$ , to equal the area of a square of side,  $s$ , requires that numbers  $r$  and  $s$  exist so that  $\pi r^2 = s^2$  or  $\pi = (s/r)^2$ . We now know that  $\pi$  can never be expressed as the ratio of two numbers.

The symbol  $\pi$ , first used in 1706, became popular after its adoption by **Leonhard Euler** in 1737 (Historical Notes).

The Babylonians, in 2000 B.C.E., are believed to have been the first to estimate  $\pi$ . **Christiaan Huygens**, an early Dutch physicist, sought to carefully construct inscribed polygons of increasing numbers of sides in circles hoping to estimate  $\pi$  from careful measurements of the radius and polygonal area.

Newspapers periodically report extensions of the value of  $\pi$ . The quest to express  $\pi$  in increasing numbers of decimal places has challenged mathematicians for centuries and continues to this day. The 1999 Guinness Book of World Records notes that a team of Tokyo researchers had computed about 206 billion digits of  $\pi$ . By 2002, the record was 1.24 trillion digits for pi. A Hitachi supercomputer, capable of over 2 trillion calculations per second, took over 400 hours to calculate the digits. Such achievements are not only mathematically noteworthy, they also have implications in cryptography, random numbers, and in the security of Internet transmissions.

◆ You can estimate  $\pi$  on a computer or programmable hand calculator using a simple Monte Carlo Method described in Chapter 11.

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**Example 1.4.1**    *A Biblical  $\pi$*

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*A biblical reference to Solomon is written, “And he made a molten sea, ten cubits from one brim to the other. It was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.”*

*Use this information, as the ancients, to estimate  $\pi$ .*

**Solution:**

*This quote implies that a 10-cubit diameter circle has a circumference of 30 cubits. Therefore,  $\pi \approx 30 \text{ cubits} / 10 \text{ cubits} = 3$ .*

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Pi appears virtually everywhere in the sciences including the DNA double helix, in disks of the moon and sun, rainbows, spreading pond ripples, Einstein’s gravitational field equation, the normal distribution, prime number distributions, waves, navigation, spectra and much more. In geometry, as you know,  $\pi$  appears in the circumference, area, and volume of circles, spheres, cones, cylinders, or circular polygons.

**The Number  $e$**

The number  $e$  (a notation credited to Leonhard Euler) is another curious number that arises from a peculiar limit. Consider the quantity

$$\left(1 + \frac{1}{n}\right)^n$$

Clearly, when  $n = 1$ , its value is 2. When  $n = 2$ , the value is 2.25. Its value increases as  $n$  increases. Is there a limit? Or does it increase beyond bound as  $n$  becomes larger and larger, tending to infinity? Actually, its value is  $e = 2.71828 \dots$ , and while it continues to increase as  $n$  increases, it is bounded above by, say, 2.72. You can easily calculate  $e$  on a computer or handheld calculator.

One of the properties of  $e$  is as the base of a system of **natural logarithms**. As such, it has a role in the calculation of compound interest, growth or decay of natural species, or situations when a quantity increases at a rate proportional to its value. The number  $e$  also appears in probability theory, statistics, trigonometry, physical laws, and many branches of engineering.

◆ {first 10-digit prime found in consecutive digits of  $e$ }.com

That message appeared on a California highway billboard and in a Cambridge, Massachusetts subway station. People who knew the correct 10-digit prime number were invited to apply for employment at Google Inc. The solution was reported on the Web by some puzzle solvers.

#### EXERCISES 1.4

1. The area of a circle is  $\pi$ . What is its diameter?
2. The circumference of a circle is  $\pi$ . What is its radius?
3. Conduct an Internet search to answer these questions:
  - a) Pi actually has a day in its honor. When is pi day?
  - b) Who was the Tokyo professor once credited with the world record for the number of digits of pi?
  - c) Who was the noted scientist born on pi day? (Hint: His initials are A.E.).
  - d) How does the illness Morbus Cyclometricus relate to pi and to squaring the circle?
  - e) How is pi known in Germany?
4. Cite five additional facts about pi from the Internet to share with your class.
5. Use the expression  $\left(1 + \frac{1}{n}\right)^n$  to approximate the value of  $e$  when
  - a)  $n = 5$
  - b)  $n = 10$
  - c)  $n = 25$ .
6. An alternative way to calculate  $e$  is the series

$$1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

Find the approximate value of these six terms.

## 1.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

## Exponential Functions

Functions of the type  $f(x) = b^x$  are **exponential functions**. Their distinguishing feature is that the variable,  $x$ , is an exponent. The base,  $b$ , is any positive real valued number (other than 1). The rates of change of exponential functions are greater than those for polynomial functions.

Recall the *Laws of Exponents* (see the following text)!

***Laws of Exponents***

1.  $b^0 = 1$

2.  $b^{-x} = \frac{1}{b^x}$

3.  $b^x b^y = b^{x+y}$

4.  $\frac{b^x}{b^y} = b^{x-y}$

5.  $(b^x)^y = b^{xy}$

6.  $(ab)^x = a^x b^x$

7.  $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

8.  $\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x = \frac{b^x}{a^x}$

**Example 1.5.1**    *Using Laws of Exponents*

*Simplify these expressions using Laws of Exponents*

a)  $(2^{3x} 2^{5x})^{1/2}$       b)  $27^{x/3} 9^{2x}$       c)  $\frac{5^{2x-1} 5^{4x+5}}{5^{3x+1}}$       d)  $\frac{14^{2x}}{7^{2x}}$

**Solution:**

a) Since the same base appears twice within the parenthesis, add the exponents to yield  $(2^{8x})^{1/2}$ . Next, multiply the exponents to yield  $2^{4x}$ .

b) There is a common base of 3, as 9 and 27 are powers of 3. Therefore,  
 $(3^3)^{x/3} (3^2)^{2x} = 3^x 3^{4x} = 3^{5x}$ .

c) First add the numerator exponents to yield  $\frac{5^{6x+4}}{5^{3x+1}}$ . Next, subtract the exponents to simplify the expression to  $5^{(6x+4)-(3x+1)} = 5^{3x+3} = 5^{3(x+1)}$ .

d) Since the exponents are the same, the expression can be rewritten as  $\left(\frac{14}{7}\right)^{2x} = 2^{2x}$ .



Often, when solving exponential equations, one either tries to equate bases or exponents. If there is a common base, exponents can be equated. Likewise, if the exponents are equal, their bases can be equated. The additional laws that follow are useful for solving exponential equations.

**Additional Laws of Exponents**

If  $b^x = b^y$  then  $x = y$ .  
 If  $a^x = b^x$  then  $a = b$ .

**Example 1.5.2 Simplifying Exponential Equations**

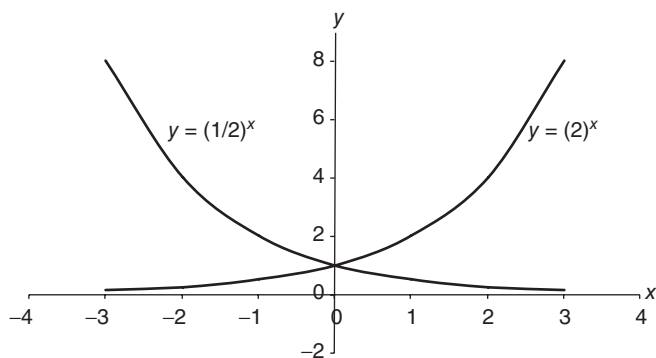
Solve for  $x$  in the following:

a)  $4^x = 128$     b)  $3^{-x} = 27$     c)  $7^{2x-3} = \sqrt[3]{49}$     d)  $125 = x^3$

**Solution:**

- a) Because the variable is an exponent, first equate bases. Rewriting the equation as  $(2^2)^x = 2^7$  or  $2^{2x} = 2^7$  yields  $2x = 7$  or  $x = 7/2$ .
- b) Rewriting the equation as  $3^{-x} = 3^3$  yields  $-x = 3$  or  $x = -3$ .
- c) First, rewrite as  $7^{2x-3} = (7^2)^{1/3} = 7^{2/3}$ . Therefore,  $2x - 3 = 2/3$  and  $x = 11/6$ .
- d) Here, we seek to equate exponents. The equation is rewritten as  $(5)^3 = (x)^3$ , which yields  $x = 5$ .

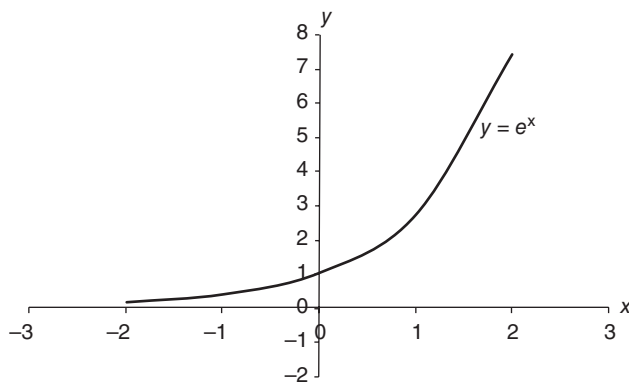
Graphs of exponential functions have several properties of interest. Their domain is the real numbers, while their range is the positive real numbers. There is no  $x$ -intercept and the  $y$ -intercept is at  $y = 1$ . There is a horizontal asymptote at  $y = 0$  (the  $x$ -axis). When the base is between 0 and 1 the function is decreasing, and when the base exceeds 1, the function is increasing. The graph of the exponential functions  $y = 2^x$  and  $y = (1/2)^x$  shows these properties.



Recall that the transcendental number  $e$  is approximately 2.718. The special exponential function  $e^x$  has the characteristics discussed.

For the graph of  $e^x$ :

- the domain is all real numbers;
- the range is positive real numbers;
- the  $x$ -axis is an asymptote;
- the  $y$ -intercept is 1;
- there is no  $x$ -intercept;
- it is an increasing function ( $e > 1$ ).



The Laws of Exponents apply to the transcendental number  $e$ , as Example 1.5.3 illustrates.

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**Example 1.5.3**    *Simplifying Exponential Exponents*

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*Simplify*

a)  $(e^3)^x$                       b)  $\frac{e^{5x-1}}{e^{x+2}}$                       c)  $\left(\frac{1}{e}\right)^{5x}$

**Solution:**

- a) *Multiplying the exponents yields  $e^{3x}$ .*  
 b) *Subtracting the exponents yields  $e^{(5x-1)-(x+2)} = e^{4x-3}$ .*  
 c) *Rewrite as  $(e^{-1})^{5x} = e^{-5x}$ .*
- 
- 

**Logarithmic Functions**

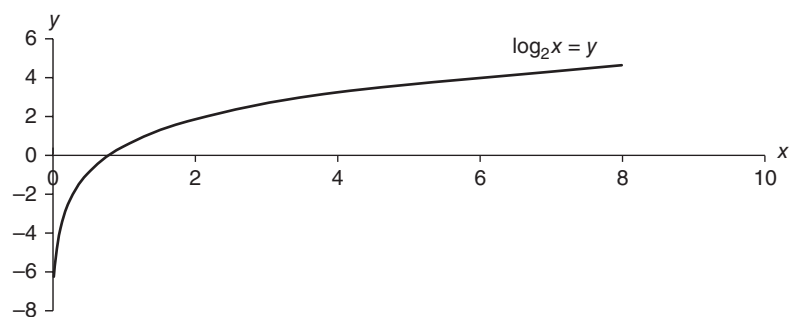
**Logarithmic functions** are inverses of exponential functions. The following formulas convert logarithmic into exponential forms and vice versa.

***Logarithmic and Exponential Forms***

**$\log_b x = y$  means  $b^y = x$  and vice versa.**

◆ **John Napier** discovered logarithms in the latter part of the sixteenth century. Logarithms are useful for constructing mathematical models and for solving equations with exponential functions.

Any logarithm can be expressed as an exponential and vice versa. For instance, to graph  $\log_2 x = y$  it is expressed equivalently as  $2^y = x$  as shown:



While the domain for logarithms is  $(0, \infty)$ , the range is all real numbers. The  $x$ -intercept of a logarithmic function is 1, while the  $y$ -intercept does not exist.

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**Example 1.5.4 Exponential and Logarithmic Forms**

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Express exponential forms in logarithmic forms and vice versa.

a)  $3^4 = 81$       b)  $2^{11} = 2048$       c)  $\log_4 1024 = 5$       d)  $\log_2 1/8 = -3$

**Solution:**

The equivalent logarithmic forms are:

a)  $\log_3 81 = 4$

b)  $\log_2 2048 = 11$

The equivalent exponential forms are:

c)  $4^5 = 1024$

d)  $2^{-3} = 1/8$

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Most calculators have two logarithm keys:  $\log x$  and  $\ln x$ . The first is for base 10 ( $\log_{10} x$ ) and the second is for base  $e$  ( $\log_e x$ ). The logarithms that are base 10 are called **common logarithms**, while the logarithms that are base  $e$  are called **natural logarithms**. For example, the common logarithm of  $x$  is written  $\log x$ , while the natural logarithm is written  $\ln x$ . Note that the bases (10 and  $e$ ) are usually omitted and distinguished by the spelling.

◆ Originally known as “*log naturalis*,” the natural logarithm acquired its name from studies of natural phenomena. Common logarithms are likely named for their association with the decimal system.

Logarithms can also have bases other than 10 and  $e$ . Useful properties of these logarithms and natural logarithms are:

### *Properties of Logarithms*

#### *Logarithms*

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$\log_b PQ = \log_b P + \log_b Q$$

$$\log_b (P/Q) = \log_b P - \log_b Q$$

$$\log_b P^r = r \log_b P$$

#### *Natural Logarithms*

$$\ln e^x = x$$

$$e^{\ln x} = x$$

$$\ln 1 = 0$$

$$\ln e = 1$$

$$\ln PQ = \ln P + \ln Q$$

$$\ln (P/Q) = \ln P - \ln Q$$

$$\ln P^r = r \ln P$$

### **Example 1.5.5**    *Simplifying Logarithms*

*Simplify these logarithms to eliminate products, quotients, or exponents.*

a)  $\log_3(x+1)(3x-5)$

c)  $\ln(x^2+2)^3(2y-1)^4(z+3)^2$

b)  $\log_7 \left( \frac{(x+2)^3(x-4)}{(x-5)^2} \right)$

**Solution:**

a) *The logarithm of a product is written as a sum of logarithms,*

$$\log_3(x+1) + \log_3(3x-5).$$

b) *First, write the logarithm as  $\log_7(x+2)^3 + \log_7(x-4) - \log_7(x-5)^2$ . Next, the exponents appear as coefficients as  $3\log_7(x+2) + \log_7(x-4) - 2\log_7(x-5)$ .*

c) *First, expand the logarithms as  $\ln(x^2+2)^3 + \ln(2y-1)^4 + \ln(z+3)^2$ . Then, using the exponents as coefficients yields  $3\ln(x^2+2) + 4\ln(2y-1) + 2\ln(z+3)$ .*

**Example 1.5.6 Simplifying Logarithmic Expressions**

Write as a single logarithm:

- a)  $\log_3(x + 1) + 2\log_3(x + 4) - 7\log_3(2x + 5)$
- b)  $5 \ln(x + 3) - 3 \ln(y + 2) - 2 \ln(z + 5)$

**Solution:**

- a) The coefficients, being exponents, yield  $\log_3(x + 1) + \log_3(x + 4)^2 - \log_3(2x + 5)^7$ . Next, the sum (difference) of logarithms becomes a product (quotient) as  $\log_3 \frac{(x + 1)(x + 4)^2}{(2x + 5)^7}$ . (Caution: combining logarithms requires a common base.)
- b) The coefficients become exponents when rewriting to yield  $\ln(x + 3)^5 - \ln(y + 2)^3 - \ln(z + 5)^2$ . The two terms with negative coefficients are placed in the denominator of the single logarithm to yield  $\ln \frac{(x + 3)^5}{(y + 2)^3(z + 5)^2}$ .

◆ The Richter scale is one example of a logarithmic measure. It was devised by Charles Richter at the California Institute of Technology in 1935 to compare magnitudes of earthquakes. An earthquake of magnitude 5 on the Richter scale is ten times as strong as a magnitude 4 on the scale and one-tenth as strong as a magnitude 6.

A major earthquake is of magnitude 7, while a magnitude 8 or more is called a great quake. The great quake can destroy entire communities. The Indian Ocean Quake of December 2004 measured 9.0 on the Richter scale. The resulting tsunami caused massive destruction in the surrounding areas.

The strongest recorded earthquake was the Great Chilean Earthquake of 1960. It measured 9.5 on the Richter scale.

Logarithms and exponentials, being inverses of each other, enable one to solve equations by exploiting common properties as shown in Examples 1.5.7 and 1.5.8.

**Example 1.5.7 Base 10 Exponents**

Solve the following using logarithms.

- a)  $10^x = 9.95$
- b)  $10^{2x+1} = 1050$

**Solution:**

- a) Since  $10^0 = 1$  and  $10^1 = 10$ , it follows that  $0 < x < 1$  and that  $x$  is very close to 1. Taking the logarithm of the equation (base 10, here) yields

$$\log_{10} 10^x = \log_{10} 9.95$$

$$x = \log_{10} 9.95 \approx 0.9978 \text{ (as obtained by calculator)}$$



which is in agreement with our estimate.

- b) Since  $10^3 = 1000$  and  $10^4 = 10,000$ , an estimate of  $x$  can be obtained from  $3 < 2x + 1 < 4$ , which yields  $1 < x < 1.5$ . Taking the logarithm of the equation (base 10, here) yields

$$\begin{aligned}\log_{10} 10^{2x+1} &= \log_{10} 1050 \quad \text{so} \\ 2x + 1 &= \log_{10} 1050 \\ x &= \frac{-1 + \log_{10} 1050}{2} \approx 1.0106\end{aligned}$$

Note this is easily estimated since  $\log_{10} 1000 = 3$ .

### Example 1.5.8 Base $e$ Exponents

Solve using logarithmic properties:

$$5e^{4x-1} = 100$$

**Solution:**

First, note that  $e^{4x-1} = 20$ . Since  $e^3 \approx 20$ , estimate  $4x - 1$  as 3 (with  $x$  near 1). Taking the natural logarithm of both sides (base is  $e$ , here) yields

$$\begin{aligned}\ln e^{4x-1} &= \ln 20 \\ 4x - 1 &= \ln 20 \\ x &= \frac{1 + \ln 20}{4} \approx 0.9989\end{aligned}$$

The value for  $x$  agrees with our preliminary estimate.

### EXERCISES 1.5

In Exercises 1–6, rewrite the expressions in the form of  $2^{kx}$  or  $3^{kx}$ .

- a)  $(8)^{3x}$                       b)  $(27)^{2x}$                       c)  $(16)^{5x}$
- a)  $(32)^{-2x}$                       b)  $(16)^{-x}$                       c)  $(81)^{-3x}$
- a)  $\left(\frac{1}{8}\right)^{-4x}$                       b)  $\left(\frac{1}{9}\right)^{6x}$                       c)  $\left(\frac{1}{27}\right)^{-2x}$
- a)  $\left(\frac{1}{4}\right)^{3x}$                       b)  $\left(\frac{1}{16}\right)^x$                       c)  $\left(\frac{1}{81}\right)^{-3x}$

$$5. \text{ a) } \frac{10^{5x}}{5^{5x}} \qquad \text{b) } \frac{32^{2x}}{16^{2x}} \qquad \text{c) } \frac{4^{3x}}{12^{3x}}$$

$$6. \text{ a) } \frac{20^{3x}}{5^{3x}} \qquad \text{b) } \frac{18^{7x}}{6^{7x}} \qquad \text{c) } \frac{6^x}{24^x}$$

*In Exercises 7–12, use the Laws of Exponents to simplify the expressions.*

$$7. \frac{7x^3x^5y^{-2}}{x^2y^4} \qquad 10. \left( \frac{5a^3b^4c^0}{10abc^2} \right)^{-3}$$

$$8. \left( \frac{2x^3}{y^4} \right)^{-2} \qquad 11. \frac{2^{5x+3}4^{x+1}}{8(2^{3x-1})}$$

$$9. \frac{x^3}{y^{-2}} \div \frac{x}{y^5} \qquad 12. \frac{9^{4x-1}27^{x+2}}{81^{x+3}}$$

*In Exercises 13–20, solve for  $x$ .*

$$13. 7^{3x} = 7^{15} \qquad 17. (1+x)5^x + (3-2x)5^x = 0$$

$$14. 10^{-3x} = 1,000,000 \qquad 18. (x^2)4^x - 9(4^x) = 0$$

$$15. 2^{7-x} = 32 \qquad 19. (x^2+4x)7^x + (x+6)7^x = 0$$

$$16. 3^{2x}3^{x+1} = 81 \qquad 20. (x^3)5^x - (7x^2)5^x + (12x)5^x = 0$$

*In Exercises 21–25, find the missing factor.*

$$21. 2^{3+h} = 2^h ( \quad ) \qquad 25. 7^{3h} - 8 = (7^h - 2) ( \quad )$$

$$22. 3^{5+h} = 3^5 ( \quad ) \qquad 26. \text{ Graph } y = \log_3 x$$

$$23. 7^{x+5} - 7^{2x} = 7^{2x} ( \quad ) \qquad 27. \text{ Graph } y = \ln x$$

$$24. 5^{2h} - 9 = (5^h - 3) ( \quad )$$

*In Exercises 28–37, evaluate the logarithms.*

$$28. \log_{10} 1,000,000 \qquad 33. \log_3 \frac{1}{81}$$

$$29. \log_3 243 \qquad 34. \ln e^3$$

$$30. \log_2 64 \qquad 35. \ln e^7$$

$$31. \log_5 125 \qquad 36. \ln e^{7.65}$$

$$32. \log_2 \frac{1}{32} \qquad 37. \ln e^{-3.4}$$

*In Exercises 38–45, evaluate the expressions.*

- |                     |                    |
|---------------------|--------------------|
| 38. $\ln(\ln e)$    | 42. $\log_4 32$    |
| 39. $e^{\ln 1}$     | 43. $\log_5 625$   |
| 40. $\log_9 27$     | 44. $\log_2 128$   |
| 41. $\log_{25} 125$ | 45. $\log_4(1/64)$ |

*In Exercises 46–53, solve for  $x$ .*

- |                            |                         |
|----------------------------|-------------------------|
| 46. $\log_x 27 = 3$        | 50. $\ln 5x = \ln 35$   |
| 47. $\log_x 64 = 3$        | 51. $e^{3x} e^{2x} = 2$ |
| 48. $\log_3(5x + 2) = 3$   | 52. $\ln(\ln 4x) = 0$   |
| 49. $\log_2(x^2 + 7x) = 3$ | 53. $\ln(7 - x) = 1/3$  |

54. Write as a set of simpler logarithms:  $\log_4 \frac{(x+1)^2(x-3)^6}{(3x+5)^3}$ .

55. Write as a set of simpler logarithms:  $\ln \frac{(x-1)^4}{(2x+3)^5(x-4)^2}$ .

56. Write as a single logarithm:  $2 \ln x - 3 \ln(y+1) + 4 \ln(z+1)$ .

57. Write as a single logarithm:  $\ln 2 - \ln 3 + \ln 7$ .

*In Exercises 58–61, solve for  $x$ .*

- |                           |                     |
|---------------------------|---------------------|
| 58. $10^{2x-1} = 105$     | 60. $3e^{x-1} = 4$  |
| 59. $10^{3x-1} = 100,100$ | 61. $e^{3x+1} = 22$ |

62. In June of 2004, an earthquake of magnitude 4.1 on the Richter scale struck northern Illinois. Its effects were felt from Wisconsin to Missouri and from western Michigan to Iowa. Another earthquake in the Midwest, the 1895 “Halloween Earthquake,” is estimated to be 6.8 on the Richter scale. How much stronger was the Halloween quake than the northern Illinois quake?

## 1.6 VARIATION

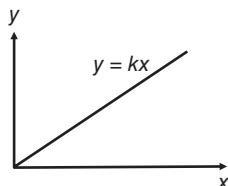
Scientific formulas can express many types of variation. In **direct variation**, two related quantities can both increase or both decrease. For example, hourly pay,  $y$ , varies directly with the number of hours worked,  $x$ . In symbols,  $y = kx$ , where  $k$ , the hourly wage, is a **proportionality constant**. A graph of  $y = kx$  is a line passing through the origin.



**Direct Variation**

$$y = kx$$

where  $k$  is a proportionality constant.

**Example 1.6.1 Summer Wages**

A student's summer job pays \$9.50 an hour. Write an equation for total wages earned. How much is earned in 30 hours?

**Solution:**

Let  $y$  represent total wages and  $x$  the hours of work. Here, the constant of proportionality is the hourly wage of \$9.50, so total wages is  $y = 9.50x$ . The student earns \$285 for 30 hours of work.

Physicists use *Hooke's law* for a spring to relate the force,  $y$ , directly to its elongation,  $x$  within its elastic limit. That is,  $y = kx$ , where  $k$ , called the *spring constant*, is the constant of proportionality. Architectural design has used proportionality since antiquity.

**Example 1.6.2 Hooke's Law**

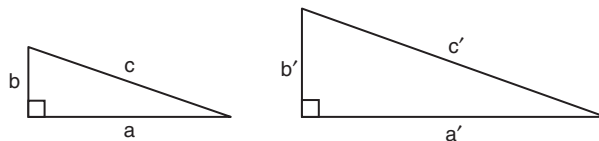
If a force of 250 Newtons stretches a spring 20 centimeters (within the elastic limit), how much will it be stretched by a 100 Newton force?

**Solution:**

Here, the force  $y$  is 250 and the elongation  $x$  is 20, so  $250 = 20k$ . The spring constant  $k$  is 12.5 Newtons per centimeter. Hooke's law for this spring is  $y = 12.5x$ . Therefore,  $100 = 12.5x$  implies that  $x = 8$  centimeters.

**Geometric similarity**, related to proportionality, can usefully simplify mathematical modeling. In geometric similarity, there is a one-to-one correspondence among objects such that the ratio of distances between their corresponding points is constant. Recall that triangles are similar if their angles have the same measure. For the similar triangles shown:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$




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**Example 1.6.3 Geometric Similarity**


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For the above similar triangles,  $a = 2$ ,  $b = 3$ ,  $b' = 9$ . Find the length  $a'$ .

**Solution:**

Here, the proportion is  $\frac{2}{a'} = \frac{3}{9}$ . Therefore,  $(a')(3) = (2)(9)$ . Solving, the length  $a' = 6$ .

---

Direct proportionality need not be linear. That is, it may **vary directly as a power** of  $x$ . A mathematical “power model” takes the form

$$y = kx^n, \quad n > 0$$

For example, the area of a circle,  $A$ , varies directly as the square of its radius,  $r$ , as  $A = \pi r^2$ .

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**Example 1.6.4 Volume of a Sphere**


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The volume of a sphere,  $V$ , varies directly as the cube of its radius,  $r$ . If  $V = 36\pi$  when  $r = 3$ , determine the proportionality constant.

**Solution:**

Since volume varies as the cube of  $r$ ,  $V = kr^3$ . Therefore,  $36\pi = k(3)^3$  implies  $k = 36\pi/27$  or  $(4/3)\pi$  (the volume of a sphere is well-known as  $V = (4/3)\pi r^3$ ).

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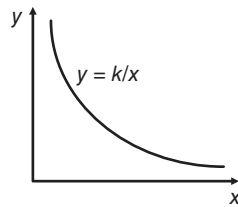
In another form of variation, **inverse variation**, one quantity increases as the other decreases, and it is expressed as  $y = k/x$  or  $xy = k$ .

**Inverse Variation**

$$y = k/x \quad (\text{or } xy = k)$$

**where  $k$  is a proportionality constant.**

While a graph of simple direct variation was a line through the origin, simple inverse variation is a *rectangular hyperbola* in the first quadrant when  $x$  and  $y$  are positive valued (shown).



As examples, for light and sound, wave frequency,  $\nu$ , varies inversely with wavelength,  $\lambda$ , so  $\lambda\nu$  is a constant; the pressure,  $P$ , of a gas at constant temperature varies inversely with its volume,  $V$ , so  $PV = R$ , a constant, expresses Boyle's law; and so on.

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**Example 1.6.5**     *Seawater Density*

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Chemists use the inverse variation of volume with mass to calculate density, defined as mass/volume. Calculate the density when 257.5 grams of seawater occupy a volume of 250 milliliters.

**Solution:**

Recall that a milliliter is equivalent to a cubic centimeter so the volume of seawater is 250 milliliters = 250 cubic centimeters. Its mass is 257.5 grams so its density is

$$\text{Density} = 257.5 \text{ grams} / 250 \text{ cubic centimeters} = 1.03 \text{ grams per cubic centimeter.}$$

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More general inverse variation, say, as the  $n^{\text{th}}$  power of  $x$ , can be expressed as

$$y = \frac{k}{x^n}, \quad n > 0.$$

Newton's law of gravitation has the force of attraction,  $F$ , between two masses varying inversely as the square of the distance between them,  $x$ . That is,  $F = k/x^2$ , where  $k$  is the constant of proportionality. Similarly, illumination from a light source varies as the inverse square of distance from the light source.

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**Example 1.6.6**     *Illumination*

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If illumination,  $I$ , is 75 units at a distance  $d = 6$  meters from a light source, express the relation between illumination and distance from the source.

**Solution:**

The inverse square relationship takes the form  $I = k/d^2$ , where  $k$  is a proportionality constant. Here  $75 = k/(6)^2 = k/36$ . Therefore,  $k = 75(36) = 2700$  illumination units per square meter. The formula relating illumination and distance here is  $I = 2700/d^2$ .

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There are other and more complicated forms of variation. Many chemical and physical relationships exhibit some type of variation. For example, the strength of a beam is related to the amount and kind of material used and to the shape of the beam. Rectangular beams add significant strength to structures so they are often used in bridges and buildings.

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**Example 1.6.7    Beam Strength**

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*The strength of a rectangular beam varies jointly as its width and the square of its depth. If the strength of a beam 2 inches wide by 8 inches deep is 768 pounds per square inch, what is the strength of a beam 4 inches wide and 6 inches deep?*

**Solution:**

*Let  $S$  represent the beam's strength,  $w$  its width, and  $d$  its depth. Accordingly, the model is  $S = kwd^2$ , where  $k$  is a proportionality constant. From the data  $768 = k(2)(8)^2$ , from which  $k = 6$ . Therefore, for the second beam,  $S = 6(4)(6)^2 = 864$  is its strength in pounds per square inch.*

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Empirical economic relationships of supply and demand; money supply and inflation; short- and long-term interest rates; and so on often have complex forms of variation.

**EXERCISES 1.6**

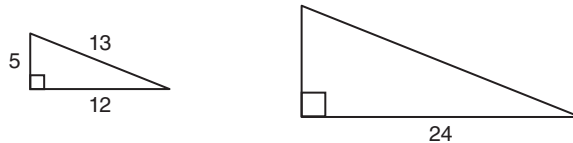
1. Describe the indicated variation:
  - a) Speed of a runner and distance traveled.
  - b) Speed of a Nascar driver and time to complete a race.
  - c) A person's adjusted gross income and the income tax rate.
  - d) The area of a circle and its diameter.
  - e) The distance between two rooms on a blueprint and their actual distance.
2. Let  $y$  vary directly as  $x$ . If  $x$  is 3 when  $y$  is 6, find  $x$  when  $y$  is 18.
3. Let  $y$  vary directly as the cube of  $r$ . If  $y$  is 32 when  $r$  is 4, find  $y$  when  $r$  is 6.
4. Let  $y$  vary inversely as the square of  $x$ . If  $y$  is 18 when  $x$  is 2, find  $y$  when  $x$  is 3.
5. Let  $y$  vary directly as  $x$ . When  $y$  is 15,  $x$  is 10. Determine  $x$  when  $y$  is 22.5.
6. Let  $y$  vary inversely as  $x$  in the previous exercise, determine  $x$  when  $y$  is 22.5.
7. Let  $x$  vary directly as  $y$  as depicted in the table. Determine the relation between  $x$  and  $y$ .

$x$	2	4	8
$y$	12.5	25	50

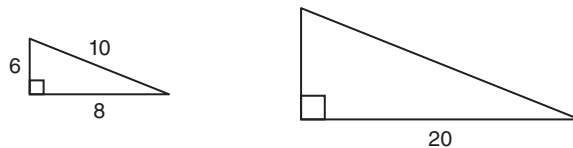
8. Let  $x$  vary inversely as  $y$ . Complete the following table.

$x$	1	2		4
$y$	36		12	

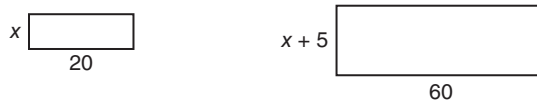
9. The area,  $A$ , of a circle is directly proportional to the square of its diameter,  $d$ . Write an equation relating area and diameter. Determine the proportionality constant if an area is  $36\pi$  square inches when the diameter is 12 inches.
10. The circumference,  $C$ , of a circle varies directly with its diameter,  $d$ . If the diameter is 9, the circumference is  $9\pi$ . What is the relation between the area of the circle and its diameter?
11. The volume,  $V$ , of a right circular cone is  $(1/3)\pi r^2 h$ . Find the volume when the height,  $h$ , is 10 centimeters and the radius,  $r$ , is 3 centimeters.
12. Suppose drug dosage is directly proportional to a patient's weight. A dose of 30 milligrams is prescribed for a patient weighing 50 kilograms.
  - a) Find the constant of proportionality.
  - b) What is the appropriate dosage for a 70-kilogram patient?
13. A force of 400 Newtons stretches a spring 10 centimeters (within the elastic limit). What is the stretch by a force of 250 Newtons?
14. The intensity of light,  $I$ , on a screen varies inversely as the square of the distance,  $d$ , from the source. If the intensity is 31.25-foot candles at a distance of 12 feet, find the intensity at 15 feet.
15. Consider the similar triangles. Find the length of the hypotenuse and the shorter side of the larger triangle.



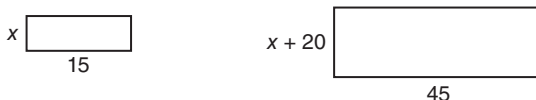
16. Consider the similar triangles. Find the length of the hypotenuse and the shorter side of the larger triangle.





17. For the pair of similar figure, determine the length of the unknown side,  $x$ .



18. For the pair of similar figure, determine the length of the unknown side,  $x$ .



19. An area of a triangle,  $A$ , varies jointly as the length of its base,  $b$ , and its height,  $h$ . If the area of a triangle is 50 square inches when the base is 20 inches and the height is 5 inches, find the area of a triangle whose base is 16 inches and height is 8 inches.
20. Identify several examples of variation encountered in your courses this semester.
-  21. Conduct an Internet search for NASA CONNECT. How does this program deal with proportionality?
-  22. How does the “principle of humanitarian law” relate to proportionality and military actions? Hint: use an Internet search.
23. Have a friend measure the distance between the tips of the middle fingers of your outstretched arms. Compare this distance to your height. Are you more a “rectangle” or more a “square”?
- When the ratio of the fingertip to height measurement is about one, a person is classified as a “square”, otherwise as a “rectangle.”
- Repeat the measurements for five individuals noting their gender. Find each person’s constant of proportionality.
  - Share the data with class members to obtain a larger data set. Calculate the average constant of proportionality for the group. What are the largest and smallest values for the group?
  - Reevaluate your class data separating individuals by gender. Does the average proportionality constant differ with gender? What about the largest and smallest values?

### 1.7 UNIT CONVERSIONS AND DIMENSIONAL ANALYSIS

While many quantities of interest can be expressed in different systems of units, each is considered to be of the same **dimension**. Mass or length, for example, can be expressed in units of grams, ounces, pounds, and so on, or centimeters, inches, feet, and so on, respectively. However, mass ( $M$ ) and length ( $L$ ) are fundamental dimensionalities. Many, if not most, physical and chemical quantities can be expressed in fundamental dimensions of mass, length, and time ( $T$ ).

Numbers, including transcendental numbers such as  $\pi$  and  $e$ , and trigonometric quantities are considered to be dimensionless.

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**Example 1.7.1**     *Dimensional Correctness*

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Show that  $I = prt$  and  $s = (1/2)gt^2$  are dimensionally correct.



**Solution:**

Interest,  $I$ , has dimensions of dollars or, in symbols, \$. Similarly, principal is in \$; and interest,  $r$ , in dollars per dollar per unit time,  $\$/\$T$

Therefore,  $I = prt$ , ( $\$ = \$ \frac{\$}{\$T}$ ) is dimensionally correct.

Similarly,  $s$  has the dimension of a length  $L$ , while gravitational acceleration  $g$  is dimensionally  $L/T^2$ . The coefficient of  $1/2$  is dimensionless. Therefore,

$$L = \frac{L}{T^2} \cdot T^2 = L,$$

and is dimensionally correct. Note that dimensional correctness does not necessarily mean that the relation is correct since it may lack dimensionless quantities.

Called **dimensional analysis**, the dimensionality of each side of an equation must be the same or the expression cannot possibly be correct. Scientists use this property of dimensional correctness to aid in the exploration and development of relationships. For example, if one has written the law of freefall as  $s = (1/2)t^2$ , it is immediately clear that this cannot be correct since the length on the left side could not possibly be equivalent to units of time squared. Note, again, that dimensional analysis, while valued and useful, cannot vouch for the complete correctness of a relation since numeric coefficients are dimensionless.

**Example 1.7.2 Speed Conversion**

Express 55 miles per hour in centimeters per second.

**Solution:**

Several conversion factors are needed:

1 mile = 5280 feet, 1 foot = 12 inches, 1 inch = 2.54 centimeters,

1 hour = 60 minutes, and 1 minute = 60 seconds. Therefore,

$$55 \text{ mph} = \frac{55 \text{ miles}}{\text{hour}} \cdot \frac{5,280 \text{ feet}}{1 \text{ mile}} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{2.54 \text{ cm}}{1 \text{ inch}} \cdot \frac{1 \text{ hour}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} = 2458.72 \frac{\text{cm}}{\text{sec}}$$

**Example 1.7.3 Time Conversion**

Express 2.5 years in hours.

**Solution:**

$$2.5 \text{ years} \cdot \frac{365 \text{ days}}{\text{year}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} = 21,900 \text{ hours}$$

Since mass is conserved in a chemical reaction, establishing a scale of atomic masses for the elements is a necessity. A *mole* of a substance contains  $6.022 \cdot 10^{23}$  atoms of the

substance. This number is *Avogadro's* number. This, of course, is an immense quantity. For example,  $6.022 \cdot 10^{23}$  spheres each about 14 centimeters in diameter would fill a volume as large as the earth.

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**Example 1.7.4**    *Atoms of Nitrogen*

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How many atoms of nitrogen (*N*) are present in 21 grams of nitrogen?

**Solution:**

The periodic table lists the weight of a mole of nitrogen as approximately 14 grams.

Changing units from grams to atoms:

$$21\text{g N} \cdot \frac{1 \text{ mole N}}{14\text{g N}} \cdot \frac{6.022 \cdot 10^{23} \text{ atoms N}}{1 \text{ mole N}} = 9.033 \cdot 10^{23} \text{ atoms of nitrogen.}$$

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**Example 1.7.5**    *Atoms of Hydrogen*

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How many atoms of Hydrogen (*H*) are present in 36 grams of water ( $H_2O$ ) molecules?

**Solution:**

From the periodic table, a mole of water weighs 18 grams since the chemical formula indicates that each molecule of water has two atoms of hydrogen and one atom of oxygen:

$$36\text{g H}_2\text{O} \cdot \frac{1 \text{ mole H}_2\text{O}}{18\text{g H}_2\text{O}} \cdot \frac{6.022 \cdot 10^{23} \text{ molecule H}_2\text{O}}{1 \text{ mole H}_2\text{O}} \cdot \frac{2 \text{ atoms H}}{1 \text{ molecule H}_2\text{O}} = 2.41 \cdot 10^{24} \text{ atoms of hydrogen.}$$



◆ In 2002, a black hole streaked through the Milky Way galaxy. In about 200 million years, it is predicted to be as close as 1000 light years from our solar system. How many miles correspond to 1000 light years? (A light year is the distance a light beam travels in 1 year.)

Felix Mirabel of the French Atomic Energy Commission estimated:

“The probability of a catastrophic event on the Earth from a black hole moving at high speed is almost zero compared with the probability of a catastrophic event from asteroids or comets.”



## EXERCISES 1.7

1. Express 5 miles in meters. (Use 1 meter = 39.37 inches, 1 mile = 5280 feet)
2. Express 1 yard in centimeters (Use 2.54 centimeters = 1 inch)
3. Express 100 kilometers per hour in miles per hour.
4. A car travels at 50 miles per hour . What is its speed in yards per second?
5. Express 50 miles per hour in centimeters per second.
6. What is the weight of 75-kilogram person in pounds? (Use 453.6 grams = 1 pound)
7. A watermelon weighs 4.3 pounds. What is its weight in grams?  
(Use 1 pound = 453.6 grams)
8. How many ounces are in a 2-liter soda bottle? (Use 1 ounce = 29.6 milliliters)
9. A block of zinc occupies a volume of 45 milliliters and weighs 320.85 grams. What is the density of zinc?
10. A gold ingot is 6 centimeters  $\times$  3.5 centimeters  $\times$  2.7 centimeters. If the ingot weighs 1095.44 grams, what is the density of gold?
11. What is the weight in pounds of a liter of mercury? The density of mercury is 13.55 grams per cubic centimeter.
12. If the density of copper is 559 pounds per cubic foot, express its density in grams per cubic centimeter.
13. How many atoms are there in 30 grams of carbon (C)?
14. How many atoms of oxygen (O) are there in 100 grams of carbon dioxide (CO<sub>2</sub>)?
15. A gas company is to store 5 tons of a gas with a density of 5.83 pounds per gallon. What tank volume is needed to store the gas?
16. Glucose is 40% C, 6.71% H, and 53.29% O (by weight). Determine its chemical formula if its weight is 180 grams.
17. Laughing gas is 63.6% nitrogen and 36.4% oxygen by weight. Determine its chemical formula if its weight is 44 grams.
18. A keg of beer contains 15.5 gallons. How many cases of twenty-four 12-ounce cans can be filled? Use dimensional analysis.
-  19. Can the linear dimensions of an acre be used to determine the number of square inches in 5 acres? Why? Hint: use the definition of an acre.
-  20. The velocity of light (in vacuum) is 186,000 miles per second. If light from another planet requires 45 minutes to reach earth, how many kilometers distant is the planet? What is the planet's name? Hint: use the Internet to find interplanetary distances.

## HISTORICAL NOTES AND COMMENTS

According to the Oxford Dictionary of the English Language (OED), the word **algebra** was widely accepted by the seventeenth century. It is believed to have origins in early Arabic

language from root words as “reunion of broken parts” and “to calculate.” The word “algebra” has contexts other than in mathematics as, for example, in the surgical treatment of fractures and bone setting and as a branch of mathematics that investigates the relations and properties of numbers by means of general symbols; and, in a more abstract sense, a calculus of symbols combined according to certain defined laws.

◆ Kinship relationships, architecture, sewing, weaving, agriculture, and spiritual or religious practices are examples of the human activities that can be expressed in mathematical expressions. The study of mathematics in the context of different cultures is called *Ethnomathematics*.

**René Descartes (1596–1650)** – Descartes, born in France in 1596, the son of an aristocrat, traveled throughout Europe studying a wide variety of subjects including mathematics, science, law, medicine, religion, and philosophy. He was greatly influenced by other thinkers of the Age of Enlightenment.

Descartes ranks as one of the most important and influential thinkers in history and is sometimes called the founder of modern philosophy. In addition to his accomplishments as a philosopher, he was an outstanding mathematician, founding analytic geometry and seeking simple universal laws that governed all physical changes.

Mathematics was probably Descartes greatest interest; building upon the works of others, he originated Cartesian coordinates and Cartesian curves. To algebra, he contributed the treatment of negative roots and the convention of exponent notation.

**Leonhard Euler (1707–1783)** – Euler is often considered one of the most prolific mathematicians of his time. Author of 866 books and papers, he won the Paris Academy Prize 12 times. Born in Switzerland, he was the son of a Lutheran minister. Originally a theology student, he changed to mathematics under the influence of Johann Bernoulli. He made significant contributions in differential calculus, mathematical analysis, and number theory. He introduced the symbols  $e$ ,  $i$ ,  $f(x)$ ,  $\pi$ , and the sigma summation sign. Remarkably, most of his publications appeared in the last 20 years of his life when he was blind.

**Christiaan Huygens (1629–1695)** – Huygens, born in the Netherlands in 1629, was the son of an important diplomat. At age 16, he entered Leiden University where he studied mathematics and law.

His first published work in 1651 displayed his geometrical skills by showing the fallacy in methods that had been proposed that claimed to square the circle. Huygens was also interested in astronomy and turned his attention to lens grinding and telescope construction. He discovered Saturn’s major satellite Titan. As work in astronomy required accurate time keeping, he took an interest and patented the first pendulum clock. Huygens’ mathematical education was influenced by Descartes occasional visits to his home.

**John Napier (1550–1617)** – born in Edinburgh (Scotland) little is known of his early years. He entered St. Andrews University at age 13. He is later believed to have completed his studies elsewhere in Europe. Napier’s study of mathematics as a hobby led to his invention of logarithms and introducing decimal notation for fractions. He is also known for his “Napier’s Bones,” used to mechanically multiply, divide, and to obtain square roots and cube roots.

◆ The web site, “The Sound of Mathematics” has files of mathematically related algorithmic music. Files for  $e$ ,  $\pi$ , and other mathematical numbers are posted.  
<http://www.geocities.com/Vienna/9349>

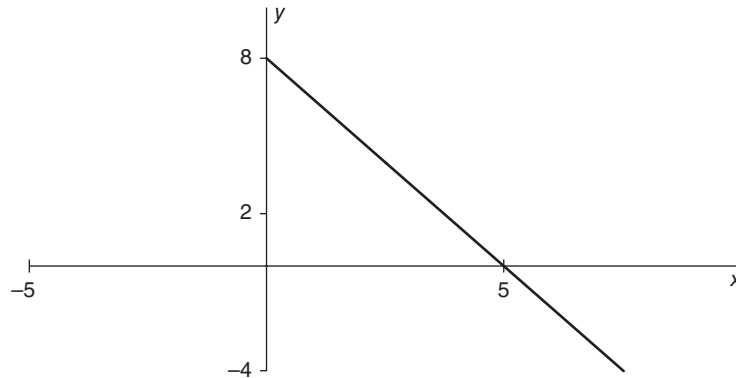
**SUPPLEMENTARY EXERCISES**

1. To earn a B grade, a student must receive an 82.6 average on three exams and a final exam score, which is weighted as two exams. If three test scores are 76, 89, and 70, what is the lowest final exam score for a B grade?
2. The price of a car including a 4% sales tax is \$19,500. What is the price excluding the tax?
3. Federal personal income tax rates depend on marital status and adjusted taxable income. Some rates for 2014 are:

Single Filers Between, \$	Married Filing Jointly, \$	Married Filing Separately, \$	Tax Rate, %
0–9,075	0–18,150	0–9,075	10
9,075–36,900	18,150–73,800	9,075–36,900	15
36,900–89,350	73,800–148,850	36,900–74,425	25
89,350–186,350	148,850–226,850	74,425–113,425	28
186,350–405,100	226,850–405,100	113,425–202,550	33
405,100–406,750	405,100–457,600	202,550–228,800	35
40,6750 or more	457,600 or more	228,800 or more	39.6

- a) What Federal tax is due for a single filer with an adjusted taxable income of \$250,000?
- b) What Federal tax is due for a married couple filing jointly with an adjusted taxable income of \$250,000?
- c) A couple’s taxable income is \$250,000. If one person earns \$200,000 and the other \$50,000, is it advantageous for them to file separately?
4. The 2014 Federal Minimum Wage is \$7.25 per hour. Oregon and Washington annually index their state’s minimum wage to inflation. Several other states also have minimum wages that exceed the Federal rate. For instance, the minimum wage in Connecticut is \$8.70, Alaska \$7.75, California \$8.00, Maine \$7.55, and Delaware \$7.75. How much more will a person who works for 37.5 hours per week at minimum wage in Connecticut earn in a year (52 weeks) than a person
  - a) in Maine?
  - b) for the Federal minimum wage?

5. Find an equation of the line below:



6. Determine the slope-intercept form for these lines:

a)  $3x + 4y = 6$ .

c)  $y - 5 = 2(x - 3)$

b)  $y - 3x = 2$

d)  $4y = 8x + 12$

7. a) Find the equation of a line passing through (3, 5) and perpendicular to the line with  $x$ -intercept of 2 and  $y$ -intercept of 5.

b) Find an equation of a horizontal line passing through (1, 4).

c) Find an equation of a line passing through (2, 7) and perpendicular to the  $x$ -axis.

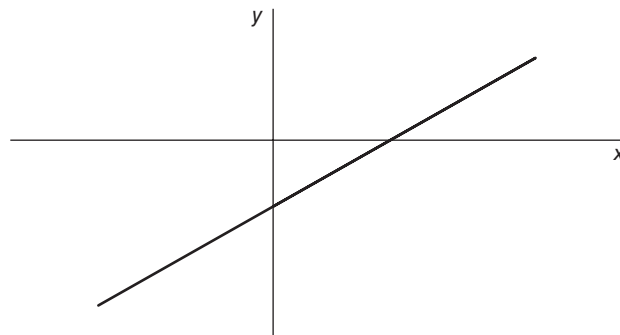
8. The graph shown is most likely which of the following? Explain your reasoning.

a)  $3x + 4$

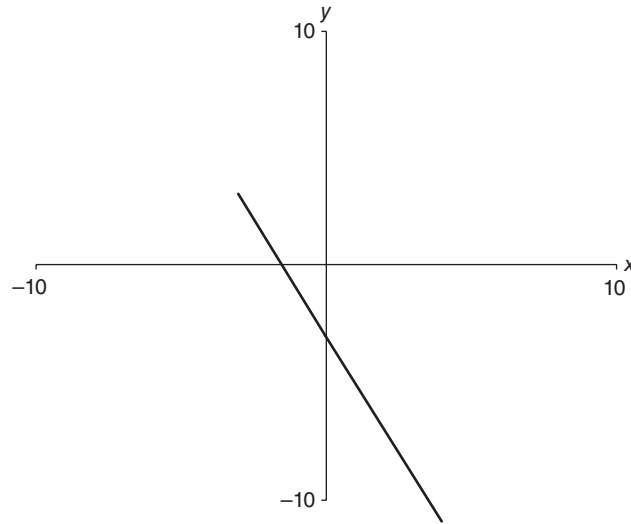
b)  $-3x + 4$

c)  $3x - 4$

d)  $-3x - 4$

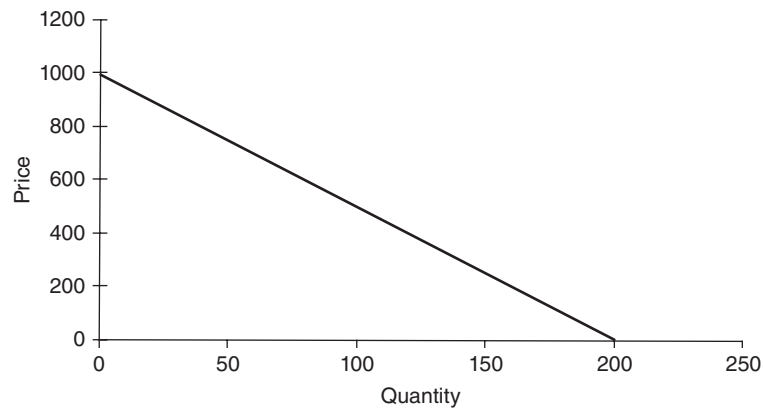


9. The graph shown is most likely which of the following? Explain your reasoning.  
 a)  $2x + 3$       b)  $-2x + 3$       c)  $2x - 3$       d)  $-2x - 3$



10. An electric utility's bill adds an energy charge of 6.45 cents per kilowatt hour to its base charge of \$7.95 per month. Write an equation for the monthly charge,  $y$ , in terms of,  $x$ , the number of kilowatt hours.
11. A water utility's quarterly bill has a fixed charge of \$40 and \$2.25 for each thousand gallons of water used. What is the bill for a customer who uses 12,000 gallons of water during the quarter?
12. A chemistry student is to make 20 milliliters of an 8% solution from two solutions that are of 12.5% and 5% concentrations. How many milliliters of each solution should be mixed?
13. How should 50 milliliters of 20% sulfuric acid be made by mixing 100% sulfuric acid with water?
14. How many liters of a 50% solution of acid should be added to 10 liters of a 20% solution to obtain a 30% solution?
15. At a charity event, 25 donated items were sold for either \$50 or \$100 each. A total of \$1650 was raised. How many more \$50 items than \$100 items were sold?
16. A concert promoter raises \$600,000 on the sale of 25,000 tickets. If the tickets were \$20 in advance sale and \$25 at the door, how many were sold in advance?
17. A business pays a monthly rent of \$2000. It costs \$2 for each item produced. The items are sold for \$6 each. Find the  
 a) revenue function;  
 b) cost function;  
 c) break-even point;  
 d) profit/loss if 1000 items are sold this month.

18. The demand function for a commodity is shown in the graph.



- Determine the demand function.
  - What is the price when demand is 50 units?
  - What quantity is demanded when the price is 600?
19. There are many interesting facts about pi on the Internet. Use websites to answer these questions.
- How many 0s, 1s, ..., 9s are in the first million digits of pi?
  - Do any of the digits seem to appear more frequently than others?
20. Rewrite  $125^{2x}$  in the form  $5^{kx}$ .
21. Use the Laws of Exponents to simplify  $\frac{125^{2x-3}25^{x+1}}{5^{4x-5}}$ .
22. Solve  $x^3(4^x) - 13x^2(4^x) + 36x(4^x) = 0$ .
23. Evaluate  $\log_4 128$ .
24. Write as a single logarithm:

$$4 \ln(x + 1) - 2 \ln(y + 3) - (1/2) \ln(z + 4).$$

25. Solve  $4e^{x-1} = 20$ .
26. Let  $y$  vary directly as the cube of  $x$ . If  $y = 128$  when  $x = 4$  find  $y$  when  $x = 5$ .
27. Let  $y$  vary inversely as  $x$ . Complete the table.

$x$	2	4		12	
$y$	24		8		48

28. Convert 5 yards to centimeters.
29. How many atoms of carbon (C) are present in 54 grams of carbon?