1

Definitions, Units, and Geometric Properties

1.1 Definitions

Acceleration	The rate of change of velocity. The second derivative of displacement with respect to time.
Added mass	The mass of fluid entrained by a vibrating structure immersed in fluid. The natural frequency of vibration of a structure surrounded by fluid is lower than that of the structure vibrating in a vacuum owing to the added mass of fluid (Tables 6.9 and 6.10).
Amplitude	The maximum excursion from the equilibrium position during a vibration cycle.
Antinode	Point of maximum vibration amplitude during free vibration in a single mode. See node.
Attenuation, acoustic	Difference in sound or vibration between two points along the path of energy propagation. Also see damping, insertion loss.
Bandwidth	The range of frequencies through which vibration energy is transferred.
Beam	Slender structure whose cross section and deflection vary along a single axis. Beams support tension, compression, and bending loads. Shear deformations are negligible compared to bending deformations in slender beams.
Boundary condition	Time-independent constraints that represent idealized structural interfaces, such as zero force, displacement, velocity, rotation, or pressure.
Broad band	A process consisting of a large number of component frequencies, none of which is dominant, distributed over a broad frequency band, usually more than one octave. Also see narrow band and tone.

Formulas for Dynamics, Acoustics and Vibration, First Edition. Robert D. Blevins.

Bulk modulus of elasticity The ratio of the hydrostatic stress, equal in all directions, to the relative change in volume it produces. $B = E/[3(1-\nu)]$ for elastic isotropic materials where E is the modulus of elasticity and ν is Poisson's ratio. $B = \rho c^2$ for fluids where ρ is the mean density and c is the speed of sound.

Cable A uniform, one-dimensional structure that can bear only tensile loads parallel to its own axis. A cable is a massive string. It has zero bending rigidity and it stretches in response to tensile loads. Also see chain.

Cable modulus The change in longitudinal stress (axial force divided by cross-sectional area) divided by the change in longitudinal strain produced by the stress. A solid rod has a cable modulus equal to the modulus of elasticity of the rod material. The cable modulus of a woven cable is typically about 50% of that of the modulus of its fibers.

Center of gravity The point on which a body balances. The center of mass. The sum of the gravitational moments created by the elements of mass is zero about the center of mass.

Center of percussion The point on a rigid body that does not accelerate when the body is impulsively loaded.

Centrifugal force Outward reaction of a mass on a rotating body away from the axis of rotation. Centrifugal (adjective) means "outward from center."

Centripetal acceleration Acceleration of a point on a rotating body toward the center of rotation. Centripetal (adjective) means "toward the axis of rotation."

Centroid Volumetric center of a volume or area. The center of mass and centroid are the same for a homogeneous body.

Chain A uniform, massive one-dimensional structure that bears only tensile loads parallel to its own axis. No bending or shear loads are borne. In contrast to cables, ideal chains do not extend under tensile loads.

Concentrated mass (point mass) A point in space with finite mass and zero rotational inertia.

Consistent units A unit system in which Newton's second law, force equals mass times acceleration, is identically satisfied without additional dimensional factors. See Table 1.2.

Coriolis acceleration Accelerations induced on a moving particle in a rotating system, after the French engineer-mathematician Gustave-Gaspard Coriolis.

Crest factor See peak-to-rms ratio.

Damping The ability of a system to absorb vibration energy. Damping limits resonant vibration amplitude and causes free vibrations to decay with time.

Damping factor A nondimensional measure of the damping of a system equal to $1/(2\pi)$ times the natural logarithm of the ratio of the amplitude of one cycle to the amplitude of the following cycle during free vibration.

Decibel (dB) Sound pressure level in decibels is 10 times the logarithm, to the base 10, of the ratio of the mean square sound pressure to a reference mean square sound pressure.

Deformation The displacement of a structure from its reference or equilibrium position.

Density Mass per unit volume.

Divergence (1) Unstable torsion motion caused by aerodynamic forces that overcome structural stiffness, and, (2) spreading of sound waves propagating from a source.

Dynamic amplification factor Also called *dynamic load factor* (*DLF*), and *magnification factor*. The maximum dynamic response amplitude of a single degree of freedom elastic system to a dynamic force divided by the static response to a steady force with the same magnitude.

Eigen German for "own characteristic." Eigenvalue is a scalar solution to a homogeneous linear equation of motion. Natural frequency is eigenvalue. Eigen vector is the associated spatial mode shape.

Elastic A material or structure whose deformations increase linearly with increasing load. Most practical structures are elastic, or approximately elastic, for loads below the onset of yielding or buckling.

Flutter Unstable divergent oscillation caused by aerodynamic forces.

Force As defined by Newton, force is proportional to mass times acceleration, Equation 1.1.

Forced vibration Vibration of a system in response to an external periodic force.

Free vibration Vibrations in the absence of external loads. Free vibrations take place after an elastic system is released from a displacement.

Frequency The number of times a periodic motion repeats itself per unit time. Vibration frequency is the number of sinusoidal periods per unit times in either units of cycles per second, called Hertz, or in cycles per minute, which is rpm, or in radians per second.

Fundamental mode The lowest natural frequency and mode shape of an elastic system.

Harmonic motion Simple harmonic motion is sinusoidal in time about an equilibrium point.

Harmonics Motion at integer multiples of a frequency.

Hertz Hertz as the unit of cycles per second was adopted by the General Conference on Weights and Measures in 1960. Its name honors Heinrich Hertz, a pioneer investigator of electromagnetic waves.

Impedance (1) Fluid mechanic impedance is ratio of pressure to fluid velocity, (2) mechanical impedance is ratio of force to velocity, (3) step impedance is the ratio of pressure differential across to velocity through a component.

Impulse The force multiplied by the time increment and integrated over the time interval during which the force acts. It has units of force-seconds. Impulse produces change in momentum. Rotational impulse is torque integrated over time and has units of force-length-seconds.

Inertial frame A set of coordinates that does not accelerate; a frame in which Newton's second law holds.

Insertion loss The change in sound pressure level between two points in an acoustic circuit when a component is inserted between the two. Also see transmission loss. Generally expressed in decibels.

Jerk The rate of change of acceleration.

Kinematics Motion within geometric constraints.

Kinetic energy The energy of mass in motion. $\frac{1}{2}MV^2$ is the kinetic energy of a mass M with velocity V.

Mass ratio The weight of a structure divided by the weight of a circumscribed cylinder of the surrounding fluid.

Membrane A thin, massive, elastic uniform sheet that can support only tensile loads in its own plane. A membrane can be flat like a drumhead or curved like a soap bubble. A one-dimensional membrane is a cable. A massless, one-dimensional, membrane is a string. The term membrane is also used for elastic systems without bending.

Modal density The number of modes of vibration with natural frequencies in a specified frequency band. See Table 6.6.

Mode shape (eigenvector) A dimensionless shape function defined over the space of a structure that describes the relative displacement of any point as the structure vibrates in a single mode. A mode shape is independent of time. There is a unique mode shape for each natural frequency of the structure. Any deformation of the structure, consistent with the boundary conditions, can be expressed as a linear sum of mode shapes.

Modulus of elasticity (Young's modulus) The ratio of normal stress to the normal strain it produces in a material. The modulus of elasticity has units of pressure. A material is *isotropic* if the modulus of elasticity is independent of direction. Also see bulk modulus.

Moment See torque.

Momentum Mass times its velocity vector. Rotational momentum is the polar mass moment of inertia of a body times its rotational velocity about an axis.

Moment of inertia of a body The sum of the products obtained by multiplying each element of mass within a body by the square of its distance from a given axis.

Moment of inertia of an area The sum of the products obtained by multiplying each element of area by the square of its distance from a given axis.

Narrow band Vibration or sound process whose frequency components fall within a narrow band, generally less than one-third octave, so that a single peak follows each zero crossing with positive slope. Also see broad band and tone.

Natural frequency (eigenvalue) The frequency at which a linear elastic structure will freely vibrate in free vibration. Continuous or multimass structures have multiple natural frequencies. The lowest of these is the fundamental natural frequency. See Section 3.3.

Neutral axis The axis of zero bending stress through the cross section of a beam. The neutral axis of homogeneous beams passes through the centroid of the cross section. See Section 4.1.

Node Point on a structure that does not deflect during vibration in a mode. Antinode is a point on a structure with maximum deflection during vibration in a single mode. Also, a point in space.

Noise Multifrequency acoustic pressure.

Nonstructural mass Mass without a corresponding stiffness. See particle.

Octave, one-third octave A logarithmic frequency scale originating in musical notation. The octave band is a frequency range where the upper frequency is twice the lower frequency. The octave bands are subdivided into three one-third-octave bands, with the ratio between the upper and lower limits of each one-third-octave band being $2^{1/3}$.

Orthotropic A material whose properties have two mutually perpendicular planes of symmetry. The material properties are direction dependent. A lamina of parallel fibers has orthotropic material properties.

Particle, point mass, concentrated mass A point in space with finite mass and zero rotational inertia.

Peak-to-rms ratio The ratio of the maximum value above the mean to the root-mean square value, about the mean, of a data time history. The peak-to-rms ratio of a sine wave is $2^{1/2}$ and the peak-to-rms ratio of a Gaussian time history approaches infinity. Also called crest factor.

Period of vibration The reciprocal of frequency. Period is the time in seconds to complete one cycle of oscillation.

Phase angle The angle, relative to 360°, at a point in time between two harmonic waves with the same frequency.

Plate A thin, flat, two-dimensional elastic structure that conforms to a two-dimensional surface. A plate has mass and supports bending loads. A plate without bending rigidity is a membrane.

Poisson's ratio The ratio of the lateral shrinkage to the longitudinal expansion of a bar of a given material that has been placed under a uniform axial load. Poisson's ratio is often near 0.3 for metals and 0.4 for rubber-like materials. It is dimensionless. A material with a Poisson's ratio of 0.5 has constant volume during loading.

Potential energy Stored energy. Potential energy is the negative of work. Mgh is the potential energy created by raising mass M by height h where g is the acceleration due to gravity.

Power spectral density The mean square value of a process, within a specified frequency band, divided by the width of that band. Also see octave.

Product of inertia of a body The sum of the products obtained by multiplying each element of mass of a body by the product of its distances from two mutually perpendicular axes. Table 1.6.

Product of inertia of an area The sum of the products obtained by multiplying each element of area of a section by the product of its distances from two mutually perpendicular axes. Table 1.5.

Radius of gyration of a body The square root of the quantity formed by dividing the mass moment of inertia of a body by the mass of the body.

Radius of gyration of an area The square root of the quantity formed by dividing the area moment of inertia of a section by the area of the section.

Random vibration A multifrequency process, described by its statistical properties.

Resonance Response to an external periodic force having the same frequency as the natural frequency of the system. The amplitude of vibration will become larger than the static response to the same force for dampine factors less than $1/2^{1/2}$.

Response The response of a system is the motion, or other output, resulting from dynamic excitation of a system.

Response spectrum Maximum response to a given transient load, often plotted as a function of the natural frequency and damping.

Restitution coefficient The ratio of the velocity of two objects after a collision to their velocity ratio before the collision, relative to the center of mass. The restitution coefficient is zero for a perfectly plastic collision. It is maximum of two for elastic collision.

Rigid body A body whose deformations are negligible.

Root-mean-square The square root of the average, over many cycles of vibration, of the square of a time history of vibration.

Rotary inertia The inertia associated with rotation of a structure about an axis. The sum of the products of elements of mass of a body times their velocity times the distance from the axis of rotation.

Rotor A body that spins about a fixed axis.

Seiching The system of waves in a harbor produced as the harbor responds sympathetically to waves in the open sea. Also see sloshing.

Sidereal day 86,164 s (23.93 h) Earth's period of rotation with respect to distant stars, which is 4 min shorter than the sun's 24-h solar day because of the earth's daily advance in its orbit about the sun.

Shear beam A beam whose deformation in shear substantially exceeds the flexural (bending) deformation.

Shear coefficient A dimensionless quantity, dependent on the shape of the cross section of a beam that is introduced into beam theory to account for the nonuniform distribution of shear stress and shear strain over the cross section. See Section 4.1, Table 4.11.

Shear modulus The rate of change in shear stress of a material that produces a unit shear strain. For isotropic elastic material the shear modulus is $G = E/[2(1-\nu)]$.

Shell A thin elastic structure defined by a curved surface. A curved plate is a shell. A shell without bending rigidity is a curved membrane.

Shock Vibration imposed suddenly and over a period of time comparable to or shorter than the natural period of vibration.

Sloshing Surface waves in a liquid-filled basin, Table 6.7.

Sonic fatigue The vibration of plate and shell structures induced by fluctuating pressure on their surfaces, Tables 7.3, 7.4, 7.5.

Sound pressure level (SPL) Twenty times the logarithm to base 10 of the rms acoustic pressure relative to a reference pressure, decibels. See Appendix C.

Specific impulse The thrust produced by a rocket motor, divided by the initial weight of fuel, times the time in seconds the fuel burns. It is a function of the fuel composition and combustion temperature. See Section 2.2.

Spectrum The distribution of vibration amplitude or energy versus frequency.

Speed of sound The speed at which small pressure fluctuations propagate through an infinite fluid or solid. See Table 6.1.

Spring constant The change in load on a linear elastic structure divided by the change in deformation that results. The torsional spring constant is the change in torque divided by the change in angular position in radians. See Table 3.2.

Spring-mass system A body or mass on a massless elastic suspension.

Stress Force on a unit area. Stress has units of pressure.

String A massless one-dimensional structure defined by a straight line that bears a uniform tension. A string cannot bear bending or compression. A massive string is a cable.

Tone (pure) A sound or vibration at a single frequency. Also see broad band and narrow band.

Torque Torque, or moment, is the vector cross product of force with a vector from a reference axis of rotation. It has units of force-length. Torque produces angular acceleration. See section 2.3.

Torsion coefficient Change in torque on a linear elastic shaft divided by the change in angular deformation, in radians, that results, per unit length of shaft. See Tables 3.2, 4.15.

Transient vibration Vibration that develops over a limited time.

Transmission loss The change in sound pressure level of a forward propagating sound wave between two points along its path. Also see insertion loss.

Vector Quantity with magnitude and direction such as velocity.

Vibration Oscillation in time. See free vibration, pure tone, and random vibration.

Viscosity The ability of a fluid to resist shearing deformation. The viscosity of a Newtonian (linear) fluid is defined as the ratio between the shear stress applied to a fluid and the shearing strain that results. Kinematic viscosity is defined as viscosity divided by fluid density; it has units of length squared over time.

Wave length The distance in space that a propagating wave travels in one period of oscillation. Wave length is the speed of propagation divided by the frequency.

White noise A noise or vibration whose spectral density is constant over all frequencies. Pink noise is distributed over a finite frequency range.

Work The integral of the scalar dot product of force vector and incremental displacement vector during the force application. Work has units of force times length (energy). Rotational work is the integral of torque times the incremental angle of rotation. See Section 2.2.

1.2 Symbols

General nomenclature for the book is in Table 1.1. In addition the heading of each table contains the nomenclature that applies to that table. These symbols and abbreviations are consistent with engineering usage and technical literature. Vectors are written in bold face type (**B**). Mode shapes have an over tilde (∼) to denote that they are independent of time.

The Greek letter λ is used for wave length and longitude; it is also used for the dimensionless natural frequency parameters of beams, plates, and shells, where it generally appears with a subscript. The symbol I is used for area moments of inertia and J is used for mass moments of inertia. The overworked symbols t and T are used for time, oscillation period, transpose of a matrix, tension, and thickness. Definitions in the tables clarify usage.

Table 1.1 Nomenclature

Symbol	Units	Definition
A	$length^2$	area
B	force / $length^2$	bulk modulus
C	(location)	center of gravity
C	$length^4$	torsion constant
c	length/time	speed of sound
e	dim'lss	2.71828 = Exp(1)
E	force / $length^2$	modulus of elasticity
f	1/time (Hertz)	frequency, f_n = natural frequency
g	length / $time^2$	acceleration due to gravity
I	$length^4$	area moment of inertia
i	dim'lss	imaginary constant $(-1)^{1/2}$
J	mass × $length^4$	mass moment of inertia
J	dimensionless	Bessel function of first kind
K	dimensionless	shear coefficient
k	force / length	deflection spring constant
k	moment / radian	torsional spring constant
L	length	Length
M	mass	Mass or Mach number
m	mass / length	mass per unit length
M	force × length	Moment
P	-	point in space
P	force	Load
p	force / $length^2$	Pressure
S	force / length	tension per unit length of edge
$S_p(f)$	rms^2 / Hz	power spectral density of quantity
T	force	Tension
T	time	period of vibration
t	time	Time
x, y, z	length	mutually orthogonal displacement
X,Y,Z	length	Displacements
Y	dimensionless	Bessel function of second kind
ω	radians	Angle
ε	length / length	Strain
γ	dimensionless	ratio of specific heats of a gas
γ	mass / $length^2$	mass per unit area
ν	dimensionless	Poisson's ratio
π	dimensionless	3.141592653
θ	radians	angle of rotation
ρ	mass / $length^3$	Density
σ	dimensionless	beam mode shape parameter
σ	force / $length^2$	Stress
ω	radians / second	circular frequency, $2\pi f$
ζ	dimensionless	fraction of critical damping

Table 1.1 Nomenclature, continued

Subscripts, superscripts, and bars

Symbol	Units	Definition
i,j,k,l,m,n	dimensionless	counting integers, 1,2,3…
rms	-	root-mean-square
Vector	(bold face)	Vector
$\overline{\text{Avg}}$	(over bar)	average over time
$\tilde{y}$	(over tilde)	mode shape, dimensionless
$[X]^T$	-	Transpose of matrix [X]
A×B	(×)	vector cross product, Eq. 1.24
A•B	(•)	vector dot product, Eq. 1.25
|X|		determinant or magnitude of X
X′(t)	-	ordinary derivative, dX/dt
$\dot{x}(t)$	over dot (.)	derivative with respect to time, t

Abbreviations

Quantity	Abbreviation	Quantity	Abbreviation
Centigrade degrees	°C	logarithm, natural base e	ln
centimeter	cm	mega-Pascal	MPa
cubic centimeters	cc	Meter	m
decibel	dB	millimeter	mm
decaNewton	daN	Newton	N
degree	deg	Kelvin degrees	°K
Fahrenheit degrees	°F	Pascal	Pa
feet	ft	Pound	lb
gram	g	power spectral density	$S_p(f)$
gravity's acceleration	g_c	Radian	rad
Hertz = cycles /second	Hz	Rankine degrees	°R
inch	in.	root-mean-square	rms
Bessel function of 1st kind	J(x)	second	s
complete elliptic integral 2nd	E(x)	Sound Pressure Level	SPL
kilogram	kg	times 10 raised to power x	E+x
logarithm base 10	$\log_{10}$	ton, metric	Te

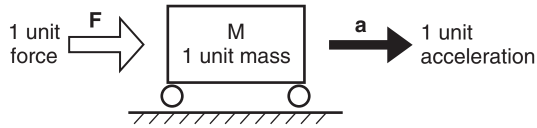


Figure 1.1 Newton's second law in consistent units, $F = Ma$

1.3 Units

The change in motion [of mass] is proportional to the motive force. – Newton [1].

The formulas presented in this book give the correct results with consistent sets of units. In *Consistent Units*, Newton's second law is identically satisfied without factors: One unit of force equals one unit of mass times one unit of acceleration, Figure 1.1.

$$F = Ma \tag{1.1}$$

Table 1.2 presents sets of consistent units that identically satisfy $F = Ma$. These units are used internationally in professional engineering. They are recommended for use with this book.

Table 1.2 Consistent sets of engineering units

System	Force	Mass	Length	Time	Pressure	Density	g_c (a)
1 m-k-s (SI)	Newton	kilogram	meter	second	Pascal (b)	$\frac{\text{kilogram}}{\text{meter}^3}$	$9.807\ \frac{\text{meter}}{\text{second}^2}$
2 c-g-s	dyne (c)	gram	centimeter	second	$\frac{\text{dyne}}{\text{centimeter}^2}$	$\frac{\text{gram}}{\text{centimeter}^3}$	$980.7\ \frac{\text{centimeter}}{\text{second}^2}$
3 cm-kg-s	kilogram	$\frac{\text{kilogram}}{g_c}$	centimeter	second	$\frac{\text{kilogram}}{\text{centimeter}^2}$	$\frac{\text{kilogram}}{g_c\text{-centimeter}^3}$	$980.7\ \frac{\text{centimeter}}{\text{second}^2}$
4 mm-N-s	Newton	metric ton (f)	millimeter	second	megaPascal (d)	$\frac{\text{metric ton}}{\text{millimeter}^3}$	$9807\ \frac{\text{millimeter}}{\text{second}^2}$
5 mm-dN-s	deca-Newton (e)	10^4 kilogram (f)	millimeter	second	$\frac{\text{decaNewton}}{\text{millimeter}^2}$	$\frac{10^4\ \text{kilogram}}{\text{millimeter}^3}$	$9807\ \frac{\text{millimeter}}{\text{second}^2}$
6 ft-lb-s	pound	slug (g)	foot	second	$\frac{\text{pound}}{\text{foot}^2}$	$\frac{\text{slug}}{\text{foot}^3}$	$32.17\ \frac{\text{foot}}{\text{second}^2}$
7 in.-lb-s	pound	$\frac{\text{pound}}{g_c}$	inch	second	$\frac{\text{pound}}{\text{inch}^2}$	$\frac{\text{pound}}{g_c\text{-inch}^3}$	$386.1\ \frac{\text{inch}}{\text{second}^2}$

(a) The General Conference of Weights and Measures defined the standard acceleration of gravity at sea level and latitude 45 degrees as 9.80665 m/s^2 (32.1740 ft/s^2) in 1901. Acceleration of gravity increases with latitude from 9.780 m/s^2 at 0 degree latitude to 9.832 m/s^2 at 90 degree latitude (Refs [2, 3, 4, 5] Section 1.4).
(b) Pascal = Newton / meter^2.
(c) dyne = 0.00001 Newton. dyne / centimenter^2 = 0.1 Pa.
(d) Mega Pascal = 10^6 Newton/m^2 = 1 Newton / mm^2 = 1000000 Pascals = 10 bar = 0.1 hectobar.
(e) 1 daNewton =1 dN = 10 Newtons. daNewton / mm^2 = 10^7 Pascals = 100 bar = 1 hectobar. Tables 1.3, 1.4.
(f) Metric ton (Te) =1000 kg =10^6 gm. 10000 kg = 10 metric tons (Te).
(g) slug = pound / g_c. It has units of lb-s^2/ft. 1 slug weighs 32.17 lb on the surface of the earth.

Mass and force have different consistent units. In SI units (Systeme International of the International Organization for Standardization), meter is the unit of length, the unit of mass is kilogram, and second is the unit of time [2, 3, 4]. The SI *consistent unit of force* called *Newton* is kilogram-meter/second squared. Substituting these units in Equation 1.1 shows that 1 N force accelerates 1 kg mass at 1 m/s^2. The General Conference of Weights and Measures defined the Newton unit of force in 1948 and the Pascal unit of pressure, 1 N/m^2, in 1971.

It is the author's experience that inconsistent units are the most common cause of errors in dynamics calculation. See Refs [5–8]. While lack of an intuitive feel for dyne, Newton, or slug may be the reason to convert the final result of a calculation to a convenient customary unit in which mass and force have same customary units, it is important to remember that formulas derived from Newton's laws discussed in this book and most engineering software require the consistent units shown in Table 1.2 to produce correct results.

One Newton force is about the weight of a small apple. If this apple is made into apple butter and spread over a table 1 m^2 then resultant pressure is 1 Pa, which is a small pressure. There is a plethora of pressure units in engineering. Zero decibels (dB) pressure at 1000 Hz is the threshold of human hearing (Section 6.1); it is 20 μPa (20×10^{-6} Pa, 2.9×10^{-9} psi), which is a very small pressure. Stress in structural materials is measured in units of ksi (1000 psi, 6.894×10^6 Pa), MPa(10^6 Pa), decaNewton/mm^2, and hectobar (both 10^7 Pa), which are all large pressures. One hectobar stress is 500 billion times greater than 0 dB pressure.

Standard prefixes for decimal unit multipliers and their abbreviations are in Table 1.3. Table 1.4 has conversion factors; ASTM Standard SI 10-2002 [3], Taylor [4], and Cardarelli [9] provide many more.

Example 1.1 Force on mass

A 1 gram mass accelerates at 1 ft/s^2. What force is on the mass?

Solution: Newton's second law (Eq. 1.1) is applied. Consistent units are required. Gram-foot-seconds is not a consistent set of units. To make the calculation in SI units, case 1 of Table 1.2, grams are converted to kilograms and feet are converted to meters. The conversion factors in Table 1.4 are 1 gram = 0.001 kg, and 1 ft = 0.3048 m so in SI 1 ft/s^2 is 0.3048 m/s^2. Equation 1.1 gives the force that accelerates 1 g at 1 ft/s^2.

$$F = Ma = 0.001\ \text{kg}\ (0.3048\ \text{m/s}^2) = 0.0003048\ \text{kg–m/s}^2 = 0.0003048\ \text{N}$$

Table 1.3 Decimal unit multipliers

femto (f)	pico (p)	nano (n)	micro (μ)	milli (m)	centi (c)	deci (d)	deca (da)	hecto (h)	kilo (k)	mega (M)	giga (G)	tera (T)
10^{-15}	10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{-2}	10^{-1}	10^{1}	10^{2}	10^{3}	10^{6}	10^{9}	10^{12}

Example: 1 hectobar = 1 hbar = 100 bar = 100000 mbar = 10 megaPascal = 10 MPa. Ref. [3–5]

Table 1.4 Conversion factors

cc = cubic centimeter; DecaNewton = 10 Newtons; mile = US statute mile, unless otherwise noted; Pascal = 1 N/m^2; pound mass = pound, avoirdupois. Refs [3, 4, 5, 9]. See Table 1.2 for consistent sets of units.

Mass Units		
To convert from	to	multiply by
gram	kilogram	1. E-3
pound	kilogram	0.45359237
slug	kilogram	14.59390
slinch (=lbf-sec^2/in)	kilogram	175.1319
ton, long	kilogram	1016.047
ton, short	kilogram	907.1847
ton, metric (=tonne)	kilogram	1000
pound	gram	453.59237
gram	pound	2.204624 E-3
kilogram	pound	2.204624
slug (=lbf-sec^2/ft)	pound	32.17
slinch (=lbf-sec^2/in)	pound	386.1
ton, long	pound	2240
ton, metric (=tonne)	pound	2204.622
ton, short	pound	2000
10^4 kg (=10 metric tons)	pound	22046.22
Force Units		
To convert from	to	multiply by
decaNewton	Newton	10
kilogram	Newton	9.806650
pound	Newton	1/0.22481
pound	decaNewton	1/2.2481
dyne	Newton	1. E-5
decaNewton	pound	2.2481
dyne	pound	2.22481 E-6
gram	dyne	980.6
kilogram	pound	2.204624
Newton	pound	0.22481
Length Units		
To convert from	to	multiply by
inch	centimeter	2.54
inch	millimeter	25.4
centimeter	inch	0.3937
millimeter	inch	0.03937
centimeter	foot	0.03208
inch	foot	1/12

Table 1.4 Conversion factors, continued

Length Units (continued)

To convert from	to	multiply by
meter	foot	3.280833
mile, nautical	foot	6076.12
mile, U.S. statute	foot	5280
millimeter	foot	0.0032808
yard	foot	3
foot	inch	12
meter	inch	39.37
millimeter	inch	1/25.4
centimeter	meter	0.01
foot	meter	0.3048
inch	meter	0.0254
kilometer	meter	1000
mile, nautical	meter	1852
mile, U.S. statute	meter	1609.347
mile, U.S. statute	kilometer	1.609347
millimeter	meter	0.001
yard	meter	0.9144

Density Units

To convert from	to	multiply by
gram / cc	kilogram / meter^3	1000
gram / millimeter^3	kilogram / meter^3	10^6
gram /millimeter^3	gram/ cubic centimeter	1000
Te(=10^3 kg))/ millimeter^3	kilogram / meter^3	10^{12}
pound / foot^3	kilogram / meter^3	16.018346
pound / inch^3	kilogram / meter^3	2.76799 E+4
pound / inch^3	gm / cc	1/0.03612729
slug / foot^3	kilogram / meter^3	5.153788
metric ton/ mm^3	kilogram/mm^3	10E-9
gram / cc	pound / foot^3	62.42879
kilogram / meter^3	pound / foot^3	0.06242879
Te (=10^3 kg)/ millimeter^3	pound / foot^3	6.242879 E10
pound / inch^3	pound / foot^3	1728
slug / foot^3	pound / foot^3	32.17
gram / cc	pound / inch^3	0.03612729

Table 1.4 Conversion factors, continued

Density Units, continued		
To convert from	to	multiply by
kilogram / meter^3	pound / inch^3	3.612729 E-5
Te(10^3 kg) / millimeter^3	pound / inch^3	3.612729 E7
pound / foot^3	pound / inch^3	1/1728
slug / foot^3	pound / inch^3	1.186169 E-2
slinch / inch^3	pound / inch^3	386.1
Pressure Units		
To convert from	to	multiply by
atmosphere	kilogram / centimeter^2	1.0333
bar	kilogram / centimeter^2	10.1971
Pascal	kilogram / centimeter^2	1.019716 E-5
pound / foot^2	kilogram / centimeter^2	0.00048825
pound / inch^2	kilogram / centimeter^2	0.070309
kilogram / millimeter^2	kilogram / centimeter^2	100
pound / inch^2	dyne / centimeter^2	68947.57
atmosphere, standard	Pascal = 1 N/meter^2	1.013250 E+5
Bar	Pascal	1. E +5
dyne / centimeter ^{2}	Pascal	0.1
hectobar	Pascal	1. E +7
kilogram / centimeter^2	Pascal	9.806650 E+4
kilogram / meter^2	Pascal	9.80665
kilogram / millimeter^2	Pascal	9.806650 E+6
megaPascal (MPa)	Pascal	1 E+6
megaPascal (MPa)	Pound/ inch^2	145.0377
millibar	Pascal	100
Newton / meter^2	Pascal	1
Newton / millimeter^2	Pascal	1E+6
column H2O one mm high	Pascal	9.80665
column H2O one mm high	kilogram / meter^2	1
column Hg one mm high	Pascal	133.322
column one inch H2O high	Pascal	249.0889
decaNewton / millimeter^2	Pascal	1. E +7
pound / foot^2	Pascal	47.88026
pound / inch^2	Pascal	6894.757
Torr	Pascal	133.322
decaNewton / millimeter^2	kilogram / millimeter^2	1/0.9806650
atmosphere, standard	pound / foot^2	2116.215
Bar	MPa	0.1
1000 pound / inch^2 (ksi)	MPa	6.894757

Table 1.4 Conversion factors, continued

Pressure Units, continued		
To convert from	to	multiply by
Bar	pound / foot^2	2088.542
kilogram / meter^2	pound / foot^2	1/4.882428
Pascal	pound / foot^2	0.02088544
atmosphere, standard	pound / inch^2	14.69594
Bar	pound / inch^2	14.50377
column H2O one inch high	pound / inch^2	0.036127
column Hg one inch Hg	pound / inch^2	0.49115
column Hg one mm high	pound / inch^2	0.01933661
decaNewton / centimeter^2	pound / inch^2	14.50355
kilogram / mm^2	pound / inch^2	1422.3
Millibar	pound / inch^2	0.01450377
megaPascal (MPa)	pound / inch^2	145.0377
megaPascal (MPa)	1000 lb/ inch^2 (ksi)	0.1450377
Newton / millimeter^2	pound / inch^2	145.0377
Pascal	pound / inch^2	1.450377 E-4
pound / foot^2	pound / inch^2	1/144
hectobar	pound / inch^2	1450.377
Pressure Units (decibels)	Also see Table 6-1	
To convert from	to	Formula
decibels rel. 20 micro Pa	Pascal, rms	20E-6 Pa$\times 10^{dB/20}$
decibels rel. 20 micro Pa	psi, rms	2.9E-9 psi$\times 10^{dB/20}$
Pascals, root-mean-square	decibels, dB	$20 \log_{10}(p_{rms} / 20\times10^{-6}$ Pa)
Velocity Units		
To convert from	to	multiply by
foot / second	meter / second	1/3.280833
kilometer / hour	meter / second	1/3.6
meter / second	foot / second	3.280833
kilometer / hour	foot / second	0.9113425
knots	meter / second	0.5144444
knots	feet / second	1.687810
miles / hour	kilometer / hour	1.609347
miles / hour	meter / second	0.44704
miles / hour	feet / second	1.466667
miles / hour	knots	0.86897

Table 1.4 Conversion factors, continued

Temperature Units		
To convert from	to	Formula
degrees Fahrenheit, °F	degrees Centigrade, °C	°C = (5/9)(°F – 32)
degrees Centigrade, °C	degrees Fahrenheit, °F	°F = (9/5) °C + 32
degrees Centigrade, °C	degrees Kelvin, °K	°K = °C + 273.15
degrees Fahrenheit, °F	degrees Rankine, °R	°R = °F + 459.69
degrees Kelvin, °K	degrees Centigrade, °C	°C = °K – 273.15
degrees Rankine, °R	degrees Fahrenheit, °F	°F = °R – 459.69
degrees Kelvin, °K	degrees Fahrenheit, °F	°F = (9/5) (°K–273.16) +32
Energy Units		
To convert from	to	multiply by
inch-pound	Newton-meter (Joule)	0.1129848
foot-pound	Newton-meter	1.355818
Newton-meter	foot-pound	0.73756
BTU (international)	Newton-meter	1055.056
calorie	Joule = 1 Newton-meter	4.184
Newton-meter	Joule	1
horse power (US)	ft-lb/s	550
horse power (US)	watt = 1 N-m/s	745.6999
watt	Joule/s	1
Volume Units		
To convert from	to	multiply by
cubic centimeter (cc)	liter	0.001
cubic centimeter (cc)	cubic inch	0.061024
liter	cubic centimeter (cc)	1000
cubic meter	liter	1000
barrel (US petroleum)	cubic meter	0.158987
barrel (US petroleum)	cubic inch	9702
gallon US	cubic inch	231
gallon US	cubic meter	0.003785412
gallon US	liter	3.785412
cubic foot	cubic inch	1728
cubic foot	liter	28.316
Standard Earth Acceleration, g = 9.80665 m^2/s		
To convert from	to	multiply by
acceleration, m^2/s	g	1/9.80665
acceleration, ft^2/s	g	1/32.174
acceleration, cm^2/s	g	980.0665
acceleration, in^2/s	g	1/386.089

For calculation in US customary lb-ft-s units, case 6 in Table 1.2, grams are converted to slugs by converting grams to pounds then pounds to slugs.

$$1\ \text{g} = 0.0022045\ \text{lb}\ (1\ \text{slug} / 32.17\ \text{lb}) = 0.00006852\ \text{slug}$$

$$F = Ma = 0.00006852\ \text{slug}\ (1\ \text{ft}/\text{s}^2) = 0.00006852\ \text{lb}$$

The results imply the relationship between Newtons and pounds: 1 N/1 lb = 0.00006852/ 0.0003049 = 0.2248. See Table 1.4.

1.4 Motion on the Surface of the Earth

The earth can be modeled as a spinning globe with a 6380 km (3960 miles) equatorial radius (Figure 1.2) that revolves daily about its polar axis. (Geophysical models of earth are discussed in Refs [5] and [10–14].) Owing to its rotation, the earth's surface is not an inertial frame of reference. As one walks in a line on the surface of the earth, one is actually walking along a circular arc because the surface of the earth is curved. Further, the earth is rotating under one's feet. Accelerations induced by the earth's curvature and rotations are important for predicting weather, weighing gold, and launching projectiles.

Point P is on the surface of the earth at radius $r = 6{,}380{,}000$ m, longitude λ and polar angle θ as shown in Fig. 1.2. Its circumferential angular velocity is the sum of the rotation of the earth about polar axis ($\Omega = 7.272 \times 10^{-5}$ rad/s) and $d\lambda/dt$. When P is stationary with respect to the earth's surface, $d\lambda/dt = d\theta/dt = 0$. The velocity and acceleration of P with respect to the center of the earth are given in spherical coordinates in case 4 of Table 2.1 with these values.

$$
\begin{aligned}
\mathbf{v} &= r(d\theta/dt)\mathbf{n}_\theta + r(\Omega + d\lambda/dt)\sin\theta\,\mathbf{n}_\lambda \\
\mathbf{v}_p &= 464.0\ \sin\theta\,\mathbf{n}_\lambda,\ \text{m/s} \\
\mathbf{a} &= -[r(d\theta/dt)^2 + r(\Omega + d\lambda/dt)^2\sin^2\theta]\mathbf{n}_r \\
&\quad + [rd^2\theta/dt^2 - r(\Omega + d\lambda/dt)^2\sin\theta\cos\theta]\mathbf{n}_\theta \\
&\quad + [rd^2\lambda/dt^2\sin\theta + 2r(\Omega + d\lambda/dt)\cos\theta\, d\theta/dt]\mathbf{n}_\lambda \\
\mathbf{a}_p &= -0.03374\ \sin^2\theta\,\mathbf{n}_r - 0.01687\sin 2\theta\,\mathbf{n}_\theta,\ \text{m/s}^2
\end{aligned}
\tag{1.2}
$$

On the equator, $\theta = 90$ degrees, the earth's surface velocity is 464.0 m/s (1670 km/hr, 1520 ft/s, 1038 mph). The inward radial acceleration of 0.03374 m/s^2 towards the center of the earth results in a 0.34% reduction in gravity. This explains the popularity of the equator for launching satellites. Objects weigh less on the equation than near the poles. For example, gold weighs 0.12% more at the mine in Nome Alaska (65.4 degrees N latitude, $\theta = 24.6$ deg) than at the bank in San Francisco (35.7 degrees N latitude, $\theta = 54.3$ deg).

Now consider that particle P moves freely at constant radius with an initial west-to-east velocity $\mathbf{v}_\lambda = r\ d\lambda/dt\ \mathbf{n}_\lambda$ and north-to-south velocity $\mathbf{v}_\theta = r\ d\theta/dt\ \mathbf{n}_\theta$, with respect to the surface of the earth. The polar and latitudinal components of accelerations are set to zero,

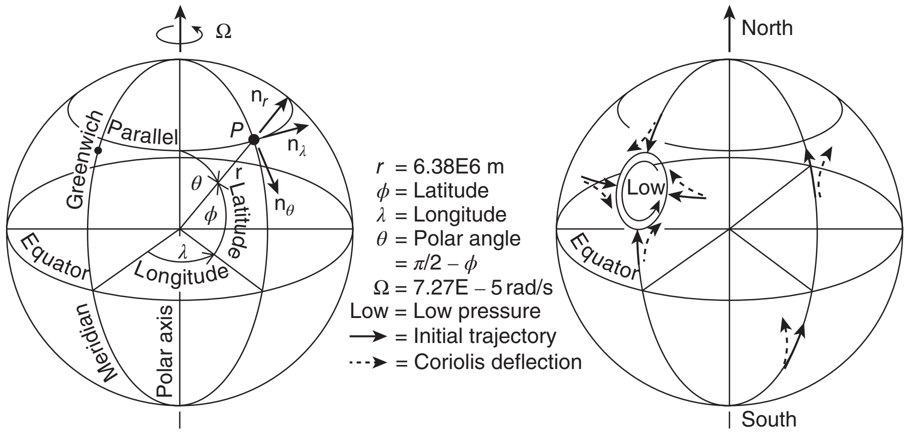


Figure 1.2 Motion of the surface of rotating earth and Coriolis deflection of moving particles relative to the earth. Latitude is zero at the equator.

$\mathbf{a}_\theta = \mathbf{a}_\lambda = 0$ for the freely moving particle. The previous kinematic equations are solved for the latitudinal and longitudinal angular accelerations relative to the earth.

$$
\begin{aligned}
d^2\theta/dt^2 &= (\Omega + d\lambda/dt)^2 \sin\theta\cos\theta \\
d^2\lambda/dt^2 &= -2(\Omega + d\lambda/dt)(d\theta/dt)\cos\theta/\sin\theta
\end{aligned}
\tag{1.3}
$$

These equations show that an initial west-to-east angular velocity ($d\lambda/dt$) induces a north-to-south acceleration ($d^2\theta/dt^2$) and a north-to-south velocity ($d\theta/dt$) produces an east-to-west acceleration ($-d^2\lambda/dt^2$) in the northern hemisphere. Thus, freely moving particles veer to their right in the Northern Hemisphere ($\theta < 90°$) and to the left in the Southern Hemisphere ($\theta > 0°$), as seen by an earth-based observer. These induced motions are named *Coriolis* accelerations, after the French engineer-mathematician Gustave-Gaspard Coriolis.

Figure 1.2 shows Coriolis accelerations spin air flowing inwards toward a region of low pressure. As a result hurricanes spin counterclockwise in the northern hemisphere and typhoons spin clockwise in the southern hemisphere. Coriolis forces affect ocean currents. Coriolis deflected tides carry migrating shad fish counterclockwise around Canada's Bay of Fundy [15].

1.5 Geometric Properties of Plane Areas

Table 1.5 has formulas for the geometric properties of plane areas [16–19]. Figure 1.3a shows a bounded plane area in the x–y plane with *area A* and *centroid C* at point x_c, y_c.

$$
A = \int_A dx\,dy, \qquad x_c = \frac{1}{A}\int_A x\,dA, \qquad y_c = \frac{1}{A}\int_A y\,dA \tag{1.4}
$$

Table 1.5 Properties of plane sections

Notation: A = cross-sectional area; C = centroid of area; K(x), E(x) = complete elliptical integrals of first and second kind; I_x = area moment of inertia about x-axis; I_y = area moment of inertia about y-axis; I_{xy} = area product of inertia about x and y axes; $I_p = I_x + I_y$ = polar area moment of inertia about z-axis; I_{xc}, I_{yc}, I_{xcyc} = area moment of inertia about axes through centroid; t = thickness; P = perimeter; x_C = distance from x-axis to centroid; y_C = distance from y-axis to centroid; θ = angle, radian. Also see Table 1.6 and Eqs 1.4 through 1.11 Refs [16–19].

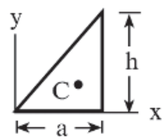
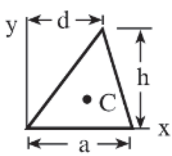
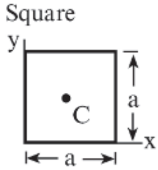
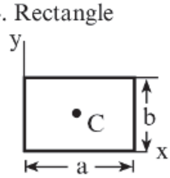
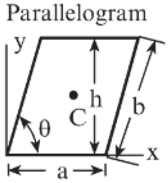
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
1. Right Triangle 	$x_C = \frac{2}{3}a$ $y_C = \frac{1}{3}h$ $A = \frac{1}{2}ah$	$I_{x_C} = \frac{ah^3}{36}$ $\quad I_{y_C} = \frac{a^3h}{36}$ $I_x = \frac{ah^3}{12}$ $\quad I_y = \frac{a^3h}{4}$ $I_{p_C} = I_{x_C} + I_{y_C} = \frac{ah^3 + a^3h}{36}$	$I_{x_C y_C} = \frac{a^2h^2}{72}$ $I_{xy} = \frac{a^2h^2}{8}$ Also see Table 1.7.
2. General Triangle 	$x_C = \frac{1}{3}(a+d)$ $y_C = \frac{1}{3}h$ $A = \frac{1}{2}ah$ For both d>a and a>d.	$I_{x_C} = \frac{ah^3}{36}$ $I_{y_C} = \frac{ah}{36}\left(h^2 - ad + d^2\right)$ $I_x = \frac{ah^3}{12}$ $I_y = \frac{ah}{12}\left(a^2 + ad + d^2\right)$	$I_{x_C y_C} =$ $\frac{ah^2}{72}(2d - a)$ $I_{xy} =$ $\frac{ah^2}{24}(a + 2d)$ see Table 1.7.
3. Square 	$x_C = \frac{1}{2}a$ $y_C = \frac{1}{2}a$ $A = a^2$	$I_{x_C} = I_{y_C} = \frac{a^4}{12}, I_{pc} = \frac{a^4}{6}$ $I_x = I_y = \frac{a^4}{3}, \quad I_p = \frac{2a^4}{3}$	$I_{x_C y_C} = 0$ $I_{xy} = \frac{a^4}{4}$
4. Rectangle 	$x_C = \frac{1}{2}a$ $y_C = \frac{1}{2}b$ $A = ah$	$I_{x_C} = \frac{ab^3}{12}$ $\quad I_{y_C} = \frac{a^3b}{12}$ $I_x = \frac{ab^3}{3}$ $\quad I_y = \frac{a^3b}{3}$ $I_{p_C} = I_{x_C} + I_{x_C} = \frac{ab^3 + a^3b}{12}$	$I_{x_C y_C} = 0$ $I_{xy} = \frac{a^2b^2}{4}$
5. Parallelogram 	$x_C = \frac{1}{2}(a + b\cos\theta)$ $y_C = \frac{b}{2}\sin\theta = \frac{h}{2}$ $A = ab\cos\theta = ah$ $\theta = \arcsin\frac{h}{b}$	$I_{x_C} = ab^3\sin^3\theta/12 = ah^3/12$ $I_{y_C} = \frac{ab}{12}\sin\theta(a^2 + b^2\cos^2\theta)$ $I_x = (ab^3\sin^3\theta)/3 = ah^3/3$ $I_y = ab\sin\theta(a + b\cos\theta)^2/3$ $-(a^2b^2\sin\theta\cos\theta)/6$	$I_{x_C y_C} =$ $\frac{ab^3}{12}\sin^2\theta\cos\theta$ $h = b\sin\theta$

Table 1.5 Properties of plane sections, continued

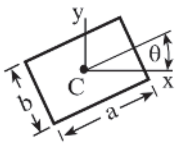
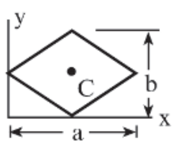
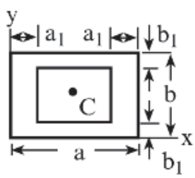
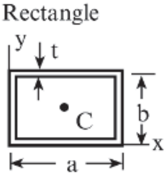
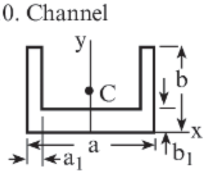
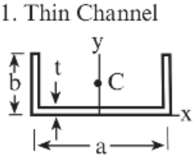
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
6. Inclined Rectangle 	$x_C = 0$ $y_C = 0$ $A = ab$ Also see case 47.	$I_{x_C} = \frac{ab}{12}(a^2 \sin^2\theta + b^2 \cos^2\theta)$ $I_{y_C} = \frac{ab}{12}(a^2 \cos^2\theta + b^2 \sin^2\theta)$ $I_{pC} = I_{x_C} + I_{y_C} = \frac{a^3b + ab^3}{12}$	$I_{x_C y_C} = \frac{ab}{24}(b^2 - a^2)\sin 2\theta$
7. Diamond 	$x_C = \frac{a}{2}$ $y_C = \frac{b}{2}$ $A = \frac{ab}{2}$	$I_{x_C} = \frac{ab^3}{48}$ $I_{y_C} = \frac{a^3b}{48}$	$I_{x_C y_C} = 0$
8. Hollow Rectangle 	$x_C = \frac{1}{2}a$ $y_C = \frac{1}{2}b$ $A = 2ab_1 + 2a_1b - 4a_1b_1$	$I_{x_C} = \frac{ab^3 - (a-2a_1)(b-2b_1)^3}{12}$ $I_{y_C} = \frac{a^3b - (a-2a_1)^3(b-2b_1)}{12}$	$I_{x_C y_C} = 0$
9. Thin Hollow Rectangle 	$x_C = \frac{1}{2}a$ $y_C = \frac{1}{2}b$ $A = 2(a+b)t$ $a, b >> t$	$I_{x_C} = \frac{tb^3}{6} + \frac{tab^2}{2}$ $I_{y_C} = \frac{ta^3}{6} + \frac{ta^2b}{2}$, $I_{pC} = t(a+b)^3$ $I_x = \frac{2tb^3}{3} + tab^2$	$I_{x_C y_C} = 0$ $I_{pc} = \frac{t(a^3 + b^3)}{6} + \frac{tab(a^2 + b^2)}{2}$
10. Channel 	$x_C = 0$ $y_C = \frac{2a_1b^2 + ab_1^2 - 2a_1b_1^2}{2A}$ $A = 2a_1b + ab_1 - 2a_1b_1$	$I_x = \frac{2}{3}a_1b^3 + \frac{1}{3}(a - 2a_1)b_1^3$ $I_y = I_{y_C} = \frac{a^3b\text{-}(b\text{-}b_1)(a\text{-}2a_1)^3}{12}$	$I_{x_C y_C} = I_{xy} = 0$
11. Thin Channel 	$x_C = 0$ $y_C = \frac{b^2}{a + 2b}$ $A = (a + 2b)t$ $a, b >> t$	$I_x = \frac{2tb^3}{3}$ $I_y = I_{y_C} = \frac{ta^3}{12} + \frac{ta^2b}{2}$ $I_p = I_x + I_y$	$I_{x_C y_C} = I_{xy} = 0$

Table 1.5 Properties of plane sections, continued

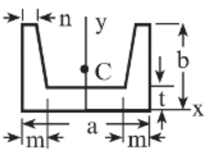
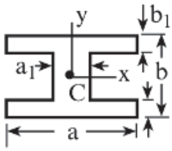
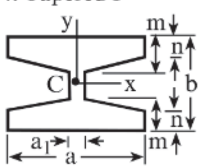
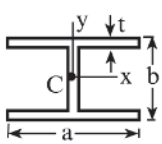
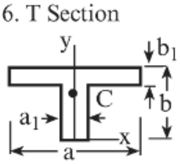
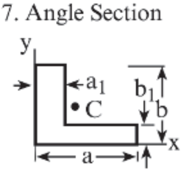
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iyy	Area Products of Inertia Ixy
12. Tapered Channel 	$x_C = 0$ $y_C = \frac{1}{6A}(6b^2n + 2(b-t)(m-n)(b+2t) + 3(a-2m)t^2)$ $A = at + (b-t)(m+n)$	$I_x = \frac{1}{3}\left(2nb^3 + (a-2m)t^3 + \frac{m-n}{2}(b-t) \bullet (b^2+2bt+3t^2)\right)$ $I_y = \frac{a^3t}{12} + \frac{b-t}{6}\left(n^3 + 3n(a-n)^2 + \frac{m-n}{6} \bullet (2(m-n)^2 + (3a-2m-4n)^2)\right)$ $I_p = I_x + I_y, \quad I_{xy} = 0$	
13. I Section 	$x_C = 0$ $y_C = 0$ $A = 2ab_1 + a_1b - 2a_1b_1$	$I_{x_C} = \frac{1}{12}\left(ab^3 - (a-a_1)(b-2b_1)^3\right)$ $I_{y_C} = \frac{1}{12}\left(2a^3b_1 + (b-2b_1)a_1^3\right)$ $I_{x_Cy_C} = 0, \; I_p = I_x + I_y,$	
14. Tapered I 	$x_C = 0$ $y_C = 0$ $A = a_1b + (a-a_1)(m+n)$	$I_{x_C} = \frac{1}{12}\left(ab^3 - \frac{(a-a_1)}{8(m-n)}(c^4 - e^4)\right)$ $I_{y_C} = \frac{1}{12}\left(2a^3n + ea_1{}^3 + \frac{(m-n)}{2(a-a_1)}(a^4 - a_1^4)\right)$ where $c = b-2n$, $e = b-2m$. $I_{x_Cy_C} = 0$	
15. Thin I Section 	$x_C = 0$ $y_C = 0$ $A = (2a+b)t$ $a, b >> t$	$I_{x_C} = \frac{1}{12}tb^3 + \frac{1}{2}tab^2$ $I_{y_C} = ta^3/6$ $I_{p_C} = \frac{1}{12}tb^3 + \frac{1}{2}tab^2 + \frac{1}{6}ta^3$	$I_{x_Cy_C} = 0$
16. T Section 	$x_C = 0$ $y_C = b - \frac{a_1b^2 + ab_1^2 - a_1b_1^2}{2A}$ $A = a_1b + ab_1 - a_1b_1$	$I_x = \frac{1}{3}(ab^3 - (a-a_1)(b-b_1)^3)$ $I_y = I_{y_C} = \frac{a^3b_1 + (b-b_1)a_1{}^3}{12}$ $I_p = I_x + I_y$	$I_{x_Cy_C} = 0$
17. Angle Section 	$x_C = \frac{a^2b_1 + a_1{}^2(b-b_1)}{2A}$ $y_C = \frac{a_1b^2 + b_1^2(a-a_1)}{2A}$ $A = a_1b + ab_1 - a_1b_1$	$I_x = \frac{1}{3}((a-a_1)b_1^3 + a_1b^3)$ $I_y = \frac{1}{3}(a^3b_1 + (b-b_1)a_1^3)$ $I_{x_C} = \frac{1}{3}(ay_C^3 - (a-a_1) \bullet (y_C - b_1)^3 + (b-y_C)^3a_1)$	$I_{xy} = \frac{1}{4}(a^2b_1^2 + a_1^2b^2 - a_1^2b_1^2)$ $I_{x_Cy_C} = I_{xy} - \frac{a^2b^2A}{4(b+b}$

Table 1.5 Properties of plane sections, continued

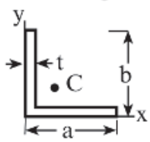
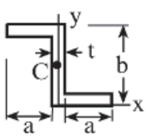
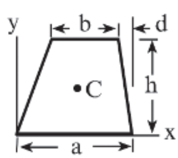
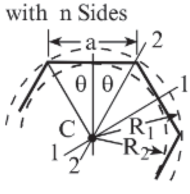
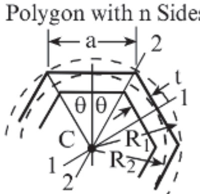
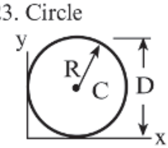
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
18. Thin Angle Section 	$x_C = \frac{a^2}{2(a+b)}$ $y_C = \frac{b^2}{2(a+b)}$ $A = (a+b)t$ $a, b >> t$	$I_{x_C} = b^3t/3 - tb^4/(4(a+b))$ $I_{y_C} = a^3t/3 - ta^4/(4(a+b))$ $I_x = \frac{1}{3}b^3t$ $I_y = \frac{1}{3}a^3t,\ I_p = \frac{t}{3}(a^3+b^3)$	$I_{xy} = \frac{1}{4}(a^2+b^2)t^2$ $I_{x_Cy_C} = -\frac{a^2b^2t}{4(a+b)}$
19. Z Section 	$x_C = 0$ $y_C = b/2$ $A = bt + 2at$ for thin section, a,b >> t neglect t^2, t^3 in formulas.	$I_{x_C} = t(b^3 + 6ab^2 - 12abt + 8at^2)/12$ $I_{y_C} = I_y = t(8a^3 + 12a^2t + 6at^2 + bt^2)/12$ $I_x = t(b^3 + 3ab^2 - 3abt + 2at^2)/3$ $I_{x_Cy_C} = -a^2bt - (b-a)at^2 + at^3$	
20. Trapezoid 	$x_C = \frac{1}{3(a+b)}(2a^2 + 2ab - ad - 2bd - b^2)$ $y_C = \frac{h}{3}\left(\frac{a+2b}{a+b}\right)$ $A = \frac{h}{2}(a+b)$	$I_{x_C} = h^3(a^2 + 4ab + b^2)/[36(a+b)]$ $I_{y_C} = h[a^4 + b^4 + 2ab(a^2+b^2)/[36(a+b)]$ $-d(a^3 + 3a^2b - 3ab^2 - b^3) + d^2(a^2 + 4ab + b^2)]$ $I_{x_Cy_C} = h^2[b(3a^2 - 3ab - b^2) + a^3$ $- d(2a^2 + 8ab + 2b^2)]/[72(a+b)]$ $I_x = h^3(a+3b)/12$	
21. Regular Polygon with n Sides 	$R_1 = \frac{a}{2\sin\theta}$ $R_2 = \frac{a}{2\tan\theta}$ $\theta = 180/n$, degree $A = \frac{1}{4}a^2 n \cot\theta$	$I_1 = \frac{A(6R_1^2 - a^2)}{24}$ $I_2 = \frac{A(12R_2^2 + a^2)}{48}$	n = number sides = 3,4,5.. θ =180 /n, degrees For n=3, θ=60° $R_1 = a/3^{1/2}$, $R_2 = a/12^{1/2}$, $A = a^2 3^{1/2}/4$, $I_1 = I_2 = a^4 3^{1/2}/96$
22. Hollow Regular Polygon with n Sides 	$R_1 = \frac{a}{2\sin\theta}$ $R_2 = \frac{a}{2\tan\theta}$ $\theta = 180/n$, (deg) $A = nat\left(1 - \frac{t}{a}\tan\theta\right)$	$I_1 = I_2 = \frac{na^3t}{8}\left(\frac{1}{3} + \frac{1}{\tan^2\theta}\right) \cdot \left[1 - \frac{3t}{a}\tan\theta + 4\left(\frac{t}{a}\tan\theta\right)^2 - 2\left(\frac{t}{a}\tan\theta\right)^3\right]$	n = number of sides = 3,4,5..
23. Circle 	$x_C = R = D/2$ $y_C = R = D/2$ $A = \pi R^2 = \frac{\pi}{4}D^2$ $P = 2\pi R = \pi D$	$I_{x_C} = I_{y_C} = \frac{1}{4}\pi R^4 = \frac{1}{64}\pi D^4$ $I_{pC} = \frac{1}{2}\pi R^4 = \frac{1}{32}\pi D^4$ $I_x = I_y = \frac{5}{4}\pi R^4 = \frac{5}{64}\pi D^4$	$I_{x_Cy_C} = 0$ $I_{xy} = \pi R^4 = \frac{\pi}{16}D^4$

Table 1.5 Properties of plane sections, continued

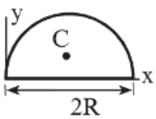
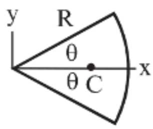
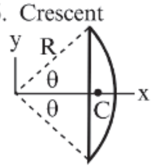
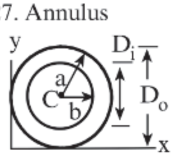
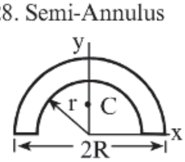
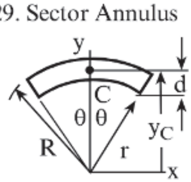
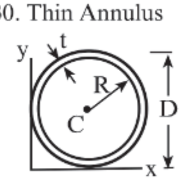
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
24. Semicircle 	$x_C = R$ $y_C = \frac{4R}{3\pi}$ $A = \frac{1}{2}\pi R^2$	$I_{x_C} = \frac{R^4(9\pi^2 - 64)}{72\pi}$ $I_{y_C} = \frac{\pi R^4}{8}$ $I_x = \frac{\pi R^4}{8}$ $I_y = \frac{5\pi R^4}{8}$	$I_{x_C y_C} = 0$ $I_{xy} = \frac{2}{3}R^4$
25. Circular Sector 	$x_C = \frac{2R}{3}\frac{\sin\theta}{\theta}$ $y_C = 0$ $A = R^2\theta$	$I_x = \frac{R^4}{4}(\theta - \sin\theta\cos\theta)$ $I_y = \frac{R^4}{4}(\theta + \sin\theta\cos\theta)$	$I_{x_C y_C} = 0$ $I_{xy} = 0$
26. Crescent 	$x_C = \frac{2R}{3}\left(\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\right)$ $y_C = 0$ $A = R^2(\theta - \sin\theta\cos\theta)$	$I_x = \frac{R^4}{4}(\theta - \sin\theta\cos\theta - \frac{2}{3}\sin^3\theta\cos\theta)$ $I_y = \frac{R^4}{4}(\theta - \sin\theta\cos\theta + 2\sin^3\theta\cos\theta)$ $I_{x_C y_C} = 0,\ I_{xy} = 0,\ I_p = I_x + I_y$	
27. Annulus 	$x_C = a$ $y_C = a$ $A = \pi(a^2 - b^2)$ $= \frac{\pi}{4}(D_o^2 - D_i^2)$	$I_{x_C} = I_{y_C} = \frac{\pi}{4}(a^4 - b^4)$ $= \frac{\pi}{64}(D_o^4 - D_i^4)$ $I_x = I_y = \frac{5}{4}\pi a^4 - \pi a^2 b^2 - \frac{\pi}{4}b^4$	$I_{x_C y_C} = 0$ $I_{xy} = \pi a^2(a^2 - b^2)$ $I_{pC} = \frac{\pi}{2}(a^4 - b^4)$
28. Semi-Annulus 	$x_C = 0$ $y_C = \frac{4R}{3\pi}\left(\frac{r}{R} + \frac{R}{R+r}\right)$ $A = \frac{\pi}{2}(R^2 - r^2)$	$I_x = \frac{\pi}{8}(R^4 - r^4)$ $I_y = I_x$ $I_p = \frac{\pi}{4}(R^4 - r^4)$	$I_{x_C y_C} = 0$ $I_{xy} = 0$
29. Sector Annulus 	$x_C = 0$ $y_C = \frac{2\sin\theta}{3\theta}\left(\frac{R^3 - r^3}{R^2 - r^2}\right)$ $d = y_C - r\sin\theta$ $A = (R^2 - r^2)\theta$	$I_{x_C} = \frac{1}{4}(R^4 - r^4)(\theta + \sin\theta\cos\theta) - \frac{4}{9}\frac{(R^3 - r^3)^2}{(R^2 - r^2)}\frac{\sin^2\theta}{\theta}$ $I_x = \frac{1}{4}(R^4 - r^4)(\theta + \sin\theta\cos\theta)$ $I_{y_C} = I_y = \frac{1}{4}(R^4 - r^4)(\theta - \sin\theta\cos\theta)$	
30. Thin Annulus 	$x_C = R = D/2$ $y_C = R = D/2$ $A = 2\pi Rt = \pi Dt$ $D = 2R$ $R >> t$	$I_{x_C} = I_{y_C} = \pi R^3 t = \frac{\pi}{8}D^3 t$ $I_x = I_y = 3\pi R^3 t = \frac{3\pi}{8}D^3 t$ $I_{pC} = 2\pi R^3 t = \frac{\pi}{4}D^3 t$	$I_{x_C y_C} = 0$ $I_{xy} = 2\pi R^3 t = \frac{\pi}{4}D^3 t$

Table 1.5 Properties of plane sections, continued

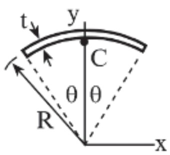
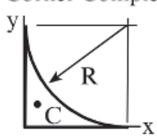
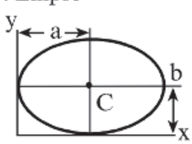
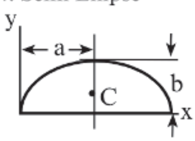
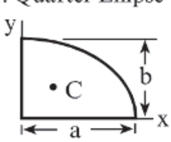
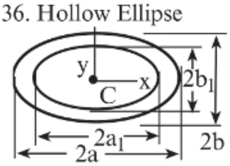
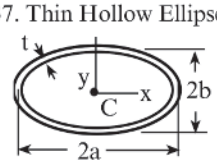
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
31. Sector Thin Annulus 	$x_C = 0$ $y_C = R\dfrac{\sin\theta}{\theta}$ $A = 2\theta Rt$ $R >> t$	$I_{x_C} = R^3t(\theta + \sin\theta\cos\theta - (2\sin^2\theta)/\theta)$ $I_x = R^3t(\theta + \sin\theta\cos\theta)$ $I_{y_C} = I_y = R^3t\ (\theta - \sin\theta\cos\theta)$	$I_{x_Cy_C} = 0$ $I_{xy} = 0$ $I_p = R^3t\theta$
32. Corner Complement 	$x_C = \dfrac{10-3\pi}{3(4-\pi)}R$ $y_C = \dfrac{10-3\pi}{3(4-\pi)}R$ $A = (1-\pi/4)R^2$	$I_{x_C} = I_{y_C} = \dfrac{9\pi^2 - 84\pi + 176}{144(4-\pi)}R^4$ $I_x = I_y = \left(1 - \dfrac{5\pi}{16}\right)R^4$	$I_{xy} = \left(\dfrac{19}{24} - \dfrac{\pi}{4}\right)R^4$
33. Ellipse 	$x_C = a \quad A = \pi ab \quad I_{x_C} = \dfrac{\pi}{4}ab^3 \quad I_{y_C} = \dfrac{\pi}{4}a^3b \quad I_{x_Cy_C} = 0 \quad I_{pc} = \dfrac{\pi}{4}ab(a^2+b^2)$ $y_C = b \qquad I_x = \dfrac{5\pi}{4}ab^3 \quad I_y = \dfrac{5\pi}{4}a^3b \quad I_{xy} = \pi a^2b^2$ Ref. [16] Perimeter $P = 4aE\left[\sqrt{1-b^2/a^2}\right] \approx \pi\left(3(a+b) - \sqrt{(3a+b)(a+3b)}\right)$, $a > b$		
34. Semi Ellipse 	$x_C = a$ $y_C = \dfrac{4b}{3\pi}$ $A = \dfrac{\pi}{2}ab$	$I_{x_C} = \dfrac{ab^3}{72\pi}(9\pi^2 - 64)$ $I_{y_C} = \dfrac{\pi}{8}a^3b$ $I_x = \dfrac{\pi}{8}ab^3 \quad I_y = \dfrac{5\pi}{8}a^3b$	$I_{x_Cy_C} = 0$ $I_{xy} = \dfrac{2}{3}a^2b^2$ $I_p = I_x + I_y$
35. Quarter Ellipse 	$x_C = \dfrac{4}{3\pi}a$ $y_C = \dfrac{4}{3\pi}b$ $A = \pi ab/4$	$I_{x_C} = ab^3\left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)$ $I_{y_C} = a^3b\left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)$ $I_x = \pi ab^3/16,\ I_y = \pi a^3b/16$	$I_{xy} = \dfrac{a^2b^2}{4}$ $I_p = I_x + I_y$
36. Hollow Ellipse 	$x_C = 0$ $y_C = 0$ $A = \pi(ab - a_1b_1)$	$I_{x_C} = \dfrac{\pi}{4}\left(ab^3 - a_1b_1^{\,3}\right)$ $I_{y_C} = \dfrac{\pi}{4}\left(a^3b - a_1^{\,3}b_1\right)$	$I_{x_Cy_C} = 0$ $I_{P_C} = \dfrac{\pi}{4}(ab(a^2+b^2) - a_1b_1(a_1^2+b_1^2))$
37. Thin Hollow Ellipse 	$x_C = 0$ $y_C = 0$ $A = \pi t(a+b)$ $a, b >> t$	$I_{x_C} = I_x = \dfrac{\pi b^2t}{4}(a+3b)$ $I_{y_C} = I_y = \dfrac{\pi a^2t}{4}(b+3a)$	$I_{x_Cy_C} = 0$ $I_{P_C} = I_{x_C} + I_{y_C}$

Table 1.5 Properties of plane sections, continued

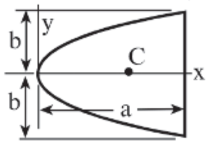
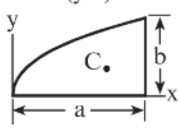
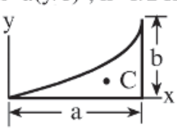
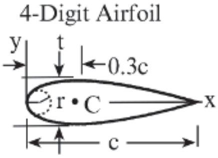
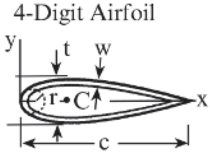
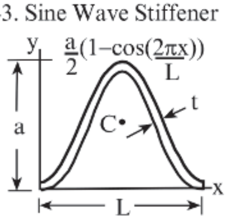
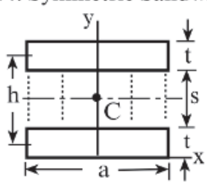
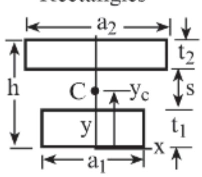
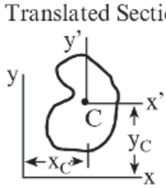
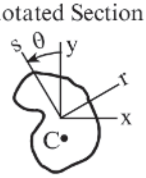
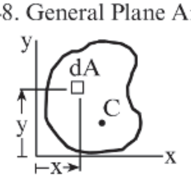
Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
38. Parabola, $x = a(y/b)^2$ 	$x_C = \frac{3}{5}a$ $y_C = 0$ $A = \frac{4}{3}ab$	$I_{x_C} = I_x = \frac{4}{15}ab^3$ $I_{y_C} = \frac{16}{175}a^3b$ $I_y = \frac{4a^3b}{7}$	$I_{x_C y_C} = 0$ $I_p = I_x + I_y$
39. Semi-Parabola $x = a(y/b)^2$ 	$x_C = \frac{3}{5}a$ $y_C = \frac{3}{8}b$ $A = \frac{2}{3}ab$	$I_{xC} = \frac{19}{480}ab^3$ $I_x = \frac{2}{15}ab^3$ $I_y = \frac{2}{7}a^3b$	$I_{xy} = \frac{1}{6}a^2b^2$
40. n^{th} Semi-Parabola $x=a(y/b)^n$, n=1/2 shown 	$x_C = \frac{n+1}{n+2}a$ $y_C = \frac{n+1}{2(2n+1)}b$ $A = \frac{1}{n+1}ab$	$I_x = \frac{1}{3(3n+1)}ab^3$ $I_y = \frac{1}{n+3}a^3b$	$I_{xy} = \frac{1}{4n+4}a^2b^2$ n = 1/2 for parabola
41. NACA Symmetric 4-Digit Airfoil 	$x_C = 0.4204c$ $y_C = 0$ $A = 0.685tc$ thickness t at $x = 0.30c$ radius $r = 1.1019t^2/c$	$I_{x_C} = I_x = 0.03941ct^3$ $I_{y_C} = 0.03782c^3t$ $I_y = 0.1589c^3t$ $I_{xy} = 0$	airfoil profile Ref. [18], $y = \pm 5t(0.2969X^{1/2}$ $-0.1260X - 0.3516X^2 +$ $0.2843X^3 - 0.1015X^4)$ where $X = x/c$.
42. NACA Symmetric 4-Digit Airfoil 	For thin skin airfoil, c>>t>>w $x_C = 0.5c$ $y_C = 0$ $A = Pw,\ P \approx 2c + t^2/c$	$I_{x_C} = I_x = 0.2754wct^2$ $I_{y_C} = (1/6)c^3w$ $I_y = (2/3)c^3w$	Profile is given above. $I_{xy} = 0$ Approximate formulas
43. Sine Wave Stiffener 	For thin skin, L >> t $x_C = L/2,\ y_C = a/2$ $A = \frac{2tL}{\pi}E\left(-\frac{\pi^2a^2}{L^2}\right)$ $\approx tL[1 + 4.4(a/L)^2]^{1/2}$	$I_x = \frac{tL^3}{6\pi^3}\left[(1+\frac{\pi^2a^2}{L^2})K\left(-\frac{\pi^2a^2}{L^2}\right) - (1-\frac{\pi^2a^2}{L^2})E\left(-\frac{4\pi^2a^2}{L^2}\right)\right] = \alpha_1 tL^3$ $\alpha_1 \approx 0.124(a/L)^{1.5} + 0.664(a/L)^3,\ I_{x_C y_C} = 0$ K(x), E(x) = complete elliptic integrals	

Table 1.5 Properties of plane sections, continued

Section	Area A and Centroid C	Area Moments of Inertia Ix, Iy	Area Products of Inertia Ixy
44. Symmetric Sandwich 	$x_C = 0$ $y_C = t + s/2$ $A = 2at$ $h = s + t$ The two identical rectangles are separated by a spacer.	$I_{x_C} = at\left(\frac{1}{2}s^2 + st + \frac{2}{3}t^2\right) = at\left(\frac{1}{2}h^2 + \frac{1}{6}t^2\right)$ $I_{y_C} = ta^3/6$ $I_x = at\left(s^2 + 3st + \frac{8}{3}t^2\right) = at\left(h^2 + th + \frac{2}{3}t^2\right)$ $I_{xy} = 0$	
45. Unequal Parallel Rectangles 	$x_C = 0,\ y_C = \dfrac{a_1t_1^2/2 + a_2t_2(h - t_2/2)}{A}$ $A = a_1t_1 + a_2t_2$ $s = h - t_1 - t_2$ Section is symmetric about y axis.	$I_{x_C} = \frac{a_1t_1^3}{12} + a_1t_1(y_C - t_1/2)^2 + \frac{a_2t_2^3}{12} + a_2t_2(h - y_C - t_2/2)^2$ $I_{y_C} = \frac{1}{12}(t_1a_1^3 + t_2a_2^3)$ $I_x = \frac{1}{3}a_1t_1^3 + \frac{1}{12}a_2t_2^3 + a_2t_2(h - t_2/2)^2,\ I_{xy} = 0$	
46. Translated Section 	Area = A x' and y' axes are through centroid C and they are parallel to x and y axes, respectively.	$I_x = I_{x_C} + y_C^2A$ $I_y = I_{y_C} + x_C^2A$ I_x and I_y about x-y axes, I_{xc} and I_{yc} about x'-y' axes.	$I_{xy} = I_{x_Cy_C} + x_Cy_CA$
47. Rotated Section 	The r and s axes are rotated counterclockwise with respect to the x and y axes. Also see Eq. 1.8.	$I_x = I_r\cos^2\theta + I_s\sin^2\theta + I_{rs}\sin 2\theta$ $I_y = I_r\sin^2\theta + I_s\cos^2\theta - I_{rs}\sin 2\theta$ $I_p = I_x + I_y = I_r + I_s$	$I_{xy} = I_{rs}\cos 2\theta + \frac{1}{2}(I_s - I_r)\sin 2\theta$ Note: $I_{xy} = 0$ and I_x is minimum for $2\theta = \arctan 2I_{rs}/(I_r - I_s)$
48. General Plane Area 	$x_C = \frac{1}{A}\int_A x\,dA$ $y_C = \frac{1}{A}\int_A y\,dA$ $A = \int_A dx\,dy$	$I_x = \int_A y^2dA$ $I_y = \int_A x^2dA$ $I_p = I_x + I_y = \int_A (x^2 + y^2)dA$	$I_{xy} = \int_A xy\,dA$ Note mass per unit length m = ρ A where ρ = material density

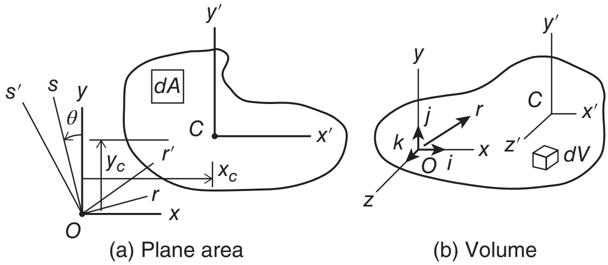


Figure 1.3 A plane section with centroid (C) and rotated and translated coordinate systems and a solid body with a translated coordinate system and a rotated vector **r**

The z-axis is perpendicular to the x–y plane. The element of area is $dA = dxdy$. *Area moments of inertia* about the x-axis (I_x) and y-axis (I_y), the *area product of inertia* (I_{xy}) about the x–y-axes, and the *polar area moment of inertia* about the z-axis ($I_p = I_{zz}$) are integrals over the area.

$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA, \quad I_{xy} = \int_A xy\, dA,$$
$$I_p = I_{zz} = I_x + I_y = \int_A (x^2 + y^2)\, dA \tag{1.5}$$

The symbol I is used for area moments of inertia and J is used for mass moments of inertia. I_x is the integral of the square of distance (y^2) along the y-axis from the x-axis times the elemental area. The *radius of gyration* for each axis is defined.

$$r_x = (I_x/A)^{1/2}, \ r_y = (I_y/A)^{1/2} \tag{1.6}$$

I_{xc}, I_{yc}, and I_{pc} are the area moments of inertia about axes with origin at the centroid C, Figure 1.3a. $I_{xy} = 0$ if the body is symmetric about either axis.

Parallel axis theorem transforms moments of inertia about the centroid, I_{xc}, I_{yc}, and I_{xcyc}, to moments of inertia about the offset parallel axes x, y, Figure 1.3a.

$$I_x = I_{xc} + y_c^2 A, \quad I_y = I_{yc} + x_c^2 A, \quad I_{xy} = I_{xcyc} + x_c y_c A$$
$$I_p = I_x + I_y = I_{pc} + (x_c^2 + y_c^2) A \tag{1.7}$$

Translation away from the centroid increases the moment I_x and I_y [20].

The r-axis and the orthogonal s-axis shown in Figure 1.3a are rotated counterclockwise by the angle θ with respect to the x- and y-axes. The area moments of inertia about *rotated axes* are,

$$I_r = I_x \cos^2\theta + I_y \sin^2\theta - I_{xy} \sin 2\theta$$
$$I_s = I_y \cos^2\theta + I_x \sin^2\theta + I_{xy} \sin 2\theta \tag{1.8}$$
$$I_{rs} = I_{xy} \cos 2\theta - (1/2)(I_{yy} - I_{xx}) \sin 2\theta$$

The sum of the area moments of inertia about two perpendicular axes is independent of the rotation of the axes.

$$I_z = I_p = I_x + I_y = I_r + I_s \tag{1.9}$$

More coordinate transformations are in Table 2.2.

Principal axes are two mutually perpendicular axes about which the product of inertia is zero. The angle of the principal axes is found from Equation 1.8 with $I_{rs} = 0$.

$$\theta = \left(\frac{1}{2}\right) \arctan\left[\frac{2I_{xy}}{\left(I_y - I_x\right)}\right] \tag{1.10}$$

Substituting this θ into Equation 1.8 gives the *principal area moments of inertia,* which are the maximum and minimum moments of inertia about rotated axes.

$$\begin{aligned} I_{\max} &= \left(\frac{1}{2}\right)(I_x + I_y) + \left(\frac{1}{2}\right)[(I_x - I_y)^2 + 4I_{xy}^2]^{1/2} \\ I_{\max} &= \left(\frac{1}{2}\right)(I_x + I_y) - \left(\frac{1}{2}\right)[(I_x - I_y)^2 + 4I_{xy}^2]^{1/2} \end{aligned} \tag{1.11}$$

If the two principal area moments of inertia are equal, then the moments of inertia are independent of axis rotation.

Geometric properties of complex areas are obtained by subdividing (meshing) the areas into elementary sections, usually triangles or rectangles, and summing their properties in space about the global coordinate system. See Equations 1.7, 1.8, and Example 1.3. Areas and moments of inertia of standardized aluminum, steel, and timber sections are presented in Refs [21–23]. Section 4.1 discusses application to beam bending theory.

Example 1.2 Area and moment of inertia of pipe section

Compute the area and area moment of inertia about the axis *A-A* through the centroid of the pipe section on the left-hand side of Figure 1.4.

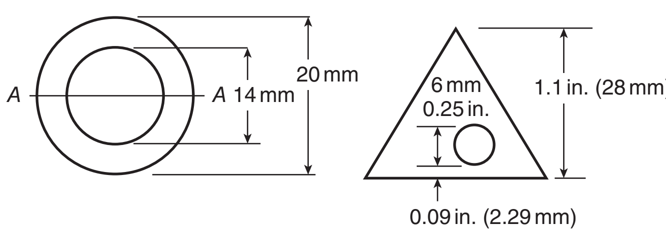


Figure 1.4 Geometric sections for Examples 1.3 and 1.4

Solution: The area and moment of inertia of the pipe section with an axis through its centroid are in case 27 of Table 1.5.

$$A = \pi(a^2 - b^2), \quad I_{xc} = \left(\frac{\pi}{4}\right)(a^4 - b^4)$$

Figure 1.4 shows the outer diameter $a = 20$ mm (0.7874 in.) and the inner diameter $b = 14$ mm (0.5512 in.). The radii are half these values, $a = 10$ mm (0.3937 in.), $b = 7$ mm (0.275 in.). Substituting these values into the above formulas gives,

$$A = 160.2\ \text{mm}^2\ (0.2483\ \text{in.}^2), I_{xc} = 5968\ \text{mm}^4 (0.01434\ \text{in.}^4)$$

The pipe section can also be computed with the thin annulus approximation given in case 30 of Table 1.5.

$$A = 2\pi Rt, \qquad I_{x_C} = \pi R^3 t$$

The thickness $t = a - b = 3$ mm is half the difference in the diameters. The average radius $R = (a + b)/2 = 8.5$ mm (0.335 in.). Substituting into the previous formulas gives approximate values: $A = 150.2$ mm^2 (0.2328 in.^{2}), $I_{xc} = 5788$ mm^4 (0.0139 in.^{4}).

Example 1.3 Area and moment of inertia of a triangle

A triangle with a hole in it is shown on the right-hand side of Figure 1.4. Compute the cross-sectional area and area moment of inertia about the axis along the base of the triangle.

Solution: The area and area moment of inertia of the triangle with a hole are equal to the area and area moment of inertia of the triangle less the area and area moment of inertia of the hole. The area and moment of inertia of the triangle for an axis along its base are in case 2 of Table 1.5.

$$A = (1/2)bh^2, \qquad I_x = (1/12)bh^3$$

For our case, $b = 0.9$ in. (22.80 mm) and $h = 1.1$ in. (27.94 mm),

$$A = 0.4950\ \text{in.}^2\ (287.7\ \text{mm}^2), \qquad I_{xc} = 0.09982\ \text{in.}^4\ (41548\ \text{mm}^4)$$

The area of the 0.25 in. (6.35 mm) diameter hole is computed from case 23 of Table 1.5 using a radius $R = 0.25$ in./2 = 0.125 in.: $A_{\text{hole}} = \pi R^2 = 0.04909$ in.^{2} (31.67 mm^2). The net area is the difference between the triangle and the hole: $A_{\text{net}} = 0.4950 - 0.04909 = 0.4459$ in.^{2} (287.7 mm^2).

The area moment of inertia of the triangle with the hole is the area moment of inertia of the triangle less the area moment of inertia of the hole, which is offset by $y_C = 0.09 + 0.125 = 0.215$ in. (5.46 mm), cases 23 and 45 of Table 1.5.

$$I_{\text{xhole}} = I_{x_C} + y_c^2 A = \left(\frac{1}{4}\right)\pi R^4 + y_c^2 \pi R^2 = 0.0001917 + 0.002269 = 0.002461\ \text{in.}^4$$

$$I_{\text{net}} = 0.09982 - 0.002461\ \text{in.}^4 = 0.09736\ \text{in.}^4\ (40520\ \text{mm}^4)$$

1.6 Geometric Properties of Rigid Bodies

Table 1.6 [16–18] has formulas for geometric properties of homogeneous rigid bodies. These are based upon classical solutions, such as Ref. 16; also see Section 1.7.

Table 1.6 Properties of homogeneous solids

Notation: A = cross sectional area; C = centroid (center of mass); J_x = mass moment of inertia about axis parallel to x axis; J_y = mass moment of inertia about axis parallel to y axis; J_z = mass moment of inertia about axis parallel to z axis through center of mass; J_{xy} = mass product moment of inertia; J_{xc}, J_{yc}, J_{zc}, J_{xcyc} = mass moments of inertia about axes through centroid; M = mass = ρV; P = perimeter of section; t = thickness; S = lateral surface area; V = volume; x_C = distance from x axis to center of mass; y_C = distance from y axis to center of mass; z_C = distance from z axis to center of mass; t = mass density. Also see Table 1.7
Refs [16, 17, 19].

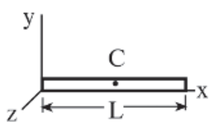
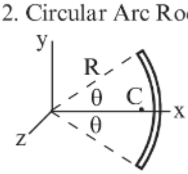
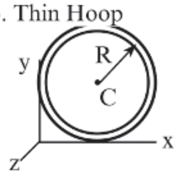
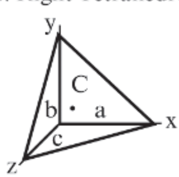
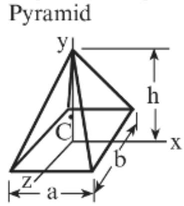
Solid Geometry	Volume V and Center of Mass C	Mass Moments Inertia Jx, Jy, Jz	Mass Products of Inertia Jxy
1. Thin Rod 	$x_C = \frac{L}{2}$ $y_C = 0$ $V = M/\rho = AL$ A = area cross section	$J_{x_C} = J_x = 0$ $J_{y_C} = J_{z_C} = \frac{1}{12}ML^2$ $J_y = J_z = \frac{1}{3}ML^2$	$J_{x_C y_C,\,etc} = 0$ $J_{xy,\,etc} = 0$ Also see Table 1-7.
2. Circular Arc Rod 	$x_C = \frac{\sin\theta}{\theta}R$ $y_C = 0$ $z_C = 0$ $V = M/\rho = 2\theta AR$ A = area cross section	$J_{x_C} = J_x = \frac{\theta - \sin\theta\cos\theta}{2\theta}MR^2$ $J_y = \frac{\theta + \sin\theta\cos\theta}{2\theta}MR^2$ $J_z = MR^2$	$J_{x_C y_C,\,etc} = 0$ $J_{xy,\,etc} = 0$
3. Thin Hoop 	$x_C = R$ $y_C = R$ $z_C = 0$ $V = M/\rho = 2\pi AR$ A = area of section	$J_{x_C} = J_{y_C} = MR^2/2$ $J_{z_C} = MR^2$ $J_x = J_y = 3MR^2/2$ $J_z = 3\,MR^2$	$J_{x_C y_C,\,etc} = 0$ $J_{xy} = MR^2$ $J_{xz} = J_{yz} = 0$
4. Right Tetrahedron  See case 4 Table 1-7.	$x_C = \frac{a}{4}$ $y_C = \frac{b}{c}$ $z_C = \frac{c}{4}$ $V = M/\rho = \frac{1}{6}abc$	$J_{x_C} = \frac{3}{80}M(b^2 + c^2)$ $J_{z_C} = \frac{3}{80}M(a^2 + b^2)$ $J_x = \frac{1}{10}M(b^2 + c^2)$ $J_z = \frac{1}{10}M(a^2 + b^2)$	$J_{y_C} = \frac{3}{80}M(a^2 + c^2)$ $J_{x_C z_C} = -\frac{1}{80}Mac$ $J_y = \frac{1}{10}M(a^2 + c^2)$ $J_{xz} = \frac{1}{20}Mac$
5. Right Rectangular Pyramid 	$x_C = 0$ $y_C = \frac{h}{4}$ $z_C = 0$ $S = ab + a\sqrt{h^2 + (b/2)^2}$ $+ b\sqrt{h^2 + (a/2)^2}$ $V = M/\rho = abh/3$	$J_{x_C} = \frac{M}{80}(4b^2 + 3h^2)$ $J_{y_C} = J_y = \frac{M}{20}(a^2 + b^2)$ $J_x = \frac{M}{20}(b^2 + 2h^2)$ $J_z = \frac{M}{20}(a^2 + 2h^2)$	$J_{x_C y_C} = J_{x_C z_C} = 0$ $J_{y_C z_C} = 0,$ $J_{xy} = J_{xz} = J_{yz} = 0$

Table 1.6 Properties of homogeneous solids, continued

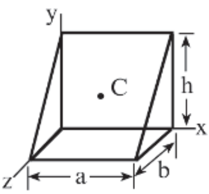
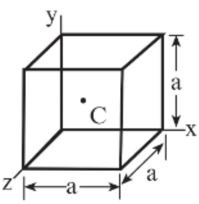
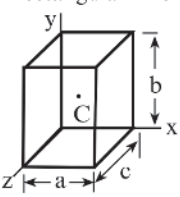
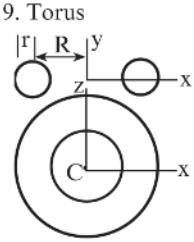
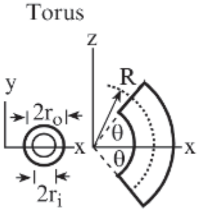
Solid Geometry	Volume V, Mass M Center of Mass C	Mass Moments Inertia Jx, Jy, Jz	Mass Products Inertia Jxy
6. Right Angle Wedge 	$x_C = a/2$ $y_C = h/3$ $z_C = b/3$ $V = \frac{M}{\rho} = \frac{1}{2}abh$	$J_x = \frac{M}{6}(b^2 + h^2)$ $J_y = \frac{M}{6}(a^2 + 2b^2)$ $J_z = \frac{M}{6}(2a^2 + h^2)$	$J_{xz} = \frac{M}{12}bh$ $J_{xy} = \frac{M}{4}ah$ $J_{yz} = \frac{M}{4}ab$ Also see Table 1.7
7. Cube 	$x_C = \frac{a}{2}$ $y_C = \frac{a}{2}$ $z_C = \frac{a}{2}$ $V = M/\rho = a^3$	$J_{x_C} = J_{y_C} = J_{z_C}$ $= \frac{1}{6}Ma^2$ $J_x = J_y = J_z = \frac{2}{3}Ma^2$	$J_{x_C y_C} = J_{x_C z_C} = 0$ $J_{y_C z_C} = 0,$ $J_{xz} = \frac{1}{4}Ma^2$ $= J_{xy} = J_{yz}$
8. Rectangular Prism 	$x_C = \frac{a}{2}$ $y_C = \frac{b}{2}$ $z_C = \frac{c}{2}$ $V = abc,\ M = \rho V$	$J_{x_C} = \frac{M}{12}(b^2 + c^2)$ $J_{y_C} = \frac{M}{12}(a^2 + c^2)$ $J_{z_C} = \frac{M}{12}(a^2 + b^2)$ $J_x = \frac{M}{3}(b^2 + c^2)$	$J_{x_C y_C} = J_{x_C z_C} = 0$ $J_{xy} = \frac{1}{4}Mab$ $J_{xz} = \frac{1}{4}Mac$ $J_{yz} = \frac{1}{4}Mbc$
9. Torus 	$x_C = 0$ $y_C = 0$ $z_C = 0$ $S = 4\pi^2 R r$ $V = M/\rho = 2\pi^2 R r^2$	$J_{x_C} = J_{z_C}$ $= \frac{M}{2}(R^2 + \frac{5}{4}r^2)$ $J_{y_C} = J_y$ $= \frac{M}{4}(4R^2 + 3r^2)$	$J_{x_C y_C} = J_{x_C z_C} = 0$ $J_{y_C z_C} = 0,$ $J_{xy} = J_{xz} = J_{yz} = 0$
10. Sector of Hollow Torus 	$x_C = \frac{\sin\theta}{\theta}\left(R + \frac{r_o^2 + r_i^2}{4R}\right)$ $y_C = 0$ $z_C = 0$ $S = 2\pi\theta R r_o$ $V = 2\pi R\theta(r_o^2 - r_i^2)$ $M = \rho V,$ Ref. [16]	$J_{x_C} = \frac{M}{16\theta}\left(4R^2(2\theta - \sin 2\theta) + (r_o^2 + r_i^2)(10\theta - 3\sin 2\theta)\right)$ $J_{y_C} = \frac{M}{4\theta}\left(4R^2\theta + 3(r_o^2 + r_i^2)\theta - K\right),\quad J_{z_C} =$ $\frac{M}{16\theta}\left[4R^2(2\theta + \sin 2\theta) + (r_o^2 + r_i^2)(10\theta + 3\sin 2\theta) - 4K\right]$ where $K = \frac{2\sin^2\theta}{\theta}\left(2R^2 + r_o^2 + r_i^2 + \frac{(r_o^2 + r_i^2)^2}{8R^2}\right)$	

Table 1.6 Properties of homogeneous solids, continued

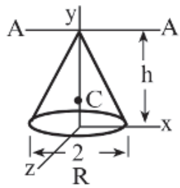
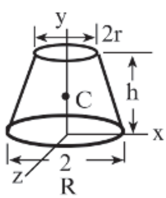
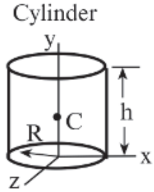
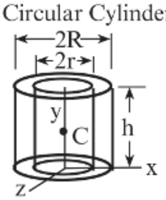
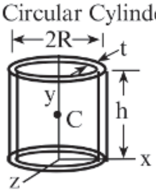
Solid Geometry	Volume V, Mass M, Center of Mass C	Mass Moments of Inertia Jx, Jy, Jz	Mass Products of Inertia Jxy
11Right Circular Cone 	$x_C = 0$ $y_C = \frac{h}{4}$ $z_C = 0$ $S = \pi R(R + \sqrt{R^2 + h^2})$ $V = \frac{M}{\rho} = \frac{\pi}{3}\ R^2 h$	$J_{x_C} = J_{z_C} = \frac{3}{80} M(4R^2 + h^2)$ $J_{y_C} = J_y = \frac{3}{10} MR^2$ $J_x = J_z = \frac{1}{20} M(3R^2 + 2h^2)$ $J_{AA} = \frac{3}{20} M(R^2 + 4h^2)$	$J_{x_C y_C} = 0$ $J_{x_C z_C} = 0$ $J_{y_C z_C} = 0$ $J_{xy} = 0$ $J_{xz} = 0$ $J_{yz} = 0$
12. Frustrum of Cone 	$x_C = 0$ $y_C = \frac{h}{4} \frac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2}$ $z_C = 0$ $S =$ $\pi(R + r)\sqrt{(R - r)^2 + h^2}$ $V = \frac{M}{\rho} = \frac{\pi h}{3}(R^2 + Rr + r^2)$	$J_{y_C} = J_y = \frac{3M}{10}\left(\frac{R^5 - r^5}{R^3 - r^3}\right)$ $J_z = J_x =$ $\frac{M}{20}\left(\frac{2h^2(R^2 + 3Rr + 6r^2)}{R^2 + Rr + r^2} + \frac{3(R^4 + R^3 r + R^2 r^2 + Rr^3 + r^4)}{R^2 + Rr + r^2}\right)$	$J_{x_C y_C} = 0$ $J_{x_C z_C} = 0$ $J_{y_C z_C} = 0$ $J_{xy} = J_{xz} = 0$ $J_{yz} = 0$ $M = \rho V$
13. Right Circular Cylinder 	$x_C = 0$ $y_C = \frac{h}{2}$ $z_C = 0$ $S = 2\pi Rh$ $V = M / \rho = \pi R^2 h$	$J_{x_C} = J_{z_C} = \frac{M}{12}(3R^2 + h^2)$ $J_{y_C} = J_y = \frac{M}{2} R^2$ $J_x = J_z = \frac{M}{12}(3R^2 + 4h^2)$	$J_{x_C y_C} = 0$ $J_{x_C z_C} = 0$ $J_{y_C z_C} = 0$ $J_{xy} = 0$ $J_{xz} = J_{yz} = 0$
14. Hollow Right Circular Cylinder 	$x_C = 0$ $y_C = \frac{h}{2}$ $z_C = 0$ $V = M / \rho = \pi(R^2 - r^2)h$	$J_{x_C} = J_{z_C} = \frac{M}{12}(3R^2 + 3r^2 + h^2)$ $J_{y_C} = J_y = \frac{M}{2}(R^2 + r^2)$ $J_x = J_z = \frac{M}{12}(3R^2 + 3r^2 + 4h^2)$	$J_{x_C y_C} = J_{x_C z_C} = 0$ $J_{y_C z_C} = 0$ $J_{xy} = J_{xz} = J_{yz} = 0$
15. Thin Hollow Right Circular Cylinder 	$x_C = 0$ $y_C = \frac{h}{2}$ $z_C = 0$ $V = M / \rho = 2\pi Rht$ $R >> t$	$J_{x_C} = J_{z_C} = \frac{M}{12}(6R^2 + h^2)$ $J_{y_C} = J_y = MR^2$ $J_x = J_z = \frac{M}{6}(3R^2 + 2h^2)$	$J_{x_C y_C} = J_{x_C z_C} = 0$ $J_{x_C y_C} = 0$ $J_{xy} = J_{xz} = J_{yz} = 0$

Table 1.6 Properties of homogeneous solids, continued

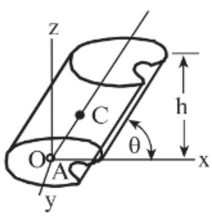
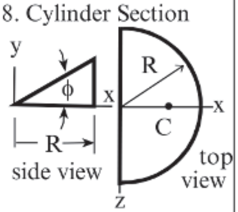
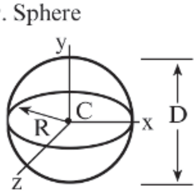
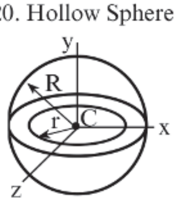
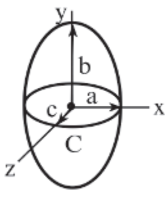
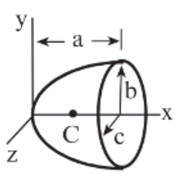
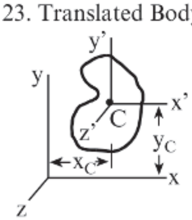
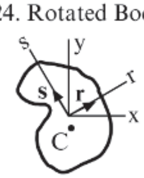
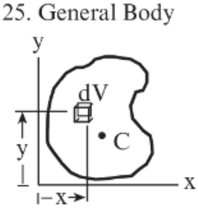
Solid Geometry	Volume, Mass M Center of Mass C	Mass Moments Inertia Jx, Jy, Jz	Mass Products of Inertia Jxy
16. Extruded Section From z=0 to z=h 	x_C = Section x_C y_C = Section y_C $z_C = h/2$ $V = M/\rho = Ah$ A = Section Area	$J_x = MI_x/A + Mh^2/3$ $J_y = MI_y/A + Mh^3/3$ $J_z = M(I_x + I_y)/A$ I_x, I_y = Area moments of inertia of cross section in x-y plane, Table 2.1	$J_{xy} = MI_{xy}/A$ $J_{xz} = Mhx_C$ $J_{yz} = Mhy_C$
17. Inclined Extrusion 	$x_C = (h/2)\sin\theta$ $y_C = 0$ $z_C = h/2$ $S = Ph$ $V = M/\rho = Ah$ A= x-y section area O is centroid of area P is perimeter of area A	$J_x = \rho h I_{xC} + \rho A h^3/3$ $J_y = \rho h I_{yC} +$ $\rho A h^3(1+\tan^2\theta)/3$ $J_z = \rho h(I_{x_C} + I_{y_C}) +$ $\rho A h^3 \tan^2\theta/3$ I_{xC} and I_{yC} are area moments inertia in x-y plane, Table 1.5, about area centroid O.	$J_{xz} =$ $\frac{\rho}{3}h^3 A\tan\theta$ $J_{xz} = J_{yz} = 0$ Note: Ends are cut parallel to x-y plane. θ is angle of extrusion.
18. Cylinder Section 	$x_C = \frac{3}{16}\pi R$ $y_C = \frac{3}{16}\pi R\tan\phi$ $z_C = 0$ $V = (2/3)R^3\tan\phi$	$J_x = \frac{M}{5}R^2\left(1+2\tan^2\phi\right)$ $J_y = \frac{3M}{5}R^2$ $J_z = \frac{2M}{5}R^2\left(1+\tan^2\phi\right)$	$J_{xy} = \frac{3M}{20}R^2\tan\phi$ $J_{xz} = J_{yz} = 0$
19. Sphere 	$x_C = 0$ $y_C = 0$ $z_C = 0$ $S = 4\pi R^2 = \pi D^2$ $V = \frac{M}{\rho} = \frac{4\pi}{3}R^3 = \frac{\pi}{6}D^3$	$J_{x_C} = J_{y_C} = J_{z_C} = \frac{2}{5}MR^2$ $= \frac{1}{10}MD^2$	$J_{x_C y_C} = 0$ $J_{x_C z_C} = 0$ $J_{y_C z_C} = 0$
20. Hollow Sphere 	$x_C = 0$ $y_C = 0$ $z_C = 0$ $V = \frac{M}{\rho} = \frac{4\pi}{3}(R^3 - r^3)$ $= 4\pi R^2 t,\ R - r << R$	$J_{x_C} = J_{z_C} = J_{y_C} =$ $= \frac{2}{5}M\left(\frac{R^5 - r^5}{R^3 - r^3}\right)$ $= \frac{2}{3}MRt,\ t = R - r$ for $R >> R - r$	$J_{x_C z_C} = 0$ $J_{x_C y_C} = 0$ $J_{y_C z_C} = 0$

Table 1.6 Properties of homogeneous solids, continued

Solid Geometry	Volume V, Mass M Center of Mass C	Mass Moments Inertia Jx, Jy, Jz	Mass Products of Inertia Jxy
21. Ellipsoid 	$x_C = 0$ $y_C = 0$ $z_C = 0$ $V = \frac{M}{\rho} = \frac{4}{3}\pi abc$ S = See Ref. [5], 9.148. sphere if a = b = c	$J_{x_C} = \frac{M}{5}(b^2 + c^2)$ $J_{y_C} = \frac{M}{5}(a^2 + b^2)$ $J_{z_C} = \frac{M}{5}(a^2 + b^2)$	$J_{xy} = J_{xz} = 0$ $J_{yz} = 0$ Equation of surface: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
22. Elliptic Paraboloid 	$x_C = \frac{2}{3}a$ $y_C = 0$ $z_C = 0$ $V = \frac{M}{\rho} = \frac{\pi}{2}abc$	$J_{x_C} = J_x = \frac{M}{6}(b^2 + c^2)$ $J_{y_C} = \frac{M}{18}(a^2 + 3c^2)$ $J_{z_C} = \frac{M}{18}(a^2 + 3b^2)$ $J_y = \frac{M}{6}(3a^2 + c^2)$ $J_z = (M/6)(3a^2 + b^2)$	$J_{xy} = J_{xz} = 0$ $J_{yz} = 0$ Equation of surface: $\frac{x}{a} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$
23. Translated Body 	Mass = M x', y', and z' axes are through centroid C. They are parallel to x, y, and z axes, respectively.	$J_x = J_{x_C} + M(y_C^2 + z_C^2)$ $J_y = J_{y_C} + M(x_C^2 + z_C^2)$ $J_z = J_{z_C} + M(x_C^2 + y_C^2)$	$J_{xy} = J_{x_C y_C} + M x_C y_C$ $J_{yz} = J_{y_C z_C} + M y_C z_C$ $J_{xz} = J_{x_C z_C} + M x_C z_C$
24. Rotated Body 	r and s axes are perpendicular and they pass through the origin of the x, y, and z axes. Their directions are specified by unit vectors **r** and **s**. $\mathbf{r} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ $\mathbf{s} = l'\mathbf{i} + m'\mathbf{j} + n'\mathbf{k}$	$J_r = l^2 J_x + m^2 J_y + n^2 J_z - 2lm J_{xy} - 2ln J_{xz} - 2mn J_{yz}$ $J_{rs} = -ll'J_x - mm'J_y - nn'J_z + (lm' + ml')J_{xy}$ $+ (ln' + l'n)J_{xz} + (mn' + nm')J_{yz}$ l, m, n are scalars that specify the direction of the unit vector **r** in the direction of the r axis. l', m', n' are scalars that specify the direction of the s axis Ref. [19]	
25. General Body 	$x_C = \frac{1}{V}\int_V x\,dV$ $y_C = \frac{1}{V}\int_V y\,dV$ $z_C = \frac{1}{V}\int_V z\,dV$ $V = \frac{M}{\rho} = \int_V dx\,dy\,dz$	$J_x = \rho\int_V (y^2 + z^2)dV$ $J_y = \rho\int_V (x^2 + z^2)dV$ $J_z = \rho\int_V (x^2 + y^2)dV$	$J_{xy} = \rho\int_V xy\,dV$ $J_{xz} = \rho\int_V xz\,dV$ $J_{yz} = \rho\int_V yz\,dV$

The *mass M*, *volume V*, and location of the *center of mass C*, also the *center of gravity*, or centroid, are found by integrating element of mass $dM = \rho dV = \rho dxdydz$ over the body.

$$M = \int_V \rho\, dV, \qquad V = \int_V dV = \int_V dxdydx \tag{1.12a}$$

$$x_c = \frac{1}{M}\int_V \rho x\, dV, \quad y_c = \frac{1}{M}\int_V \rho y\, dV, \quad z_c = \frac{1}{M}\int_V \rho z\, dV \tag{1.12b}$$

The *mass moments of inertia* J_{xx}, J_{yy}, J_{zz} and the *mass products of inertia* about the x-, y-, and z-axes are

$$J_x = \int \rho(y^2 + z^2)dV \quad J_y = \int_V \rho(x^2 + z^2)dV \quad J_z = \int_V \rho\,(x^2 + y^2)dV \tag{1.13a}$$

$$J_{xy} = \int_V \rho\, xy\, dV \quad J_{xz} = \int_V \rho\, xz\, dV \quad J_{yz} = \int_V \rho\, yz\, dV \tag{1.13b}$$

The reader is cautioned that some authors (Refs [12, 13] and see Eq. 2.36) define the mass products of inertia (J_{xy}, J_{xz}, J_{yz}) as the negative of these expressions. The density ρ has units of mass per unit volume. If ρ is constant, as is the case in Tables 1.5, 1.6, and 1.7, the center of mass coincides with the centroid of the volume, density can be taken outside the integrals, the centroid is center of mass and mass is density times volume, $M = \rho V$.

Radius of gyration is the square root of the moment of inertia divided by the mass.

$$r_x = \left(\frac{J_x}{M}\right)^{1/2}, \quad r_y = \left(\frac{J_y}{M}\right)^{1/2}, \quad r_z = \left(\frac{J_z}{M}\right)^{1/2} \tag{1.14}$$

Parallel axis theorem transforms mass moments of inertia about the center of mass, J_{xc}, J_{yc}, J_{zc}, Figure 1.3b, to mass moments of inertia about the parallel axes offset by x_c, y_c, z_c.

$$J_x = J_{x_c} + M(y_c^2 + z_c^2), \quad J_y = J_{y_c} + M(x_c^2 + z_c^2), \quad J_z = J_{z_c} + M(x_c^2 + y_c^2) \tag{1.15a}$$

$$J_{xy} = J_{x_c y_c} + M x_c y_c, \qquad J_{xz} = J_{x_c z_c} + M x_c z_c, \quad J_{yz} = J_{y_c z_c} + M y_c z_c \tag{1.15b}$$

Translation of axes away from the center of mass increases the moments of inertia J_x, J_y, J_z, (Eq. 1.15a); products of inertia (Eq. 1.15b) may increase or decrease.

Rotated unit vector **r** with origin O (Fig. 1.3b) is defined relative to unit-magnitude vectors **i**, **j**, and **k** in the x-, y-, and z-directions, respectively.

$$\mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \quad a_x^2 + a_y^2 + a_z^2 = 1 \tag{1.16}$$

The coefficients a_x, a_y, a_z are cosines of the angles between the rotated vector and the base coordinates. Look ahead to Equation 1.22. Mass moments of inertia about the rotated axis are

$$J_a = a_x^2 J_x + a_y^2 J_y + a_z^2 J_z - 2a_x a_y\, J_{xy} - 2a_x a_z\, J_{xz} - 2a_y a_z J_{yz} \tag{1.17}$$

The unit magnitude vector **s**,

$$\mathbf{s} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}, \quad b_x^2 + b_y^2 + b_z^2 = 1 \tag{1.18}$$

Table 1.7 Properties of elements described by points and vectors

Notation: A = cross sectional area; C = centroid; **OC** = vector from origin to center of mass (centroid of volume); $\mathbf{J_n}$ = mass moment of inertia about axis through origin in the direction of the unit vector **n**; M = mass, ρV; **n** = unit length vector through point O in direction of reference axis; O = origin of axes; **OP** = vector from point O to point P, etc.; P, Q, R, T, etc. = nodes (points) in space with Cartesian coordinates (x_P, y_P, z_P) etc.; S = surface area; t = thickness of plate; V = volume of body; |X| = magnitude or determinant of X; ρ = mass density; • = vector dot product, Eq. 1.25; × = vector cross product, Eq. 1.24 Refs [16, 24–27]

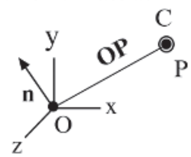
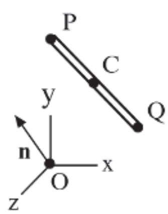
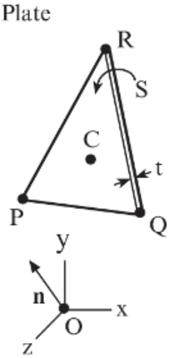
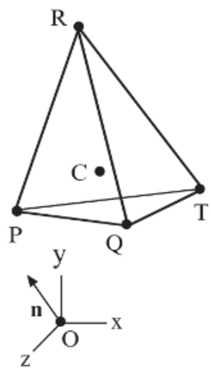
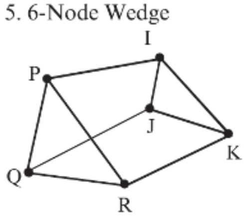
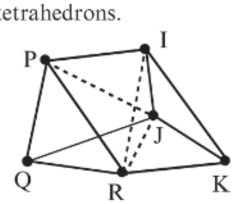
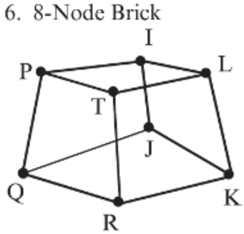
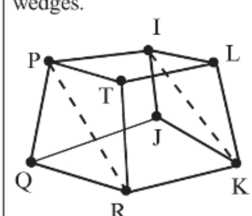
Geometry	Properties with Point Coordinates	Properties with Vectors
1. Point Mass at P  	$x_C = x_P,\ y_C = y_P,\ z_C = z_P$ Mass = M, $\mathbf{OP} = x_p\mathbf{i} + y_p\mathbf{j} + z_p\mathbf{k}$ Mass moments of inertia about O, $J_x = M(y_P^2 + z_P^2)$ $J_{xy} = Mx_Py_P$ $J_y = M(x_P^2 + z_P^2)$ $J_{xz} = Mx_Px_p$ $J_z = M(x_P^2 + y_P^2)$ $J_{yz} = My_Pz_p$	Center of mass, **OC = OP** Mass moment of inertia about an axis through origin in the direction of the unit vector **n**, $J_n = M\|\mathbf{OP}\times\mathbf{i}\|^2$
2. Straight Rod  	$M = \rho AL$, where length L = $\left((x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2\right)^{1/2}$ A = cross sectional area, $A << L^2$ $x_C = (x_P + x_Q)/2,\ \ y_C = (y_P + y_Q)/2$ $z_C = (z_P + z_Q)/2$ $J_x = \frac{M}{3}(y_P^2 + y_Py_Q + y_Q^2 + z_P^2 + z_Pz_Q + z_Q^2)$ $J_{xy} = \frac{M}{6}(2x_Py_P + 2x_Qy_Q + x_Py_Q + x_Qy_P)$ J_y, J_z, J_{yz}, J_{xz} can be obtained from above by substituting x, y, or z for x and/or y.	$\mathbf{OC} = \frac{1}{2}(\mathbf{OP} + \mathbf{OQ}),\quad M = \rho A\|\mathbf{PQ}\|$ $J_n = \frac{M}{24}\bullet$ $\left[\ \|\left((3^{1/2}-1)\mathbf{OP} + (3^{1/2}+1)\mathbf{OQ}\right)\times\mathbf{n}\|^2\right.$ $\left.+\|\left((3^{1/2}+1)\mathbf{OP} + (3^{1/2}-1)\mathbf{OQ}\right)\times\mathbf{n}\|^2\ \right]$ Note: mass and inertia of a rod can be represented by two masses, mass M/2, located $\pm\ L/\sqrt{12}$ along the rod from the center of the rod where L is the length of the rod.
3. Thin Triangular Plate  S = area of one side M = mass of plate	Mass = ρSt, where t = thickness $x_C = (x_P + x_Q + x_R)/3$ $y_C = (y_P + y_Q + y_R)/3$ $z_C = (z_P + z_Q + z_R)/3$ $S = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_Q - x_P & y_Q - y_P & z_Q - z_P \\ x_R - x_P & y_R - y_P & z_R - z_P \end{vmatrix}$ $J_x = \frac{M}{12}((y_P + y_Q)^2 + (y_P + y_R)^2 +$ $(y_R + y_Q)^2 + (z_P + z_Q)^2 +$ $(z_P + z_R)^2 + (z_R + z_Q)^2)$ $J_{xy} = \frac{M}{12}((x_P + x_Q)(y_P + y_Q) +$ $(x_P + x_R)(y_P + y_R) +$ $(x_Q + x_R)(y_Q + y_R))$	$\mathbf{OC} = \frac{1}{3}(\mathbf{OP} + \mathbf{OQ} + \mathbf{OR}),\ M = \rho tS$ $S = \frac{1}{2}\|\mathbf{PQ}\times\mathbf{PR}\|$, each side $J_\mathbf{n} = \frac{M}{12}\left(\|(\mathbf{OP} + \mathbf{OR})\times\mathbf{n}\|^2\right.$ $+\|(\mathbf{OP} + \mathbf{OQ})\times\mathbf{n}\|^2$ $\left.+\|(\mathbf{OQ} + \mathbf{OR})\times\mathbf{n}\|^2\right)$ Mass and moment of inertia of thin triangle can be represented by three masses of size M/3 located midway between vertices Ref. [24]. J_y, J_z, J_{xz} can be obtained from the formulas for J_x and J_{xy} by substituting x, y, or z for x and/or y.

Table 1.7 Properties of elements described by points and vectors, continued

Geometry	Properties with Point Coordinates	Properties with Vectors
4. Tetrahedron (4-nodes)  x_P, y_P, z_P are Cartesian coordinates of point P. **OP** is vector from point O to point P.	$M=\rho V$ $x_C=(x_P+x_Q+x_R+x_T)/4$ $y_C=(y_P+y_Q+y_R+y_T)/4$ $z_C=(z_P+z_Q+z_R+z_T)/4$ $S=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ x_Q-x_P & y_Q-y_P & z_Q-z_P\\ x_R-x_P & y_R-y_P & z_R-z_P\end{vmatrix}$ $V=\frac{1}{6}\begin{vmatrix}x_T-x_P & y_T-y_P & z_T-z_P\\ x_Q-x_P & y_Q-y_P & z_Q-z_P\\ x_R-x_P & y_R-y_P & z_R-z_P\end{vmatrix}$ $J_x=\frac{M}{10}(y_P^2+y_Q^2+y_R^2+y_T^2+y_Py_Q+y_Py_R$ $+y_Qy_R+y_Py_T+y_Qy_T+y_Ry_T+$ $z_P^2+z_Q^2+z_R^2+z_T^2+z_Pz_Q+z_Pz_R$ $+z_Qz_R+z_Pz_T+z_Qz_T+z_Rz_T)$ $J_{xy}=\frac{M}{20}(2x_Py_P+2x_Qy_Q+2x_Ry_R+$ $2x_Ty_T+x_Qy_P+x_Ry_P+x_Ty_P+x_Py_Q$ $+x_Ry_Q+x_Ty_Q+x_Py_R+x_Qy_R$ $+x_Ty_R+x_Py_T+x_Qy_T+x_Ry_T)$	$\mathbf{OC}=\frac{1}{4}(\mathbf{OP}+\mathbf{OQ}+\mathbf{OR}+\mathbf{OT})$ $S=\frac{1}{2}\lvert\mathbf{PR}\times\mathbf{PT}\rvert+\frac{1}{2}\lvert\mathbf{PR}\times\mathbf{PQ}\rvert$ $+\frac{1}{2}\lvert\mathbf{PT}\times\mathbf{PQ}\rvert+\frac{1}{2}\lvert\mathbf{QR}\times\mathbf{QT}\rvert$ $V=\frac{1}{6}\lvert(\mathbf{PQP}\times\mathbf{R})\bullet\mathbf{PT}\rvert$ $J_n=\frac{M}{10}\big(\lvert\mathbf{OP}\times\mathbf{n}\rvert^2+\lvert\mathbf{OQ}\times\mathbf{n}\rvert^2$ $+\lvert\mathbf{OR}\times\mathbf{n}\rvert^2+\lvert\mathbf{OT}\times\mathbf{n}\rvert^2$ $+(\mathbf{OP}\times\mathbf{n})\bullet(\mathbf{OT}\times\mathbf{n})$ $+(\mathbf{OQ}\times\mathbf{n})\bullet(\mathbf{OR}\times\mathbf{n})$ $+(\mathbf{OQ}\times\mathbf{n})\bullet(\mathbf{OT}\times\mathbf{n})$ $+(\mathbf{OP}\times\mathbf{n})\bullet(\mathbf{OQ}\times\mathbf{n})$ $+(\mathbf{OP}\times\mathbf{n})\bullet(\mathbf{OR}\times\mathbf{n})$ $+(\mathbf{OQ}\times\mathbf{n})\bullet(\mathbf{OR}\times\mathbf{n})$ $+(\mathbf{OP}\times\mathbf{n})\bullet(\mathbf{OT}\times\mathbf{n})\big)$ J_n is about **n** axis. J_y, J_z, J_{yz}, J_{xz} can be obtained from formulas for J_x and J_{xy} by substituting y and/or z for x and y Refs [25–27]
5. 6-Node Wedge  	The 6-node wedge is subdivided into 3 tetrahedrons, PQRJ, PRJI, and RKIJ. The properties of each tetrahedron are computed by the formulas in case 4. Their moments of inertia and volumes are summed in space to give the moments of inertia and volume of the wedge, case 6 of Table 1.6.	Wedge divided into 3 tetrahedrons. 
6. 8-Node Brick  	The 8-node brick is subdivided into two 6-node wedges, PQRIJK, PTRILK. The properties of each wedge are computed from 3 tetrahedrons, cases 4 and 5. The results are summed in space to give the moments of inertia and volume of the brick Ref. [27].	Brick divided into two wedges. 

is perpendicular to **a** so **r** • **s** = $a_x b_z + a_y b_y + a_z b_z = 0$ (see Eq. 1.25). The mass product of inertia with respect to the rotated **r–s**-axes is [19].

$$J_{rs} = -a_x b_x J_x - a_y b_y J_y - a_z b_z J_z + (a_x b_y + a_y b_x)\, J_{xy} + (a_x b_z + a_z b_x)\, J_{xz} + (a_y b_z + a_z b_y) J_{yz} \tag{1.19}$$

The general expression for the six mass moments of inertia in a rotated coordinate system (Eqs 1.19 and 1.22) is a tensor summation [19].

Principal axes of a solid body ([19], p. 558) are found by solution of the cubic equation that results from setting the determinant (|..|) of the matrix on the left-hand side of the following eigenvalue problem to zero.

$$\begin{bmatrix} (J_x - J) & J_{xy} & J_{xz} \\ J_{yx} & (J_y - J) & J_{yz} \\ J_{zx} & J_{yz} & (J_z - J) \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = 0 \tag{1.20}$$

The three principal moments of inertia are J_1, J_2, and J_3. Products of inertia are zero about principal axes. Substituting these into the equation and solving gives the associated axis vector direction cosines (Eq. 1.16) of the principal axis.

The mass moments of inertia of thin planar slices from a homogeneous body are equal to their two-dimensional area moments of inertia (Table 1.5) times density. Two-dimensional areas times density extrude into the third dimension to create mass moments of inertia, cases 16 and 17 of Table 1.6 and the following example. Also see Section 1.7.

Example 1.4 Geometric properties of a steel pipe

A steel pipe made with density $\rho = 8$ g/cc (0.289 lb/in.^{3}) extends along the x-axis from $x = 0$ to $x = 250$ mm (9.843 in.). The center of the cross section, shown in Figure 1.4, coincides with the x-axis. Compute the pipe mass and its mass moments of inertia about its center of gravity.

Solution: Case 14 of Table 1.6 has the mass and mass moments of inertia of a tube that extends out the y-axis in terms of its outer radius $R = 10$ mm (0.3934 in.), the inner diameter, $r = 7$ mm (0.276 in.), and length $h = 250$ mm (9.84 in.). Substituting the x-axis for the y-axis, these formulas give

$$M = \rho\pi(R^2 - r^2)h = 320.4 \text{ g} = 0.3204 \text{ kg} = 0.706 \text{ lb}$$

$$J_x = \frac{M}{2}(R^2 + r^2) = 23870 \text{ gm} - \text{mm}^2 = 2.387 \times 10^{-5} \text{ kg} - \text{m}^2 = 0.08152 \text{ lb} - \text{in}^2$$

$$J_y = \frac{M}{12}(3R^2 + 3r^2 + h^2) = 1.681 \times 10^6 \text{ g} - \text{mm}^2 = 1.681 \times 10^{-3} \text{ kg} - \text{m}^2 = 5.744 \text{ lb} - \text{in}^2$$

The pipe extends out the x-axis from $x = y = 0$; y is perpendicular to x. This solution is in case 16 of Table 1.6.

1.7 Geometric Properties Defined by Vectors

Table 1.7 has the geometric properties of straight-sided polygonal planar areas and homogeneous volumes in terms of vectors to *nodes* at their vertices rather than relative dimensions. These formulas are based on Refs [16, 24–27].

A one-node object is a point mass. A two-node object is a rod. A three-node object is a triangle. A four-node solid is a tetrahedron, which is a pyramid-like solid with four vertices (nodes) and four triangular sides. A wedge is a six-node solid. A brick is an eight-node solid. The brick can be subdivided into two wedges each of which is further subdivided into three tetrahedrons (cases 4–6 of Table 1.7 [27]).

Vectors have magnitude and direction. Boldface (**r**) denotes vector. **i**, **j**, and **k** are the unit vectors in the x-, y-, and z-directions, respectively. The vector **R** can be defined by its starting coordinates (x_1, y_1, z_1) and ending coordinates (x_2, y_3, z_2), by its Cartesian components (r_x, r_y, r_z), or by its magnitude $(d^2 = r_x{}^2 + r_y{}^2 + r_z{}^2)$ times a unit vector **r**.

$$\mathbf{R} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k} \tag{1.21}$$

$$\begin{aligned}\mathbf{r} = \frac{\mathbf{R}}{d} &= \left(\frac{r_x}{d}\right)\mathbf{i} + \left(\frac{r_y}{d}\right)j + \left(\frac{r_z}{d}\right)\mathbf{k} \\ &= \frac{(x_2 - x_1)}{d}\mathbf{i} + \frac{(y_2 - y_1)}{d}\mathbf{j} + \frac{(z_2 - z_1)}{d}\mathbf{k} \\ &= \cos\alpha\ \mathbf{i} + \cos\beta\ \mathbf{j} + \cos\gamma\ \mathbf{k}\end{aligned} \tag{1.22}$$

R and **r** make angles α, β, and γ with respect to the positive x-, y-, and z-axes, respectively. *Direction cosines* are the cosines of these angles and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

The vectors **R** and **S** are defined by their Cartesian components.

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}, \quad \mathbf{S} = S_x\mathbf{i} + S_y\mathbf{j} + S_z\mathbf{k} \tag{1.23}$$

Cross product of two vectors is a vector [25].

$$\mathbf{R} \times \mathbf{S} = (R_yS_z - R_zS_y)\mathbf{i} + (R_zS_x - R_xS_z)\mathbf{j} + (R_xS_y - R_yS_x)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R_x & R_y & R_z \\ S_x & S_y & S_z \end{vmatrix} \tag{1.24}$$

Here "$|\cdot|$" means determinant. The magnitude of the vector cross product is $|\mathbf{r}||\mathbf{s}|\ \sin\ \theta$, where θ is the acute angle between vectors; it is the area of the parallelogram defined by bringing the bases of the vectors together. The cross product direction is perpendicular to the plane of the two vectors using the right-hand rule: vectors **r**, **s**, and $\mathbf{r} \times \mathbf{s}$ are a right-handed orthogonal triad $\mathbf{r} \times \mathbf{s} = -\mathbf{s} \times \mathbf{r}$. If $\mathbf{r} \times \mathbf{s} = 0$ then **r** and **s** are parallel.

Scalar product, or *dot product*, of two vectors is the scalar sum of the product of their components [12].

$$\mathbf{R} \bullet \mathbf{S} = R_xS_x + R_yS_y + R_zS_z \tag{1.25}$$

The magnitude of the dot product is $|\mathbf{r}| \cdot |\mathbf{s}| \cos\theta$, where θ is the acute angle between the two vectors; if $\mathbf{R} \bullet \mathbf{S} = 0$ then the vectors are perpendicular. The dot product is communicative, $\mathbf{R} \bullet \mathbf{S} = \mathbf{S} \bullet \mathbf{R}$. The magnitude of **R** is $(\mathbf{R} \bullet \mathbf{R})^{1/2}$.

The angle between two vectors is $\theta = \cos^{-1} (\mathbf{R} \bullet \mathbf{S})/(|\mathbf{R}||\mathbf{S}|)$.

Straight line is defined by the Cartesian coordinates (x,y,z) of a point x_0, y_0, z_0 on the line and the vector $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ in the direction of the line. Equations of the line are ([16], p. 243).

$$\begin{array}{ll} \text{canonical} & (x - x_0)/a_x = (y - y_0)/a_y = (z - z_0)/a_x \\ \text{parametric} & x - x_0 = a_x t,\ y - y_0 = a_y t,\ z - z_0 = a_z t,\ \ -\infty < t < \infty \\ \text{vector} & \mathbf{r} - \mathbf{r}_0 = \mathbf{a}t,\ \textit{with}\ \mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} \quad -\infty < t < \infty \end{array} \tag{1.26}$$

The minimum distance from a straight line to a point off the line is $|\mathbf{u}\times\mathbf{a}|/|\mathbf{a}|$ ([16], p. 245), where $\mathbf{a}$ is the line vector with direction cosines a_x, a_y, a_y, and $\mathbf{u}$ is a vector from a point on the line to the point off the line.

Equation of a Plane. If nonparallel vectors **a**, **b**, and **c** lie in a plane then the vector equation of the plane is $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$. If vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, are to points on the plane then $\mathbf{n} = (\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{x}_2 - \mathbf{x}_3)$ is a vector perpendicular to the plane. If $\mathbf{r}$ is a vector on the plane then equation of the plane is $\mathbf{r} \bullet \mathbf{n} = D$, which is equivalent to the scalar equation $ax + by + cz = d$. The minimum distance from a plane to an off-plane point is $D = \mathbf{n} \bullet (\mathbf{x}_o - \mathbf{x})/|\mathbf{n}|$ where $\mathbf{x}$ is a vector to a point on the plane and $\mathbf{x}_\text{o}$ is the vector to the off-plane point [16].

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