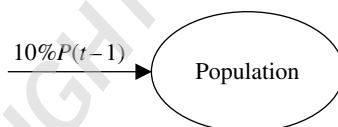

1

DENSITY INDEPENDENT POPULATION MODELS

1.1 EXPONENTIAL GROWTH

1 Consider the flow diagram in Text Figure 1.21.



TEXT FIGURE 1.21 Flow diagram for Exercise 1.1.1.

a. Find the corresponding DDS.

The flow diagram tells us how the population changes from one year to the next. Inward pointing arrows represent additions while outward pointing arrows represent subtractions. Here there is only one arrow, and it represents an addition. Thus the DDS is given by

$$P(t) = P(t-1) + 10\%P(t-1).$$

We can also write the DDS as $P(t) = P(t-1) + 0.10P(t-1)$, or $P(t) = 1.10P(t-1)$.

Solutions Manual to Accompany Models for Life: An Introduction to Discrete Mathematical Modeling with Microsoft® Office Excel®, First Edition. Jeffrey T. Barton.

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Companion website: www.wiley.com/go/barton/solutionsmanual_modelsforlife

- b. Use a calculator to predict the population after 2 years if $P(0) = 50$.

If $P(0) = 50$, then by using the DDS we can predict the population 1 year later:

$$\begin{aligned} P(1) &= P(0) + 10\%P(0) \\ &= 50 + 0.10 \cdot 50 \\ &= 55. \end{aligned}$$

Applying the DDS once more gives us the model prediction for year 2:

$$\begin{aligned} P(2) &= P(1) + 10\%P(1) \\ &= 55 + 0.10 \cdot 55 \\ &= 60.5. \end{aligned}$$

After 2 years we predict the population will be 60.5.

- c. Use Excel to project the population in year 10.

Since the model in this problem is the exponential growth model, we can save time by using the same spreadsheet we created for the Yellowstone grizzly population. We only need to change the growth rate to $r = 10\%$ and the initial population to $P(0) = 50$. Figure 1.1 shows the result with the projection for year 10 highlighted. The model predicts a population of about 129.7 in year 10.

	A	B	C
1	Exercise 1.1.1		
2			
3	Growth rate, $r =$		10%
4			
5	t	Population	
6	0	50	
7	1	55.0	
8	2	60.5	
9	3	66.6	
10	4	73.2	
11	5	80.5	
12	6	88.6	
13	7	97.4	
14	8	107.2	
15	9	117.9	
16	10	129.7	

FIGURE 1.1 Excel output for Exercise 1.1.1.

- 3 Consider the flow diagram in Text Figure 1.23.



TEXT FIGURE 1.23 Flow diagram for Exercise 1.1.3.

- a. Find the corresponding DDS.

The flow diagram tells us how the population changes from one year to the next. Inward pointing arrows represent additions while outward pointing arrows represent subtractions. Here we have two arrows: one an addition and one a subtraction. Thus the DDS is given by

$$\begin{aligned} P(t) &= P(t-1) + 8\%P(t-1) - 5\%P(t-1) \\ &= P(t-1) + 3\%P(t-1). \end{aligned}$$

We can also write the DDS as $P(t) = P(t-1) + 0.03P(t-1)$, or $P(t) = 1.03P(t-1)$.

- b. Use a calculator to predict the population after 2 years if $P(0) = 100$.

If $P(0) = 100$, then by using the DDS we can predict the population 1 year later:

$$\begin{aligned} P(1) &= P(0) + 3\%P(0) \\ &= 100 + 0.03 \cdot 100 \\ &= 103. \end{aligned}$$

Applying the DDS once more gives us the model prediction for year 2:

$$\begin{aligned} P(2) &= P(1) + 3\%P(1) \\ &= 103 + 0.03 \cdot 103 \\ &= 106.09. \end{aligned}$$

After 2 years we predict the population will be about 106.1.

- c. Use Excel to project the population in year 10.

We see from the DDS that this model is still an exponential growth model with $r = 3\%$. Thus we can use the Yellowstone grizzly spreadsheet with the new growth rate and the initial population set to 100. The result is given in Figure 1.2 with the projection for year 10 highlighted. The model predicts a population of about 134.4 in year 10.

	A	B	C
1	Exercise 1.1.3		
2			
3	Growth rate, $r=$		3%
4			
5	t	Population	
6	0	100	
7	1	103.0	
8	2	106.1	
9	3	109.3	
10	4	112.6	
11	5	115.9	
12	6	119.4	
13	7	123.0	
14	8	126.7	
15	9	130.5	
16	10	134.4	

FIGURE 1.2 Excel output for Exercise 1.1.3.

- 5 Draw a flow diagram that corresponds to the following DDS:

$$P(t) = P(t-1) + 4\%P(t-1).$$

The addition of 4% of the previous year's population is represented by an inward pointing arrow in the flow diagram, given in Figure 1.3.

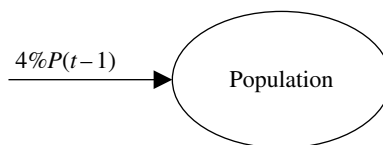


FIGURE 1.3 Flow diagram for Exercise 1.1.5.

- 7 Draw a flow diagram that corresponds to the following DDS:

$$P(t) = P(t-1) - 0.30P(t-1).$$

The DDS indicates a subtraction of 30% of the previous year's population. We account for this subtraction with an outward pointing arrow in the flow diagram, given in Figure 1.4. Note that there is no minus sign in front of the arrow label.

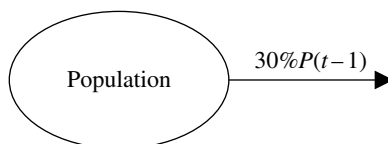


FIGURE 1.4 Flow diagram for Exercise 1.1.7.

- 9 Give the flow diagram and corresponding DDS for a grizzly population that is growing by 8% per year and has 5 bears illegally poached annually.

We represent the 8% growth by an inward pointing arrow and the poaching by an outward pointing arrow. The result is Figure 1.5. Note that there is no minus sign in front of the 5.

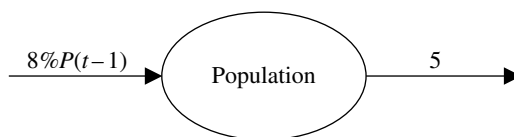


FIGURE 1.5 Flow diagram for Exercise 1.1.9.

The corresponding DDS is given by $P(t) = P(t-1) + 0.08 \cdot P(t-1) - 5$.

- 11 Suppose you know that the DDS for a population is given by

$$P(t) = P(t-1) + 3\%P(t-1) - 50.$$

- a. Draw a flow diagram that would lead to this DDS.

The 3% increase is represented by an inward pointing arrow while the removal of 50 from the population is represented by an outward pointing arrow. The result is given in Figure 1.6.



FIGURE 1.6 Flow diagram for Exercise 1.1.11.

- b. Explain in a complete sentence how the population is changing from year to year.

The population is experiencing growth of 3% of the previous year's population while at the same time 50 members of the population are leaving each year.

- 13** Suppose that the *1993 Grizzly Bear Recovery Plan* had never been implemented and that the 1993 estimate of a 1% growth rate continued to hold. How long would it have taken for the population to reach 416 bears?

We use the Yellowstone grizzly population Excel model with $r=1\%$ and $P(0)=197$. We are looking for the year that the population reaches 416 bears, so we drag the model formulas down until we see the population meet or exceed 416 for the first time. This happens 76 years from the initial population estimate, and the population of bears is projected to be about 419.7 at that time.

- 15** Suppose that the numbers of adult females with cubs sighted in Yellowstone were 52 in 2003, 60 in 2004, and 65 in 2005. Estimate the total grizzly population in 2005.

The 3-year total of adult female grizzlies is $52 + 60 + 65 = 177$. No known deaths are mentioned, so we assume 0 known deaths. Thus we have 177 adult females, representing about 27.4% of the total population of bears. This total is given by $\frac{177}{0.274} \approx 645.99$, or about 646 bears.

- 17** Text Table 1.2 contains more population data for the wild California condor population from the 1996 *Recovery Plan for the California Condor* (U.S. Fish and Wildlife Service, 1996).

TEXT TABLE 1.2 The Number of California Condors Remaining in the Wild between 1982 and 1985 (U.S. Fish and Wildlife Service, 1996)

Year	Number Wild California Condors
1982	21
1983	19
1984	15
1985	9

- a.** Compare the population values in the table to what our model would predict using the rate of decline found in Example 1.5 and an initial population of 50 condors. In general, how well did our model do?

Here we use the California condor Excel spreadsheet that we already created, where $P(0)=50$ in 1968, and the rate of decline from Example 1.5 is $r=6.6\%$. Next we drag the model formulas down until we reach the year 1985, or $t=17$. The projected values for years 1982-5 for our model are 19, 18, 17, and 16. We compare the model projections to the data in Text Table 1.2, which recorded condor populations of 21, 19, 15, and 9 for the years 1982-5. Our model seems to have done reasonably well, though from the data it appears as though something happened in 1985 that caused a larger than predicted decline in the population.

- b. Can you think of possible reasons for any discrepancies?

As noted above, the most striking difference between our model projections and the actual population data seems to be for the year 1985. There could be any number of reasons for the larger than predicted decline in 1985, including accidents, poaching, or disease.

- 19** Recall that our estimate for the California condor's rate of decline was based on the lower population estimates given by Sibley, Mailed, and Wilbur. Re-estimate the rate of decline from 1968 to 1978 using three other combinations from the population estimates:

- a. The lower value from 1960's and the higher value from 1978.

The range of values for the California condor population was given as 50–60 in the late 1960's and 25–30 in 1978. Taking the lower value from the 1960's (with the assumption of 1968 for our starting year), we use $P(0) = 50$. Using the higher estimate, 30 condors, in 1978 gives us $P(10) = 30$. Thus we repeat the trial-and-error approach from Example 1.5 in order to estimate the rate of decline from 1968 to 1978. We use the already created California condor Excel model and type in different values for r until we get 30 condors in 1978. The result is shown in Figure 1.7 with the value for r highlighted. Our new estimate for the rate of decline is about 5% per year. Note that it makes sense for the rate of decline to be lower than in Example 1.5 because the assumed population in 1978 is higher – there was less of an assumed decline.

- b. The higher value from 1960's and the lower value from 1978.

	A	B	C
1	Exercise 1.19		
2			
3	Rate of decline, $r =$		5.0%
4			
5	t	Population	
6	0	50	
7	1	48	
8	2	45	
9	3	43	
10	4	41	
11	5	39	
12	6	37	
13	7	35	
14	8	33	
15	9	32	
16	10	30	

FIGURE 1.7 Excel output for Exercise 1.1.19.

Here we need to use $P(0) = 60$ and $P(10) = 25$. Repeating the trial-and-error exercise from part a. gives us the estimate $r = 8.4\%$. Note that it makes sense for the rate of decline to be higher than in Example 1.5 because the assumed population in 1968 is higher – there is more of an assumed decline to 1978.

- c. The higher value from 1960's and the higher value from 1978.

Here we need to use $P(0) = 60$ and $P(10) = 30$. Repeating the trial-and-error exercise from part a. gives us the estimate $r = 6.6\%$. Note that it makes sense for the rate of decline to be the same as in Example 1.5 because the assumed population declines by 50% from 1968 to 1978, just as it did in Example 1.5 when the values used were 50 in 1968 and 25 in 1978.

- d. How much difference do you see in r ?

The range of values for r is $5\% - 8.4\%$, depending on the values used from the given data estimates. The value we used in the text, $r = 6.6\%$, falls squarely in that range and thus seems a reasonable choice.

1.2 EXPONENTIAL GROWTH WITH STOCKING OR HARVESTING

- 1 Consider the flow diagram in Text Figure 1.33.

- a. Find the corresponding DDS.



TEXT FIGURE 1.33 Flow diagram for Exercise 1.2.1.

Recalling that inward pointing arrows represent additions and outward pointing arrows subtractions, the DDS is given by $P(t) = P(t-1) + 0.08P(t-1) - 50$.

- b. Use a calculator to predict the population after 2 years if $P(0) = 650$.

According to the DDS we find the population after 1 year by calculating

$$\begin{aligned}
 P(1) &= P(0) + 0.08P(0) - 50 \\
 &= 650 + 0.08 \cdot 650 - 50 \\
 &= 652.
 \end{aligned}$$

Applying the DDS once more give us our projection for year 2:

$$\begin{aligned}
 P(2) &= P(1) + 0.08P(1) - 50 \\
 &= 652 + 0.08 \cdot 652 - 50 \\
 &= 654.16.
 \end{aligned}$$

Thus after 2 years our model predicts a population of about 654.

- c. Use Excel to project the population after 15 years.

The model in this problem is in the form of an exponential growth model with harvesting. Thus we can use the white-tailed deer Excel model, being careful to enter the appropriate parameter values. Once we have entered the correct parameter values we drag the model formulas down to year 15 and record the result. The result is a population of about 704 as shown in Figure 1.8.

	A	B	C
1	Exercise 1.2.1		
2			
3	Growth rate, $r =$		8.0%
4	Harvest num., $a =$		50
5			
6	t	Population	
7	0	650	
8	1	652.0	
9	2	654.2	
10	3	656.5	
11	4	659.0	
12	5	661.7	
13	6	664.7	
14	7	667.8	
15	8	671.3	
16	9	675.0	
17	10	679.0	
18	11	683.3	
19	12	688.0	
20	13	693.0	
21	14	698.4	
22	15	704.3	

FIGURE 1.8 Excel output for Exercise 1.2.1.

- 3 Draw a flow diagram that corresponds to the following DDS:

$$P(t) = P(t-1) + 4\%P(t-1) - 40.$$

The addition of 4% of the previous year's population is represented by an inward pointing arrow while the subtraction of 40 is represented by an outward pointing arrow. The result is given in Figure 1.9. Note that there is no minus sign in front of the "40."

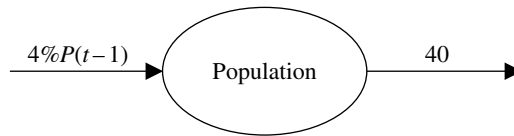


FIGURE 1.9 Flow diagram for Exercise 1.2.3.

- 5** *Extension:* In Example 1.8 we determined that an average of 12.23 Mississippi sandhill cranes were captive-reared and released annually between 1981 and 1993. Text Table 1.3 gives the actual numbers, taken from Table 2 in Valentine and Lohoefer (1991), for the years 1981–1990.

TEXT TABLE 1.3 The Number of Mississippi Sandhill Cranes Captive-Reared and Released between 1981 and 1990 (Valentine & Lohoefer, 1991)

Year of Release	Number Captive-released
1981	9
1982	4
1983	8
1984	4
1985	10
1986	7
1987	2
1988	10
1989	13
1990	29

- a.** How does 12.23 compare to the average number of cranes that were actually released?

Here we find the average of the values given in Text Table 1.3:

$$\frac{(9 + 4 + 8 + 4 + 10 + 7 + 2 + 10 + 13 + 29)}{10} = 9.6.$$

Thus the actual average number of cranes released from 1981–90 was 9.6 – about 2.6 less than the average number we estimated for 1981–93.

- b.** What factor(s) might account for the difference?

One possibility is that the table of data only extends from 1981 to 1990, so there could have been large cohorts of cranes released in 1991, 1992, and 1993. It is also possible that the assumed rate of decline used in Example 1.8 (6%) turned out to be higher than what the population actually experienced, thus causing us to overestimate the number of cranes released.

- c. Using a rate of decline of 6%, and the actual release values from 1981 to 1990, estimate how many cranes there were in 1990. (Note: this exercise will require a significant modification of the crane Excel spreadsheet.)

The difference in this problem versus what we have done previously is that there is no longer a single stocking number that we can refer to for all years. Instead, we have a different stocking number each year that must be added. We can arrange this by including a separate column in our Excel model where we store the release data, and then referring to that column for the stocking number each year. The set-up is given in Figure 1.10. Note how the population formula now refers to column C for the stocking number each year. Figure 1.11 shows the numerical result. The model predicts about 107 cranes in 1990.

	A	B	C
1	Exercise 1.2.5		
2			
3	Rate of decline, $r =$		0.06
4	Stocking num., $a =$		12.23
5			
6	t	Population	Release Data
7	0	50	
8	$=A7+1$	$=B7-\$C\$3*B7+C8$	9
9	$=A8+1$	$=B8-\$C\$3*B8+C9$	4
10	$=A9+1$	$=B9-\$C\$3*B9+C10$	8
11	$=A10+1$	$=B10-\$C\$3*B10+C11$	4
12	$=A11+1$	$=B11-\$C\$3*B11+C12$	10
13	$=A12+1$	$=B12-\$C\$3*B12+C13$	7
14	$=A13+1$	$=B13-\$C\$3*B13+C14$	2
15	$=A14+1$	$=B14-\$C\$3*B14+C15$	10
16	$=A15+1$	$=B15-\$C\$3*B15+C16$	13
17	$=A16+1$	$=B16-\$C\$3*B16+C17$	29

FIGURE 1.10 Excel set-up for Exercise 1.2.5.

- d. Determine, on average, how many cranes must have been released in 1991, 1992, and 1993 in order to end up with 135 cranes in 1993.

After the year 1990 we no longer have release data, so from that year on we refer to the stocking parameter that we kept in cell C4. The formula version of the spreadsheet is given in Figure 1.12. Our task is to experiment with different stocking numbers until we end up with 135 cranes in 1993. The result is a stocking number of about 16.3 cranes per year.

	A	B	C
1	Exercise 1.2.5		
2			
3	Rate of decline, $r =$		6.0%
4	Stocking num., $a =$		12.23
5			
6	t	Population	Release Data
7	0	50	
8	1	56.0	9
9	2	56.6	4
10	3	61.2	8
11	4	61.6	4
12	5	67.9	10
13	6	70.8	7
14	7	68.6	2
15	8	74.4	10
16	9	83.0	13
17	10	107.0	29

FIGURE 1.11 Excel output for Exercise 1.2.5.

	A	B	C
1	Exercise 1.2.5		
2			
3	Rate of decline, $r =$		0.06
4	Stocking num., $a =$		12.23
5			
6	t	Population	Release Data
7	0	50	
8	=A7+1	=B7-\$C\$3*B7+C8	9
9	=A8+1	=B8-\$C\$3*B8+C9	4
10	=A9+1	=B9-\$C\$3*B9+C10	8
11	=A10+1	=B10-\$C\$3*B10+C11	4
12	=A11+1	=B11-\$C\$3*B11+C12	10
13	=A12+1	=B12-\$C\$3*B12+C13	7
14	=A13+1	=B13-\$C\$3*B13+C14	2
15	=A14+1	=B14-\$C\$3*B14+C15	10
16	=A15+1	=B15-\$C\$3*B15+C16	13
17	=A16+1	=B16-\$C\$3*B16+C17	29
18	=A17+1	=B17-\$C\$3*B17+\$C\$4	
19	=A18+1	=B18-\$C\$3*B18+\$C\$4	
20	=A19+1	=B19-\$C\$3*B19+\$C\$4	

FIGURE 1.12 Excel set-up for Exercise 1.2.5d.

- 7 When deciding on a real harvesting strategy, it is not just the total number harvested that is important. Rather, the sex ratio of the harvest number is also very important. In fact, Palmer and Storm write, “During the past decade, deer populations in the Northeast have continued to increase except in states that harvested markedly more antlerless than antlered deer. (Palmer & Storm, 1995)” In a sentence or two, discuss why the sex ratio of deer harvests should play such an important role in controlling deer populations.

Imagine two populations of deer: each has 1,000,000 male deer and 1,000,000 female deer for a total of 2,000,000 in each population. Suppose that from Population 900,000 male deer are harvested, while from Population 900,000 female deer are harvested. We would expect that even though the same number of deer were harvested from each, Population 1 will experience much more growth in the following year than Population 2 because all of the female deer still have the potential to produce offspring.

1.3 TWO FUNDAMENTAL EXCEL TECHNIQUES

- 1 Consider the DDS below:

$$P(t) = P(t-1) + 0.10P(t-1) - 50.$$

- a. Graph the population over a period of 10 years if $P(0) = 550$.

This is a straightforward graphing exercise where we have a model representing an exponentially growing population that is undergoing harvesting. We copy the formula down to year 10, select both the time and population columns including the column headings, and insert an X,Y-Scatter graph. The result is shown in Figure 1.13.

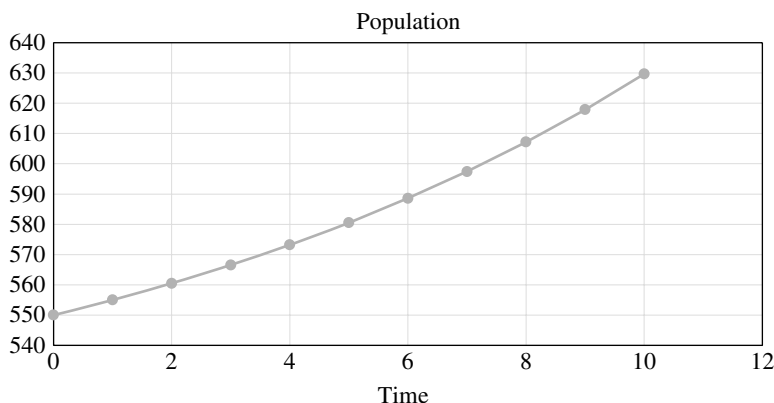


FIGURE 1.13 Excel graph for Exercise 1.3.1a.

- b. Graph the population over a period of 10 years if $P(0) = 450$.

Here we need only change the initial population from 550 to 450, and Excel will automatically update our graph to give the one shown in Figure 1.14.

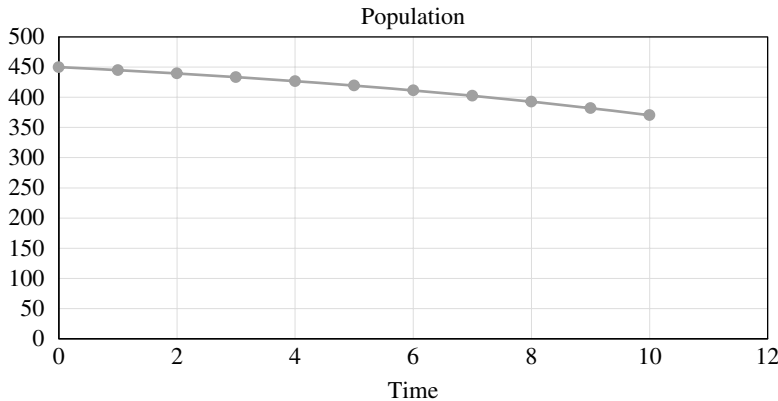


FIGURE 1.14 Excel graph for Exercise 1.3.1b.

- c. Describe the difference in the behavior of the population in the two cases.

In the first case we see an increasing population while in the second case the population is decreasing. With an initial population of 450 the 10% growth rate is not enough to overcome the effects of a harvesting level of 50.

- 3 Describe the difference in population behavior between Exercises 1.3.1 and 1.3.2.

In Exercise 1.3.1, the first population increases without bound while the second population crashes to extinction. Neither of these behaviors is present in Exercise 1.3.2. In Exercise 1.3.2, the decreasing population decreases less and less over time, while the increasing population increases less and less over time. Both of the populations in Exercise 1.3.2 will eventually level off, or stabilize, at a positive population value, and we can find that value by dragging the model formula down far enough that the population is no longer changing.

- 5 Sketch a graph by hand for each of the following situations.

- a. A population where $P(0) = 100$ and the population is increasing over time at an increasing rate.

An example sketch is provided in Figure 1.15.

- b. A population where $P(0) = 200$ and the population is increasing over time at a decreasing rate.

An example sketch is provided in Figure 1.16.

- c. A population where $P(0) = 50$ and the population is decreasing over time at a decreasing rate.

An example sketch is provided in Figure 1.17.

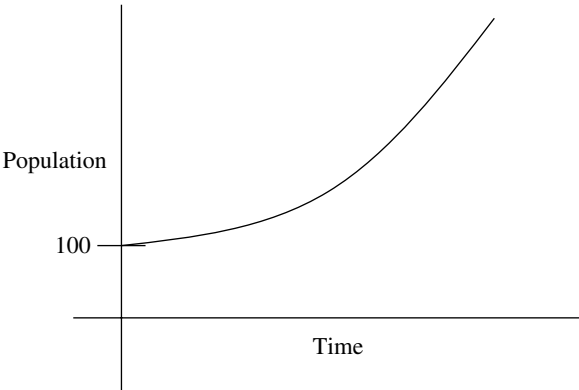


FIGURE 1.15 Example graph for Exercise 1.3.5a.

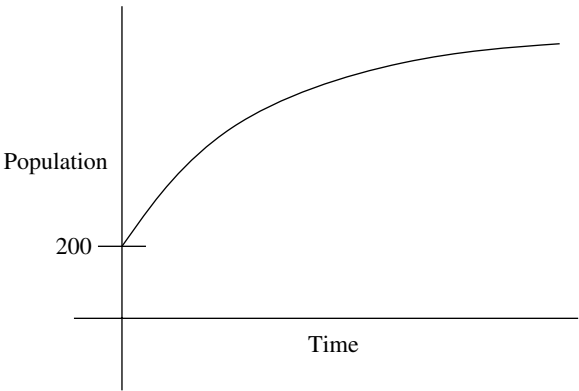


FIGURE 1.16 Example graph for Exercise 1.3.5b.

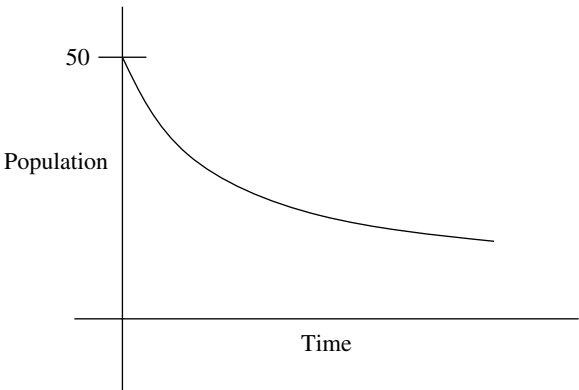


FIGURE 1.17 Example graph for Exercise 1.3.5c.

- d. A population where $P(0) = 100$ and the population is decreasing over time at an increasing rate.

An example sketch is provided in Figure 1.18.

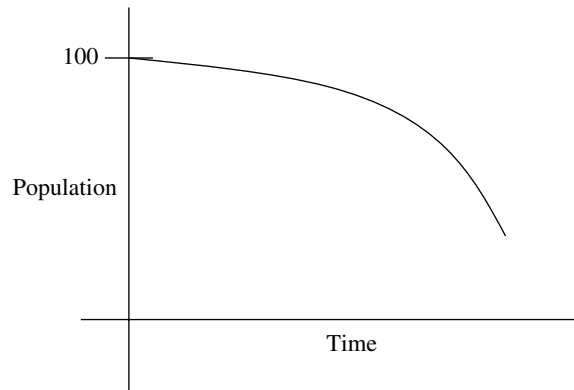


FIGURE 1.18 Example graph for Exercise 1.3.5d.

- 7 Consider the DDS given by $P(t) = P(t-1) + rP(t-1) - 50$. If the initial population is 400, use Goal Seek to determine the value for r that results in a population of 800 10 years later.

Here we use the Excel model for a population that is growing exponentially and undergoing harvesting. Since the growth rate is unknown, we use a stand-in value of $r = 5\%$ initially. We also set the harvesting number to 50 and the initial population to 400. We need to copy the formula down to year 10 and then use Goal Seek to find the growth rate that produces a population of 800 in year 10. The set-up just before Goal Seek is run is shown in Figure 1.19. The result is a growth rate of about 17%.

- 9 Regarding white-tailed deer, recall the Curtis and Sullivan (Curtis & Sullivan, 2001) estimate that deer populations can double every 2–3 years. In Examples 1.9 and 1.10 we based our Excel work on a doubling time of 3 years.

- a. Use Goal Seek to rework Examples 1.9 and 1.10 in the text, this time assuming a doubling time of 2 years.

Reworking Example 1.9 means we need to find the growth rate, r , that would result in a doubling of the deer population in 2 years. We can start with any initial deer population, and we use our deer Excel model to find the correct r (making sure that the harvest number is set to 0 before proceeding). The set-up just before Goal Seek is run is given in Figure 1.20. The result is an annual growth rate of approximately 41.4%.

Reworking Example 1.10 can also be with Goal Seek. The initial number of deer is 3,000,000, and we use Goal Seek to find the new harvesting number that keeps

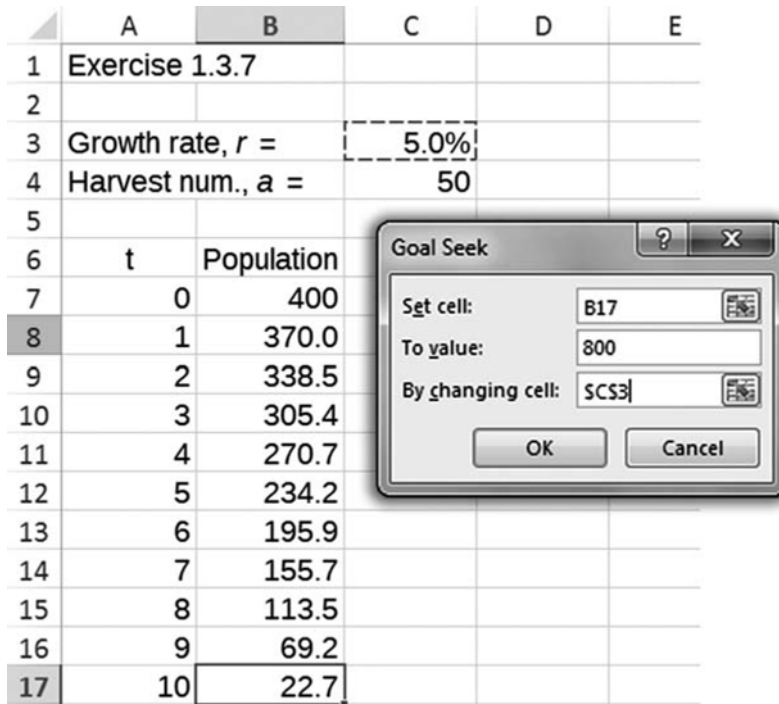


FIGURE 1.19 Excel set-up for Exercise 1.3.7.

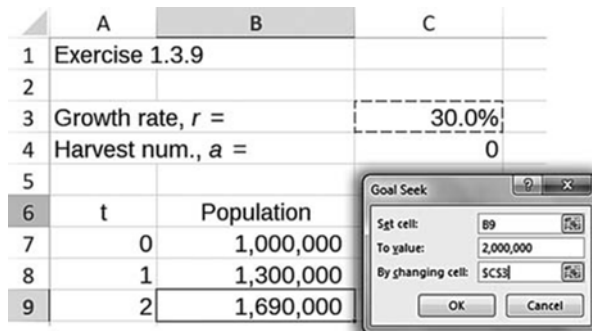


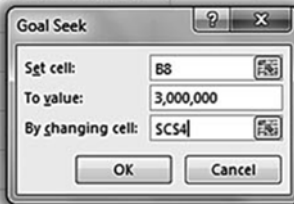
FIGURE 1.20 Excel set-up for Example 1.9 in Exercise 1.3.9.

the population from growing. The set-up just before Goal Seek is run is given in Figure 1.21. The resulting harvesting number is about 1,242,641 deer per year.

- b. How does the new harvesting number compare to the previous estimate of 780,000?

The harvesting number turns out to be higher than the original 780,000.

	A	B	C	D
1	Exercise 1.3.9			
2				
3	Growth rate, $r =$		41.4%	
4	Harvest num., $a =$		0	
5				
6	t	Population		
7	0	3,000,000		
8	1	4,242,641		
9	2	6,000,000		



The Goal Seek dialog box is open, showing the following settings:

- Set cell: B8
- To value: 3,000,000
- By changing cell: C54
- Buttons: OK, Cancel

FIGURE 1.21 Excel set-up for Example 1.10 in Exercise 1.3.9.

- c. Explain why the answer for b. makes sense in the context of the problem.

Because we assume a shorter doubling time and get an increased growth rate as a result, we should expect to have to harvest more deer to keep the population from growing.

1.4 EXPLICIT FORMULAS

For all of the exercises below, use the appropriate explicit formula to find the solution.

- 1 Consider the DDS given by $P(t) = P(t-1) + 0.10P(t-1)$. Determine the population in year 5 if the initial population is 400.

We use the explicit formula for exponential growth and plug in the relevant parameters. With $t = 5$, $P(0) = 400$, and $r = 0.10$, we have

$$\begin{aligned}
 P(t) &= (1 + r)^t P(0) \\
 P(5) &= (1 + .10)^5 400 \\
 &= 644.204.
 \end{aligned}$$

The population in year 5 is projected to be about 644.

- 3 Consider the DDS given by $P(t) = P(t-1) + 0.10P(t-1) - 5$. Determine the population in year 5 if the initial population is 400.

We use the explicit formula for exponential growth with harvesting and plug in the relevant parameters. With $t = 5$, $P(0) = 400$, $r = 0.10$, and $a = -5$, we have

$$\begin{aligned}
 P(t) &= (1+r)^t P(0) + a \frac{(1+r)^t - 1}{r} \\
 P(5) &= (1+.10)^5 400 - 5 \frac{(1+.10)^5 - 1}{.10} \\
 &= 644.204 - 30.5255 \\
 &\approx 613.68.
 \end{aligned}$$

The population in year 5 is projected to be about 614.

- 5** Consider the DDS given by $P(t) = P(t-1) + 0.10P(t-1) - a$. If the initial population is 500, determine the value for a that results in a population of 600 12 years later.

Here we use the explicit formula for exponential growth with harvesting, and we have to solve for the unknown a . Plugging in all known parameters gives

$$\begin{aligned}
 P(t) &= (1+r)^t P(0) + a \frac{(1+r)^t - 1}{r} \\
 P(12) &= (1+.10)^{12} 500 - a \frac{(1+.10)^{12} - 1}{.10}.
 \end{aligned}$$

We have introduced a minus sign in front of the a in order to maintain consistency with the problem statement. Next we note that if the population in year 12 is to be 600 then we know $P(12) = 600$. Thus we have

$$\begin{aligned}
 600 &= (1+.10)^{12} 500 - a \frac{(1+.10)^{12} - 1}{.10} \\
 600 &\approx 1569.21 - a \cdot 21.38 \\
 21.38a &\approx 969.21 \\
 a &\approx \frac{969.21}{21.38} \approx 45.33.
 \end{aligned}$$

The harvesting number would need to be about 45.33 per year in order for the population to be 600 in year 12.

- 7** Given the initial population estimate of 197 Yellowstone grizzlies in 1993 and the later estimate of 416 Yellowstone grizzlies in 2002, we found that the population grew by about 8.65% per year.

- a.** Using the 8.65% growth rate, what would your model predict for the population in the year 2193?

Here we use the explicit formula for exponential growth with $r = 0.0865$, $P(0) = 197$, and $t = 200$:

$$P(t) = (1 + r)^t P(0)$$

$$P(200) = (1.0865)^{200} 197$$

$$P(200) = 3,165,281,631.27.$$

The population in the year 2193 is predicted to be about 3.2 billion bears.

- b.** Does your answer in part a. seem reasonable? Why or why not?

No, 3.2 billion is unreasonably large for the Yellowstone grizzly population.

- c.** Suppose that the 1993 Grizzly Bear Recovery Plan had never been implemented and that the 1993 estimate of a 1% growth rate continued to hold. How long would it have taken for the population to reach 416 bears?

Here we use the explicit formula for exponential growth where we know the values of all parameters except time, t . With $r = 0.01$, $P(0) = 197$, and a goal population of 416 bears, we have

$$P(t) = (1 + r)^t P(0)$$

$$416 = (1.01)^t 197$$

$$2.112 \approx (1.01)^t.$$

Here we can use trial and error with a calculator to find t , or we can take the natural logarithm of both sides, which has the effect of bringing any exponents down in front of the logarithm. We get

$$\ln(2.112) \approx \ln(1.01^t)$$

$$0.7476 \approx t \cdot \ln(1.01)$$

$$0.7476 \approx t \cdot 0.00995$$

$$\frac{0.7476}{0.00995} \approx t$$

$$75.14 \approx t.$$

It would have taken just over 75 years for the grizzly population to reach 416 bears had the growth rate remained at 1% each year.

- 9** Consider the explicit formula for our harvesting/stocking model. Show that if there is no stocking or harvesting, then the formula is the same as the explicit formula for plain exponential growth.

Here we begin with the explicit formula for exponential growth with harvesting or stocking, and we plug in 0 for the harvesting/stocking number:

$$P(t) = (1+r)^t P(0) + a \frac{(1+r)^t - 1}{r}$$

$$P(t) = (1+r)^t P(0) + 0 \frac{(1+r)^t - 1}{r}$$

$$P(t) = (1+r)^t P(0).$$

The result is the explicit formula for exponential growth.

- 11** The U.S. Census in 2000 (see www.census.gov) estimated the population of the United States to be 281.4 million. Without immigration, the population would grow by approximately 0.6% each year. Data available at www.census.gov indicates that approximately 1,000,000 immigrants enter the U.S. each year.

- a.** Create a flow diagram for the U.S. population.

A flow diagram for the U.S. population is given in Figure 1.22.

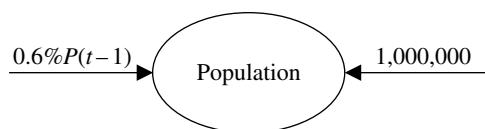


FIGURE 1.22 Flow diagram for Exercise 1.4.11.

- b.** From the flow diagram, give the DDS.

Each inward pointing arrow represents an addition to the previous year's population, so the DDS is given by

$$P(t) = P(t-1) + 0.006 \cdot P(t-1) + 1,000,000.$$

- c.** Use the explicit formula to predict the U.S. population in the year 2050.

With 281,400,000 as our initial population in the year 2000, $t = 50$. Thus we have

$$P(50) = (1.006)^{50} 281,400,000 + 1,000,000 \frac{(1.006)^{50} - 1}{.006}$$

$$P(50) \approx 379,509,917 + 58,108,219$$

$$P(50) \approx 437,618,136.$$

Under the assumed conditions the U.S. population will be approximately 437.6 million in the year 2050.

- d. How does your projection compare to the 419.9 million projected for 2050 by the U.S. Census?

Our projection is about 4% higher than the U.S. Census projection. This indicates that either our assumed growth rate is too high for the years 2000–2050, our assumed immigration rate is too high, or both.

- 13 *Extension:* Following the spirit of the derivation for the explicit formula for an affine model, find the explicit formula for the general model described in Exercise 1.4.12. Confirm your result in 1.4.12d by using the explicit formula.

The DDS for the general model in Exercise 1.4.12 is given by

$$\begin{aligned} P(t) &= P(t-1) + rP(t-1) + a_0(1+s)^{t-1} \\ &= (1+r)P(t-1) + a_0(1+s)^{t-1}, \end{aligned}$$

where r , a_0 , and s are all parameters that can be positive or negative. Our task is to work step-by-step through a few iterations of the DDS until we can see a pattern.

For $t = 1$ we get

$$\begin{aligned} P(1) &= (1+r)P(0) + a_0(1+s)^{1-1} \\ &= (1+r)P(0) + a_0(1+s)^0 \\ &= (1+r)P(0) + a_0. \end{aligned}$$

For $t = 2$ we get

$$\begin{aligned} P(2) &= (1+r)P(1) + a_0(1+s)^{2-1} \\ &= (1+r)P(1) + a_0(1+s). \end{aligned}$$

After substituting for $P(1)$ we get

$$\begin{aligned} P(2) &= (1+r)[(1+r)P(0) + a_0] + a_0(1+s) \\ &= (1+r)^2P(0) + (1+r)a_0 + a_0(1+s). \end{aligned}$$

For $t = 3$ we get

$$\begin{aligned} P(3) &= (1+r)P(2) + a_0(1+s)^{3-1} \\ &= (1+r)P(2) + a_0(1+s)^2. \end{aligned}$$

After substituting for $P(2)$ we get

$$\begin{aligned} P(3) &= (1+r) \left[(1+r)^2P(0) + (1+r)a_0 + a_0(1+s) \right] + a_0(1+s)^2 \\ &= (1+r)^3P(0) + (1+r)^2a_0 + a_0(1+r)(1+s) + a_0(1+s)^2. \end{aligned}$$

For $t=4$ we get

$$\begin{aligned} P(4) &= (1+r)P(3) + a_0(1+s)^{4-1} \\ &= (1+r)P(3) + a_0(1+s)^3. \end{aligned}$$

After substituting for $P(3)$ we get

$$\begin{aligned} P(4) &= (1+r) \left[(1+r)^3 P(0) + (1+r)^2 a_0 + a_0(1+r)(1+s) + a_0(1+s)^2 \right] + a_0(1+s)^3 \\ &= (1+r)^4 P(0) + (1+r)^3 a_0 + a_0(1+r)^2(1+s) + a_0(1+r)(1+s)^2 + a_0(1+s)^3. \end{aligned}$$

The pattern seems to be that for any t we have

$$P(t) = (1+r)^t P(0) + a_0 \left[(1+r)^{t-1} + (1+r)^{t-2}(1+s) + \cdots + (1+r)(1+s)^{t-2} + (1+s)^{t-1} \right].$$

Next we need to do some algebra on the sum inside the brackets. For starters, we factor out the term $(1+r)^{t-1}$ to get

$$\begin{aligned} P(t) &= (1+r)^t P(0) + a_0(1+r)^{t-1} \left[1 + (1+r)^{-1}(1+s) + \cdots + (1+r)^{-t+2}(1+s)^{t-2} \right. \\ &\quad \left. + (1+r)^{-t+1}(1+s)^{t-1} \right]. \end{aligned}$$

Placing terms with negative exponents in denominators gives

$$P(t) = (1+r)^t P(0) + a_0(1+r)^{t-1} \left[1 + \frac{(1+s)}{(1+r)} + \cdots + \frac{(1+s)^{t-2}}{(1+r)^{t-2}} + \frac{(1+s)^{t-1}}{(1+r)^{t-1}} \right].$$

Then we have

$$P(t) = (1+r)^t P(0) + a_0(1+r)^{t-1} \left[1 + \left(\frac{1+s}{1+r} \right) + \cdots + \left(\frac{1+s}{1+r} \right)^{t-2} + \left(\frac{1+s}{1+r} \right)^{t-1} \right].$$

We can now recognize the sum inside the brackets as a geometric series with ratio $x = \frac{1+s}{1+r}$. This observation allows us to rewrite the sum using the geometric series formula to get

$$P(t) = (1+r)^t P(0) + a_0(1+r)^{t-1} \left[\frac{\left(\frac{1+s}{1+r} \right)^t - 1}{\left(\frac{1+s}{1+r} \right) - 1} \right].$$

We can use the formula as is, or we can employ some further simplification with exponents to yield the more succinct version

$$P(t) = (1+r)^t P(0) + a_0 \left[\frac{(1+s)^t - (1+r)^t}{s-r} \right].$$

The last task is to confirm the result of Exercise 1.4.12d by plugging in all required parameters. With $r = -0.06$, $a_0 = 100$, $s = 0.10$, $P(0) = 400$, and $t = 10$, we get

$$\begin{aligned} P(t) &= (1+r)^t P(0) + a_0 \left[\frac{(1+s)^t - (1+r)^t}{s-r} \right] \\ P(10) &= (.94)^{10} 400 + 100 \left[\frac{(1.10)^{10} - (.94)^{10}}{0.10 - (-.06)} \right] \\ &\approx 215.45 + 1284.45 \\ &\approx 1499.9. \end{aligned}$$

The result is about 1500 cranes after 10 years, and this agrees with the Excel result from 12d.

1.5 EQUILIBRIUM VALUES AND STABILITY

1 Consider the DDS given by $P(t) = P(t-1) + .05P(t-1) - 10$.

a. Find all equilibrium values for the DDS.

We need to find P^* such that

$$P^* = P^* + .05P^* - 10.$$

We have

$$0 = .05P^* - 10$$

$$10 = .05P^*$$

$$200 = P^*.$$

Thus the only equilibrium value is $P^* = 200$.

b. Use Excel to confirm that the values found in a. are in fact equilibrium values.

To confirm that $P^* = 200$ is an equilibrium value in Excel we need to enter the DDS and verify that if the DDS starts at 200, it stays at 200. Figure 1.23 shows the formula for the DDS along with the verification we need.

	A	B
1	Exercise 1.5.1	
2		
3	t	P(t)
4	0	200
5	1	=B4+0.05*B4-10
6	2	200
7	3	200
8	4	200
9	5	200
10	6	200
11	7	200
12	8	200

FIGURE 1.23 Excel confirmation for Exercise 1.5.1.

- c. Determine the stability of any equilibrium values found in a. by producing an appropriate Excel graph.

We graph the DDS for different starting points near the equilibrium value all on the same axes. Figure 1.24 shows the result. Because populations that start off of the equilibrium value continue to get farther away from it, the equilibrium at 200 is unstable. We also note that the horizontal line at 200 provides graphical confirmation that it is in fact an equilibrium value.

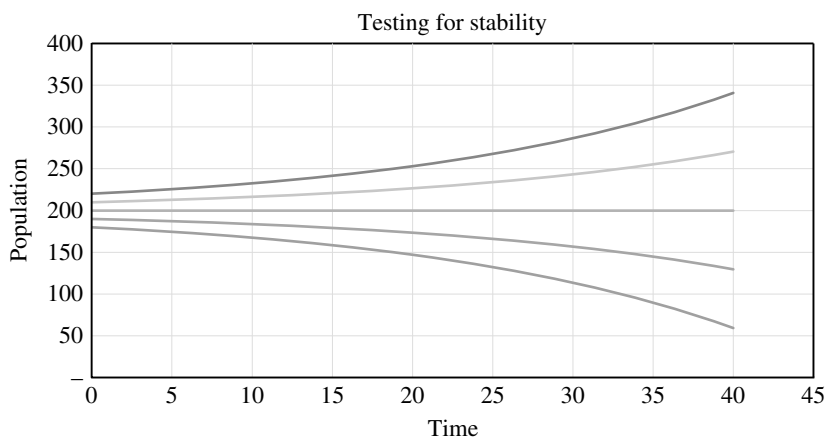


FIGURE 1.24 Excel graph testing for stability in Exercise 1.5.1.

- 3 *Extension:* Consider the DDS given by $P(t) = P(t-1) + 0.004(5 - P(t-1)) \cdot P(t-1)$.
- a. Find all equilibrium values for the DDS.

We need to find all P^* such that

$$P^* = P^* + 0.004(5 - P^*) \cdot P^*.$$

We have

$$0 = 0.004(5 - P^*) \cdot P^*.$$

A product of real numbers is only zero if at least one of the factors is zero, so we must have either $P^* = 0$ or $5 - P^* = 0$. Thus our two equilibrium values are $P^* = 0$ and $P^* = 5$.

- b. Use Excel to confirm that the values found in a. are in fact equilibrium values.

First we need to implement the DDS in Excel. Figure 1.25 shows the correct Excel formula. To verify that 0 and 5 are equilibrium values, we plug those values in for the initial population and note that the population does not change from those values. Figure 1.26 shows the verification for the equilibrium value at 5.

	A	B
1	Exercise 1.5.3	
2		
3	t	P(t)
4	0	100.00
5	1	=B4+0.004*(5-B4)*B4
6	2	47.86

FIGURE 1.25 Excel set-up for Exercise 1.5.3.

	A	B
1	Exercise 1.5.3	
2		
3	t	P(t)
4	0	5.00
5	1	5.00
6	2	5.00
7	3	5.00
8	4	5.00
9	5	5.00
10	6	5.00
11	7	5.00
12	8	5.00

FIGURE 1.26 Excel confirmation of equilibrium value for Exercise 1.5.3.

- c. Determine the stability of any equilibrium values found in a. by producing an appropriate Excel graph.

For each equilibrium value we graph the DDS for different nearby starting points all on the same axes. Figure 1.27 shows the result. Populations that start a little above 0 are moving away from 0, but populations that start off of 5 are moving toward 5. Thus we say that the equilibrium at 0 is unstable and that the equilibrium at 5 is stable. We also note that the horizontal line at 5 provides graphical confirmation that it is in fact an equilibrium value.

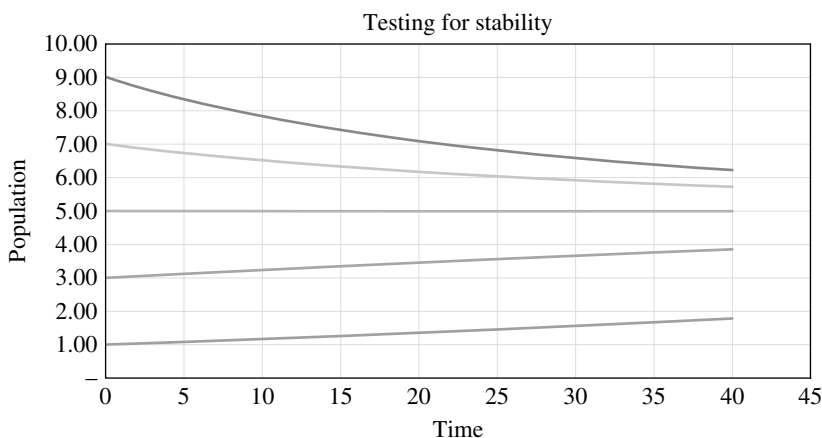


FIGURE 1.27 Excel graph testing for stability in Exercise 1.5.3.

5 Extension: Consider the DDS given by

$$P(t) = P(t-1) + 0.05 \left(1 - \frac{P(t-1)}{10,000} \right) P(t-1) - 125.$$

a. Find all equilibrium values for the DDS.

We need to find all P^* such that

$$P^* = P^* + 0.05 \left(1 - \frac{P^*}{10,000} \right) P^* - 125.$$

We have

$$0 = 0.05 \left(1 - \frac{P^*}{10,000} \right) P^* - 125.$$

On the right-hand side of the equation we have a quadratic:

$$0 = 0.05 \left(1 - \frac{P^*}{10,000} \right) P^* - 125$$

$$0 = (0.05 - 0.000005P^*)P^* - 125$$

$$0 = -0.000005(P^*)^2 + 0.05P^* - 125.$$

Dividing both sides of the equation by the coefficient on the squared term yields the equation

$$0 = (P^*)^2 - 10,000P^* + 25,000,000.$$

We use the quadratic formula to solve:

$$\begin{aligned} P^* &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{10,000 \pm \sqrt{(-10,000)^2 - 4 \cdot 1 \cdot 25,000,000}}{2} \\ &= \frac{10,000 \pm 0}{2} \\ &= 5,000. \end{aligned}$$

Because the discriminant turns out to be zero, we end up with only one solution. Our sole equilibrium value is $P^* = 5,000$.

b. Use Excel to confirm that the values found in a. are in fact equilibrium values.

To confirm that $P^* = 5,000$ is an equilibrium value in Excel we need to enter the DDS and verify that if the DDS starts at 5,000, it stays at 5,000. Figure 1.28 shows the formula for the DDS along with the verification we need.

	A	B
1	Exercise 1.5.5	
2		
3	t	P(t)
4	0	5,000.00
5	1	=B4+0.05*(1-B4/10000)*B4-125
6	2	5,000.00
7	3	5,000.00
8	4	5,000.00
9	5	5,000.00
10	6	5,000.00
11	7	5,000.00
12	8	5,000.00

FIGURE 1.28 Excel confirmation of equilibrium value for Exercise 1.5.5.

- c. Determine the stability of any equilibrium values found in a. by producing an appropriate Excel graph.

We graph the DDS for different starting points near the equilibrium value all on the same axes. Figure 1.29 shows the result. The behavior displayed in the figure is very different than what we are used to seeing. It appears that populations that start above 5,000 decrease back toward 5,000, but populations that start below 5,000 continue to decrease away from 5,000. This is an example of a **semi-stable** equilibrium value, which are discussed in Chapter 5. We also note that the horizontal line at 5,000 provides graphical confirmation that it is in fact an equilibrium value.

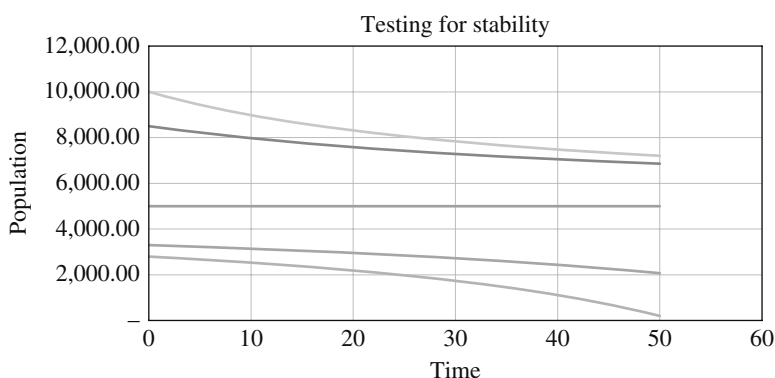


FIGURE 1.29 Excel graph testing for stability in Exercise 1.5.5.

- 7 Show that for *any* exponential model where there is no harvesting or stocking the only equilibrium value is 0.

The DDS for the general exponential model is given by $P(t) = P(t-1) + rP(t-1)$. We find all P^* such that $P^* = P^* + rP^*$. Then $0 = rP^*$, and we see that when $r \neq 0$, the only solution is $P^* = 0$. We can safely exclude the case where $r = 0$ since in that case we do not have a proper exponential model.

- 9 The U.S. Census in 2000 (see www.census.gov) estimated the population of the United States to be 281.4 million. Without immigration, the population would grow by approximately 0.6% each year. Data available at www.census.gov indicates that approximately 1,000,000 immigrants enter the U.S. each year.

- a. Suppose that instead of growing by 0.6% per year the U.S. population was declining by 0.6% each year. Give the DDS for this situation.

The DDS is given by $P(t) = P(t-1) - 0.006P(t-1) + 1,000,000$.

- b. At what value would the U.S. population stabilize in the long run?

Since this is an affine model the population would stabilize at the stable equilibrium value given by

$$P^* = -\frac{a}{r} = -\frac{1,000,000}{-0.006} \approx 166,666,667.$$

The population would stabilize at about 167 million people.

- c. Produce a graph that indicates the U.S. population would stabilize at this value no matter where it started.

This amounts to producing a graph that shows the equilibrium value is stable. Such a graph is given in Figure 1.30.

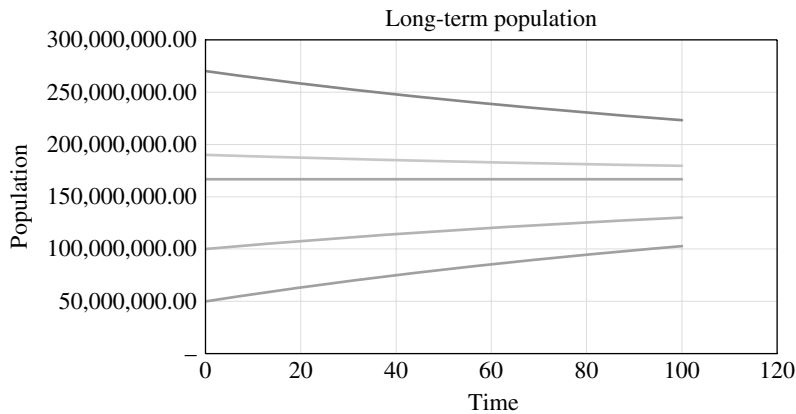


FIGURE 1.30 Excel graph testing for stability in Exercise 1.5.9.

- d. If the U.S. government wanted to stabilize the population at 400,000,000, how many people should it allow to immigrate each year?

Here we have to arrange for the equilibrium value to be equal to 400,000,000. Assuming the same growth rate of -0.6% , we solve for a :

$$\begin{aligned} P^* &= -\frac{a}{r} \\ 400,000,000 &= -\frac{a}{-0.006} \\ 400,000,000 &= \frac{a}{0.006} \\ 2,400,000 &= a. \end{aligned}$$

We have found that if the U.S. population were declining by 0.6% each year and the U.S. government wanted a long-term population of 400,000,000, then it should allow approximately 2,400,000 people to immigrate to the U.S. each year.