1 INTRODUCTION TO *N*-DIMENSIONAL GEOMETRY

1.1 FIGURES IN *N*-DIMENSIONS

There is much to be said for a hands-on approach in geometry. We enhance our understanding in two dimensions by drawing plane figures or by experimenting with polygons cut out of paper. In three dimensions, we can construct polyhedrons out of cardboard, wire, or plastic straws. In some cases, we may use a computer program to provide a visual representation. Although everyone knows that a picture does not constitute a "proof," there is no doubt that a decent diagram can be utterly convincing.

However, we cannot make a full-dimensional model of an object that has more than three dimensions. We cannot visualize a cube of four or five dimensions, at least not in the usual sense. Nevertheless, if you can imagine that "hypercubes" of four or five dimensions exist, you might suspect that they are a lot like three-dimensional cubes. You would be correct. Geometry in four or higher dimensions is quite similar to geometry in two and three dimensions, and although we cannot visualize space of any more than three dimensions, we *can* build up a fairly reliable intuition about what such a space is like.

In this section, we examine some n-dimensional objects and learn how to work with them. We begin with the higher dimensional analogs of points, lines, planes, and spheres. We find out that these can be described in a *dimension-free* way and that their geometric properties are as they should be.

We also examine how geometric notions such as perpendicular and parallel lines extend to higher dimensions. The emphasis will be on how n-dimensional space is like two- or three-dimensional space, but we also examine some of the ways in which they differ.

Geometry of Convex Sets, First Edition. I. E. Leonard and J. E. Lewis.

Geometry at this level necessarily involves proofs. Even if you have already had a geometry course, you will find that some of the techniques are new. In this first section, we spend some time explaining how a proof proceeds before actually carrying it out. We apologize to anyone who is already familiar with the methods.

1.2 POINTS, VECTORS, AND PARALLEL LINES

1.2.1 Points and Vectors

We denote the usual two-dimensional coordinate space by $\mathbb{R}^2$, three-dimensional coordinate space by $\mathbb{R}^3$, four-dimensional coordinate space by $\mathbb{R}^4$, and so on. In general, $\mathbb{R}^n$ denotes n-dimensional space (the n is not a variable—it stands for some fixed nonnegative integer) and is the set of all n-tuples of real numbers, that is,

$$\mathbb{R}^n = \{\, (x_1, x_2, \ldots, x_n) \;:\; x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n \,\}.$$

Points in $\mathbb{R}^n$ will be denoted by italic letters such as x, y, and z. Numbers are usually (but not always) denoted with lowercase Greek letters, α, β, γ, and so on.*

A ***point*** in $\mathbb{R}^n$ is just an n-tuple and can also be described by giving its coordinates. In $\mathbb{R}^4$, for example, the ordered 4-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ denotes the point whose ith coordinate is α_i. The ***origin*** $(0, 0, 0, 0)$ is denoted by $\overline{0}$ (the bar is used to avoid confusion with the real number 0). The symbol $\overline{0}$ is also used to denote the origin in any space $\mathbb{R}^n$.

Points from the same space can be added together, subtracted from each other, and multiplied by scalars (that is, real numbers), and these operations are performed coordinatewise.

* The Greek alphabet is as follows:

A	α	Alpha	I	ι	Iota	P	ρ, ϱ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ, ς	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upsilon
E	ϵ, ε	Epsilon	N	ν	Nu	Φ	ϕ, φ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π, ϖ	Pi	Ω	ω	Omega

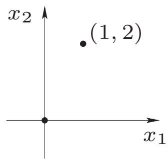
The alternate pi (ϖ), sigma (ς), and upsilon (υ) are very seldom used.

If $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$ are two points in $\mathbb{R}^n$ and if ρ is any number, then we define

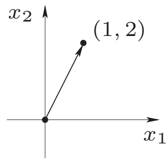
- $x = y$ if and only if $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, n$ (***equality***)
- $x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$ (***addition***)
- $\rho x = (\rho\alpha_1, \rho\alpha_2, \ldots, \rho\alpha_n)$ (***scalar multiplication***).

With these definitions of addition and scalar multiplication, $\mathbb{R}^n$ becomes an n-dimensional vector space.*

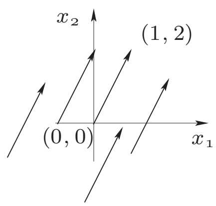
In other words, ***points*** behave algebraically as though they were ***vectors***. As a consequence, a notation such as $(1, 2)$ can be interpreted as a vector as well as a point. The two interpretations may be tied together geometrically by thinking of the vector as an arrow whose tail is at the origin and whose tip is at the point $(1, 2)$, as in the figure below.



or



A more abstract notion of a vector is as a class of arrows (an equivalence class, to be more exact), each arrow in the class having the same length and pointing in the same direction. Any particular arrow from the class of a given vector is called a ***representative*** of the vector. The arrow, in the example above, whose tail is at the point $(0, 0)$ and whose tip is at the point $(1, 2)$, is but one of infinitely many representatives of the vector, as in the figure below.



Note. Representatives of a given vector are called ***free vectors***, since apart from their length and direction, there is no restriction as to their position.

Although a given n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ has infinitely many free vectors associated with it, the n-tuple always represents *one and only one point* in $\mathbb{R}^n$. When we refer to *the* point associated with a given vector, we mean that point.

* Unlike the situation in synthetic geometry, the notions of point and line are no longer primitive or undefined terms but are defined in terms of n-tuples (coordinatized space).

1.2.2 Lines

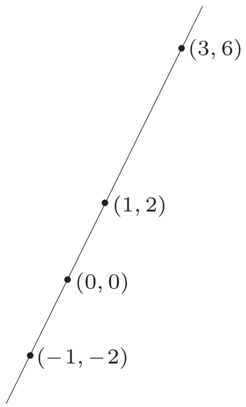
If we were to multiply the point $v = (1, 2)$ by the numbers

$$-1, \qquad \tfrac{1}{2}, \qquad \text{and} \qquad 3,$$

we would get the points

$$(-1, -2), \qquad (\tfrac{1}{2}, 1), \qquad \text{and} \qquad (3, 6),$$

respectively. The three points all lie on the straight line through $\overline{0}$ and v, as in the figure below.



If μ is *any* number, then the point μv also lies on the straight line through $\overline{0}$ and v. This is not just a property of $\mathbb{R}^2$: if v is any point other than $\overline{0}$ in $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$, then the point μv also lies on the straight line through $\overline{0}$ and v.

In fact, we can describe the line L through $\overline{0}$ and v algebraically as the set of all multiples of the point v:

$$L = \{\, \mu v \;:\; -\infty < \mu < \infty \,\}.$$

The line L passes through the origin.

To obtain a line M parallel to L, but passing through a given point p, we simply add p to every point of L:

$$M = \{\, p + \mu v \;:\; -\infty < \mu < \infty \,\},$$

since p is on M (take $\mu = 0$) and the vector v is parallel to L.

The above equations for L and M are also used to describe lines in spaces of dimension $n > 3$. These equations, in fact, define what is meant by straight lines in higher dimensions.

Often a line is described by a ***vector equation***. For example, a point x is on M if and only if

$$x = p + \mu v,$$

for some real number μ. In this vector equation, p and v are fixed, and x is a variable point; thus, the ***line passing through the point*** p ***in the direction of*** v can be written as

$$M = \{\, x \in \mathbb{R}^n \,:\, x = p + \mu v,\ -\infty < \mu < \infty \,\}.$$

To give a concrete example, suppose that $p = (1, -2, 4, 1)$ and $v = (7, 8, -6, 5)$ are points in $\mathbb{R}^4$. Denoting x by (x_1, x_2, x_3, x_4), we can describe the vector equation for the line through p parallel to the vector v as the set of all points $x \in \mathbb{R}^4$ such that

$$(x_1, x_2, x_3, x_4) = (1, -2, 4, 1) + \mu(7, 8, -6, 5),\ -\infty < \mu < \infty,$$

which is the same as

$$(x_1, x_2, x_3, x_4) = (1 + 7\mu, -2 + 8\mu, 4 - 6\mu, 1 + 5\mu),\ -\infty < \mu < \infty.$$

This is sometimes written in ***parametric form***:

$$\begin{cases} x_1 = 1 + 7\mu \\ x_2 = -2 + 8\mu \\ x_3 = 4 - 6\mu \\ x_4 = 1 + 5\mu, \end{cases}$$

where $-\infty < \mu < \infty$.

Perhaps some words of caution are worthwhile at this point. Giving a name to something does not endow it with any special properties. Simply because we have called an object in $\mathbb{R}^n$ by the same name as something from $\mathbb{R}^2$ or $\mathbb{R}^3$, it does not follow that the n-dimensional object has the same properties as its namesake in $\mathbb{R}^2$ or $\mathbb{R}^3$. We cannot assume that what we are calling straight lines in $\mathbb{R}^n$ automatically have the same properties as do straight lines in two or three dimensions. This has to be proved, which is what the next few theorems do.

Theorem 1.2.1. *In the definition of the line*

$$M = \{\, x \in \mathbb{R}^n \,:\, x = p + \mu v,\ -\infty < \mu < \infty \,\},$$

the point p may be replaced by any point on M.

Proof. It is often worthwhile to rephrase a statement to make sure that we understand what it means. This theorem is saying that if p' is any point on the line M and if we replace p with p' in the equation of M, then we will get exactly the same line.

In a situation similar to this, it helps to give different names to the potentially different lines. We will use M' to denote the line that we get when we replace p by p':

$$M' = \{\, x \in \mathbb{R}^n \,:\, x = p' + \mu v,\ -\infty < \mu < \infty \,\}.$$

Our task is to show that M and M' must be the same line.

How do we show that two lines are the same? Well, lines are just special types of sets, and we will use the standard strategy for showing that two sets are equal—namely, we will show that every point in the first set also belongs to the second set and vice versa.

(i) Note that a typical point x on M' can be written as

$$x = p' + \mu v$$

for some number μ. We claim that x is also on M. To show that this is the case, we have to show that there is some number λ such that x can be written as $p + \lambda v$, which is a point on M.
Now, since p' is on M, there must be some number β such that

$$p' = p + \beta v,$$

and therefore,

$$\begin{aligned} x &= p' + \mu v \\ &= (p + \beta v) + \mu v \\ &= p + (\beta + \mu)v. \end{aligned}$$

Setting $\lambda = \beta + \mu$ shows that x is on M, and since x was an arbitrary point of M', then $M' \subseteq M$.

(ii) Conversely, we will show that every point on M is also on M'. If we let y be a typical point on M, then

$$y = p + \mu v$$

for some number μ. Using the fact that p' is on M, we have

$$p' = p + \beta v$$

for some number β, so that $p = p' - \beta v$. Therefore,

$$\begin{aligned} y &= p + \mu v \\ &= (p' - \beta v) + \mu v \\ &= p' + (-\beta + \mu)v, \end{aligned}$$

which shows that y is on M', and since y was an arbitrary point of M, then $M \subseteq M'$.

Thus, $M = M'$, which completes the proof.

□

The proof of the next theorem is somewhat similar and is left as an exercise.

Theorem 1.2.2. *In the definition of the line*

$$M = \{\, x \in \mathbb{R}^n \; : \; x = p + \mu v, \; -\infty < \mu < \infty \,\},$$

the vector v may be replaced by any nonzero multiple of v.

Two nonzero vectors are said to be ***parallel*** if one is a multiple of the other. With this terminology, the previous theorems can be combined as follows.

Corollary 1.2.3. *In the definition of the line*

$$M = \{\, x \in \mathbb{R}^n \; : \; x = p + \mu v, \; -\infty < \mu < \infty \,\},$$

the point p may be replaced by any point on the line and the vector v may be replaced by any vector parallel to v.

In the next theorem, we would like to show that if one line is a subset of another line, then the two lines must be the same. Before proving this, we should convince ourselves that we actually have something to prove, since it looks like this might be just another way of stating that a point and vector determine a unique straight line. Perhaps the two statements

(i) "A point and a vector determine a unique straight line"
(ii) "If one line is contained in another, then the two lines must coincide"

are logically equivalent in the sense that one follows from the other without really invoking any geometry.

To see that this is not the case, try replacing the word "line" by the words "solid ball." It is true that a point and a vector determine a unique solid ball, namely, the ball with the point as its center and with a radius equal to the length of the vector. It is clearly *not true* that two solid balls must coincide if one is a subset of the other.

Having talked ourselves into believing that there is something to prove, the next problem is to find a way to carry it out. This is how we will do it:

We will first suppose that M_1 passes through the point p_1 and is parallel to the nonzero vector v_1 and that M_2 passes through the point p_2 and is parallel to the nonzero vector v_2. The assumption is that every point on M_1 is contained in M_2. We will show that this means that v_1 and v_2 are parallel, which is what our intuition suggests should be the case. We will then use the previous theorems to finish the proof.

Theorem 1.2.4. *If M_1 and M_2 are straight lines in $\mathbb{R}^n$, and if $M_1 \subset M_2$, then $M_1 = M_2$.*

Proof. We may suppose that M_1 and M_2 are the lines

$$M_1 = \{\, x \in \mathbb{R}^n \ : \ x = p_1 + \mu v_1, \ -\infty < \mu < \infty \,\}$$
$$M_2 = \{\, x \in \mathbb{R}^n \ : \ x = p_2 + \mu v_2, \ -\infty < \mu < \infty \,\}.$$

Since M_1 is a subset of M_2, then $p_1 \in M_1 \subset M_2$, and we can replace p_2 in M_2 with the point p_1. We can then write M_2 as

$$M_2 = \{\, x \in \mathbb{R}^n \ : \ x = p_1 + \mu v_2, \ -\infty < \mu < \infty \,\}.$$

Again, using the fact that $M_1 \subset M_2$, the point $p_1 + v_1$ of M_1 must also belong to M_2, and from the previous equation, there must be a number μ_0 such that

$$p_1 + v_1 = p_1 + \mu_0 v_2,$$

so that $v_1 = \mu_0 v_2$. But this means that in the last equation for M_2, we can replace the vector v_2 with the parallel vector v_1, that is,

$$M_2 = \{\, x \in \mathbb{R}^n \ : \ x = p_1 + \mu v_1, \ -\infty < \mu < \infty \,\},$$

and the right-hand side of this equation is precisely M_1.

□

Theorem 1.2.5. *Let M be the straight line given by*

$$M = \{\, x \in \mathbb{R}^n \ : \ x = p + \mu v, \ -\infty < \mu < \infty \,\}.$$

If x_1 and x_2 are distinct points on M, then the vector $x_1 - x_2$ is parallel to v.

Proof. To prove this, one only needs to write down what x_1 and x_2 are in terms of p and v and then perform the subtraction. Since each of the two points is on M, there must be numbers μ_1 and μ_2 such that

$$x_1 = p + \mu_1 v \qquad \text{and} \qquad x_2 = p + \mu_2 v,$$

and, therefore,

$$\begin{aligned} x_1 - x_2 &= (p + \mu_1 v) - (p + \mu_2 v) \\ &= (\mu_1 - \mu_2)v. \end{aligned}$$

Since x_1 and x_2 are distinct, $(\mu_1 - \mu_2)v \neq \overline{0}$, and if $\mu_1 = \mu_2$, then

$$(\mu_1 - \mu_2)v = 0 \cdot v = \overline{0}.$$

Therefore, $\mu_1 \neq \mu_2$, so that $x_1 - x_2$ is a nonzero multiple of v, that is, $x_1 - x_2$ is parallel to v.

□

Another way to describe a line is to specify points on it.

Theorem 1.2.6. *A straight line in $\mathbb{R}^n$ is completely determined by any two distinct points on the line.*

Proof. This is proved using Theorem 1.2.5 and Corollary 1.2.3.

If p and q are distinct points on M, then M must be parallel to the vector $q - p$ by Theorem 1.2.5. Since p is on M, Corollary 1.2.3 now implies that M must be the line

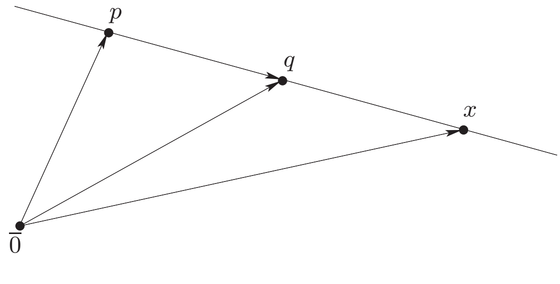
$$\{\, x \in \mathbb{R}^n \,:\, x = p + \mu(q - p),\ -\infty < \mu < \infty \,\}.$$

In other words, p and q completely determine M.

□

In the proof of the previous theorem, we showed that if p and q are two distinct points in $\mathbb{R}^n$, then the line determined by p and q will have the equation

$$x = p + \mu(q - p),\ -\infty < \mu < \infty.$$



This equation may be rearranged to get the more usual form

$$x = (1-\mu)p + \mu q, \ -\infty < \mu < \infty.$$

Note that if $\mu = 0$, then $x = p$, while if $\mu = 1$, then $x = q$. Since p and q are on the line, then the vector $q - p$ must be parallel to the line.

Other equations that also describe the same line are

$$x = \eta p + \mu q, \quad \eta + \mu = 1$$

and

$$x = \mu p + (1-\mu)q, \ \ -\infty < \mu < \infty.$$

Example 1.2.7. *If $p = (-1,-1)$ and $q = (1,1)$, find the equation of the line passing through p and q.*

Solution. The equation of the line M passing through p and q can be written as

$$\begin{aligned}(x_1, x_2) &= (1-\mu)(-1,-1) + \mu(1,1)\\ &= (-1+2\mu, -1+2\mu)\end{aligned}$$

or parametrically as

$$\begin{aligned}x_1 &= -1 + \lambda\\ x_2 &= -1 + \lambda\end{aligned}$$

for $-\infty < \lambda < \infty$.

□

As in $\mathbb{R}^2$ or $\mathbb{R}^3$, two distinct lines in $\mathbb{R}^n$ either do not intersect or they meet in exactly one point.

Theorem 1.2.8. *Two distinct lines in $\mathbb{R}^n$ meet in at most one point.*

Proof. Suppose that

$$M_1 = \{\, p + \mu v \ : \ -\infty < \mu < \infty \,\}$$

and

$$M_2 = \{\, q + \mu w \ : \ -\infty < \mu < \infty \,\}$$

are distinct lines in $\mathbb{R}^n$ and $M_1 \cap M_2 \neq \emptyset$.

We may assume that the vectors v and w are not parallel. If they are parallel and $M_1 \cap M_2 \neq \emptyset$, then by Theorem 1.2.2, M_1 and M_2 coincide, which contradicts the fact that they are distinct.

Suppose that x_0 and x_1 are in $M_1 \cap M_2$, then

$$\begin{aligned} x_0 - p &= \lambda_0 v \\ x_0 - q &= \mu_0 w \end{aligned}$$

for some real numbers λ_0 and μ_0. Also,

$$\begin{aligned} x_1 - p &= \lambda_1 v \\ x_1 - q &= \mu_1 w \end{aligned}$$

for some numbers λ_1 and μ_1.

Therefore,

$$p - q = -\lambda_0 v + \mu_0 w = \mu_1 w - \lambda_1 v,$$

so that

$$(\mu_1 - \mu_0) w = (\lambda_1 - \lambda_0) v.$$

If $\mu_0 \neq \mu_1$, then

$$w = \left(\frac{\lambda_1 - \lambda_0}{\mu_1 - \mu_0}\right) v,$$

which is a contradiction. Thus, $\mu_0 = \mu_1$.

Similarly, if $\lambda_0 \neq \lambda_1$, then

$$v = \left(\frac{\mu_1 - \mu_0}{\lambda_1 - \lambda_0}\right) w,$$

which is a contradiction. Thus, $\lambda_0 = \lambda_1$.

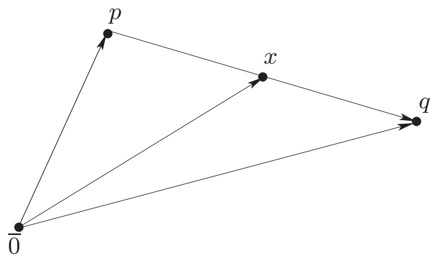
Therefore,

$$x_1 = p + \lambda_1 v = p + \lambda_0 v = x_0.$$

□

1.2.3 Segments

The part of a straight line between two distinct points p and q in $\mathbb{R}^n$ is called a ***straight line segment***, or simply a ***line segment***.



- $[p, q]$ denotes the ***closed line segment*** joining p and q

$$[p, q] = \{\, x \in \mathbb{R}^n \,:\, x = (1 - \mu)p + \mu q \,:\, 0 \le \mu \le 1 \,\}.$$

- (p, q) denotes the ***open line segment*** joining p and q

$$(p, q) = \{\, x \in \mathbb{R}^n \,:\, x = (1 - \mu)p + \mu q \,:\, 0 < \mu < 1 \,\}.$$

- $[p, q)$ and $(p, q]$ denote the ***half open line segments*** joining p and q

$$[p, q) = \{\, x \in \mathbb{R}^n \,:\, x = (1 - \mu)p + \mu q \,:\, 0 \le \mu < 1 \,\},$$
$$(p, q] = \{\, x \in \mathbb{R}^n \,:\, x = (1 - \mu)p + \mu q \,:\, 0 < \mu \le 1 \,\}.$$

Note that as the scalar increases from $\mu = 0$ to $\mu = 1$, the point x moves along the line segment from $x = p$ to $x = q$.

1.2.4 Examples

Example 1.2.9. *Show that the points*

$$a = (-2, -2), \qquad b = (-1, 1), \qquad c = (1, 7)$$

are collinear.

Solution. Note that

$$b - a = (-1, 1) - (-2, -2) = (1, 3)$$
$$c - a = (1, 7) - (-2, -2) = (3, 9) = 3(1, 3)$$

so that $c - a = 3(b - a)$ and $b - a$ and $c - a$ are parallel.

Therefore,

$$x = a + \lambda(b - a), \ -\infty < \lambda < \infty$$

and

$$x = a + \lambda(c - a), \ -\infty < \lambda < \infty$$

are the same line.

For $\lambda = 0$, $x = a$ is on the line from the first equation, while for $\lambda = 1$, $x = b$ and $x = c$ are on the line from the first and second equation, respectively. Therefore, a, b, and c are collinear.

$\square$

Example 1.2.10. *Let a and b be distinct points in $\mathbb{R}^n$ and let μ and ν be scalars such that $\mu + \nu = 1$. Show that the point $c = \mu a + \nu b$ is on the line through a and b.*

Solution. Note that

$$\begin{aligned} c - a &= \mu a + \nu b - a \\ &= \nu b - (1 - \mu)a \\ &= \nu b - \nu a \\ &= \nu(b - a). \end{aligned}$$

Therefore, $c - a$ is parallel to $b - a$, and the points a, b, and c are collinear.

$\square$

Example 1.2.11. *Given distinct point a_1 and a_2 in $\mathbb{R}^n$, the midpoint of the segment $[a_1, a_2]$ is given by*

$$\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2.$$

Solution. Since $\frac{1}{2} + \frac{1}{2} = 1$, then the point

$$\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2$$

is on the line joining a_1 and a_2.

Also,

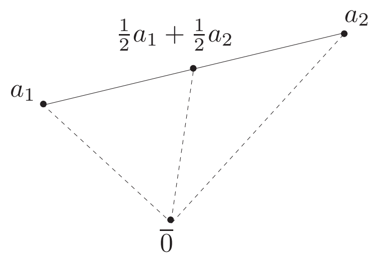
$$\begin{aligned} \tfrac{1}{2}a_1 + \tfrac{1}{2}a_2 - a_1 &= \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 \\ &= \tfrac{1}{2}(a_2 - a_1) \end{aligned}$$

so that

$$\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2 = a_1 + \tfrac{1}{2}(a_2 - a_1).$$

Also,

$$\begin{aligned} a_2 - \left(\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2\right) &= \tfrac{1}{2}a_2 - \tfrac{1}{2}a_1 \\ &= \tfrac{1}{2}(a_2 - a_1). \end{aligned}$$



Therefore,

$$\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2$$

is the midpoint of the segment $[a_1, a_2]$.

$\square$

Exercise 1.2.12. *Given three noncollinear points a, b, and c in $\mathbb{R}^n$, show that the medians of the triangle with vertices a, b, and c intersect at a point G, the familiar centroid from synthetic geometry, and that*

$$G = \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c.$$

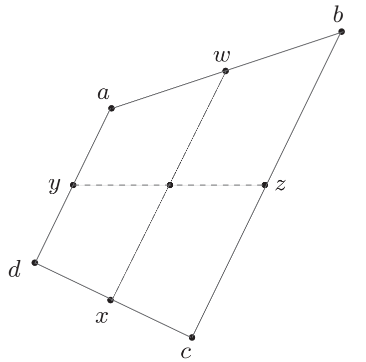
In general, if a_1, a_2, $\ldots$, a_k are k points in $\mathbb{R}^n$, where $n > 2$, we may define the point

$$\tfrac{1}{k}a_1 + \tfrac{1}{k}a_2 + \cdots + \tfrac{1}{k}a_k$$

to be the ***centroid*** of the set $\{\, a_1, a_2, \ldots, a_k \,\}$. It is then obvious from the previous examples that this definition agrees with the notion of the centroid of a segment or a triangle, that is, for $n = 1$ or $n = 2$.

Example 1.2.13. *Show that the point of intersection of the lines joining the midpoints of the opposite sides of a plane quadrilateral is the centroid of the vertices of the quadrilateral.*

Solution. Let a, b, c, and d be the vertices of the quadrilateral.



The centroid of the vertices is $\frac{1}{4}(a+b+c+d)$.

The midpoints of the sides, in opposite pairs, are

$$
\begin{aligned}
w &= \tfrac{1}{2}a + \tfrac{1}{2}b, \qquad & x &= \tfrac{1}{2}c + \tfrac{1}{2}d \\
y &= \tfrac{1}{2}a + \tfrac{1}{2}d, \qquad & z &= \tfrac{1}{2}b + \tfrac{1}{2}c.
\end{aligned}
$$

The midpoint of the segment $[w, x]$ is

$$\tfrac{1}{2}w + \tfrac{1}{2}x = \tfrac{1}{4}(a+b+c+d),$$

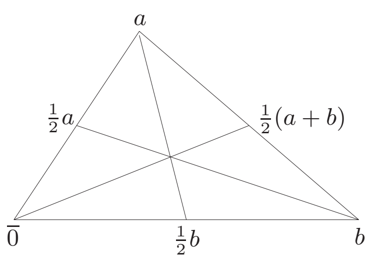
and the midpoint of the segment $[y, z]$ is

$$\tfrac{1}{2}y + \tfrac{1}{2}z = \tfrac{1}{4}(a+b+c+d).$$

Thus, the segments $[w, x]$ and $[y, z]$ intersect at $\frac{1}{4}(a+b+c+d)$, the centroid of the vertices of the quadrilateral.

$\square$

Example 1.2.14. *Given a triangle with vertices* $\overline{0}$, a, b, *as shown in the figure below,*



show that there exists a triangle whose sides are equal in length and parallel to the medians of the triangle.

Solution. The medians of the triangle are the following line segments:

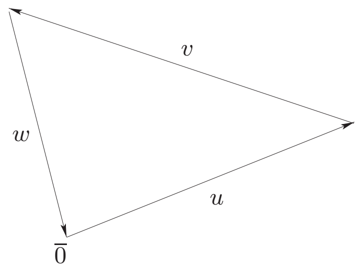
$$\text{from } \overline{0} \text{ to } \tfrac{1}{2}(a+b), \qquad \text{from } b \text{ to } \tfrac{1}{2}a, \qquad \text{from } a \text{ to } \tfrac{1}{2}b$$

and the (free) vectors corresponding to these sides are

$$u = \tfrac{1}{2}(a+b), \qquad v = \tfrac{1}{2}a - b, \qquad w = \tfrac{1}{2}b - a$$

and we need only show that these vectors can be positioned to form a triangle.

Let one vertex of the triangle be $\overline{0}$, let a second vertex be the point u, and let the third vertex be the point $u + v$.



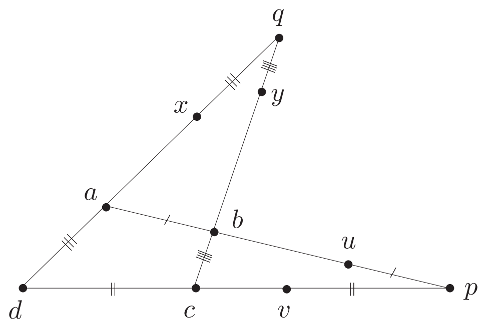
- Clearly, the edge from $\overline{0}$ to u is parallel to and equal in length to u, the first median.
- The second edge from u to $u+v$ is parallel to and equal in length to v, the second median.
- The third edge from $u+v$ to $\overline{0}$ is parallel to and equal in length to the vector $u+v$, but

  $$u + v = \tfrac{1}{2}(a+b) + \tfrac{1}{2}a - b = a - \tfrac{1}{2}b = -w,$$

  so that the third side of the triangle is parallel to and equal in length to w, the third median.

□

Example 1.2.15. *Given the quadrilateral $[a,\, b,\, c,\, d]$ in the plane, the sides $[a, b]$ and $[c, d]$, when extended, meet at the point p. The sides $[b, c]$ and $[a, d]$, when extended, meet at the point q. On the rays from p through b and c are points u and v so that $[p, u]$ and $[p, v]$ are of the same lengths as $[a, b]$ and $[c, d]$, respectively. On the rays from q through a and b are points x and y so that $[q, x]$ and $[q, y]$ are of the same lengths as $[a, d]$ and $[b, c]$, respectively.*



Show that $[u, v]$ *is parallel to* $[x, y]$.

Solution. In the figure, we have

$$\begin{aligned} u &= p + (a - b) \\ v &= p + (d - c) \\ x &= q + (a - d) \\ y &= q + (b - c), \end{aligned}$$

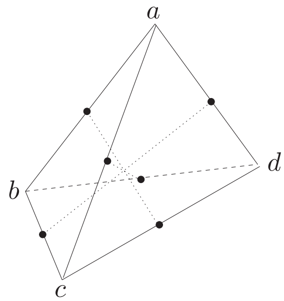
so that

$$\begin{aligned} u - v &= (a - b) - (d - c) = a + c - (b + d) \\ x - y &= (a - d) - (b - c) = a + c - (b + d), \end{aligned}$$

and $[u, v]$ is parallel to $[x, y]$.

□

Example 1.2.16. *Let* a, b, c, *and* d *be the vertices of a tetrahedron in* $\mathbb{R}^3$.



Show that the three lines through the midpoints of the opposite sides are concurrent.

Solution. Note that in each case, the centroid of the tetrahedron

$$\tfrac{1}{4}(a+b+c+d)$$

is the midpoint of the segment.

For example, consider the opposite edges $[a, b]$ and $[c, d]$, their midpoints are $\frac{1}{2}(a+b)$ and $\frac{1}{2}(c+d)$, and the midpoint of the segment $\left[\frac{1}{2}(a+b), \frac{1}{2}(c+d)\right]$ is

$$\tfrac{1}{2}\left(\tfrac{1}{2}(a+b)+\tfrac{1}{2}(c+d)\right)=\tfrac{1}{4}(a+b+c+d).$$

□

1.2.5 Problems

A remark about the exercises is necessary. Certain questions are phrased as statements to avoid the incessant use of "prove that." See Problem 1, for example. Such statements are supposed to be proved. Other questions have a "true–false" or "yes–no" quality. The point of such questions is not to guess, but to justify your answer. Questions marked with $*$ are considered to be more challenging. Hints are given for some problems. Of course, a hint may contain statements that must be proved.

1. Let S be a nonempty set in $\mathbb{R}^n$. If every three points of S are collinear, then S is collinear.
2. In $\mathbb{R}^2$, there are two different types of equations that describe a straight line:

   (a) A vector equation: $(x_1, x_2) = (\alpha_1, \alpha_2) + \mu(\beta_1, \beta_2)$.
   (b) A linear equation: $\mu_1 x_1 + \mu_2 x_2 = \delta$.

   Given that the line L has the vector equation

   $$(x_1, x_2) = (4, 5) + \mu(-3, 2),$$

   find a linear equation for L.
3. Given that the line L has the linear equation

   $$\mu_1 x_1 + \mu_2 x_2 = \delta,$$

   show that the point

   $$\left(\frac{\mu_1\delta}{\mu_1^2+\mu_2^2}, \frac{\mu_2\delta}{\mu_1^2+\mu_2^2}\right)$$

   is on the line and that the vector $(-\mu_2, \mu_1)$ is parallel to the line.
   Hint. If p is on the line and if $p+v$ is also on the line, then v must be parallel to the line.

4. Prove Theorem 1.2.2. In the definition of the line

$$M = \{ x \in \mathbb{R}^n : x = p + \mu v, \ -\infty < \mu < \infty \}$$

the vector v may be replaced by any nonzero multiple of v.

5. The centroid of three noncollinear points a, b, and c in $\mathbb{R}^n$ is defined to be

$$G = \tfrac{1}{3}(a + b + c).$$

Show that this definition of the centroid yields the synthetic definition of the centroid of the triangle with vertices a, b, and c, namely, the point at which the three medians of the triangle intersect. Prove also that the medians do indeed intersect at a common point.

1.3 DISTANCE IN *N*-SPACE

1.3.1 Metrics

In $\mathbb{R}^n$, several distance functions are used, the most common being the Euclidean distance.

If $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$, then the ***Euclidean distance*** between x and y is given by

$$d(x, y) = \sqrt{(\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2 + \cdots + (\alpha_n - \beta_n)^2}.$$

When $\mathbb{R}^n$ is equipped with the Euclidean distance, it is called ***Euclidean n-space***.

The word "metric" is synonymous with "distance function." A metric has the three basic properties that one expects a distance to have, namely, it is never negative, it is symmetric (the distance from Calgary to Edmonton is the same as the distance from Edmonton to Calgary), and the triangle inequality holds (the total length of two sides of a triangle is never smaller than the length of the third side).

Formally, a ***metric*** on a set X is any mapping $d(\,\cdot\,,\cdot\,)$ from $X \times X$ into $\mathbb{R}$ that has the following properties:

(M_1) $d(x, y) \geq 0$, with equality if and only if $x = y$ ***(nonnegativity)***
(M_2) $d(x, y) = d(y, x)$ ***(symmetry)***
(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ ***(triangle inequality)***

for all $x, y, z \in X$. The pair (X, d) is called a ***metric space***.

Besides the ℓ_2 or ***Euclidean metric*** defined above, two other metrics that are occasionally used in $\mathbb{R}^n$ are:

- the ℓ_1 or "Manhattan" metric given by

$$d(x,y) = |\alpha_1 - \beta_1| + |\alpha_2 - \beta_2| + \cdots + |\alpha_n - \beta_n|,$$

- the ℓ_∞ or "sup" or "supremum" metric given by

$$d(x,y) = \max\{\, |\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|, \ldots, |\alpha_n - \beta_n| \,\}.$$

The metrics that are used on the linear space $\mathbb{R}^n$ are almost always derived from a norm.

1.3.2 Norms

A ***norm*** on a linear space X is any mapping $\|\cdot\|$ from X to $\mathbb{R}$ that has the following properties:

(N_1) $\|x\| \geq 0$, with equality if and only if $x = 0$ ***(nonnegativity)***
(N_2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ ***(positive homogeneity)***
(N_3) $\|x + y\| \leq \|x\| + \|y\|$ ***(triangle inequality)***

for all $x, y \in X$ and $\lambda \in \mathbb{R}$. The pair $(X, \|\cdot\|)$ is called a ***normed linear space***. The norm of a vector is always thought of as being the length of the vector.

If $x = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, the norms of x corresponding to the three metrics that we defined above are:

- $\|x\|_2 = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$ ***Euclidean norm*** or ℓ_2 ***norm***
- $\|x\|_1 = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$ ℓ_1 ***norm***
- $\|x\|_\infty = \max\{\, |\alpha_1|, |\alpha_2|, \ldots, |\alpha_n| \,\}$ ℓ_∞ ***norm***.

Note. Every norm $\|\cdot\|$ on a linear space always has a corresponding metric (although the converse is *not* true). The metric is derived from the norm by defining

$$d(x,y) = \|x - y\|.$$

In addition to the three properties of nonnegativity, symmetry, and the triangle inequality, a metric that is derived from a norm satisfies the following:

(M_1) $d(x, y) \geq 0$, with equality if and only if $x = y$ ***(nonnegativity)***
(M_2) $d(x, y) = d(y, x)$ ***(symmetry)***
(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ ***(triangle inequality)***
(M_4) $d(x + v, y + v) = d(x, y)$ ***(translation invariance)***
(M_5) $d(\lambda x, \lambda y) = |\lambda| \cdot d(x, y)$ ***(positive homogeneity)***

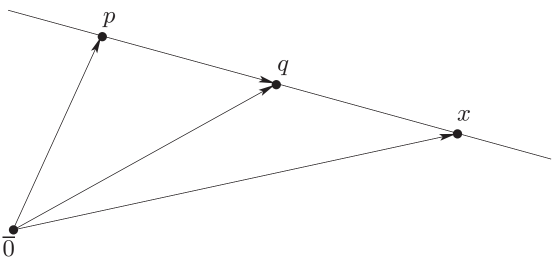
for all $x, y, z, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

The property of translation invariance means that a "ruler" does not stretch or shrink when it is moved parallel to itself from one part of the space to another.*

Example 1.3.1. *If p and q are points in $\mathbb{R}^n$ with $\|p - q\| = \delta$, and if*

$$x = (1 - \lambda)p + \lambda q,$$

find the distance between x and p and the distance between x and q.



Solution. Note that we do not have to specify which distance function we are using, since the points p, q, and x are collinear.

The distance from x to p is

$$\begin{aligned} \|x - p\| &= \|(1 - \lambda)p + \lambda q - p\| = \|p - \lambda p + \lambda q - p\| \\ &= \|\lambda(q - p)\| = |\lambda| \cdot \|q - p\| \\ &= |\lambda| \cdot \delta. \end{aligned}$$

* However, you might wonder what happens to a ruler when it is rotated, especially when a non-Euclidean norm is being used.

The distance from x to q is

$$\begin{aligned}\|x-q\| &= \|(1-\lambda)p+\lambda q-q\| = \|p-\lambda p+\lambda q-q\|\\ &= \|(1-\lambda)p+(-1+\lambda)q\| = \|(1-\lambda)p-(1-\lambda)q\|\\ &= \|(1-\lambda)(p-q)\| = |(1-\lambda)|\cdot\|-p+q\|\\ &= |1-\lambda|\cdot\|q-p\| = |1-\lambda|\cdot\delta.\end{aligned}$$

Note that if $0 \le \lambda \le 1$, then the point x is on the line segment between p and q; if $\lambda < 0$, then the point x is on the line joining p and q beyond p; while if $\lambda > 1$, the point x is on the line joining p and q beyond q.

□

Example 1.3.2. *Find all points on the line through p and q that are twice as far from p as they are from q.*

Solution. A point x on the line through p and q can be expressed as

$$x = (1-\lambda)p + \lambda q$$

for some scalar λ.

From the previous example,

$$\|x-p\| = |\lambda|\cdot\|q-p\| \qquad \text{and} \qquad \|x-q\| = |1-\lambda|\cdot\|q-p\|,$$

so we must have

$$|\lambda|\cdot\|q-p\| = 2|1-\lambda|\cdot\|q-p\|.$$

It follows that

$$|\lambda| = 2|1-\lambda|,$$

which implies that

$$\lambda^2 = (2(1-\lambda))^2\,.$$

Expanding and rearranging, we have

$$3\lambda^2 - 8\lambda + 4 = 0$$

or

$$(3\lambda-2)(\lambda-2) = 0.$$

The solutions

$$\lambda = \tfrac{2}{3} \qquad \text{and} \qquad \lambda = 2,$$

give us two points that are twice as far from p as they are from q:

$$x_1 = \tfrac{1}{3}p + \tfrac{2}{3}q \qquad \text{and} \qquad x_2 = -p + 2q.$$

□

Remark. As we noted earlier, in the previous two examples, the answers did not depend on the actual distance function but only on the general properties of the norm. When working with distances and norms, one should use the general properties whenever possible. Of course, the actual numerical value of $\|x\|$ depends on the particular norm that is involved, as illustrated by the next example.

Example 1.3.3. *Find the distance between the points* $p = (1, 1)$ *and* $q = (-1, 2)$ *using the three different metrics described earlier.*

Solution. This is just a matter of straightforward computation.

For the ℓ_2 or Euclidean norm, we have

$$\|p - q\|_2 = \sqrt{(1 - (-1))^2 + (1 - 2)^2} = \sqrt{5}.$$

For the ℓ_1 norm, we have

$$\|p - q\|_1 = |1 - (-1)| + |1 - 2| = 3.$$

For the ℓ_∞ or supremum norm, we have

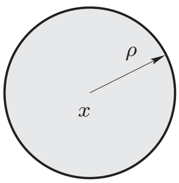
$$\|p - q\|_\infty = \max\{\, |1 - (-1)|, |1 - 2|\,\} = 2.$$

□

1.3.3 Balls and Spheres

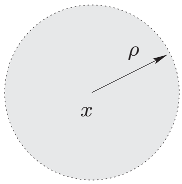
In $\mathbb{R}^n$ with the ℓ_2 norm, the ***closed ball*** centered at x with radius ρ is the set

$$\overline{B}(x, \rho) = \{\, y \in \mathbb{R}^n \,:\, \|x - y\|_2 \le \rho \,\}.$$



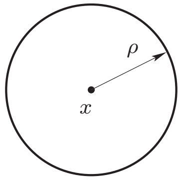
If we omit the boundary, the ***open ball*** centered at x with radius ρ is the set

$$B(x, \rho) = \{\, y \in \mathbb{R}^n \,:\, \|x - y\|_2 < \rho \,\}.$$



The boundary itself is called a ***sphere*** (or a ***circle*** in $\mathbb{R}^2$) centered at x with radius ρ and is the set

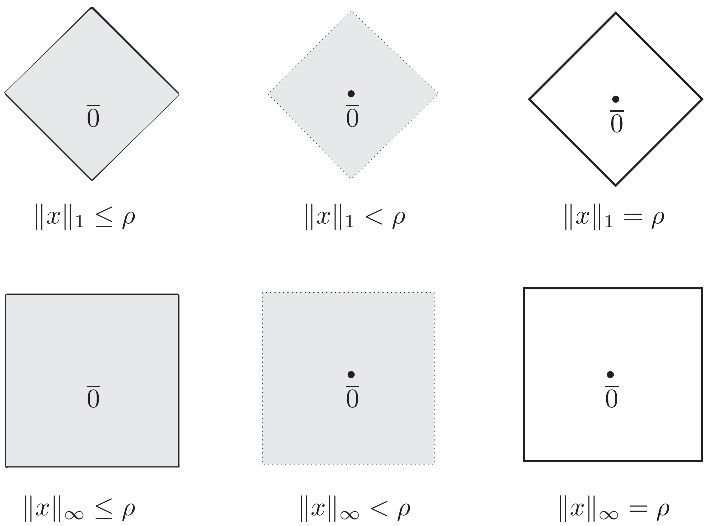
$$S(x, \rho) = \{\, y \in \mathbb{R}^n \;:\; \|x - y\|_2 = \rho \,\}.$$



For the particular case when $x = \overline{0}$ and $\rho = 1$, the sets are called the ***closed unit ball***, the ***open unit ball***, and the ***unit sphere***, respectively.

The shape of a ball will depend on the norm being used. The previous three figures show a closed ball, an open ball, and a sphere in $\mathbb{R}^2$ using the Euclidean norm.

The figure below shows the corresponding balls and sphere in $\mathbb{R}^2$ centered at $\overline{0}$ with radius ρ in the ℓ_1 norm and the ℓ_∞ norm.



It should be mentioned that the closed unit ball in the ℓ_∞ norm is often called the ***unit cube***, that is, the unit cube is the set

$$S(\overline{0}, 1) = \{\, (\alpha_1, \alpha_2, \ldots, \alpha_n) \;:\; |\alpha_i| \leq 1,\; i = 1, 2, \ldots, n \,\}.$$

Example 1.3.4. *Find the points where the line through the origin parallel to the vector* $v = (2, 0, -3, 6)$ *intersects the unit sphere.*

Note. Whenever a specific distance is to be calculated but no distance function is specified, we always assume that the Euclidean distance is to be used.

Solution. The line has the equation

$$x = \mu v, \ -\infty < \mu < \infty,$$

and for $x = (x_1, x_2, x_3, x_4)$, this can be written as

$$(x_1, x_2, x_3, x_4) = \mu(2, 0, -3, 6), \ -\infty < \mu < \infty.$$

This line intersects the unit sphere at the points where

$$\|(x_1, x_2, x_3, x_4)\| = \|\mu(2, 0, -3, 6)\| = 1,$$

that is, where

$$|\mu| \cdot \|(2, 0, -3, 6)\| = |\mu|\sqrt{4 + 0 + 9 + 36} = 7|\mu| = 1,$$

so we must have $\mu = \frac{1}{7}, -\frac{1}{7}$. Therefore, the line intersects the unit sphere at

$$x_1 = \left(\tfrac{2}{7}, 0, -\tfrac{3}{7}, \tfrac{6}{7}\right) \qquad \text{and} \qquad x_2 = \left(-\tfrac{2}{7}, 0, \tfrac{3}{7}, -\tfrac{6}{7}\right).$$

□

If A is a subset of $\mathbb{R}^n$ and if v is a vector from $\mathbb{R}^n$, the set $A + v$ defined by

$$A + v = \{\, x + v \in \mathbb{R}^n \ : \ x \in A \,\}$$

is called a ***translate*** of A.

Note that the set $A + v$ is obtained from the set A by adding the vector v to every point in A. The key to working with translates is to use the fact that

$$x \in A + v \qquad \text{if and only if} \qquad x - v \in A.$$

Example 1.3.5. *Show that the translate of a closed ball is a closed ball.*

Note that in this case, no specific distance has to be calculated, so we will attempt to solve the problem using only the general properties of the norm.

Solution. Let

$$A = \overline{B}(q, \rho) = \{\, x \in \mathbb{R}^n \ : \ \|x - q\| \le \rho \,\}.$$

If $A + v$ is indeed a closed ball, then we should expect that it has the same radius as A and that its center is at $q + v$. Thus, we will try to show that

$$A + v = \overline{B}(q + v, \rho) = \{\, x \in \mathbb{R}^n \ : \ \|x - (q + v)\| \le \rho \,\}.$$

As usual, we will show that the two sets are equal by showing that each is a subset of the other.

(i) We will show first that $\overline{B}(q+v,\rho) \subseteq A+v$, so we let $x \in \overline{B}(q+v,\rho)$, then

$$\|x-(q+v)\| \leq \rho,$$

so that

$$\|(x-v)-q\| \leq \rho,$$

that is,

$$z = x - v \in A = \overline{B}(q,\rho).$$

Therefore,

$$x = z + v \in A + v,$$

and since $x \in \overline{B}(q+v,\rho)$ is arbitrary, then $\overline{B}(q+v,\rho) \subseteq A+v$.

(ii) Conversely, if $x \in A+v = \overline{B}(q,\rho)+v$, then $x = z+v$, where $z \in \overline{B}(q,\rho)$, so that

$$\|z-q\| \leq \rho,$$

that is,

$$\|(x-v)-q\| \leq \rho,$$

so that

$$\|x-(q+v)\| \leq \rho,$$

and $x \in \overline{B}(q+v,\rho)$. Since $x \in A+v$ is arbitrary, then $A+v \subseteq \overline{B}(q+v,\rho)$.

Therefore,

$$A+v = \overline{B}(q+v,\rho) = \{\, x \in \mathbb{R}^n \,:\, \|x-(q+v)\| \leq \rho \,\}.$$

$\square$

Exercise 1.3.6. *Show that*

$$B(q,\rho)+v = B(q+v,\rho)$$

and

$$S(q,\rho)+v = S(q+v,\rho).$$

If A is a subset of $\mathbb{R}^n$ and λ is a real number with $\lambda > 0$, the set

$$\lambda A = \{\, z \in \mathbb{R}^n \,:\, z = \lambda x, \text{ where } x \in A \,\}$$

is called a ***positive homothet*** of A.

Note that the set λA is obtained from the set A by multiplying each vector in A by the positive scalar λ. In fact, the definition of λA applies to all scalars $\lambda \in \mathbb{R}$, not just $\lambda > 0$.

Theorem 1.3.7. *The positive homothet of a closed ball is a closed ball.*

The proof of this theorem is left as an exercise. What about a positive homothet of an open ball? A sphere?

The ℓ_1, ℓ_2, and ℓ_∞ norms are not the only useful norms on $\mathbb{R}^n$. One sometimes encounters the ℓ_p norm.

Given any real number p with $1 < p < \infty$, the ℓ_p ***norm*** on $\mathbb{R}^n$ is defined to be

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Even though the proof that the ℓ_p norms are indeed norms is beyond the scope of this text, we can still determine what the balls and spheres look like in $\mathbb{R}^2$.

Example 1.3.8. *Sketch the closed ball $\overline{B}(\overline{0}, a)$ in $\mathbb{R}^2$ when it is equipped with the ℓ_p norm, where $1 < p < \infty$.*

Solution. The closed ball is the set

$$\overline{B}(\overline{0}, a) = \{\, (x_1, x_2) \in \mathbb{R}^2 \;:\; (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \leq a \,\},$$

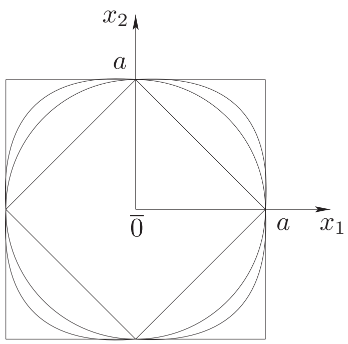
and since this is symmetric about the x_1 and x_2 axes, it is sufficient to sketch the curve

$$x_1^p + x_2^p = a^p$$

for $0 \leq x_1 \leq a$ and $0 \leq x_2 \leq a$.

Differentiating implicitly, it is easy to see that this curve has a horizontal tangent at the point $(0, a)$ and a vertical tangent at the point $(a, 0)$ and is concave down on the interval $0 < x_1 < a$.

Using the symmetry of the curve, we get the figure below.



The figure shows the ℓ_1, ℓ_2, ℓ_p, and ℓ_∞ balls. In the figure, $2 < p < \infty$. For the case $1 < p < 2$, the ℓ_p ball would be contained in the ℓ_2 ball and would contain the ℓ_1 ball.

As p increases without bound, it would appear that the ℓ_p balls expand to approach the ℓ_∞ ball, and this is indeed the case, as the next example shows.

Example 1.3.9. *Show that*

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$

for each $x \in \mathbb{R}^n$.

Solution. If we let $x = (x_1, x_2, \ldots, x_n)$ be any point in $\mathbb{R}^n$, then there is an index n_0 such that

$$|x_{n_0}| = \max\{\,|x_k| \;:\; 1 \le k \le n\,\} = \|x\|_\infty,$$

and, therefore,

$$\|x\|_\infty = |x_{n_0}| = (|x_{n_0}|^p)^{\frac{1}{p}} \le \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} = \|x\|_p,$$

that is,

$$\|x\|_\infty \le \|x\|_p$$

for all $p \ge 1$.

Also,

$$\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^n \left|\frac{x_k}{\|x\|_\infty}\right|^p\right)^{\frac{1}{p}} \cdot \|x\|_\infty,$$

and since $|x_k| \le \|x\|_\infty$ for $1 \le k \le n$, then

$$\|x\|_p \le \|x\|_\infty \cdot n^{\frac{1}{p}}$$

for all $p \ge 1$.

Combining these two inequalities, we obtain

$$\|x\|_\infty \le \|x\|_p \le \|x\|_\infty \cdot n^{\frac{1}{p}}$$

for all $p \ge 1$.

Letting $p \to \infty$, since

$$\lim_{p\to\infty} n^{\frac{1}{p}} = \lim_{p\to\infty} e^{\frac{1}{p}\log n} = e^0 = 1.$$

Since $\|x\|_p$ is stuck between two quantities approaching $\|x\|_\infty$, the limit exists and

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$

for all $x \in \mathbb{R}^n$.

□

1.4 INNER PRODUCT AND ORTHOGONALITY

An ***inner product**** on $\mathbb{R}^n$ is a mapping $\langle \cdot\ ,\ \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that for all x, y, z in $\mathbb{R}^n$ and all λ in $\mathbb{R}$,

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = \overline{0}$,
(ii) $\langle x, y\rangle = \langle y, x\rangle$,
(iii) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ and $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
(iv) $\langle \lambda x, y\rangle = \langle x, \lambda y\rangle = \lambda\langle x, y\rangle$.

The ***standard*** or ***Euclidean inner product*** on $\mathbb{R}^n$ is given by

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$.

Exercise 1.4.1. *Show that the Euclidean inner product is an actual inner product as defined above.*

The standard inner product is intimately related to the Euclidean norm, since it is immediately apparent from the definition of the inner product that

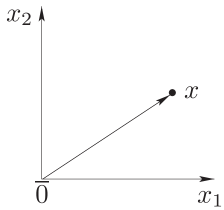
$$\|x\|_2^2 = \langle x, x\rangle,$$

that is,

$$\|x\|_2 = \sqrt{\langle x, x\rangle} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}.$$

This is just the ***Euclidean norm*** on $\mathbb{R}^n$ and is the usual Euclidean distance from the origin $\overline{0}$ to the point x, that is, it is the Euclidean length of the vector x.

* Also called a ***scalar product*** or a ***dot product*** on $\mathbb{R}^n$, and from properties (i), (ii), (iii), and (iv), it is also called a ***nondegenerate, positive definite, symmetric, bilinear mapping***.



Note that in the definition of the inner product, property (i) says that $\|x\| = 0$ if and only if $x = \overline{0}$.

It is very common to approach the notion of distance in $\mathbb{R}^n$ by first discussing the inner product. However, we have elected to discuss the notion of distance first because there are many different distance functions for $\mathbb{R}^n$ that cannot be derived from an inner product.

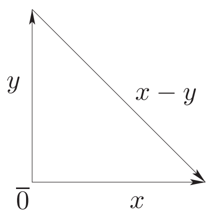
In two or three dimensions, orthogonality is synonymous with perpendicularity. In fact, in the Euclidean plane, we have the following notion of orthogonality or perpendicularity.

Theorem 1.4.2. *If x and y are vectors in $\mathbb{R}^2$, then $x \perp y$ if and only if $\langle x, y \rangle = 0$.*

Proof.

If x and y are perpendicular vectors in the plane, then from the Pythagorean theorem and its converse, we have $x \perp y$ if and only if

$$\|x - y\|_2^2 = \|x\|_2^2 + \|y\|_2^2.$$



However,

$$\langle x - y, x - y \rangle = \|x\|_2^2 + \|y\|_2^2$$

if and only if

$$\langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle = \|x\|_2^2 + \|y\|_2^2,$$

that is, if and only if

$$\|x\|_2^2 - 2\langle x, y \rangle + \|y\|_2^2 = \|x\|_2^2 + \|y\|_2^2,$$

if and only if

$$\langle x, y \rangle = 0.$$

□

A similar argument holds for orthogonal vectors in $\mathbb{R}^3$, so we may use this as a definition of orthogonality in higher dimensions.

In $\mathbb{R}^n$, the vectors x and y are ***orthogonal***,* denoted by $x \perp y$, if and only if $\langle x, y \rangle = 0$. Note that this is the *standard* or *Euclidean* inner product on $\mathbb{R}^n$.

The Pythagorean theorem is easily extended to $\mathbb{R}^n$, as shown in the following theorem.

Theorem 1.4.3. *If $x, y \in \mathbb{R}^n$, then $x \perp y$ if and only if $\|x+y\|_2^2 = \|x\|_2^2+\|y\|_2^2$.*

Proof. The points $\overline{0}$, x, and y form a triangle in $\mathbb{R}^n$ which lies in a plane (that is, a two-dimensional subspace of $\mathbb{R}^n$). Now the usual Pythagorean theorem in this plane gives the result.

$\square$

Note that we could have stated the previous theorem as $x \perp y$ if and only if

$$\|x - y\|_2^2 = \|x\|_2^2 + \|y\|_2^2.$$

Example 1.4.4. *(Packing the Elephant)*
Show that a three-dimensional elephant can be packed inside the unit cube of $\mathbb{R}^n$ if n is sufficiently large. In fact, our sun can be packed inside the unit cube if n is large enough.

Solution. We assume that distance in $\mathbb{R}^n$ and $\mathbb{R}^3$ is being used with the same units. For example, we assume that the length of the vector $(1, 0, 0, \ldots, 0)$ is 1 m in both $\mathbb{R}^n$ and $\mathbb{R}^3$.

Let p_i, for $i = 1, 2, 3$, be the following three points in $\mathbb{R}^{3k}$:

$$p_1 = (\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, \underbrace{0, 0, \ldots, 0}_{2k \text{ times}})$$

$$p_2 = (\underbrace{0, 0, \ldots, 0}_{k \text{ times}} \underbrace{1, 1, \ldots, 1}_{k \text{ times}}, \underbrace{0, 0, \ldots, 0}_{k \text{ times}})$$

$$p_3 = (\underbrace{0, 0, \ldots, 0}_{2k \text{ times}}, \underbrace{1, 1, \ldots, 1}_{k \text{ times}}).$$

* Some people like to exclude the zero vector from considerations of perpendicularity. From the definition of orthogonality, the zero vector is ***always*** orthogonal to every other vector.

It is easily checked that

$$\langle p_1, p_2\rangle = \langle p_1, p_3\rangle = \langle p_2, p_3\rangle = 0,$$

so that the segments

$$[\overline{0}, p_1], \quad [\overline{0}, p_2], \text{ and } [\overline{0}, p_3]$$

are mutually orthogonal. Thus, they form three of the edges of a three-dimensional tetrahedron with vertices $\overline{0}$, p_1, p_2, and p_3.

This tetrahedron is entirely contained in the unit cube of $\mathbb{R}^{3k}$, and so it suffices to show that for sufficiently large k, we can pack an elephant inside this tetrahedron.

For $i = 1, 2, 3,$ the length of each of the edges $[\overline{0}, p_i]$ is $\sqrt{k}$. Thus, we can make the tetrahedron as large as we like by simply increasing the magnitude of k.

□

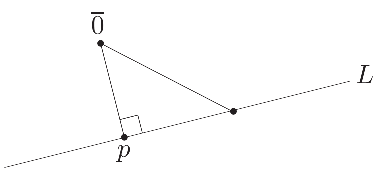
The notion of orthogonality can also be extended to arbitrary sets.

We say that a vector v is ***orthogonal to a set*** A if v is orthogonal to every vector determined by every pair of points p and q in A, that is, for every p and q in A, we have $\langle v, p - q\rangle = 0$.

Note that if a vector v is orthogonal to a set A, then every multiple of v is also orthogonal to A and v is orthogonal to every translate of A.

1.4.1 Nearest Points

With the Euclidean metric on $\mathbb{R}^2$, we know from the Pythagorean theorem that the hypotenuse of a right triangle is the longest side of the triangle. Thus, the line L is perpendicular (that is, orthogonal) to the segment joining $\overline{0}$ to its nearest point p in L, as in the figure below.



In fact, this is true no matter what the dimension of the space is.

Theorem 1.4.5. *Let L be a line in $\mathbb{R}^n$ and let p be a point on L, then p is the closest point of L to $\overline{0}$ if and only if p is orthogonal to $q-p$ for every other point q on L.*

Proof. If we let q be any point on L with $q \neq p$, then L is the line through p parallel to $v = q - p$, that is, the line through p whose equation is

$$x = p + \lambda v, \ -\infty < \lambda < \infty.$$

Note that

$$\|p\|_2 \leq \|p + \lambda v\|_2 \ \text{ for all } \lambda$$

if and only if

$$\|p\|_2^2 \leq \|p + \lambda v\|_2^2 \ \text{ for all } \lambda,$$

that is, if and only if

$$\|p\|_2^2 \leq \|p\|_2^2 + 2\lambda\langle p, v\rangle + \lambda^2\|v\|_2^2 \ \text{ for all } \lambda.$$

Therefore,

$$\|p\|_2 \leq \|p + \lambda v\|_2 \ \text{ for all } \lambda$$

if and only if

$$0 \leq 2\lambda\langle p, v\rangle + \lambda^2\|v\|_2^2 \ \text{ for all } \lambda. \tag{$*$}$$

Thus, we need to show that the last inequality is true if and only if $p \perp v$.

(i) Suppose first that $\langle p, v\rangle = 0$, then $(*)$ becomes

$$\lambda^2\|v\|_2^2 \geq 0,$$

which is true for all λ.

(ii) Conversely, suppose that $\langle p, v\rangle \neq 0$, then if we let

$$\lambda = -\frac{\langle p, v\rangle}{\|v\|_2^2},$$

then

$$2\lambda\langle p, v\rangle + \lambda^2\|v\|_2^2 = -\frac{|\langle p, v\rangle|^2}{\|v\|_2^2} < 0$$

and $(*)$ is not true. Hence, if $(*)$ is true, then $\langle p, v\rangle = 0$.

□

Corollary 1.4.6. *A line in $\mathbb{R}^n$ has a unique point of minimum Euclidean norm.*

Proof. If both p and q were points on the line with minimum norm, the previous theorem implies that

$$\langle q, q-p\rangle = 0 \qquad \text{and} \qquad \langle p, q-p\rangle = 0.$$

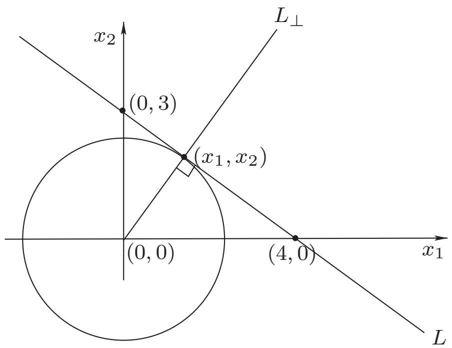
Subtracting, we would get $\langle q-p, q-p\rangle = 0$, which means that $\|q-p\|_2 = 0$ so that $p = q$.

□

Example 1.4.7. *Given the line $L: \ 3x_1 + 4x_2 = 12$ in $\mathbb{R}^2$, find the closest point on the line L to the origin using*

(a) the ℓ_2 norm,
(b) the ℓ_1 norm,
(c) the ℓ_∞ norm.

Solution. As can be seen from the figure below,



the vector equation of the line L is

$$x = (x_1, x_2) = (0,3) + \mu\left((4,0) - (0,3)\right) = (0,3) + \mu(4,-3)$$

for $-\infty < \mu < \infty$.

(a) Using the ℓ_2 norm, the point on the line L closest to the origin is the point of intersection of L with the line $L_\perp$ through the origin in the direction $(3,4)$. The vector equation of $L_\perp$ is

$$x = (x_1, x_2) = \overline{0} + \mu(3,4) = \mu(3,4),\ -\infty < \mu < \infty,$$

and these lines intersect when

$$(0,3) + \mu(4,-3) = \lambda(3,4),$$

that is, when

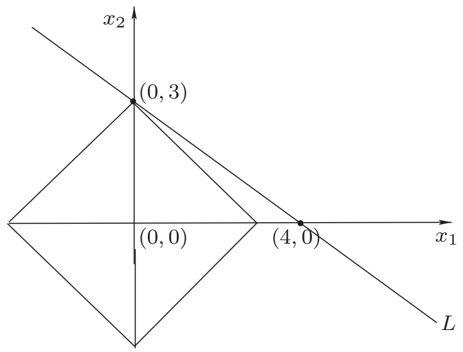
$$4\mu = 3\lambda \quad \text{and} \quad 3 - 3\mu = 4\lambda.$$

Solving this system of equations, we get $\lambda = \frac{12}{25}$, and therefore, the point

$$(x_1, x_2) = \tfrac{12}{25}(3,4) = \left(\tfrac{36}{25}, \tfrac{48}{25}\right)$$

is the point on the line L, which is closest to the origin when we use the Euclidean norm.

(b) Using the ℓ_1 norm, the closest point on L to the origin is the point at which the ℓ_1 balls centered at $(0,0)$ first hit the line. That is, the ℓ_1 ball centered at $\overline{0}$ with radius 3 touches L at the point $(0,3)$ and nowhere else, and any other ℓ_1 ball centered at $\overline{0}$ with a different radius either misses L or else contains $(0,3)$.

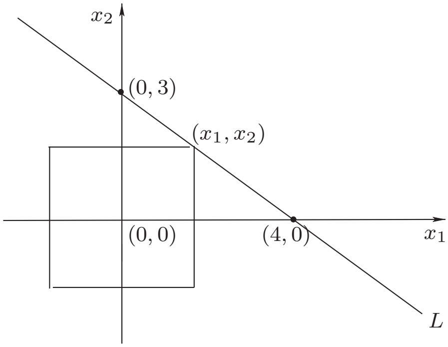


In this case, the closest point is $(0,3)$.

(c) Using the ℓ_∞ norm, the closest point on L to the origin is the point where the ℓ_∞ balls centered at $(0,0)$ first hit the line. That is, the smallest ℓ_∞ ball centered at $\overline{0}$ touches L at the point (x_1, x_2) where $x_1 = x_2$, as shown in the following figure. In other words, the nearest point is where the line

$$M = \{\, x \in \mathbb{R}^2 \,:\, x = \lambda(1,1),\ -\infty < \lambda < \infty \,\}$$

meets L.



In this case, the closest point is $\left(\frac{12}{7}, \frac{12}{7}\right)$.

□

From the examples given above, it should be clear now that the geometry in $\mathbb{R}^n$ depends heavily on the norm used to measure distances, and many of the familiar notions from Euclidean geometry may not look so familiar now. However, in the majority of this text, we will use the Euclidean norm, and we will see that many geometrical properties of $\mathbb{R}^n$ for $n > 2$ are the same as the familiar properties for $\mathbb{R}^2$ or $\mathbb{R}^3$.

1.4.2 Cauchy–Schwarz Inequality

An approach similar to that used in the proof of Theorem 1.4.5 yields one of the most useful relationships between the inner product and the Euclidean distance.

Theorem 1.4.8. *(The Cauchy–Schwarz Inequality)**
If u and v are vectors in $\mathbb{R}^n$, then

$$|\langle u, v\rangle| \leq \|u\|_2\,\|v\|_2,$$

and equality holds if and only if one of u or v is a multiple of the other.

Proof. If $|\langle u, v\rangle| = 0$, we have

$$0 = |\langle u, v\rangle| \leq \|u\|_2\,\|v\|_2$$

with equality if and only if at least one of u or v is zero. Thus, the Cauchy inequality is true in this case.

* Also known as the ***Cauchy*** inequality, the ***Schwarz*** inequality, the ***Bunyakovsky*** inequality, or as the ***Cauchy–Bunyakovsky–Schwarz*** inequality.

If $|\langle u, v\rangle| \neq 0$, then for any scalar λ, we have

$$0 \leq \|u + \lambda v\|_2^2 = \|u\|_2^2 + 2\lambda\langle u, v\rangle + \lambda^2\|v\|_2^2,$$

and taking

$$\lambda = -\frac{\langle u, v\rangle}{\|v\|_2^2},$$

we get

$$0 \leq \|u\|_2^2 - 2\frac{|\langle u, v\rangle|^2}{\|v\|_2^2} + \frac{|\langle u, v\rangle|^2}{\|v\|_2^2} = \|u\|_2^2 - \frac{|\langle u, v\rangle|^2}{\|v\|_2^2},$$

so that

$$|\langle u, v\rangle|^2 \leq \|u\|_2^2\,\|v\|_2^2.$$

Cauchy's inequality follows by taking nonnegative square roots.

Now note that equality holds if and only if $u + \lambda v = \overline{0}$, that is, if and only if one of u or v is a multiple of the other.

□

Remark. The equality is sometimes stated as

$$\langle u, v\rangle \;\leq \|u\|_2\,\|v\|_2,$$

and in this case, equality holds if and only if one of u or v is a nonnegative multiple of the other.

Now that we have the Cauchy–Schwarz inequality, we can give a simple proof of the ***triangle inequality*** in $\mathbb{R}^n$ with the Euclidean distance.

Theorem 1.4.9. *(The Triangle Inequality)**
For any vectors u and v in $\mathbb{R}^n$, we have

$$\|u + v\|_2 \leq \|u\|_2 + \|v\|_2$$

with equality if and only if one of u or v is a nonnegative multiple of the other.

Proof. If u and v are vectors from $\mathbb{R}^n$, then

$$\begin{aligned}\|u + v\|_2^2 &= \|u\|_2^2 + 2\langle u, v\rangle + \;\|v\|_2^2\\ &\leq \|u\|_2^2 + 2\|u\|_2\,\|v\|_2 + \|v\|_2^2\\ &= \left(\|u\|_2 + \|v\|_2\right)^2,\end{aligned}$$

* Sometimes known as ***Minkowski's*** inequality.

so that

$$\|u+v\|_2 \le \|u\|_2 + \|v\|_2.$$

Note that equality holds in the inequality above if and only if equality holds in the Cauchy inequality, that is, if and only if one of u or v is a nonnegative multiple of the other.

$\square$

Note. If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are points in $\mathbb{R}^n$, then the coordinatized versions of the Cauchy–Schwarz inequality and Minkowski's inequality are given by

Cauchy–Schwarz Inequality:

$$\sum_{k=1}^{n} |u_k v_k| \le \left(\sum_{k=1}^{n} |u_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |v_k|^2\right)^{\frac{1}{2}}.$$

Triangle Inequality:

$$\left(\sum_{k=1}^{n} |u_k + v_k|^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{n} |u_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} |v_k|^2\right)^{\frac{1}{2}}.$$

The next example shows when we can have equality in the triangle inequality in each of the usual norms.

Example 1.4.10. *Given $p = (1,1)$ and $q = (-1,-1)$, find all points x in $\mathbb{R}^2$ such that*

(a) $\|p-x\|_2 + \|q-x\|_2 = \|p-q\|_2$,
(b) $\|p-x\|_1 + \|q-x\|_1 = \|p-q\|_1$,
(c) $\|p-x\|_\infty + \|q-x\|_\infty = \|p-q\|_\infty$.

Solution.

(a) Clearly, if x is any point on the line segment joining p and q, then

$$x = (1-\mu)p + \mu q$$

for some $0 < \mu < 1$, and, therefore,

$$\|p - x\|_2 + \|q - x\|_2 = \mu\|p - q\|_2 + (1 - \mu)\|p - q\|_2 = \|p - q\|_2.$$

Conversely, suppose that $x \in \mathbb{R}^2$ is a point in the plane for which

$$\|p - q\|_2 = \|p - x + x - q\|_2 = \|p - x\|_2 + \|q - x\|_2,$$

then we have equality in the triangle inequality, which can happen if and only if one of $p - x$ and $x - q$ is a nonnegative multiple of the other, that is, if and only if

$$p - x = \mu(x - q)$$

for some $\mu \geq 0$, that is, if and only if

$$x = \frac{1}{1+\mu}p + \frac{\mu}{1+\mu}q.$$

Thus, we have equality in the triangle inequality if and only if x is on the line segment between p and q, so the set we want is the line segment $[p, q]$.

(b) If $x = (x_1, x_2) \in \mathbb{R}^2$ is any point with $p_1 \leq x_1 \leq q_1$ and $p_2 \leq x_2 \leq q_2$, then

$$\begin{aligned}
\|p - x\|_1 + \|q - x\|_1 &= |p_1 - x_1| + |p_2 - x_2| + |q_1 - x_1| + |q_2 - x_2| \\
&= (x_1 - p_1) + (x_2 - p_2) + (q_1 - x_1) + (q_2 - x_2) \\
&= q_1 - p_1 + q_2 - p_2 \\
&= |p_1 - q_1| + |p_2 - q_2| \\
&= \|p - q\|_1,
\end{aligned}$$

that is,

$$\|p - x\|_1 + \|q - x\|_1 = \|p - q\|_1.$$

On the other hand, if $x = (x_1, x_2) \in \mathbb{R}^2$ is any point such that

$$\|p - x\|_1 + \|q - x\|_1 = \|p - q\|_1,$$

then from the triangle inequality for real numbers, we have

$$|p_1 - q_1| \leq |p_1 - x_1| + |q_1 - x_1| \tag{$*$}$$

and

$$|p_2 - q_2| \leq |p_2 - x_2| + |q_2 - x_2|. \tag{$**$}$$

Suppose that either $x_1 \notin [p_1, q_1]$ or $x_2 \notin [p_2, q_2]$, then at least one of the following is true (the reader should check this!)

$$\begin{aligned}
&|p_1 - x_1| > |p_1 - q_1|, \qquad &|q_1 - x_1| > |p_1 - q_1|, \\
&|p_2 - x_2| > |p_2 - q_2|, \qquad &|q_2 - x_2| > |p_2 - q_2|,
\end{aligned}$$

and we would have strict inequality in at least one (or both) of the inequalities $(*)$ and $(**)$ above. Adding $(*)$ and $(**)$, we would have

$$|p_1 - q_1| + |p_2 - q_2| < |p_1 - x_1| + |p_2 - x_2| + |q_1 - x_1| + |q_2 - x_2|,$$

that is, $\|p - q\|_1 < \|p - x\|_1 + \|q - x\|_1$, which is a contradiction. Therefore, $x_1 \in [p_1, q_1]$ and $x_2 \in [p_2, q_2]$, and in this case, the set we want is the closed box

$$B = \{x = (x_1, x_2) \,:\, p_1 \le x_1 \le q_1,\, p_2 \le x_2 \le q_2\}.$$

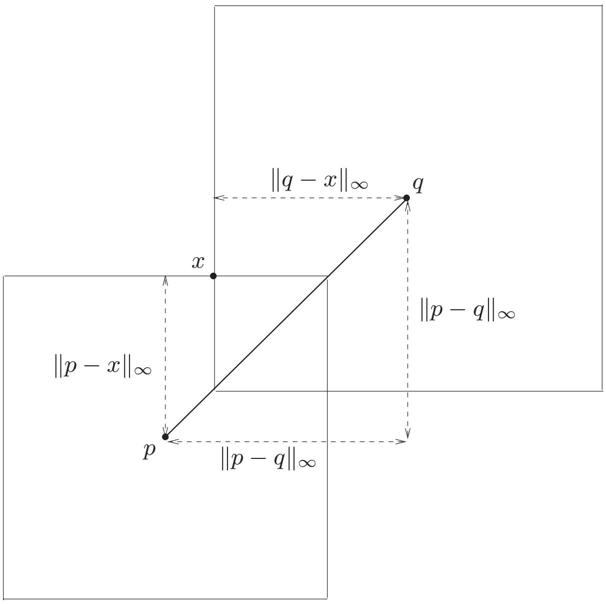
(c) If $x \in [p, q]$, then $x = (1 - \mu)p + \mu q$ for some $0 < \mu < 1$, so that

$$\|x - p\|_\infty = \mu\|p - q\|_\infty \qquad \text{and} \qquad \|x - q\|_\infty = (1 - \mu)\|p - q\|_\infty,$$

and adding, we get

$$\|x - p\|_\infty + \|x - q\|_\infty = \mu\|p - q\|_\infty + (1 - \mu)\|p - q\|_\infty = \|p - q\|_\infty.$$

Conversely, if $x \notin [p, q]$, since the line joining $p = (-1, -1)$ and $q = (1, 1)$ is the line $x_2 = x_1$, then the smallest ℓ_∞ balls centered at p and q containing x overlap, as in the figure below.



It is clear from the figure that

$$\|p - x\|_\infty + \|q - x\|_\infty > \|p - q\|_\infty.$$

In fact,

$$\|p - x\|_\infty + \|q - x\|_\infty = \|p - q\|_\infty$$

if and only if the point x lies on the line segment between p and q, that is,

$$x = (1 - \mu)p + \mu q$$

for some $0 \leq \mu \leq 1$. The set we want is the line segment $[p, q]$.

$\square$

1.4.3 Problems

In the following exercises, assume that "distance" means "Euclidean distance" unless otherwise stated.

1. (a) The ***unit cube*** in $\mathbb{R}^n$ is the set of points

$$\{\, x = (\alpha_1, \alpha_2, \ldots, \alpha_n) \,:\, |\alpha_i| \leq 1,\; i = 1, 2, \ldots, n \,\}.$$

   Draw the unit cube in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

   (b) What is the length of the longest line segment that you can place in the unit cube of $\mathbb{R}^n$?

   (c) What is the radius of the smallest Euclidean ball that contains the unit cube of $\mathbb{R}^n$?

2. (a) Let L be the straight line through $\overline{0}$ parallel to the vector $v = (1, 3, -1, 2)$. Find the two points where the line enters and exits the unit cube.

   Hint. Solve a similar problem in $\mathbb{R}^2$ first.

   $*$(b) Let L be the straight line through the point $p = (\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{2})$ parallel to the vector $v = (2, -1, 2, 1)$. Find the two points where the line enters and exits the unit cube.

3. Find the distance between the points $(1, -2)$ and $(-2, 3)$ using

   (a) the ℓ_1 metric,

   (b) the "sup" metric,

   (c) the Euclidean metric.

4. Let $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ denote, respectively, the ℓ_1, Euclidean, and "sup" norms. Identify all those points x in $\mathbb{R}^n$ that have the property

$$\|x\|_1 = \|x\|_2 = \|x\|_\infty.$$

   Hint. Try this for $\mathbb{R}^2$ first.

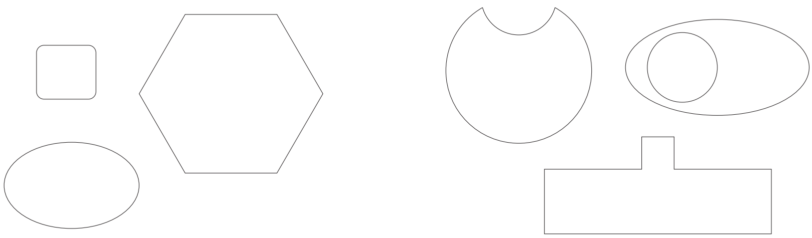
5. Show that a positive homothet of a closed ball is a closed ball.

1.5 CONVEX SETS

In this section, we give a brief introduction to convex sets, plus some examples.

A subset A of $\mathbb{R}^n$ is said to be ***convex*** if and only if whenever x and y are two points from A, then the entire segment $[x, y]$ is a subset of A; equivalently, A is convex if and only if $x, y \in A$ and $0 < \lambda < 1$ imply that $(1-\lambda)x + \lambda y \in A$ also.

Note that a convex set does not have any holes, dimples, or bumps.



Convex

Not convex

The sets depicted on the left in the above figure are convex, while those on the right are not convex.

Note. In order to verify that a set is convex using the definition, we use the following strategy:

Choose two arbitrary points in the set and show that the segment joining the points is in the set.

Example 1.5.1. *The closed unit ball in* $\mathbb{R}^n$

$$\overline{B}(\overline{0}, 1) = \{\, x \in \mathbb{R}^n \,:\, \|x\| \le 1 \,\}$$

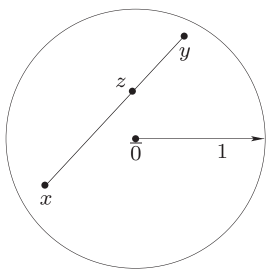
is a convex set.

Proof. Note that we did not specify any particular norm, so we should be able to prove this for *any* norm on $\mathbb{R}^n$.

Let $x, y \in \overline{B}(\overline{0}, 1)$, then for any point $z \in [x, y]$, we have

$$z = (1-\lambda)x + \lambda y$$

for some scalar λ with $0 \le \lambda \le 1$.
We also want to show that $z \in \overline{B}(\overline{0}, 1)$; that is, $\|z\| \le 1$.



$\overline{B}(\overline{0}, 1)$

This follows from the triangle inequality:

$$\begin{aligned}\|z\| &= \|(1-\lambda)x + \lambda y\| \\ &\le \|(1-\lambda)x\| + \|\lambda y\| \\ &= |1-\lambda| \cdot \|x\| + |\lambda| \cdot \|y\|,\end{aligned}$$

and since $0 \le \lambda \le 1$, then $|1-\lambda| = 1-\lambda$ and $|\lambda| = \lambda$, so that

$$\|z\| \le (1-\lambda)\|x\| + \lambda\|y\| \le (1-\lambda)\cdot 1 + \lambda \cdot 1 = 1.$$

□

Exercise 1.5.2. *Show that the open unit ball in* $\mathbb{R}^n$

$$B(\overline{0}, 1) = \{\, x \in \mathbb{R}^n \;:\; \|x\| < 1 \,\}$$

is a convex set.

Note. It is not difficult to see that the two extreme cases $A = \emptyset$ and $A = \mathbb{R}^n$ are both convex. In fact, if $\emptyset$ were not convex, then from the definition, we could find $x, y \in \emptyset$ such that $[x, y] \not\subseteq \emptyset$. Since there are no points in $\emptyset$, we cannot find such an x and y, so $\emptyset$ must be convex. We say that $\emptyset$ is convex ***vacuously***.

The next theorem shows that convexity is preserved under intersection and will give us another method for showing that a set is convex. Note that the family $\mathcal{F}$ in the theorem may be finite or infinite.

Theorem 1.5.3. *Suppose that $\mathcal{F}$ is a nonempty family of convex subsets of $\mathbb{R}^n$, then the set*

$$C = \bigcap \{\, A \;:\; A \in \mathcal{F} \,\}$$

is convex.

Proof. Let $x, y \in C$. We will show that $[x, y] \subset C$.

Since $x \in C$, then $x \in A$ for every $A \in \mathcal{F}$, and since $y \in C$, then $y \in A$ for every $A \in \mathcal{F}$.

Since each $A \in \mathcal{F}$ is convex, then $[x, y] \subset A$ for each $A \in \mathcal{F}$, and, therefore,

$$[x, y] \subset \bigcap \{\, A \;:\; A \in \mathcal{F} \,\}.$$

□

Remark. As mentioned earlier, this theorem provides another method of showing that a set is convex, namely, we show that the set is the intersection of a family of sets, all of which are known to be convex. At this point, illustrative examples would be highly contrived, and we give one such example.

Example 1.5.4. *If p and q are distinct points in $\mathbb{R}^n$, show that the segment*

$$[p, q] = \{\, x \in \mathbb{R}^n \,:\, x = (1-\lambda)p + \lambda q,\ 0 \le \lambda \le 1 \,\}$$

is convex.

Proof. Let X be the subset of $\mathbb{R}^n$ formed by intersecting all of the closed Euclidean balls that contain p and q. We will show that $X = [p, q]$.

Suppose that p and q are contained in a closed Euclidean ball centered at x_0 with radius $r > 0$, and let

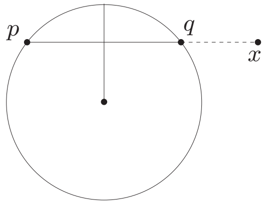
$$z = (1-\mu)p + \mu q,$$

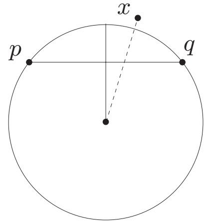
where $0 \le \mu \le 1$, be any point on the line segment joining p and q, then

$$\begin{aligned}
\|z - x_0\|_2 &= \|(1-\mu)p + \mu q - x_0\|_2 \\
&= \|(1-\mu)p + \mu q - (1-\mu)x_0 - \mu x_0\|_2 \\
&\le (1-\mu)\|p - x_0\|_2 + \mu\|q - x_0\|_2 \\
&\le (1-\mu)r + \mu r \\
&= r.
\end{aligned}$$

That is, z is in the closed ball also. Therefore, any closed Euclidean ball that contains p and q also contains the line segment joining p and q, that is, $[p, q]$. Hence, the intersection of all such closed Euclidean balls contains the entire segment $[p, q]$, and $[p, q] \subseteq X$.

Now suppose that x is a point that is not on the line segment $[p, q]$. If p, q, and x are collinear, then by taking the center on a line bisecting $[p, q]$ and a small enough radius, as in the figure below on the left, we can find a closed Euclidean ball containing p and q, but not containing x. Similarly, if p, q, and x are not collinear, then by taking the center on a line in a plane bisecting $[p, q]$ and a large enough radius, as in the figure below on the right, we can find a closed Euclidean ball containing p and q, but not containing x.





In either case, since x is not in this closed ball, it cannot possibly be in the intersection of all closed Euclidean balls containing p and q. Thus, if $x \notin [p, q]$, then $x \notin X$, and the contrapositive of this statement is true, that is, if $x \in X$, then $x \in [p, q]$. Therefore, $X \subseteq [p, q]$.

Combining these two containments, we get $X = [p, q]$, and since X is the intersection of a family of convex sets, then $[p, q]$ is convex.

□

Exercise 1.5.5. *Show $[p, q]$ is convex directly from the definition of convexity.*

1.6 HYPERPLANES AND LINEAR FUNCTIONALS

1.6.1 Linear Functionals

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then we say that f is a ***linear functional*** on $\mathbb{R}^n$ if and only if

(i) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$, and
(ii) $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Thus, a linear functional is a real-valued function on $\mathbb{R}^n$ that is both ***additive*** and ***homogeneous***.

Note. The notation $f = 0$ means that $f(x) = 0$ for all $x \in \mathbb{R}^n$, that is, f is identically zero. Thus, if f is a linear functional such that $f \neq 0$, then there is an $a \in \mathbb{R}^n$ such that $f(a) \neq 0$.

Example 1.6.1. *The following are examples of linear functionals:*

(1) $f(x_1, x_2) = 3x_1 + 4x_2$
(2) $f(x, y, z) = 3x + \sqrt{2}y - 52z$
(3) $f(u_1, u_2, u_3, u_4) = -1.0u_1 + 2.5u_2 - 7.3u_3 - 9.9u_4$
(4) $f(w, x, y, z) = 0$
(5) $f(x_1, x_2, x_3, x_4) = x_1 - x_3$.

Any missing terms are understood as being zero, so (5) is the same as

$$f(x_1, x_2, x_3, x_4) = x_1 + 0x_2 - x_3 + 0x_4.$$

The following are *not* linear functionals:

(1) $f(x_1, x_2) = 3x_1 + 4\sqrt{x_2}$

(2) $f(x, y, z) = \dfrac{3}{x} + \dfrac{\sqrt{2}}{y} - \dfrac{52}{z}$

(3) $f(u_1, u_2, u_3, u_4) = -1.0u_1^2 + 2.5u_2^2 - 7.3u_3^2 - 9.9u_4^2$

(4) $f(x_1, x_2, x_3, x_4) = x_1(1 - x_3)$.

$\square$

Representation of Linear Functionals

The notion of a linear functional occurs in many places in mathematics, and it is often very important to describe every possible type of linear functional that can arise in a given setting. Clearly, any function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form

$$f(x_1, x_2, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, is a linear functional on $\mathbb{R}^n$. We will show in the following theorem* that these are the *only* ones!

Theorem 1.6.2. *If f is a nonzero linear functional on $\mathbb{R}^n$, then there is a unique vector $a \in \mathbb{R}^n$, $a \neq \overline{0}$, such that*

$$f(x) = \langle a, x \rangle$$

for all $x \in \mathbb{R}^n$.

Proof. Let $\{ e_1, e_2, \ldots, e_n \}$ be the standard basis vectors for $\mathbb{R}^n$, that is,

$$e_1 = (1, 0, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ e_n = (0, 0, 0, \ldots, 1),$$

where for $k = 1, 2, \ldots, n$, the basis vector e_k has a 1 in the kth coordinate and 0 elsewhere.

Let $x = (x_1, x_2, \ldots, x_n)$ be an arbitrary vector in $\mathbb{R}^n$, then we can write

$$x = \sum_{k=1}^{n} x_k e_k,$$

and since f is a linear functional, then

$$f(x) = \sum_{k=1}^{n} f(e_k) x_k = \langle a, x \rangle,$$

* This theorem is called the ***Riesz Representation Theorem***.

where a is the vector

$$a = (f(e_1), f(e_2), \dots, f(e_n)).$$

Also, since f is nonzero, then at least one of the scalars $f(e_k)$ is nonzero, that is, $a \neq \overline{0}$.

Now we note that the vector a depends only on the linear functional f, so we have

$$f(x) = \langle a, x \rangle$$

for all $x \in \mathbb{R}^n$.

The fact that this representation of f is unique, that is, there is only one vector a that represents it, stems from the fact that there is only one way to write a vector x as a linear combination of the basis vectors.

□

It is usual to identify a linear functional with the vector of constant terms that are used to define it. For example, if f is defined by the equation

$$f(x_1, x_2, x_3) = 3x_1 - 2x_2 + 7x_3,$$

then we will say that f is ***represented*** by $(3, -2, 7)$. If g is the linear functional

$$g(u_1, u_2, u_3, u_4) = u_1 - 2u_3 + u_4,$$

then we would say that g is represented by $(1, 0, 2, 1)$.

Thus, every n-dimensional vector $(\alpha_1, \alpha_2, \dots, \alpha_n)$ gives rise to a unique linear functional on $\mathbb{R}^n$, and conversely, given any linear functional on $\mathbb{R}^n$, there is a unique n-dimensional vector that represents it.

Sums and Multiples of Linear Functionals

If f and g are two linear functionals on $\mathbb{R}^n$, we define the ***sum*** of f and g to be the function $f + g$ whose value is

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in \mathbb{R}^n$, and if λ is a scalar, we define the ***scalar multiple*** λf to be the function whose value is

$$(\lambda f)(x) = \lambda \cdot f(x)$$

for all $x \in \mathbb{R}^n$.

It is easily verified that $f + g$ and λf are linear functionals on $\mathbb{R}^n$ and that with these pointwise definitions of addition and scalar multiplication, the set of all linear functionals on $\mathbb{R}^n$ is a real vector space (called the ***dual space***). The next theorem shows that we can identify it with $\mathbb{R}^n$, and the proof is left as an exercise.

Theorem 1.6.3. *If the linear functional f is represented by $a \in \mathbb{R}^n$ and the linear functional g is represented by $b \in \mathbb{R}^n$, and if λ is any scalar, then the linear functional $f+g$ is represented by $a+b$ and the linear functional λf is represented by λa.*

Example 1.6.4. *If f and g are defined by*

$$\begin{aligned} f(x,y,z) &= 3x + y - 4x \\ g(x,y,z) &= -x + 2y + z, \end{aligned}$$

find the vectors that represent the functionals f, g, and $2f - g$.

Solution. From the definitions of f and g,

f is represented by $(3, 1, -4)$ and
g is represented by $(-1, 2, 1)$,

so that $2f - g$ is represented by

$$\begin{aligned} 2(3,1,-4) - (-1,2,1) &= (6,2,-8) - (-1,2,1) \\ &= (6-(-1), 2-2, -8-1) \\ &= (7,0,-9). \end{aligned}$$

□

Because the identification between vectors and linear functionals is so strong, it is usual to abuse the language and say that f *is* $(3,1,-4)$ instead of saying that f *is represented by* the vector $(3,1,-4)$.*

Example 1.6.5. *Let S be the set of all points in $\mathbb{R}^4$ whose fourth coordinate is zero. Find a linear functional f on $\mathbb{R}^4$ and a scalar β such that x is in S if and only if $f(x) = \beta$.*

Solution. We have to produce the linear functional

$$f(x_1, x_2, x_3, x_4) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$$

and the scalar β such that $x \in S$ if and only if $f(x) = \beta$. We will try to guess what f and β should be, and then we will show that our guess is correct.

* Woe is us! Now we can think of $(3,1,-4)$ in *three* different ways.

First, we note that the zero vector is in S because its fourth coordinate is 0. Thus, no matter what f we try, we will always get $f(\overline{0}) = 0$, and it seems reasonable to guess that $\beta = 0$.

Next we are going to guess what each α_i should be. Notice that S contains the vector $(1,0,0,0)$ so that $f(1,0,0,0) = \alpha_1$. Since $f(x) = 0$ for all x in S, we have $\alpha_1 = 0$. If we repeat this argument using the points $(0,1,0,0)$ and $(0,0,1,0)$, we find that α_2 and α_3 must also be zero. However, we cannot use the same argument for α_4 because $(0,0,0,1)$ is not in S. In fact, we cannot determine what α_4 must be, so we will guess that it can be any nonzero number.

Thus, we conclude that

$$f(x_1, x_2, x_3, x_4) = \alpha_4 x_4$$

where α_4 is any nonzero number and that $\beta = 0$.

This does not finish the solution—all we have done so far is to produce what seems like a reasonable guess. To complete the solution, we have to show that the guess is correct.

Note. A comment is worthwhile here. When you are actually writing a solution, you do not need to tell the reader how you guessed the answer. You are only obliged to show that your guess is correct. Here is a completely acceptable solution.

We will show that $f = (0,0,0,\alpha_4)$ and that $\beta = 0$. To check this, note that every point in S is of the form $(x_1, x_2, x_3, 0)$, so that

$$f(x_1, x_2, x_3, 0) = 0x_1 + 0x_2 + 0x_3 + \alpha_4 \cdot 0 = 0$$

for any real number α_4, which completes the proof.

□

Geometry of Linear Functionals

If f is a linear functional on $\mathbb{R}^n$ and α is a scalar, then the set of all points $x \in \mathbb{R}^n$ such that $f(x) = \alpha$ is denoted by $f^{-1}(\alpha)$, that is,

$$f^{-1}(\alpha) = \{\, x \in \mathbb{R}^n \,:\, f(x) = \alpha \,\}.$$

The set $f^{-1}(\alpha)$ is called the ***counterimage*** of α under f or the ***inverse image*** of α under f.

Note that $f^{-1}(\alpha)$ consists of *all* points $x \in \mathbb{R}^n$ such that $f(x) = \alpha$, and it should be stressed that $f^{-1}(\alpha)$ is a *set*. The notation is ***not*** meant to imply that f is invertible! In particular, one must *avoid thinking that* $f^{-1}(f(x)) = x$.*

If f is a nonzero linear functional on $\mathbb{R}^n$, then the ***kernel*** of f, denoted by $\ker(f)$, is the set of all $x \in \mathbb{R}^n$ such that $f(x) = 0$, that is,

$$\ker(f) = f^{-1}(0) = \{\, x \in \mathbb{R}^n \,:\, f(x) = 0 \,\}.$$

The next theorem shows that this is actually a subspace of $\mathbb{R}^n$.

Theorem 1.6.6. *If f is a linear functional on $\mathbb{R}^n$ and $f \neq 0$, then*

$$f^{-1}(0) = \ker(f) = \{\, x \in \mathbb{R}^n \,:\, f(x) = 0 \,\}$$

is a subspace of $\mathbb{R}^n$.

Proof. If $x, y \in f^{-1}(0)$, then

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0,$$

so $x + y \in f^{-1}(0)$.

If $\lambda \in \mathbb{R}$ and $x \in f^{-1}(0)$, then

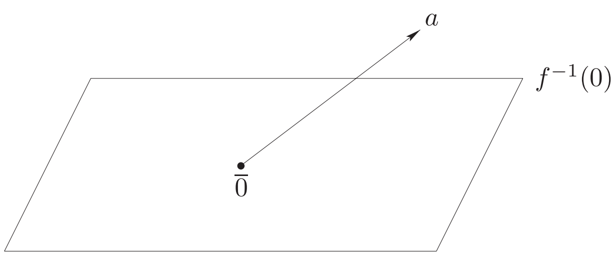
$$f(\lambda x) = \lambda f(x) = \lambda \cdot 0 = 0,$$

so $\lambda x \in f^{-1}(0)$.

Therefore, $f^{-1}(0)$ is closed under addition and scalar multiplication and hence is a subspace.

□

Note. If f is a linear functional on $\mathbb{R}^n$ and $f \neq 0$, then there is an $a \in \mathbb{R}^n$ such that $f(a) \neq 0$, so that $a \notin f^{-1}(0)$ and $f^{-1}(0) \subsetneq \mathbb{R}^n$, that is, $f^{-1}(0)$ is a ***proper subspace*** of $\mathbb{R}^n$.



* The only thing that you can say is that $x \in f^{-1}(f(x))$.

The next theorem is reminiscent of the orthogonal projection theorem in linear algebra.

Theorem 1.6.7. *Let f be a nonzero linear functional on $\mathbb{R}^n$ and let $a \in \mathbb{R}^n$ be a vector such that $f(a) \neq 0$, then any point $p \in \mathbb{R}^n$ can be written uniquely as*

$$p = \lambda a + x$$

where $\lambda \in \mathbb{R}$ and $x \in f^{-1}(0)$.

Proof. Consider the vector $x = p - \lambda a$. Since f is a linear functional,

$$f(x) = f(p) - \lambda f(a),$$

and since $f(a) \neq 0$, then $f(x) = 0$ when $\lambda = \dfrac{f(p)}{f(a)}$. Therefore,

$$x_0 = p - \frac{f(p)}{f(a)} a \in f^{-1}(0),$$

so that

$$p = \frac{f(p)}{f(a)} a + x_0,$$

where $x_0 \in f^{-1}(0)$.

To show that this representation is unique, suppose that

$$p = x_1 + \mu a,$$

where $\mu \in \mathbb{R}$ and $x_1 \in f^{-1}(0)$, then

$$f(p) = f(x_1) + \mu f(a) = 0 + \mu f(a) = \mu f(a),$$

so that

$$\mu = \frac{f(p)}{f(a)} \qquad \text{and} \qquad \mu = \lambda.$$

Therefore,

$$p = x_1 + \frac{f(p)}{f(a)} a,$$

and since

$$p = x_0 + \frac{f(p)}{f(a)} a,$$

then $x_1 = x_0$.

□

Note that in the theorem, a can be *any* fixed vector with $f(a) \neq 0$.

Now let

$$\mathcal{B} = \{ e_1, e_2, \ldots, e_k \}$$

be a basis for the subspace $f^{-1}(0)$, then the set

$$\mathcal{B}' = \{ a, e_1, e_2, \ldots, e_k \}$$

is linearly independent since $a \notin f^{-1}(0)$, and the set also spans $\mathbb{R}^n$ by the previous theorem. Therefore, it forms a basis for $\mathbb{R}^n$, and hence, $n = k + 1$, that is, $k = n - 1$. Thus, we have shown the following theorem.

Theorem 1.6.8. *If f is a nonzero linear functional on $\mathbb{R}^n$, then $f^{-1}(0)$ is a subspace of dimension $n - 1$, that is, $f^{-1}(0)$ is a subspace of* **codimension** 1.

1.6.2 Hyperplanes

In $\mathbb{R}^2$, the set of points satisfying a linear equation such as

$$3x_1 + 4x_2 = 7$$

is a straight line.

In $\mathbb{R}^3$, the set of points satisfying a linear equation such as

$$2x_1 - 6x_2 - x_3 = 5$$

is a plane.

More generally, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ and β are constant scalars and if at least one α_i is nonzero, then the set of all points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ that satisfy the linear equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \beta \qquad (*)$$

is called a ***hyperplane*** in $\mathbb{R}^n$.

In other words, a hyperplane is a set H_β in $\mathbb{R}^n$ defined by

$$H_\beta = \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,:\, \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \beta \}. \qquad (**)$$

In terms of linear functionals, we can rewrite equation $(*)$ as

$$f(x) = \beta$$

and we can rewrite equation $(**)$ as

$$H_\beta = f^{-1}(\beta) = \{ x \in \mathbb{R}^n \,:\, f(x) = \beta \}$$

where x is the point $(x_1, x_2, \ldots, x_n)$ and f is a nonzero linear functional on $\mathbb{R}^n$ represented by the vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.*

Thus, in $\mathbb{R}^2$, a hyperplane is a line, and in $\mathbb{R}^3$, a hyperplane is a plane. The hyperplane in Example 1.6.5 is readily identified with $\mathbb{R}^3$.

In all three cases, the hyperplane is one dimension less than the dimension of the space in which it lives. We know from Theorem 1.6.8 that in $\mathbb{R}^n$, the hyperplane

$$f^{-1}(0) = \{(x_1, x_2, \ldots, x_n) \,:\, \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \}$$

is actually a subspace of dimension $n-1$.

Note. We can also see this as follows: in $\mathbb{R}^n$, let H be the hyperplane $f^{-1}(0)$, where f is a nonzero linear functional.

To see that the subspace H has dimension $n-1$, note that by definition, H is the set of all solutions $(x_1, x_2, \ldots, x_n)$ to the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0,$$

where at least one α_k is nonzero.

However, we know from the theory of linear equations in linear algebra that the solution space to this equation contains $n-1$ linearly independent vectors and no more than $n-1$ linearly independent vectors, which is to say that H has dimension $n-1$.

Example 1.6.9. *Show that if $\mathcal{S} = \{v_1, v_2, \ldots, v_{n-1}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then the linear subspace V spanned by $\mathcal{S}$ is a hyperplane.*

Solution. Perhaps the quickest way to see this is to use determinants. Suppose that the vectors v_k are

$$\begin{aligned} v_1 &= (\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n}) \\ v_2 &= (\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,n}) \\ &\vdots \\ v_{n-1} &= (\alpha_{n-1,1}, \alpha_{n-1,2}, \ldots, \alpha_{n-1,n}). \end{aligned}$$

* Note that the hyperplane H_β depends on both the vector $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *and* the scalar β.

Let x be the vector $(x_1, x_2, \ldots, x_n)$, and consider the determinantal equation

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \cdots & \alpha_{n-1,n} \end{vmatrix} = 0. \tag{$*$}$$

When we expand this in terms of cofactors of the first row, we obtain

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = 0,$$

where A_{ij} denotes the cofactor of the ith row and jth column. Since the cofactors A_{ij} are constants, we recognize $(*)$ as the equation of a hyperplane. It is clear that this hyperplane contains the vectors v_k. For example, to see that v_1 satisfies $(*)$, we substitute v_1 for x in the left side of $(*)$ and get

$$\begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \cdots & \alpha_{n-1,n} \end{vmatrix}$$

which must be zero, since it contains two identical rows.

□

In the next theorem, we prove that the hyperplane

$$H_\beta = f^{-1}(\beta),$$

where $\beta \neq 0$, is a translate of the subspace $f^{-1}(0)$ and so $f^{-1}(\beta)$ must have the same "dimension" as $f^{-1}(0)$.

Theorem 1.6.10. *Let f be a nonzero linear functional on $\mathbb{R}^n$, $\beta \in \mathbb{R}$, with $\beta \neq 0$, and let*

$$H_\beta = f^{-1}(\beta).$$

If $a \in \mathbb{R}^n$ is any point such that $f(a) = \beta$, then

$$H_\beta = a + f^{-1}(0),^*$$

that is, the hyperplane H_β is just a translate of a subspace of dimension $n-1$.

* Because $\{a\} \cap f^{-1}(0) = \emptyset$, this is sometimes written as $a \oplus f^{-1}(0)$ and called a ***direct sum***.

Proof. Let $x \in H_\beta$, then

$$f(x) = \beta = f(a),$$

so that

$$f(x - a) = \beta - \beta = 0,$$

that is,

$$x - a \in f^{-1}(0).$$

Therefore, $H_\beta \subseteq a + f^{-1}(0)$.

Conversely, if $x \in a + f^{-1}(0)$, then

$$x = a + z,$$

where $z \in f^{-1}(0)$, that is, $f(z) = 0$. Therefore,

$$f(x) = f(a) = \beta.$$

Hence, $x \in H_\beta$ and $a + f^{-1}(0) \subseteq H_\beta$.

$\square$

Remark. Hyperplanes in infinite dimensional spaces are defined as being ***maximal proper subspaces***, or ***translates of maximal proper subspaces***. Theorems 1.6.8 and 1.6.10 show that in $\mathbb{R}^n$ at least, this definition and the one we gave are equivalent.

Note. Summarizing all this, any hyperplane

$$H_\beta = \{\, x \in \mathbb{R}^n \,:\, f(x) = \beta \,\}$$

can be written as

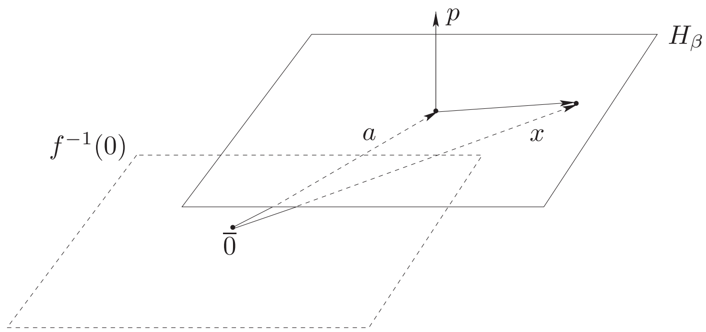
$$H_\beta = \{\, x \in \mathbb{R}^n \,:\, \langle p, x \rangle = \beta \,\}$$

for some fixed $p \neq \overline{0}$ in $\mathbb{R}^n$,

and

$$f^{-1}(0) = \{\, x \in \mathbb{R}^n \,:\, \langle p, x \rangle = 0 \,\}.$$

Thus, $f^{-1}(0)$ is the subspace of all vectors orthogonal to p, and H_β is just a translate of this subspace, as shown in the following figure.



Here

$$H_\beta = a + f^{-1}(0),$$

where $\langle p, a\rangle = \beta$, so that $x \in H_\beta$ if and only if

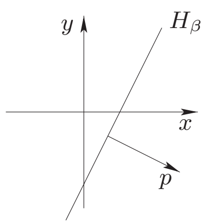
$$f(x) = f(a + x - a) = f(a) + f(x - a) = \langle p, a\rangle + \langle p, x - a\rangle = \beta + 0 = \beta.$$

Example 1.6.11. *If $p = (4, -2)$ and $\beta = 1$, sketch the hyperplane in $\mathbb{R}^2$ determined by p and β.*

Solution. The hyperplane is the set

$$H_\beta = \{\, (x, y) \in \mathbb{R}^2 \, : \, 4x - 2y = 1 \,\},$$

since in $\mathbb{R}^2$, a hyperplane is just a line, a translate of a subspace of dimension 1. The line passes through the point $a = (\frac{1}{4}, 0)$ and is perpendicular to the vector $p = (4, -2)$. The hyperplane is sketched below.



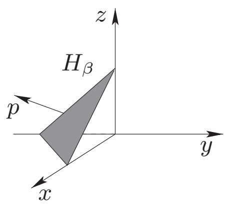
$\square$

Example 1.6.12. *If $p = (4, -2, 3)$ and $\beta = 1$, sketch the hyperplane in $\mathbb{R}^3$ determined by p and β.*

Solution. The hyperplane is the set

$$H_\beta = \{\,(x, y, z) \in \mathbb{R}^3 \;:\; 4x - 2y + 3z = 1\,\},$$

since in $\mathbb{R}^3$, a hyperplane is just a plane, a translate of a subspace of dimension 2. The plane passes through the point $a = (\frac{1}{4}, 0, 0)$ and is perpendicular to the vector $p = (4, -2, 3)$. The portion of the hyperplane in one octant is sketched below.



$\square$

Halfspaces

A hyperplane determines two ***halfspaces***, one on each side of the hyperplane, and the most fundamental property of the hyperplane is that it divides the space $\mathbb{R}^n$ into three disjoint parts:

- the hyperplane itself

$$H = \{\,x \in \mathbb{R}^n \;:\; f(x) = \beta\,\}$$

- an ***open halfspace*** to one side of the hyperplane

$$H^+ = \{\,x \in \mathbb{R}^n \;:\; f(x) > \beta\,\}$$

- the open halfspace on the other side of the hyperplane

$$H^- = \{\,x \in \mathbb{R}^n \;:\; f(x) < \beta\,\}.$$

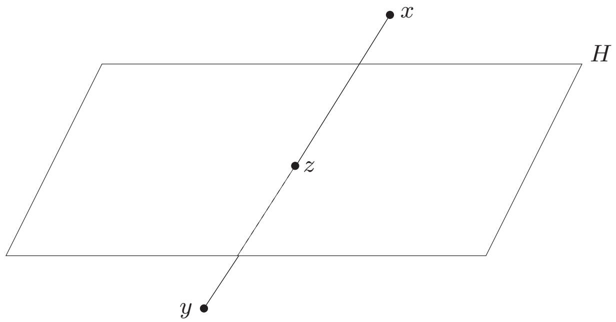
If the hyperplane is adjoined to either of the open halfspaces, the result is a set of the type

$$\{\,x \in \mathbb{R}^n \;:\; f(x) \le \beta\,\} \quad \text{or} \quad \{\,x \in \mathbb{R}^n \;:\; f(x) \ge \beta\,\}.$$

Such sets are called ***closed halfspaces***.

A hyperplane H separates $\mathbb{R}^n$ in the following sense.

Theorem 1.6.13. *If the point x is in one of the open halfspaces determined by a hyperplane H and y is in the other open halfspace, then the line segment (x, y) intersects the hyperplane H at precisely one point z.*



Proof. Suppose that the hyperplane H is given by

$$H = \{ z \in \mathbb{R}^n \ : \ \langle p, z \rangle = \lambda \}$$

for some scalar λ and some nonzero vector $p \in \mathbb{R}^n$. Since x and y are in different halfspaces, we may assume that

$$\alpha = \langle p, x \rangle < \lambda < \langle p, y \rangle = \beta.$$

We have to show that there is a point $w \in (x, y)$ such that $\langle p, w \rangle = \lambda$.

Let $w = (1 - \mu)x + \mu y$ be a typical point on the line through x and y. We want to find a scalar μ with $0 < \mu < 1$ such that

$$\langle p, (1 - \mu)x + \mu y \rangle = \lambda,$$

that is,

$$(1 - \mu)\langle p, x \rangle + \mu \langle p, y \rangle = \lambda,$$

or

$$(1 - \mu)\alpha + \mu\beta = \lambda.$$

Solving this equation for μ, we have

$$\mu = \frac{\lambda - \alpha}{\beta - \alpha},$$

and since $\alpha < \lambda < \beta$, then $0 < \mu < 1$, that is, w is in the line segment (x, y).

□

Example 1.6.14. *Given* $f = (1, -2, -1, 4)$, *show that the hyperplane* $f^{-1}(10)$ *misses the closed unit ball.*

Solution. Since the norm has not been specified, we assume that we are dealing with the Euclidean ball. We will solve this problem by showing that the ball lies to one side of the hyperplane, that is, we will show that the ball is contained in the open halfspace $\{ x \in \mathbb{R}^4 \, : \, f(x) < 10 \}$.*

Now, a typical point of the unit ball $x = (x_1, x_2, x_3, x_4)$ has

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1.$$

Consequently, we know for certain that for each coordinate x_k, the absolute value $|x_k|$ must be no greater than 1. Thus,

$$\begin{aligned} f(x) &= x_1 - 2x_2 - x_3 + 4x_4 \\ &\leq |x_1| + 2|x_2| + |x_3| + 4|x_4| \\ &\leq 1 + 2 + 1 + 4, \end{aligned}$$

which shows that $f(x) < 10$ and

$$\overline{B}(\overline{0}, 1) \subsetneq \{ x \in \mathbb{R}^4 \, : \, f(x) < 10 \}.$$

$\square$

Example 1.6.15. *Given that* $f = (3, -1, 0, 2)$, *find the point where the line through* $(1, 0, 0, 2)$ *parallel to* $(1, 1, -1, 3)$ *intersects the hyperplane* $f^{-1}(1)$.

Solution. The problem asks us to find the point x on the line for which $f(x) = 1$.

A typical point $x = (x_1, x_2, x_3, x_4)$ on the line can be written as

$$(x_1, x_2, x_3, x_4) = (1, 0, 0, 2) + \mu(1, 1, -1, 3) = (1 + \mu, \mu, -\mu, 2 + 3\mu)$$

for some scalar μ.

Therefore,

$$f(x) = f(1 + \mu, \mu, -\mu, 2 + 3\mu) = 3(1 + \mu) - 1\mu + 0 + 2(2 + 3\mu) = 7 + 6\mu,$$

and setting this equal to 1 yields $\mu = -1$. Hence, the point of intersection of the line and hyperplane is

$$x = (1 + \mu, \mu, -\mu, 2 + 3\mu) = (0, -1, 1, -1).$$

$\square$

* Can you see why we know immediately that the unit ball cannot be contained in the other open halfspace?

Example 1.6.16. *Given that x and y are points in $\mathbb{R}^n$ and that f is a linear functional with $f(x) = 3$ and $f(y) = 9$, find the hyperplane determined by f that contains the midpoint of the straight line segment joining x and y.*

Solution. The midpoint of the segment is

$$z = \tfrac{1}{2}x + \tfrac{1}{2}y,$$

and hence, the value of the linear functional f at the midpoint is

$$f(z) = f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) = \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) = \tfrac{1}{2}\cdot 3 + \tfrac{1}{2}\cdot 9 = 6.$$

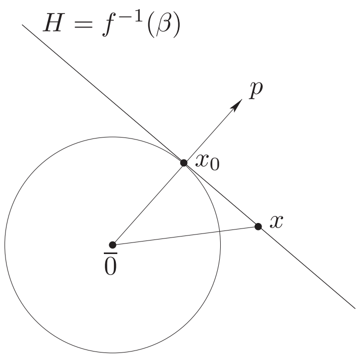
Therefore, the hyperplane is $f^{-1}(6)$.

□

We will need the following theorem. Although it is possible to prove it at this point, a simpler proof will be given later in the text.

Theorem 1.6.17. *Given the hyperplane $H = f^{-1}(\beta)$ in $\mathbb{R}^n$, where f is represented by p, the point on H that has minimum Euclidean norm is the point where the straight line through $\overline{0}$ in the direction of p intersects H.*

Proof. Suppose that the line through $\overline{0}$ in the direction of p intersects the hyperplane H at x_0, as depicted in the figure below.



Let x be an arbitrary point in H, then

$$\langle p, x - x_0\rangle = \langle p, x\rangle - \langle p, x_0\rangle = \beta - \beta = 0,$$

so that the vector $x - x_0$ is orthogonal to p, and from the Pythagorean theorem,

$$\|x\|_2^2 = \|x_0\|_2^2 + \|x - x_0\|_2^2 \geq \|x_0\|_2^2,$$

so that

$$\|x\|_2 \geq \|x_0\|_2$$

for all $x \in H$.

Therefore, x_0 is the point in the hyperplane H that has minimum Euclidean norm.

$\square$

Corollary 1.6.18. *In the hyperplane $H = f^{-1}(\beta)$, where f is represented by p, the point closest to $\overline{0}$ is*

$$x_0 = \frac{\beta}{\|p\|_2^2}\, p$$

and $\|x_0\|_2 = \dfrac{|\beta|}{\|p\|_2}$.

Proof. Since

$$f(x_0) = \frac{\beta}{\|p\|_2^2}\, f(p) = \frac{\beta}{\|p\|_2^2}\, \langle p, p\rangle = \beta,$$

by the previous theorem, the point x_0 is precisely the point in $H = f^{-1}(\beta)$, which is closest to $\overline{0}$.

Also,

$$\|x_0\|_2 = \frac{|\beta|}{\|p\|_2^2}\, \|p\|_2 = \frac{|\beta|}{\|p\|_2}.$$

$\square$

The word "normal" in geometry is also synonymous with perpendicularity. If H is the hyperplane $f^{-1}(\beta)$, the vector p representing the nonzero linear functional f is often called a ***normal vector*** to the hyperplane. Of course, any nonzero multiple of p is also a normal vector.

For example, recall that for the plane

$$ax + by + cz = d$$

in $\mathbb{R}^3$, the distance from the plane to the origin is given by the formula

$$\rho = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}},$$

as in the corollary above. You may also recall that the equation

$$\frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

is sometimes called the *normal form* of the equation of the plane $ax + by + cz = d$.

Example 1.6.19. *Find the point $z_0 = (x_0, y_0)$ on the line*

$$L = \{ z = (x, y) \in \mathbb{R}^2 : 3x - 2y = 1 \}$$

in $\mathbb{R}^2$ that is closest to the origin and find the norm of z_0.

Solution. Since no distance function is specified, we assume that the Euclidean distance is to be used, and, hence, the previous theorem will apply.

The line L is really the hyperplane $H = f^{-1}(1)$, where f is represented by the vector $p = (3, -2)$. According to the theorem, the point z in question is where the line through $\overline{0} = (0, 0)$ and $p = (3, -2)$ intersects H.

Now, a typical point on the line through $(0, 0)$ and $(3, -2)$ is

$$z = (x, y) = (0, 0) + \mu(3, -2) = (3\mu, -2\mu)$$

and we are looking for a value of $\mu \in \mathbb{R}$ such that

$$f(x, y) = \langle p, z \rangle = 3 \cdot (3\mu) - 2 \cdot (-2\mu) = 13\mu = 1.$$

This yields $\mu = \frac{1}{13}$, and the point z_0 we want is

$$z_0 = (3\mu, -2\mu) = \left(\tfrac{3}{13}, -\tfrac{2}{13}\right),$$

with norm

$$\|z_0\|_2 = \left\|\left(\tfrac{3}{13}, -\tfrac{2}{13}\right)\right\| = \sqrt{\left(\tfrac{3}{13}\right)^2 + \left(-\tfrac{2}{13}\right)^2} = \tfrac{1}{\sqrt{13}}.$$

Remark. In the preceding example, the distance from z_0 to $\overline{0}$ is $\frac{1}{\sqrt{13}}$. This means that the hyperplane H is tangent to the closed ball $\overline{B}(\overline{0}, \frac{1}{\sqrt{13}})$ at the point $\left(\frac{3}{13}, -\frac{2}{13}\right)$. The next example uses the correspondence between tangency and points of minimum ℓ_2 norm.

Example 1.6.20. *Let f be the linear functional on $\mathbb{R}^4$ with the Euclidean norm represented by the vector $p = (-1, 2, -1, 1)$. Find the point on the closed unit ball in $\mathbb{R}^4$, where f attains its maximum value and find that maximum value.*

Solution. First, we make sure the question is clear. For every point x in $\overline{B}(\overline{0}, 1)$, $f(x)$ has a specific value. What we want to find is the point where this value is a maximum.

Next, we interpret this geometrically. Suppose that the maximum happens to be β and that it occurs at the point x_0. Then $f(x_0) = \beta$, and for every other point y in $\overline{B}(\overline{0}, 1)$, we must have $f(y) < \beta$.

However, this would mean that the hyperplane H, whose equation is $f(x) = \beta$, touches the closed unit ball at precisely x_0. Furthermore, we know that in this situation, the point x_0 is where the line through $\overline{0}$ would intersect the hyperplane H. Thus, if we can find x_0, then we will have found the solution.

There is something else about x_0 that we can exploit. Since x_0 is the point of tangency to the unit ball, it cannot be inside the ball, and therefore, $\|x_0\|_2 = 1$. Thus, the problem reduces to the following: find the point where the line through $\overline{0}$ and p intersects the unit sphere. There will be two such points, and one of them will be the one we want.

Now we proceed with the solution. We want to find the points x in $\mathbb{R}^4$ where the line through $\overline{0} = (0, 0, 0, 0)$ and $p = (-1, 2, -1, 1)$ pierces the unit sphere

$$S = \{\, x \in \mathbb{R}^4 \,:\, \|x\|_2 = 1 \,\}.$$

The line has the equation

$$x = (1 - \mu)\overline{0} + \mu(-1, 2, -1, 1) = \mu(-1, 2, -1, 1), \quad -\infty < \mu < \infty$$

and we want to find values of μ such that $\|x\|_2 = 1$.

Now,

$$\|x\|_2 = \|\mu(-1, 2, -1, 1)\| = |\mu|\sqrt{(-1)^2 + 2^2 + (-1)^2 + 1^2}$$

so that

$$\|x\|_2 = \sqrt{7}\,|\mu|,$$

and, therefore, $\|x\|_2 = 1$ when $|\mu| = \frac{1}{\sqrt{7}}$. The two places where the line intersects the sphere are

$$\frac{1}{\sqrt{7}}(-1, 2, -1, 1) \quad \text{and} \quad -\frac{1}{\sqrt{7}}(-1, 2, -1, 1).$$

Checking the value of f at each of these points, we have

$$f\left(\frac{1}{\sqrt{7}}(-1, 2, -1, 1)\right) = \sqrt{7} \quad \text{and} \quad f\left(-\frac{1}{\sqrt{7}}(-1, 2, -1, 1)\right) = -\sqrt{7}.$$

Therefore, f attains a maximum value of $\sqrt{7}$ on the closed unit ball, and it attains that value at the point

$$x_0 = \frac{1}{\sqrt{7}}(-1, 2, -1, 1).$$

$\square$

Example 1.6.21. *If p is a point in the unit sphere in $\mathbb{R}^n$, then the hyperplane H whose equation is $\langle p, x\rangle = 1$ is tangent to the unit sphere at p, and only at the point p (here the Euclidean norm is being used).*

Solution. Since p is on the sphere, then $\|p\|_2 = 1$. Taking $x = p$ in the equation of the hyperplane, we have

$$\langle p, x\rangle = \langle p, p\rangle = \|p\|^2 = 1,$$

and thus $p \in H$.

Now let $q \in S\left(\overline{0}, 1\right)$, where $q \neq p$. We want to show that $x = q$ does not satisfy the equation for H. We recall that in the triangle inequality

$$\langle x, y\rangle \leq \|x\|_2 \cdot \|y\|_2,$$

and equality holds if and only if one of x or y is a nonnegative multiple of the other.

The only nonnegative multiple of p that belongs to $S\left(\overline{0}, 1\right)$ is p itself, and hence, neither p nor q is a nonnegative multiple of the other, so that

$$\langle p, q\rangle < \|p\|_2 \cdot \|q\|_2 = 1 \cdot 1 = 1.$$

Therefore, $q \notin H$.

$\square$

Example 1.6.22. *Let S be the unit sphere (in the Euclidean norm) in $\mathbb{R}^n$. Suppose that H is the hyperplane whose equation is $\langle p, x\rangle = \alpha$, where $0 < \alpha < 1$. Let $T = S \cap H$. Show that T is a sphere in H, that is, show that there is some point q in H and some constant δ such that $\|x - q\|_2 = \delta$ for all $x \in T$.*

Solution. The vector p is orthogonal to the hyperplane H, and if $x \in T = S \cap H$, then

$$\alpha = \langle p, x\rangle \leq \|p\|_2 \cdot \|x\|_2 = \|p\|_2,$$

so that $\|p\|_2 \geq \alpha$.

The equation of the line L through $\overline{0}$ in the direction of p is

$$L: \quad x = \mu p, \quad 0 < \mu < \infty$$

and this line intersects H at the point $x = \mu p$ where

$$\langle p, x\rangle = \mu\langle p, p\rangle = \alpha,$$

so that

$$\mu = \frac{\alpha}{\langle p, p\rangle} = \frac{\alpha}{\|p\|_2^2}.$$

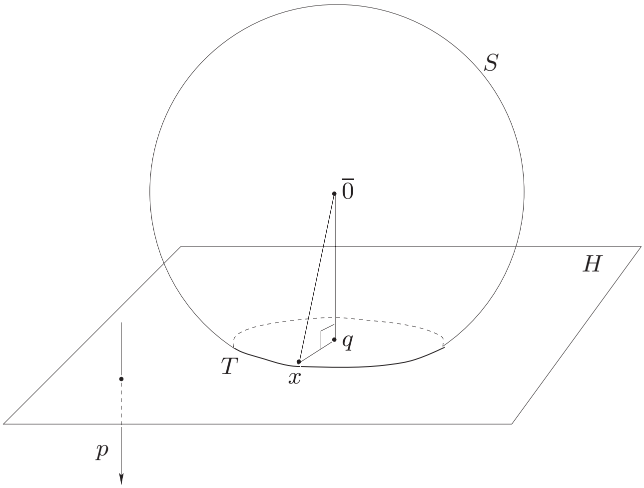
Thus, the point in H closest to $\overline{0}$ is the point

$$q = \frac{\alpha}{\|p\|_2^2} \cdot p$$

so that $\|q\|_2 = \dfrac{\alpha}{\|p\|_2}$, and hence, $\|q\|_2 \leq 1$.

Now let $x \in T = S \cap H$ and let $z = x - q$, then z is orthogonal to p, since

$$\langle z, p\rangle = \langle x - q, p\rangle = \langle x, p\rangle - \langle q, p\rangle = \alpha - \alpha = 0,$$



and from the Pythagorean theorem, we have

$$\|x - q\|_2^2 + \|q\|_2^2 = \|x\|_2^2 = 1,$$

that is, $\|x - q\|_2^2 = 1 - \dfrac{\alpha^2}{\|p\|_2^2}$ for all $x \in T = S \cap H$. Taking

$$\delta = \sqrt{1 - \frac{\alpha^2}{\|p\|_2^2}},$$

then $0 \leq \delta < 1$ and

$$\|x - q\|_2 = \delta$$

for all $x \in T = S \cap H$, that is, T is a sphere in H.

□

1.6.3 Problems

In the following exercises, unless otherwise stated, assume that the closed unit ball is the closed unit ball in the Euclidean norm.

*1. Find a hyperplane $H = f^{-1}(1)$ in $\mathbb{R}^4$ that is tangent to the unit cube at the point $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$. Verify your answer.

2. Draw each of the following hyperplanes in $\mathbb{R}^2$:

   (a) The hyperplane through the point $(1, 3)$ that is perpendicular to the line through $(0, 0)$ and $(1, 3)$,
   (b) The hyperplane that is tangent to the unit sphere at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$,
   (c) The hyperplane whose equation is $-\alpha_1 + 2\alpha_2 = 3$,
   (d) $f^{-1}(0)$ where f is represented by $p = (3, 1)$,
   (e) $f^{-1}(1)$ where f is represented by $p = (3, 1)$,
   (f) $f^{-1}(2)$ where f is represented by $p = (3, 1)$.

3. Find an equation for the hyperplane of

   (a) Problem 2 (a),
   (b) Problem 2 (b).

4. Find the point of intersection of the plane

$$H = \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \;:\; 2x_1 - 3x_2 + x_3 = 2 \,\}$$

   with the line through $(1, 0, 1)$ and $(-2, 1, 2)$.

5. Given the linear functional $f(x_1, x_2) = 4x_1 - 3x_2$, find

   (a) the point x on the closed unit ball where $f(x)$ is a maximum,
   (b) the point x in the hyperplane $f^{-1}(2)$ that is closest to the origin,
   (c) the point x in the hyperplane $f^{-1}(3)$ that is closest to the origin.

6. Given the linear functional $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + x_3$, find

   (a) the point x on the unit ball where $f(x)$ is a maximum,
   (b) the point x in the hyperplane $f^{-1}(2)$ that is closest to the origin,
   (c) the point x in the hyperplane $f^{-1}(3)$ that is closest to the origin.

7. Let f be the linear functional on $\mathbb{R}^3$ represented by the vector $p = (3, -2, -3)$ and let $\mathcal{S}$ be the set

$$\mathcal{S} = \{\, (1, 1, -2),\ (-3, 4, 1),\ (60, 10, 15),\ (-8, -2, 4),\ (0, 1, 1) \,\}.$$

   (a) Determine which points of $\mathcal{S}$ are on the same side of $f^{-1}(0)$,
   (b) Which point or points of $\mathcal{S}$ are closest to $f^{-1}(0)$?

(c) Which points of $\mathcal{S}$ are on the same side of $f^{-1}(8)$ as the origin?
(d) Find the point or points of $\mathcal{S}$ that are closest to $f^{-1}(8)$.

8. Given the line $L = \{ (x, y) : 3x + 4y = 5 \}$ in $\mathbb{R}^2$, find the point on L of minimum norm in each of the following cases, and draw a figure with the appropriate unit ball:

   (a) for the Euclidean norm,
   (b) for the ℓ_1 norm,
   (c) for the ℓ_∞ norm.

9. Given that H is the hyperplane $f^{-1}(2)$, and given that $g = 4f$, find β such that $g^{-1}(\beta)$ is exactly the same as H.

10. Let L be the line

$$L = \{ x \in \mathbb{R}^n : x = p + \mu q, \ -\infty < \mu < \infty \},$$

where p and q are distinct points in $\mathbb{R}^n$, and let f be a linear functional on $\mathbb{R}^n$ such that $f(p) = 3$ and $f(q) = -1$. Find

   (a) the point where L intersects the hyperplane $f^{-1}(1)$; in other words, find the scalar μ such that $f(p + \mu q) = 1$,
   (b) the scalar β such that the hyperplane $f^{-1}(\beta)$ intersects the line at the point $x = p + \mu q$ where $\mu = 3.4$.

11. Let L be the line

$$L = \{ x \in \mathbb{R}^n : x = \mu p + (1 - \mu) q, \ -\infty < \mu < \infty \},$$

where p and q are distinct points in $\mathbb{R}^n$, and let f be a linear functional on $\mathbb{R}^n$ such that $f(p) = 6$ and $f(q) = 1$. Find

   (a) the point where L intersects the hyperplane $f^{-1}(-2)$,
   (b) the scalar β such that the hyperplane $f^{-1}(\beta)$ passes through the midpoint of the line segment joining p and q.

12. (a) If a hyperplane in $\mathbb{R}^n$, where $n > 1$, meets a straight line at two distinct points, show that the hyperplane contains the straight line. *Hint*. Let H be the hyperplane $f^{-1}(\alpha)$ and let L be a straight line that intersects H at two distinct points p and q.
    (b) Consequently, show that a hyperplane H and a straight line L must be related in exactly one of the following ways:
       (i) H and L intersect in exactly one point,
       (ii) $L \subset H$,
       (iii) L misses H.

13. Given that $H = f^{-1}(1)$, where the linear functional f on $\mathbb{R}^4$ is represented by the vector $p = (1, 0, 1, -1)$, find

(a) a line L_1 through $\overline{0}$ that intersects H in exactly one point,
(b) a line L_2 through $\overline{0}$ that misses H.

14. If a hyperplane $f^{-1}(\alpha)$ misses the straight line L, then for some scalar β, the hyperplane $f^{-1}(\beta)$ contains L.
Hint. Let p and q be points on the line. If $f(p) \neq f(q)$, then for every real number α, there is a unique solution δ to the equation $\delta f(p) + (1-\delta) f(q) = \alpha$ (why?). Thus, $f^{-1}(\alpha)$ would intersect L (why?). Therefore, $f(p)$ and $f(q)$ must be the same. Let $\beta = f(p)$.

15. If the hyperplane $H = f^{-1}(\alpha)$ intersects the straight line L in exactly one point, then for every scalar β, the hyperplane $H_\beta = f^{-1}(\beta)$ intersects L in exactly one point.
Hint. Conclude that this must happen because of what we know from Problems 12 and 14.

*16. (a) In $\mathbb{R}^3$, the intersection of the closed unit ball with a plane is either the empty set, a single point, or a disk (which is like a closed ball from $\mathbb{R}^2$ embedded in $\mathbb{R}^3$).
(b) List (without proof) the possible intersections in $\mathbb{R}^2$ of the closed unit ball with a straight line.
(c) List (without proof) the possible intersections in $\mathbb{R}^4$ of the closed unit ball with a hyperplane.

17. Prove Theorem 1.6.3.

18. If the ℓ_1 or ℓ_∞ norm is used, show that a line may have infinitely many points of minimum norm.

19. Show that a hyperplane in $\mathbb{R}^n$ has a unique point of minimum norm.

20. Use the Cauchy–Schwarz inequality to show that the triangle inequality holds.
Hint. $\|u+v\|^2 = ?$

21. Show that if f is a linear functional on $\mathbb{R}^n$ and f is represented by the vector $p \in \mathbb{R}^n$, then

$$\|p\| = \max\{\, f(x) \,:\, x \in B \,\} = \max\{\, \langle p, x\rangle \,:\, x \in B \,\},$$

where B is the closed unit ball in $\mathbb{R}^n$.

22. Let f be the linear functional on $\mathbb{R}^4$ represented by $p = (-1, 1, 1, -3)$. Find the point of the hyperplane $f^{-1}(0)$ that is closest to the point $x = (3, -2, 2, 1)$.

23. Develop a general formula for the point q on the hyperplane

$$H_\beta = \{\, x \in \mathbb{R}^n \,:\, \langle p, x\rangle = \beta \,\}$$

that is closest to the point x_0. Assume that $p \neq \overline{0}$.