1

The real numbers

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From the considerations in the Introduction, it is clear that in order to have a firm foundation of Calculus, one needs to study the real numbers carefully. We will do this in this chapter. The plan is as follows:

(1) An intuitive, visual picture of $\mathbb{R}$: the number line. We will begin our understanding of $\mathbb{R}$ intuitively as points on the 'number line'. This way, we will have a mental picture of $\mathbb{R}$, in order to begin stating the precise properties of the real numbers that we will need in the sequel. It is a legitimate issue to worry about the actual construction of the set of real numbers, and we will say something about this in Section 1.8.

(2) Properties of $\mathbb{R}$. Having a rough feeling for the real numbers as being points of the real line, we will proceed to state the precise properties of the real numbers we will need. So we will think of $\mathbb{R}$ as an undefined set for now, and just state rigorously what properties we need this set $\mathbb{R}$ to have. These desirable properties fall under three categories:

(a) *the field axioms*, which tell us about what laws the arithmetic of the real numbers should follow,

(b) *the order axiom*, telling us that comparison of real numbers is possible with an order $>$ and what properties this order relation has, and

(c) *the Least Upper Bound Property of* $\mathbb{R}$, which tells us roughly that unlike the set of rational numbers, the real number line has 'no holes'. This last property is the most important one from the viewpoint of Calculus: it is the one which makes Calculus possible with real numbers. *If* rational numbers had this nice property, then we would not have bothered studying real numbers, and instead we would have just used rational numbers for doing Calculus.

The How and Why of One Variable Calculus, First Edition. Amol Sasane.
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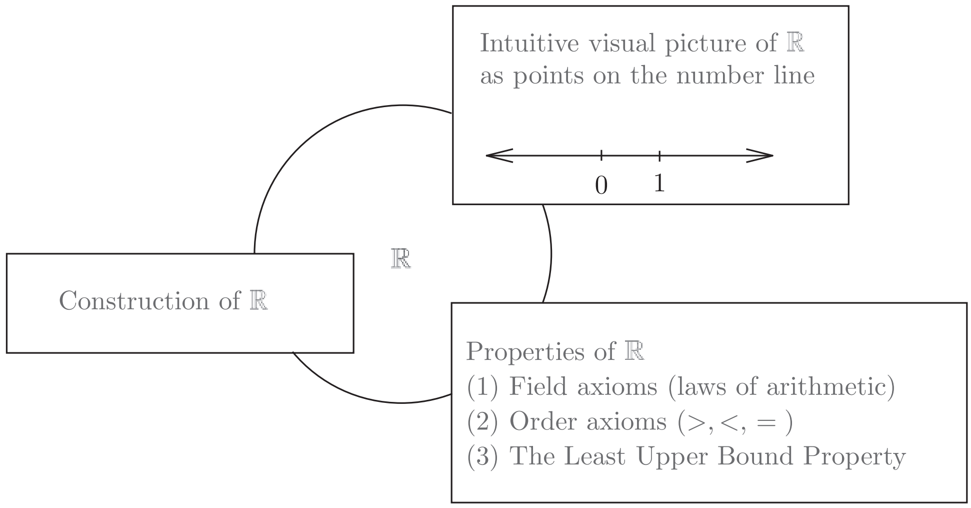
(3) The construction of $\mathbb{R}$. Although we will think of real numbers intuitively as 'numbers that can be depicted on the number line', this is not acceptable as a rigorous mathematical definition. So one can ask:

> Is there really a set $\mathbb{R}$ that can be constructed which has the stipulated properties (2)(a), (b), and (c) (and which will be detailed further in Sections 1.2, 1.3, 1.4)?

The answer is yes, and we will make some remarks about this in Section 1.8.



1.1 Intuitive picture of $\mathbb{R}$ as points on the number line

In elementary school, we learn about

$$\begin{aligned} \text{the natural numbers} \quad & \mathbb{N} := \{1, 2, 3, \cdots\} \\ \text{the integers} \quad & \mathbb{Z} := \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}, \text{ and} \\ \text{the rational numbers} \quad & \mathbb{Q} := \left\{ \left[\frac{n}{d}\right] : n, d \in \mathbb{Z}, d \neq 0 \right\}. \end{aligned}$$

Incidentally, the rationale behind denoting the rational numbers by $\mathbb{Q}$ is that it reminds us of '**q**uotient', and $\mathbb{Z}$ for integers comes from the German word '**z**ählen' (meaning 'count'). In the above,

$$\left[\frac{n}{d}\right]$$

represents a whole family of 'equivalent fractions'; for example, $\frac{2}{4} = \frac{1}{2} = \frac{-3}{-6}$ etc.

We are accustomed to visualising these numbers on the 'number line'. What is the number line? It is any line in the plane, on which we have chosen a point O as the 'origin', representing the number 0, and chosen a unit length by marking off a point on the right of O, where the number 1 is placed. In this way, we get all the positive integers, $1, 2, 3, 4, \cdots$ by repeatedly marking off successively the unit length towards the right, and all the negative integers $-1, -2, -3, \cdots$ by repeatedly marking off successively the unit length towards the left.

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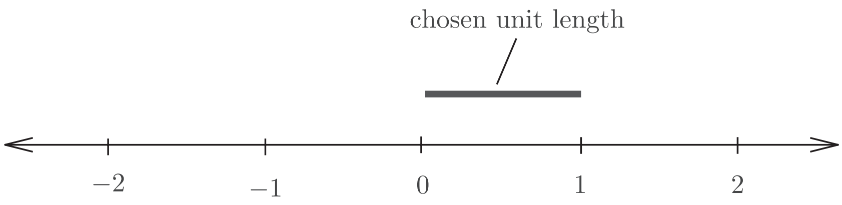

Just like the integers can be depicted on the number line, we can also depict all rational numbers on it as follows. First of all, here is a procedure for dividing a unit length on the number line into d ($\in \mathbb{N}$) equal parts, allowing us to construct the rational number $1/d$ on the number line. See Figure 1.1.

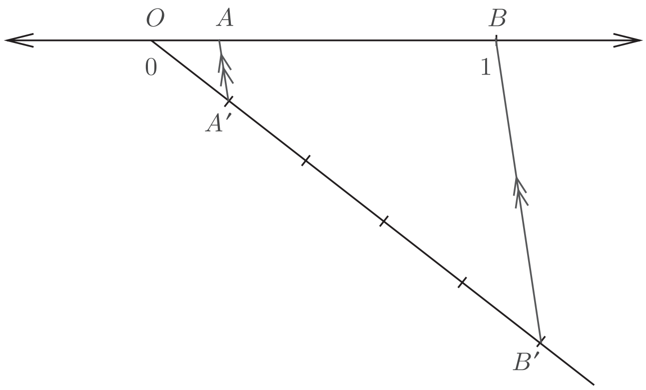

Figure 1.1 Construction of rational numbers: in the above picture, given the length 1 (that is, knowing the position of B), we can construct the length $1/5$, and so the point A corresponds to the rational number $1/5$.

The steps are as follows: Let the points O and B correspond to the numbers 0 and 1.

(1) Take any arbitrary length $\ell(OA')$ along a ray starting at O in any direction other than that of the number line itself.

(2) Let B' be a point on the ray such that $\ell(OB') = d \cdot \ell(OA')$.

(3) Draw AA' parallel to BB' to meet the number line at A.

Conclusion: From the similar triangles $\Delta OAA'$ and $\Delta OBB'$, we see that the length $\ell(OA) = 1/d$.

Having obtained $1/d$, we can now construct n/d on the number line for *any* $n \in \mathbb{Z}$, by repeating the length $1/d$ n times towards the right of 0 if $n > 0$, and towards the left $-n$ times from 0 if n is negative.

Hence, we can depict all the rational numbers on the number line. Does this exhaust the number line? That is, suppose that we start with all the points on the number line being coloured black, and suppose that at a later time, we colour all the rational ones by red: are there any black points left over? The answer is yes, and we demonstrate this below. We will show that there does 'exist', based on geometric reasoning, a point on the number line, whose square is 2, but we will also argue that this number, denoted by $\sqrt{2}$, is not a rational number.

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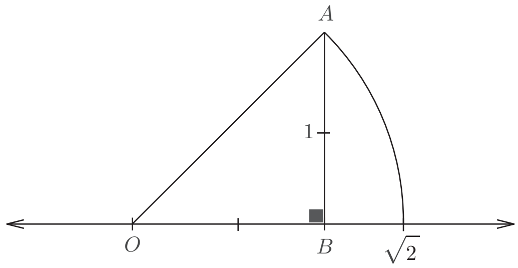
First of all, the picture below shows that $\sqrt{2}$ exists as a point on the number line. Indeed, by looking at the right angled triangle ΔOBA, Pythagoras's Theorem tells us that the length of the hypotenuse OA satisfies

$$(\ell(OA))^2 = (\ell(OB))^2 + (\ell(AB))^2 = 1^2 + 1^2 = 2,$$

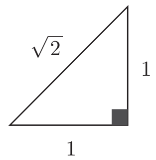
and so $\ell(OA)$ is a number, denoted say by $\sqrt{2}$, whose square is 2. By taking O as the centre and radius $\ell(OA)$, we can draw a circle using a compass that intersects the number line at a point C, corresponding to the number $\sqrt{2}$. Is $\sqrt{2}$ a rational number? We show below that it isn't!



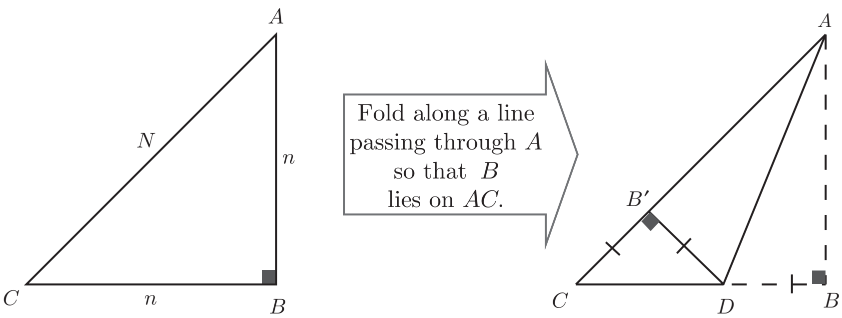
Exercise 1.1. Depict $-11/6$ and $\sqrt{3}$ on the number line.

Theorem 1.1 (An 'origami' proof of the irrationality of $\sqrt{2}$). *There is no rational number $q \in \mathbb{Q}$ such that $q^2 = 2$.*

Proof. Suppose that $\sqrt{2}$ is a rational number. Then some scaling of the triangle



by an integer will produce a similar triangle, all of whose sides are integers. Choose the smallest such triangle, say ΔABC, with integer lengths $\ell(BC) = \ell(AB) = n$, and $\ell(AC) = N$, $n, N \in \mathbb{N}$. Now do the following origami: fold along a line passing through A so that B lies on AC, giving rise to the point B' on AC. The 'crease' in the paper is actually the angle bisector AD of the angle $\angle BAC$.



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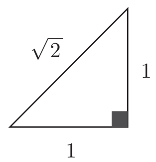
In $\Delta CB'D$, $\angle CB'D = 90^\circ$, $\angle B'CD = 45^\circ$. So $\Delta CB'D$ is an isosceles right triangle. We have $\ell(CB') = \ell(B'D) = \ell(AC) - \ell(AB') = N - n \in \mathbb{N}$, while

$$\ell(CD) = \ell(CB) - \ell(DB) = n - \ell(B'C) = n - (N - n) = 2n - N \in \mathbb{N}.$$

So $\Delta CB'D$ is similar to the triangle



has integer side lengths, and is smaller than ΔABC, contradicting the choice of ΔABC. So there is no rational number q such that $q^2 = 2$. $\square$

A different proof is given in the exercise below.

Exercise 1.2. $(*)$ We offer a different proof of the irrationality of $\sqrt{2}$, and en route learn a technique to prove the irrationality of 'surds'.[1]

(1) Prove the Rational Zeros Theorem: Let $c_0, c_1, \cdots, c_d$ be $d \geq 1$ integers such that c_0 and c_d are not zero. Let $r = p/q$ where p, q are integers having no common factor and such that $q > 0$. Suppose that r is a zero of the polynomial $c_0 + c_1x + \cdots + c_dx^d$. Then q divides c_d and p divides c_0.

(2) Show that $\sqrt{2}$ is irrational.

(3) Show that $\sqrt[3]{6}$ is irrational.

Thus, we have seen that the elements of $\mathbb{Q}$ can be depicted on the number line, and that not all the points on the number line belong to $\mathbb{Q}$. We think of $\mathbb{R}$ as *all* the points on the number line. As mentioned earlier, if we take out everything on the number line (the black points) except for the rational numbers $\mathbb{Q}$ (the red points), then there will be holes among the rational numbers (for example, there will be a missing black point where $\sqrt{2}$ lies on the number line). We can think of the real numbers as 'filling in' these holes between the rational numbers. We will say more about this when we make remarks about the construction of $\mathbb{R}$. Right now, we just have an intuitive picture of the set of real numbers as a bigger set than the rational numbers, and we think of the real numbers as points on the number line. Admittedly, this is certainly not a mathematical definition, and is extremely vague. In order to be precise, and to do Calculus rigorously, we just can't rely on this vague intuitive picture of the real numbers. So we now turn to the precise properties of the real numbers that we are allowed to use in

[1] Surds refer to irrational numbers that arise as the nth root of a natural number. The mathematician al-Khwarizmi (around 820 AD) called irrational numbers 'inaudible', which was later translated to the Latin *surdus* for 'mute'.

developing Calculus. While stating these properties, we will think of the set $\mathbb{R}$ as an (as yet) undefined set containing $\mathbb{Q}$ which will satisfy the properties of

(1) the field axioms (laws of arithmetic in $\mathbb{R}$),

(2) the order axioms (allowing us to compare real numbers with $>, <, =$), and

(3) the Least Upper Bound Property (making Calculus possible in $\mathbb{R}$),

stipulated below.

It is a pertinent question if one can construct (if there really exists) such a set $\mathbb{R}$ satisfying the above properties (1–3). The answer to this question is yes, but it is tedious. So in this first introductory course, we will not worry ourselves too much with it. It is a bit like the process of learning physics: typically one does not start with quantum mechanics and the structure of an atom, but with the familiar realm of classical mechanics. To consider another example, imagine how difficult it would be to learn a foreign language if one starts to painfully memorise systematically all the rules of grammar first; instead a much more fruitful method is to start practicing simple phrases, moving on to perhaps children's comic books, listening to pop music in that language, news, literature, and so on. Of course, along the way one picks up grammar and a formal study can be done at leisure later resulting in better comprehension. We will actually give some idea about the construction of the real numbers in Section 1.8. Right now, we just accept on faith that the construction of $\mathbb{R}$ possessing the properties we are about to learn can be done, and to have a concrete object in mind, we rely on our familiarity with the number line to think of the real numbers when we study the properties (1), (2), (3) listed above.

We also remark that property (3) (the Least Upper Bound Property) of $\mathbb{R}$ will turn out to be crucial for doing Calculus. The properties (1), (2) are also possessed by the rational number system $\mathbb{Q}$, but we will see that (3) fails for $\mathbb{Q}$.

1.2 The field axioms

The content of this section can be summarised in one sentence: $(\mathbb{R}, +, \cdot)$ forms a field. What does this mean? It is a compact way of saying the following. $\mathbb{R}$ is a set, equipped with two operations:

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

called *addition*, which sends a pair of real numbers (x, y) to their *sum* $x + y$, and the other operation is

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

called *multiplication*, which sends a pair of real numbers (x, y) to their *product* $x \cdot y$, and these two operations satisfy certain laws, called the 'field axioms'.[2] The field axioms for $\mathbb{R}$ are

[2] There are other number systems, for example, the rational numbers $\mathbb{Q}$ which also obey similar laws of arithmetic, and so $(\mathbb{Q}, +, \cdot)$ is also deemed to be a field. So the word 'field' is invented to describe the situation that one has a number system $\mathbb{F}$ with corresponding operations $+$ and $\cdot$ which obey the usual laws of arithmetic, rather than listing all of these laws.

listed below:

$$+\begin{cases} \text{(F1) (Associativity)} & \text{For all } x, y, z \in \mathbb{R},\ x + (y + z) = (x + y) + z. \\ \text{(F2) (Additive identity)} & \text{For all } x \in \mathbb{R},\ x + 0 = x = 0 + x. \\ \text{(F3) (Inverses)} & \text{For all } x \in \mathbb{R}, \text{there exists } -x \in \mathbb{R} \\ & \quad \text{such that } x + (-x) = 0 = -x + x. \\ \text{(F4) (Commutativity)} & \text{For all } x, y \in \mathbb{R},\ x + y = y + x. \end{cases}$$

$$\cdot\begin{cases} \text{(F5) (Associativity)} & \text{For all } x, y, z \in \mathbb{R},\ x \cdot (y \cdot z) = (x \cdot y) \cdot z. \\ \text{(F6) (Multiplicative identity)} & 1 \neq 0 \text{ and for all } x \in \mathbb{R},\ x \cdot 1 = x = 1 \cdot x. \\ \text{(F7) (Inverses)} & \text{For all } x \in \mathbb{R}\backslash\{0\},\ \text{there exists } x^{-1} \in \mathbb{R} \\ & \quad \text{such that } x \cdot x^{-1} = 1 = x^{-1} \cdot x. \\ \text{(F8) (Commutativity)} & \text{For all } x, y \in \mathbb{R},\ x \cdot y = y \cdot x. \end{cases}$$

$$+, \cdot \begin{cases} \text{(F9) (Distributivity)} \quad \text{For all } x, y, z \in \mathbb{R},\ x \cdot (y + z) = x \cdot y + x \cdot z. \end{cases}$$

With these axioms, it is possible to prove the usual arithmetic manipulations we are accustomed to. Here are a couple of examples.

Example 1.1. For every $a \in \mathbb{R}$, $a \cdot 0 = 0$.

Let $a \in \mathbb{R}$. Then we have $a \cdot 0 \overset{F2}{=} a \cdot (0 + 0) \overset{F9}{=} a \cdot 0 + a \cdot 0$. So with $x := a \cdot 0$, we have got $x + x = x$. Adding $-x$ on both sides (F3!), and using (F1) we obtain

$$0 = x + (-x) = (x + x) + (-x) \overset{F1}{=} x + (x + (-x)) \overset{F3}{=} x + 0 \overset{F2}{=} x = a \cdot 0,$$

completing the proof of the claim. $\diamondsuit$

Example 1.2. If $a, b \in \mathbb{R}$, and $a \cdot b = 0$, then $a = 0$ or $b = 0$.

If $a = 0$, then we are done. Suppose that $a \neq 0$. By (F7), there exists a real number a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Hence

$$b = 1 \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0.$$

So if $a \neq 0$, then $b = 0$. Thus $\Big(a, b \in \mathbb{R} \text{ such that } a \cdot b = 0\Big) \Rightarrow \Big(a = 0 \text{ or } b = 0\Big)$. $\diamondsuit$

Of course in this book, we will not do such careful justifications every time we need to manipulate real numbers. We have listed the above laws to once and for all stipulate the laws of arithmetic for real numbers that justify the usual calculational rules we are familiar with, so that we know the *source* of it all. For example, the student may wish to try his/her hand at producing a rigorous justification based on (F1) to (F9) of the following well known facts.

Exercise 1.3. $(*)$ Using the field axioms of $\mathbb{R}$, prove the following:

(1) Additive inverses are unique.

(2) For all $a \in \mathbb{R}$, $(-1) \cdot a = -a$.

(3) $(-1) \cdot (-1) = 1$.

1.3 Order axioms

We now turn to order axioms for the real numbers. This is the source of the inequality '$>$' that we are used to, enabling one to compare two real numbers. The relation $>$ between real numbers arises from a special subset $\mathbb{P}$ of the real numbers.

Order axiom. There exists a subset $\mathbb{P}$ of $\mathbb{R}$ such that

(O1) If $x, y \in \mathbb{P}$, then $x + y \in \mathbb{P}$ and $x \cdot y \in \mathbb{P}$.

(O2) For every $x \in \mathbb{R}$, *one and only one* of the following statements is true:

$\underline{1}^\circ$ $x = 0$.
$\underline{2}^\circ$ $x \in \mathbb{P}$.
$\underline{3}^\circ$ $-x \in \mathbb{P}$.

Definition 1.1 (Positive numbers). The elements of $\mathbb{P}$ are called *positive numbers*. For real numbers x, y, we say that

$x > y$ if $x - y \in \mathbb{P}$,
$x < y$ if $y - x \in \mathbb{P}$,
$x \geq y$ if $x = y$ or $x > y$,
$x \leq y$ if $x = y$ or $x < y$.

It is clear from (O2) that 0 is *not* a positive number. Also, from (O2) it follows that for real numbers x, y, *one and only one* of the following statements is true:

$\underline{1}^\circ$ $x = y$.

$\underline{2}^\circ$ $x > y$.

$\underline{3}^\circ$ $x < y$.

Why is this so? If $x \neq y$, then $x - y \neq 0$, and so by (O2), we have the mutually exclusive possibilities $x - y \in \mathbb{P}$ or $y - x = -(x - y) \in \mathbb{P}$ happening, that is, either $x > y$ or $x < y$.

Example 1.3. $1 > 0$.

We have three possible, mutually exclusive cases:

$\underline{1}^\circ$ $1 = 0$.

$\underline{2}^\circ$ $1 \in \mathbb{P}$.

$\underline{3}^\circ$ $-1 \in \mathbb{P}$.

As $1 \neq 0$, we know that 1° is not possible.

Suppose that 3° holds, that is, $-1 \in \mathbb{P}$. From Exercise 1.3(3), $(-1) \cdot (-1) = 1$. Using (O1), and the fact that $-1 \in \mathbb{P}$, it then follows that $1 = (-1) \cdot (-1) \in \mathbb{P}$. So if we assume that 3° holds, then we obtain that *both* 2° and 3° are true, which is impossible as it violates (O2).

Thus by (O2), the only remaining case, namely 2° must hold, that is, $1 \in \mathbb{P}$. $\diamondsuit$

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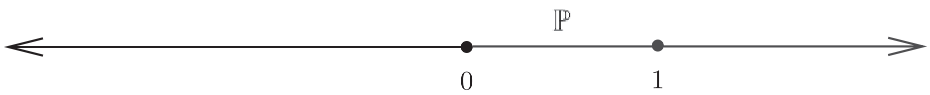
Exercise 1.4. $(*)$ Using the order axioms for $\mathbb{R}$, show the following:

(1) For all $a \in \mathbb{R}$, $a^2 \geq 0$.

(2) There is no real number x such that $x^2 + 1 = 0$.

Again, just like we can use the field axioms to justify arithmetic manipulations of real numbers, it is enough to know that if challenged, one can derive all the usual laws of manipulating inequalities among real numbers based on these order axioms, but we will not do this at every instance we meet an inequality.

From our intuitive picture of $\mathbb{R}$ as points on the number line, what is the set $\mathbb{P}$? $\mathbb{P}$ is simply the set of all points/real numbers to the right of the origin O.



Also, geometrically on the number line, the inequality $a < b$ between real numbers a, b means that b lies to the right of a on the number line.

1.4 The Least Upper Bound Property of $\mathbb{R}$

This property is crucial for proving the results of Calculus, and when studying the proofs of the key results (the Bolzano–Weierstrass Theorem, the Intermediate Value Theorem, the Extreme Value Theorem, and so on), we will gradually learn to appreciate the key role played by it.

Definition 1.2 (Upper bound of a set). Let S be a subset of $\mathbb{R}$. A real number u is said to be an *upper bound of S* if for all $x \in S$, $x \leq u$.

If we think of the set S as some blob on the number line, then u should be any point on the number line that lies to the right of the points of the blob.

Example 1.4.

(1) If $S = \{0, 1, 9, 7, 6, 1976\}$, then 1976 is an upper bound of S. In fact, any real number $u \geq 1976$ is an upper bound of S. So S has lots of upper bounds.

(2) Let $S := \{x \in \mathbb{R} : x < 1\}$. Then 1 is an upper bound of S. In fact, any real number $u \geq 1$ is an upper bound of S.

(3) If $S = \mathbb{R}$, then S has no upper bound. Why? Suppose that $u \in \mathbb{R}$ is an upper bound of $\mathbb{R}$. Consider $u + 1 \in S = \mathbb{R}$. Then

$$\underbrace{u+1}_{\in S} \leq \underbrace{u}_{\text{upper bound of } S},$$

and so $1 \leq 0$, a contradiction!

(4) Let $S = \emptyset$ (the empty set, containing no elements). *Every* $u \in \mathbb{R}$ is an upper bound. For if $u \in \mathbb{R}$ is not an upper bound of S, then there must exist an element $x \in S$ which prevents u from being an upper bound of S, that is,

$$\boxed{\text{it is not the case that } x \leq u}$$

But S has no elements at all, much less an element such that $\boxed{\cdots}$ holds.

(This is an example of a 'vacuous truth'. Consider the statement

$$\boxed{\text{Every man with 60000 legs is intelligent.}}$$

This is considered a true statement in Mathematics. The argument is: Can you show me a man with 60000 legs for which the claimed property (namely of being intelligent) is not true? No! Because there are no men with 60000 legs! By the same logic, even the statement

$$\boxed{\text{Every man with 60000 legs is not intelligent.}}$$

is true in Mathematics.) $\diamondsuit$

Definition 1.3 (Set bounded above). If $S \subset \mathbb{R}$ and S has an upper bound (that is, the set of upper bounds of S is not empty), then S is said to be *bounded above*.

Example 1.5. The set $\mathbb{R}$ is not bounded above.

Each of the sets $\{0, 1, 9, 7, 6, 1976\}$, $\emptyset$, $\{x \in \mathbb{R} : x < 1\}$ is bounded above. $\diamondsuit$

Similarly one can define the notions of a lower bound, and of a set being bounded below.

Definition 1.4 (Lower bound of a set; set bounded below). Let S be a subset of $\mathbb{R}$. A real number ℓ is said to be a *lower bound of* S if for all $x \in S$, $\ell \leq x$.

If $S \subset \mathbb{R}$ and S has a lower bound (that is, the set of lower bounds of S is not empty), then S is said to be *bounded below*.

If we think of the set S as some blob on the number line, then ℓ should be any point on the number line that lies to the left of the points of the blob.

Example 1.6.

(1) If $S = \{0, 1, 9, 7, 6, 1976\}$, then 0 is a lower bound of S. In fact, any real number $\ell \leq 0$ serves as a lower bound of S. So S is bounded below.

(2) Let $S := \{x \in \mathbb{R} : x < 1\}$. Then S is not bounded below. Let us show this. Suppose that, on the contrary, S does have a lower bound, say $\ell \in \mathbb{R}$. Let $x \in S$. Then $\ell \leq x < 1$. We have

$$\ell - 1 < \ell \leq x < 1,$$

and so $\ell - 1 < 1$. Thus $\ell - 1 \in S$, and as ℓ is a lower bound of S, we must have $\ell \leq \ell - 1$, that is, $1 < 0$, a contradiction! So our original assumption that S is bounded below must be false. Thus S is not bounded below. (This claim was intuitively obvious too, since the set of points in S on the number line is the entire ray of points on the left of 1, leaving no room for points on $\mathbb{R}$ to be on the 'left of S'.)

(3) If $S = \mathbb{R}$, then S has no lower bound. Indeed, if $\ell \in \mathbb{R}$ is a lower bound of $\mathbb{R}$, then

$$\underbrace{\ell}_{\text{lower bound of } S} \leq \underbrace{\ell - 1}_{\in S},$$

and so $1 \leq 0$, a contradiction. Thus $\mathbb{R}$ is not bounded below.

(4) Let $S = \emptyset$ (the empty set, containing no elements). Every $\ell \in \mathbb{R}$ is a lower bound. If $\ell \in \mathbb{R}$ is not a lower bound of S, then there must exist an element $x \in S$ which prevents ℓ from being a lower bound of S, that is, it is not the case that $\ell \leq x$. But as S is empty, this is impossible. So S is bounded below. ♢

Definition 1.5 (Bounded set). Let $S \subset \mathbb{R}$. S is called *bounded* if S is bounded below and bounded above.

Example 1.7.

S	An upper bound	Bounded above?	A lower bound	Bounded below?	Bounded?
$\{0, 1, 9, 7, 6, 1976\}$	1976 Any $u \geq 1976$	Yes	0 Any $\ell \leq 0$	Yes	Yes
$\{x \in \mathbb{R} : x < 1\}$	1 Any $u \geq 1$	Yes	Does not exist	No	No
$\mathbb{R}$	Does not exist	No	Does not exist	No	No
$\emptyset$	Every $u \in \mathbb{R}$	Yes	Every $\ell \in \mathbb{R}$	Yes	Yes

♢

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We now introduce the notions of a least upper bound (also called supremum) and a greatest lower bound (also called infimum) of a subset S of $\mathbb{R}$.

Definition 1.6 (Supremum and infimum). Let S be a subset of $\mathbb{R}$.

(1) $u_* \in \mathbb{R}$ is called a *least upper bound of S* (or a *supremum of S*) if

  (a) u_* is an upper bound of S, and

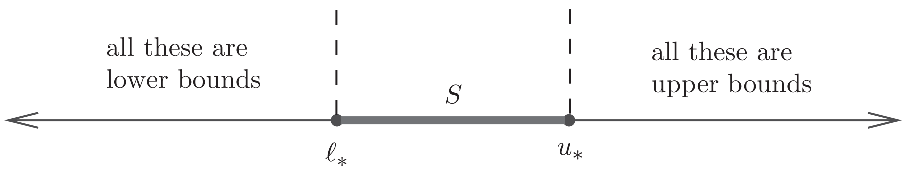
  (b) if u is an upper bound of S, then $u_* \leq u$.

(2) $\ell_* \in \mathbb{R}$ is called a *greatest lower bound of S* (or an *infimum of S*) if

  (a) ℓ_* is a lower bound of S, and

  (b) if ℓ is a lower bound of S, then $\ell \leq \ell_*$.

Pictorially, the supremum is the leftmost point among the upper bounds, and the infimum is the rightmost point among the lower bounds of a set.



Example 1.8.

(1) If $S = \{0, 1, 9, 7, 6, 1976\}$, then $u_* = 1976$ is a least upper bound of S because

  (1) 1976 is an upper bound of S, and

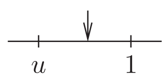
  (2) if u is an upper bound of S, then $(S \ni)\ 1976 \leq u$, that is $u_* \leq u$.

Similarly, 0 is a greatest lower bound of S.

(2) Let $S = \{x \in \mathbb{R} : x < 1\}$. Then we claim that $u_* = 1$ is a least upper bound of S. Indeed we have:

  (a) 1 is an upper bound of S: If $x \in S$, then $x < 1 = u_*$.

  (b) Let u be an upper bound of S. We want to show that $u_* = 1 \leq u$. Suppose the contrary, that is, $1 > u$. Then there is a gap between u and 1.



  (But then we see that this gap between u and 1 of course contains elements of S which are to *right* of the supposed upper bound u, and this should give the contradiction we seek.) To this end, let us consider the number $(1 + u)/2$. We have

$$\frac{1+u}{2} < \frac{1+1}{2} = 1$$

and so $(1+u)/2$ belongs to S. As u is an upper bound of S, we must have

$$\frac{1+u}{2} < u,$$

which upon rearranging gives $1 < u$, a contradiction.

S does not have a lower bound, and so certainly no greatest lower bound either (a greatest lower bound has to be first of all a lower bound!).

(3) $\mathbb{R}$ does not have a supremum, and no infimum either.

(4) $\emptyset$ has no supremum. (We intuitively expect this: every real number serves as an upper bound, but there is no smallest one among these!) Indeed, suppose on the contrary that $u_* \in \mathbb{R}$ is a supremum. Then $u_* - 1 \in \mathbb{R}$ is an upper bound of $\emptyset$ (since it is *some* real number, and we had seen that *all* real numbers are upper bounds of $\emptyset$). As u_* is a least upper bound, we must have $u_* \leq u_* - 1$, that is, $1 \leq 0$, a contradiction.

Similarly, $\emptyset$ has no infimum either. $\diamondsuit$

A set may have many upper bounds and many lower bounds, but it is intuitively clear, based on our visual number line picture, that the supremum and infimum of a set, assuming they exist, must be unique. Here is a formal proof.

Theorem 1.2. *If a subset S of $\mathbb{R}$ has a supremum, then it is unique.*

Proof. Let u_*, u_*' be two supremums of S. Then as u_*' is, in particular, an upper bound, and since u_* is a least upper bound, we must have

$$u_* \leq u_*'. \tag{1.1}$$

Similarly, since u_* is, in particular, an upper bound, and since u_*' is a least upper bound, we must also have

$$u_*' \leq u_*. \tag{1.2}$$

From (1.1) and (1.2), it now follows that $u_* = u_*'$. $\square$

So when S has **a** supremum, then it is **the** supremum. Thus we can give it special notation (since we know what it means unambiguously):

$$\sup S.$$

Similarly, if a set S has an infimum, it is unique and is denoted by

$$\inf S.$$

Example 1.9. We have

$$\sup\{0, 1, 9, 7, 6, 1976\} = 1976,$$

$$\sup\{x \in \mathbb{R} : x < 1\} = 1,$$

$$\inf\{x \in \mathbb{R} : x \geq 1\} = 1.$$

To see the last equality, we note that 1 is certainly a lower bound of the set $S := \{x \in \mathbb{R} : x \geq 1\}$, and if ℓ is any lower bound, then as 1 is an element of the set S, we have $\ell \leq 1$. $\diamondsuit$

Note that comparing the first two examples above, when $S := \{0, 1, 9, 7, 6, 1976\}$, we have

$$\sup S \in S,$$

while in the case of $S := \{x \in \mathbb{R} : x < 1\}$, we have

$$\sup S \notin S.$$

It will be convenient to keep track of when the supremum (or for that matter infimum) of a set *belongs to* the set. So we introduce the following definitions and corresponding notation.

Definition 1.7 (Maximum, minimum of a set).

(1) If $\sup S \in S$, then $\sup S$ is called a *maximum of S*, denoted by $\max S$.

(2) If $\inf S \in S$, then $\inf S$ is called a *minimum of S*, denoted by $\min S$.

Example 1.10.

S	Supremum	Maximum	Infimum	Minimum
$\{0, 1, 9, 7, 6, 1976\}$	1976	1976	0	0
$\{x \in \mathbb{R} : x < 1\}$	1	Does not exist	Does not exist	Does not exist
$\mathbb{R}$	Does not exist	Does not exist	Does not exist	Does not exist
$\emptyset$	Does not exist	Does not exist	Does not exist	Does not exist
$\{x \in \mathbb{R} : x \geq 1\}$	Does not exist	Does not exist	1	1

$\diamondsuit$

Exercise 1.5. Provide the following information about the set S

An upper bound	A lower bound	Is S bounded?	$\sup S$	$\inf S$	$\max S$	$\min S$

where S is given by:

(1) $(0, 1] := \{x \in \mathbb{R} : 0 < x \leq 1\}$

(2) $[0, 1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

(3) $(0, 1) := \{x \in \mathbb{R} : 0 < x < 1\}$.

In the above Example 1.10, we note that if S is nonempty and bounded above, then its supremum exists. In fact, this is a fundamental property of the real numbers, called the *Least Upper Bound Property* of the real numbers, which we state below:

$$\boxed{\text{If } S \subset \mathbb{R} \text{ is such that } S \neq \emptyset \text{ and } S \text{ has an upper bound, then } \sup S \text{ exists.}}$$

Example 1.11.

(1) $S = \{0, 1, 9, 7, 6, 1976\}$ is a subset of $\mathbb{R}$, it is nonempty, and it has an upper bound. So the Least Upper Bound Property of $\mathbb{R}$ tells us that this set should have a least upper bound. This is indeed true, as we had seen earlier that S has 1976 as the supremum.

(2) $S = \{x \in \mathbb{R} : x < 1\}$ is a subset of $\mathbb{R}$, it is nonempty ($0 \in S$), and it has an upper bound (for example, 2). So the Least Upper Bound Property of $\mathbb{R}$ tells us that this set should have a least upper bound. This is indeed true, as we had seen earlier that 1 is the supremum of S.

(3) $S = \mathbb{R}$ is a subset of $\mathbb{R}$, it is nonempty, and it has no supremum. So what went wrong? Well, S is not bounded above.

(4) $S = \emptyset$ is a subset of $\mathbb{R}$ and it is bounded above. But it has no supremum. There is no contradiction to the Least Upper Bound Property because S is empty! $\diamondsuit$

Example 1.12. Let $S := \{x \in \mathbb{R} : x^2 \leq 2\}$. Clearly S is a subset of $\mathbb{R}$ and it is nonempty since $1 \in S$: $1^2 = 1 \leq 2$. Let us show that S is bounded above. In fact, 2 serves as an upper bound of S. Since if $x > 2$, then $x^2 > 4 > 2$. Thus if $x \in S$, then $x^2 \leq 2$, and so $x \leq 2$.

By the Least Upper Bound Property of $\mathbb{R}$, $u_* := \sup S$ exists in $\mathbb{R}$. Moreover, one can show that this u_* satisfies $u_*^2 = 2$ by showing that the cases $u_*^2 < 2$ and $u_*^2 > 2$ are both impossible.

First of all, $u_* \geq 1$ (as u_* is in particular an upper bound of S and $1 \in S$). Now define

$$r := u_* - \frac{u_*^2 - 2}{u_* + 2} = \frac{2(u_* + 1)}{u_* + 2} > 0. \tag{1.3}$$

Then, we have

$$r^2 - 2 = \frac{2(u_*^2 - 2)}{(u_* + 2)^2}. \tag{1.4}$$

$\underline{1}^\circ$ Suppose $u_*^2 < 2$. Then (1.4) implies that $r^2 - 2 < 0$, and so $r \in S$. But from (1.3), $r > u_*$, contradicting the fact that u_* is an upper bound of S.

$\underline{2}^\circ$ Suppose that $u_*^2 > 2$. If $r' > r$ (> 0), then $r'^2 = r' \cdot r' > r \cdot r' > r \cdot r = r^2$. From (1.4), $r^2 > 2$, and so from the above, we know that $r'^2 > 2$ as well. Hence $r' \notin S$. So we have shown that if $r' \in S$, then $r' \leq r$. This means that r is an upper bound of S. But this is impossible, since (1.3) shows that $r < u_*$, and u_* is the *least* upper bound of S.

So it must be the case that $u_*^2 = 2$. Note also that u_* is nonnegative (as $u_* \geq 1 \in S$). (We will denote this nonnegative $u_* \in \mathbb{R}$ satisfying $u_*^2 = 2$ by $\sqrt{2}$.) $\diamondsuit$

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Example 1.13 ($\mathbb{Q}$ does not possess the Least Upper Bound Property). Consider the set $S := \{x \in \mathbb{Q} : x^2 \leq 2\}$. Clearly S is a subset of $\mathbb{Q}$ and it is nonempty since $1 \in S$: $1^2 = 1 \leq 2$. Let us show that S is bounded above. In fact, 2 serves as an upper bound of S. Since if $x > 2$, then $x^2 > 4 > 2$. Thus, if $x \in S$, then $x^2 \leq 2$, and so $x \leq 2$.

If $\mathbb{Q}$ has the Least Upper Bound Property, then the above nonempty subset of $\mathbb{Q}$ which is bounded above must possess a least upper bound $u_* := \sup S \in \mathbb{Q}$. Once again, just as in the previous example, we can show that this $u_* \in \mathbb{Q}$ must satisfy that $u_*^2 = 2$ (and we have given the details below). But we know that this is impossible as we had shown that there is no rational number whose square is 2.

First of all, $u_* \geq 1$ (as u_* is in particular an upper bound of S and $1 \in S$). Now define

$$r := u_* - \frac{u_*^2 - 2}{u_* + 2} = \frac{2(u_* + 1)}{u_* + 2} > 0. \tag{1.5}$$

Note also that as $u_* \in \mathbb{Q}$, the rightmost expression for r shows that $r \in \mathbb{Q}$ as well. Then, we have

$$r^2 - 2 = \frac{2(u_*^2 - 2)}{(u_* + 2)^2}. \tag{1.6}$$

$\underline{1}^\circ$ Suppose $u_*^2 < 2$. Then (1.6) implies that $r^2 - 2 < 0$, and so $r \in S$. But from (1.5), $r > u_*$, contradicting the fact that u_* is an upper bound of S.

$\underline{2}^\circ$ Suppose that $u_*^2 > 2$. If $r' > r$ (> 0), then $r'^2 = r' \cdot r' > r \cdot r' > r \cdot r = r^2$. From (1.6), $r^2 > 2$, and so from the above, we know that $r'^2 > 2$ as well. Hence $r' \notin S$. So we have shown that if $r' \in S$, then $r' \leq r$. This means that r is an upper bound of S. But this is impossible, since (1.5) shows that $r < u_*$, and u_* is the *least* upper bound of S.

So it must be the case that $u_*^2 = 2$. But as we mentioned earlier, this is impossible by Theorem 1.1. Hence $\mathbb{Q}$ does not possess the Least Upper Bound Property. ◇

In order to get the useful results in Calculus (for example, the fact that for an increasing sequence of numbers bounded above, there must be a smallest number bigger than each of the terms of the sequence—a fact needed to calculate the area of a circle via the polygons inscribed within it as described in the Introduction), it turns out to be the case that the Least Upper Bound Property is indispensable. So it makes sense that when we set up the definitions and results in Calculus, we do not work with the rational number system $\mathbb{Q}$ (which regrettably does *not* possess the Least Upper Bound Property), but rather with the larger real number system $\mathbb{R}$, which does possess the Least Upper Bound Property.

Exercise 1.6. Let $a_1, a_2, a_3, \cdots$ be an infinite list (or sequence) of real numbers such that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, that is, the sequence is increasing. Also suppose that

$$S := \{a_n : n \in \mathbb{N}\}$$

is bounded above. Show that there is a smallest real number L that is bigger than each of the a_n, $n \in \mathbb{N}$.

Exercise 1.7.

(1) Let S be a nonempty subset of real numbers, which is bounded below. Let $-S$ denote the set of all real numbers $-x$, where x belongs to S. Prove that $\inf S$ exists and $\inf S = -\sup(-S)$.

(2) Conclude from here that $\mathbb{R}$ also has the 'Greatest Lower Bound Property':

$$\boxed{\text{If } S \text{ is a nonempty subset of } \mathbb{R} \text{ having a lower bound, then } \inf S \text{ exists.}}$$

Exercise 1.8. Let S be a nonempty subset of $\mathbb{R}$, which is bounded above, and let $\alpha > 0$. Show that $\alpha \cdot S := \{\alpha x : x \in X\}$ is also bounded above and that $\sup(\alpha \cdot S) = \alpha \cdot \sup S$. Similarly, if S is a nonempty subset of $\mathbb{R}$, which is bounded below and $\alpha > 0$, then show that $\alpha \cdot S$ is bounded below, and that $\inf(\alpha \cdot S) = \alpha \cdot \inf S$.

Exercise 1.9. Let A and B be nonempty subsets of $\mathbb{R}$ that are bounded above and such that $A \subset B$. Prove that $\sup A \leq \sup B$.

Exercise 1.10. For any nonempty bounded set S, prove that $\inf S \leq \sup S$, and that the equality holds if and only if S is a singleton set (that is, a set with cardinality 1).

Exercise 1.11. Let A and B be nonempty subsets of $\mathbb{R}$ that are bounded above. Prove that $\sup(A \cup B)$ exists and that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Exercise 1.12. Determine whether the following statements are true or false.

(1) If u is an upper bound of S ($\subset \mathbb{R}$), and $u' < u$, then u' is not an upper bound of S.

(2) If u_* is the supremum of S ($\subset \mathbb{R}$), and $\epsilon > 0$, then $u_* - \epsilon$ is not an upper bound of S.

(3) Every subset of $\mathbb{R}$ has a maximum.

(4) Every subset of $\mathbb{R}$ has a supremum.

(5) Every bounded subset of $\mathbb{R}$ has a maximum.

(6) Every bounded subset of $\mathbb{R}$ has a supremum.

(7) Every bounded nonempty subset of $\mathbb{R}$ has a supremum.

(8) Every set that has a supremum is bounded above.

(9) For every set that has a maximum, the maximum belongs to the set.

(10) For every set that has a supremum, the supremum belongs to the set.

(11) For every set S that is bounded above, $|S|$ defined by $\{|x| : x \in S\}$ is bounded.

(12) For every set S that is bounded, $|S|$ defined by $\{|x| : x \in S\}$ is bounded.

(13) For every bounded set S, if $\inf S < x < \sup S$, then $x \in S$.

Exercise 1.13. Let A and B be nonempty subsets of $\mathbb{R}$ that are bounded above and define

$$A + B = \{x + y : x \in A \text{ and } y \in B\}.$$

Prove that $\sup(A + B)$ exists and that $\sup(A + B) = \sup A + \sup B$.

Exercise 1.14. Let S be a nonempty set of positive real numbers, and define

$$S^{-1} = \left\{ \frac{1}{x} : x \in S \right\}.$$

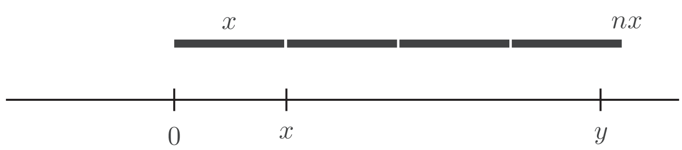
Show that S^{-1} is bounded above if and only if $\inf S > 0$. Furthermore, in case $\inf S > 0$, show that

$$\sup S^{-1} = \frac{1}{\inf S}.$$

We now prove the following theorem, which is called the *Archimedean property* of the real numbers.

Theorem 1.3 (Archimedean Property). *If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that $y < nx$.*

If $y \leq 0$ to begin with, then the above is just the trivial statement that $n \cdot x > 0 \geq y$, which works with every $n \in \mathbb{N}$. So the interesting content of the theorem is when $y > 0$. Then the above is telling us, that no matter how small x is, if we keep 'tiling' the real line with multiples of the length x, then eventually we will surpass y. Here is a picture to bear in mind.



Proof. Suppose that it is not the case that

'there exists an $n \in \mathbb{N}$ such that $nx > y$'.

Then for *every* $n \in \mathbb{N}$, we must have $nx \leq y$. Let $S := \{nx : n \in \mathbb{N}\}$. Then S is a subset of $\mathbb{R}$, $S \neq \emptyset$ (indeed, $x = 1 \cdot x \in S$), and y is an upper bound of S. Thus, by the Least Upper Bound Property of $\mathbb{R}$, $u_* := \sup S$ exists. As $x > 0$, the number $u_* - x$ is smaller than the least upper bound u_* of S. Hence $u_* - x$ cannot be an upper bound of S, which means that there is an element $mx \in S$, for some $m \in \mathbb{N}$, which prevents $u_* - x$ from being an upper bound: $mx > u_* - x$. Rearranging, we obtain $u_* < mx + x = (m + 1)x \in S$, contradicting the fact that u_* is an upper bound of S. Thus, our original claim is false. In other words, there *does exist* an $n \in \mathbb{N}$ such that $nx > y$. $\square$

Example 1.14. Let $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \cdots \right\}$. We claim that $\inf S = 0$.

Clearly 0 is a lower bound of S since all the elements of S are positive.

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Suppose that ℓ is a lower bound of S. We want to show that $\ell \leq 0$. Suppose on the contrary that $\ell > 0$. Then by the Archimedean property (with the real numbers x and y taken as $x = 1$ (> 0) and $y = 1/\ell$), there exists a $n \in \mathbb{N}$ such that

$$\frac{1}{\ell} = y < nx = n \cdot 1 = n,$$

and so

$$\frac{1}{n} < \ell,$$

contradicting the fact that ℓ is a lower bound of S. Thus, any lower bound of S must be less than or equal to 0. Hence 0 is the infimum of S. $\diamondsuit$

Exercise 1.15. Provide the following information about the set S

An upper bound	A lower bound	Is S bounded?	$\sup S$	$\inf S$	$\max S$	$\min S$

where S is given by:

(1) $\left\{ \dfrac{1}{n} : n \in \mathbb{Z} \backslash \{0\} \right\}$

(2) $\left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$

(3) $\left\{ (-1)^n \left(1 + \dfrac{1}{n} \right) : n \in \mathbb{N} \right\}$

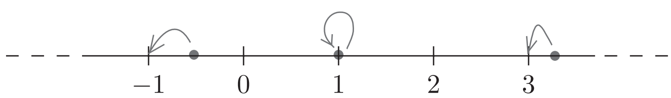
Exercise 1.16. Let $S := \{(xy-1)^2 + x^2 : (x,y) \in \mathbb{R}^2\}$.

(a) Show that S is bounded below.

(b) What is $\inf S$? *Hint:* To justify your answer, consider $(x,y) = (1/n, n)$, $n \in \mathbb{N}$.

(c) Does $\min S$ exist?

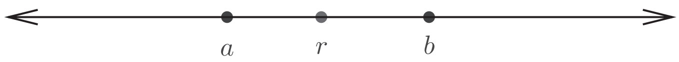
Example 1.15 (The greatest integer part $\lfloor \cdot \rfloor$ of $x \in \mathbb{R}$). If we think of the real numbers as points of the line, then we see that along it, there are 'milestones' at each of the integers. So if we take any real number, then it lies between two milestones. We take $\lfloor x \rfloor$ to be the milestone immediately to the left of x—in other words, it is the 'greatest integer less than or equal to x'. So for example $\lfloor 3.1 \rfloor = 3$, $\lfloor 0 \rfloor = 0$, $\lfloor n \rfloor = n$ for all integers n, $\lfloor -3.1 \rfloor = -4$, etc.



Using the Archimedean Property, one can give a rigorous justification of the fact that every real number *has to* belong to an interval $[n, n+1)$ for some $n \in \mathbb{Z}$ (so that this $n = \lfloor x \rfloor$). By the Archimedean Property, there exists an $m_1 \in \mathbb{N}$ such that $m_1 \cdot 1 > x$. By the Archimedean

Property, there exists an $m_2 \in \mathbb{N}$ such that $m_2 \cdot 1 > -x$. So there are integers m_1, m_2 such that $-m_2 < x < m_1$. Among the finitely many integers $k \in \mathbb{Z}$ such that $-m_2 \leq k \leq m_1$, we take as $\lfloor x \rfloor$ the largest one such that it is also $\leq x$. $\diamondsuit$

Theorem 1.4 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *If $a, b \in \mathbb{R}$, and $a < b$, then there exists a $r \in \mathbb{Q}$ such that $a < r < b$.*



This results says that '$\mathbb{Q}$ is dense in $\mathbb{R}$'. In everyday language, we may say for example, that 'These woods have a dense growth of birch trees', and the picture we then have in mind is that in any small area of the woods, we find a birch tree. A similar thing is conveyed by the above: no matter what 'patch' (described by the two numbers a and b) we take on the real line (thought of as the woods), we can find a rational number (analogous to birch trees) in that patch.

Proof. As $b - a > 0$ and since $1 \in \mathbb{R}$, by the Archimedean Property, there exists an $n \in \mathbb{N}$ such that $n(b-a) > 1$, that is, $na + 1 < nb$. Let $m := \lfloor na \rfloor + 1$. Then $\lfloor na \rfloor \leq na < \lfloor na \rfloor + 1$, that is, $m - 1 \leq na < m$. So

$$a < \frac{m}{n} \leq \frac{na+1}{n} < \frac{nb}{n} = b.$$

With $r := \dfrac{m}{n} \in \mathbb{Q}$, the proof of the theorem is complete. $\square$

Exercise 1.17 (Density of irrationals in $\mathbb{R}$). Show that if $a, b \in \mathbb{R}$ and $a < b$, then there exists an irrational number between a and b.

1.5 Rational powers of real numbers

Definition 1.8 (Integral powers of nonzero real numbers).

(1) Given $a \in \mathbb{R}$ and $n \in \mathbb{N}$, we define $a^n \in \mathbb{R}$ by $a^n := \underbrace{a \cdot a \ \cdots \ a}_{n \text{ times}}$.

(2) If $a \in \mathbb{R}$ and $a \neq 0$, then we define $a^0 := 1$.

(3) If $a \in \mathbb{R}$, $a \neq 0$ and $n \in \mathbb{N}$, then we define $a^{-n} := \left(\dfrac{1}{a}\right)^n$.

In this manner, all integral powers of nonzero real numbers is defined, and it can be checked that the following *laws of exponents* hold:

(E1) For all $a, b \in \mathbb{R}$ and all $n \in \mathbb{Z}$, with $a, b \neq 0$ if $n \leq 0$, $(ab)^n = a^n b^n$.

(E2) For all $a \in \mathbb{R}$, all $m, n \in \mathbb{Z}$, with $a \neq 0$ if $m \leq 0$ or $n \leq 0$, $(a^m)^n = a^{mn}$ and $a^{m+n} = a^m a^n$.

(E3) For all $a, b \in \mathbb{R}$ with $0 \leq a < b$ and $n \in \mathbb{N}$, $a^n < b^n$.

For example, (E3) can be shown like this: If $0 \leq a < b$, then we have

$$\begin{aligned} a^2 &= a \cdot a < a \cdot b < b \cdot b = b^2, \\ a^3 &= a^2 \cdot a < b^2 \cdot a < b^2 \cdot b = b^3, \\ a^4 &= a^3 \cdot a < b^3 \cdot a < b^3 \cdot b = b^4, \text{ and so on.} \end{aligned}$$

One can also define fractional powers of positive real numbers. First we have the following:

Theorem 1.5 (Existence of nth roots). *For every $a \in \mathbb{R}$ with $a \geq 0$ and every $n \in \mathbb{N}$, there exists a unique $b \in \mathbb{R}$ such that $b \geq 0$ and $b^n = a$.*

This unique b is called the *nth root of a*, and is denoted by $a^{1/n}$ or $\sqrt[n]{a}$.

Proof. We will skip the details of the proof, which is similar to what we did to show that $\sqrt{2}$ exists in Example 1.12: the number b we seek is the supremum u_* of the set

$$S_a := \{x \in \mathbb{R} : x^n \leq a\},$$

and u_* can be shown to exist by using the Least Upper Bound Property of $\mathbb{R}$. □

Definition 1.9 (Fractional powers of positive real numbers). If $r \in \mathbb{Q}$ and

$$r = \frac{m}{n},$$

where $m, n \in \mathbb{Z}$ and $n > 0$, then for $a \in \mathbb{R}$ such that $a > 0$, we define

$$a^r := (a^m)^{1/n}.$$

It can be shown that if

$$r = \frac{m}{n} = \frac{p}{q},$$

with $p, q \in \mathbb{Z}$ and $q > 0$, then $(a^p)^{1/q} = (a^m)^{1/n}$, so that our notion of raising to rational powers is 'well-defined', that is, it does not depend on *which* particular integers m, n we take in the representation of the rational number r.

Later on, after having studied the logarithm function in Chapter 5, we will also extend the above definitions consistently to the case of real powers of positive real numbers.

1.6 Intervals

In Calculus, we will consider real-valued functions of a real variable, and develop results about these. It will turn out while doing so that we will keep meeting certain types of subsets of the real numbers (for example, subsets of this type will often be the 'domains' of our real-valued functions for which the results of Calculus hold). These special subsets of $\mathbb{R}$ are called 'intervals', and we give the definition below. Roughly speaking, these are the 'connected subsets' of the real line, namely subsets of $\mathbb{R}$ not having any 'holes/gaps'.

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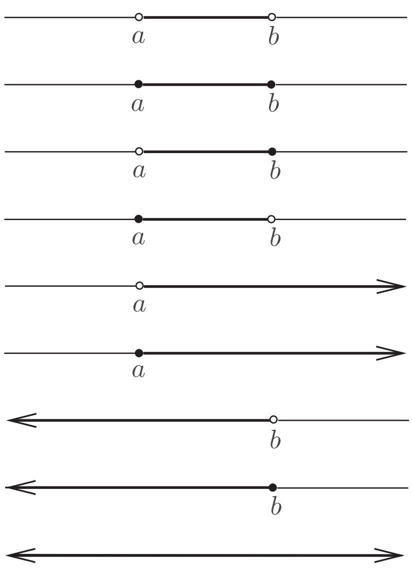
Definition 1.10 (Interval). An *interval* is a set consisting of all the real numbers between two given real numbers, or of all the real numbers on one side or the other of a given number. So an interval is a set of any of the following forms, where $a, b \in \mathbb{R}$:

$$(a,\, b) = \{x \in \mathbb{R} : a < x < b\}$$
$$[a,\, b] = \{x \in \mathbb{R} : a \le x \le b\}$$
$$(a,\, b] = \{x \in \mathbb{R} : a < x \le b\}$$
$$[a,\, b) = \{x \in \mathbb{R} : a \le x < b\}$$
$$(a,\, \infty) = \{x \in \mathbb{R} : a < x\}$$
$$[a,\, \infty) = \{x \in \mathbb{R} : a \le x\}$$
$$(-\infty,\, b) = \{x \in \mathbb{R} : x < b\}$$
$$(-\infty,\, b] = \{x \in \mathbb{R} : x \le b\}$$
$$(-\infty,\, \infty) = \mathbb{R}$$



In the above notation for intervals, a parenthesis '(' or ')' means that the respective endpoint is not included, and a square bracket '[' or ']' means that the endpoint is included. Thus $[0, 1)$ means the set of all real numbers x such that $0 \le x < 1$. (Note that the use of the symbol ∞ in the notation for intervals is simply a matter of convenience and is not be taken as suggesting that there is a number ∞.)

Also, it will be convenient to give certain types of interval a special name.

Definition 1.11 (Open interval). An interval of the form (a, b), (a, ∞), $(-\infty, b)$, or $\mathbb{R}$ is called an *open interval*.

We note that if I is an open interval, then for every member $x \in I$, there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset I$, that is, there is always some 'room' around x consisting only of elements of I.

Exercise 1.18. Show that if $a, b \in \mathbb{R}$, then the interval (a, b) has the following property:

$$\text{for every } x \in (a, b), \text{ there exists a } \delta > 0 \text{ such that } (x - \delta, x + \delta) \subset (a, b).$$

Show also that $[a, b]$ does not possess the above property.

Definition 1.12 (Compact interval). If $a, b \in \mathbb{R}$ and $a \le b$, then we call $[a, b]$ a *compact interval*.

Note that $\mathbb{R}\backslash[a, b]$ is the union of two open intervals, namely $(-\infty, a)$ and (b, ∞) and that $[a, b]$ is a bounded set.

Exercise 1.19. If A_n, $n \in \mathbb{N}$, is a collection of sets, then $\bigcap_{n\in\mathbb{N}} A_n$ denotes their intersection:

$$\bigcap_{n\in\mathbb{N}} A_n = \{x : \forall n \in \mathbb{N},\ \ x \in A_n\},$$

and $\bigcup_{n\in\mathbb{N}} A_n$ denotes their union: $\bigcup_{n\in\mathbb{N}} A_n = \{x : \exists n \in \mathbb{N} \text{ such that } x \in A_n\}$. Prove that

(1) $\emptyset = \bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right)$.

(2) $\{0\} = \bigcap_{n\in\mathbb{N}} \left[0, \frac{1}{n}\right]$.

(3) $(0, 1) = \bigcup_{n\in\mathbb{N}} \left[\frac{1}{n+2}, 1 - \frac{1}{n+2}\right]$.

(4) $[0, 1] = \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$.

1.7 Absolute value $|\cdot|$ and distance in $\mathbb{R}$

In Calculus, in order to talk about notions such as *rate of change, continuity, convergence, etc*, we will need a notion of 'closeness/distance' between real numbers. This is provided by the absolute value $|\cdot|$, and the distance between real numbers x and y is $|x - y|$. We give the definitions below.

Definition 1.13 (Absolute value and distance).

(1) The *absolute value* or *modulus* of a real number x is denoted by $|x|$, and it is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

(2) The *distance* $d(x, y)$ between two real numbers x and y is the absolute value $|x - y|$ of their difference.

Thus, $|1| = 1$, $|0| = 0$, $|-1| = 1$, and the distance between the real numbers -1 and 1 is equal to $d(-1, 1) = |-1 - 1| = |-2| = 2$. The distance gives a notion of closeness of two points, which is crucial in the formalisation of the notions of analysis. We can now specify regions comprising points close to a certain point $c \in \mathbb{R}$ in terms of inequalities in absolute values, that is, by demanding that the distance of the points of the region, to the point c, is less than a certain positive number δ, say $\delta = 0.01$ or $\delta = 0.0000001$, and so on.

Theorem 1.6. *Let $c \in \mathbb{R}$ and $\delta > 0$. Then:*

$$\boxed{d(x, c) := |x - c| < \delta} \quad \Leftrightarrow \quad \boxed{c - \delta < x < c + \delta.}$$

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Although the proof is trivial, it is worthwhile remembering Theorem 1.6, as such a manipulation will keep arising over and over again in our subsequent development of Calculus. See Figure 1.2.

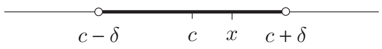

Figure 1.2 The interval $I = (c-\delta, c+\delta) = \{x \in \mathbb{R} : |x-c| < \delta\}$ is the set of all points in $\mathbb{R}$ whose distance to the point c is strictly less than δ (> 0).

Proof.
($\Rightarrow$) Suppose that $|x-c| < \delta$. Then $x - c \le |x-c| < \delta$, and $-(x-c) \le |x-c| < \delta$. So $-\delta < x - c < \delta$, that is, $c - \delta < x < c + \delta$.

($\Leftarrow$) If $c-\delta < x < c+\delta$, then $x - c < \delta$ *and* $-(x-c) = c - x < \delta$. Thus $|x-c| < \delta$, because $|x-c|$ is either $x-c$ or $-(x-c)$, and in both cases the numbers are less than δ. □

If we think of the real numbers as points on the number line, and we think about the integers as milestones, then it is clear that the distance between, say -1 and 3 should be 4 miles, and we observe that $4 = |-1-3|$. So taking $|x-y|$ as the distance between $x, y \in \mathbb{R}$ is a sensible thing to do, based on our visual picture of $\mathbb{R}$ as points on the number line (Figure 1.3).

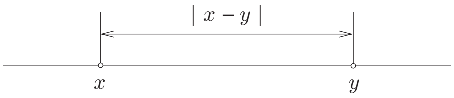

Figure 1.3 Distance between real numbers.

Exercise 1.20. Show that a subset S of $\mathbb{R}$ is bounded if and only if there exists an $M \in \mathbb{R}$ such that for all $x \in S$, $|x| \le M$.

The following properties of the absolute value will be useful in the sequel.

Theorem 1.7. *If x, y are real numbers, then*

$$|x \cdot y| = |x| \cdot |y|, \tag{1.7}$$

$$|x + y| \le |x| + |y|. \tag{1.8}$$

(1.8) is called the *triangle inequality*.

Proof. We prove (1.7) by exhausting all possible cases:

$\underline{1}^\circ$ $x = 0$ or $y = 0$. Then $|x| = 0$ or $|y| = 0$, and so $|x|\,|y| = 0$. On the other hand, as $x = 0$ or $y = 0$, it follows that $xy = 0$ and so $|xy| = 0$.

$\underline{2}^\circ$ $x > 0$ and $y > 0$. Then $|x| = x$ and $|y| = y$, and so $|x|\,|y| = xy$. On the other hand, as $x > 0$ and $y > 0$, it follows that $xy > 0$ and so $|xy| = xy$.

$\underline{3}^\circ$ $x > 0$ and $y < 0$. Then $|x| = x$ and $|y| = -y$, and so $|x|\,|y| = x(-y) = -xy$. On the other hand, as $x > 0$ and $y < 0$, it follows that $xy < 0$ and so $|xy| = -xy$.

$\underline{4}^\circ$ $x < 0$ and $y > 0$. This follows from 3° above by interchanging x and y.

$\underline{5}^\circ$ $x < 0$ and $y < 0$. Then $|x| = -x$ and $|y| = -y$, and so $|x|\ |y| = (-x)(-y) = xy$. On the other hand, as $x < 0$ and $y < 0$, it follows that $xy > 0$ and so $|xy| = xy$.

This proves (1.7).

Next we prove (1.8). First observe that from the definition of $|\cdot|$, it follows that for any real $x \in \mathbb{R}$, $|x| \geq x$: indeed if $x \geq 0$, then $|x| = x$, while if $x < 0$, then $-x > 0$, and so we have that $|x| = -x > 0 > x$.

From (1.7), we also have $|-x| = |-1 \cdot x| = |-1||x| = 1|x| = |x|$, for all $x \in \mathbb{R}$, and so it follows that $|x| = |-x| \geq -x$ for all $x \in \mathbb{R}$.

We have the following cases:

$\underline{1}^\circ$ $x + y \geq 0$. Then we have that $|x + y| = x + y$. Since $|x| \geq x$ and $|y| \geq y$, we obtain $|x| + |y| \geq x + y = |x + y|$.

$\underline{2}^\circ$ $x + y < 0$. Then $|x + y| = -(x + y)$. Since $|x| \geq -x$ and $|y| \geq -y$, it follows that $|x| + |y| \geq -x + (-y) = -(x + y) = |x + y|$.

This proves (1.8). $\square$

Using these, it is easy to check that the 'metric/distance function' defined by

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R},$$

satisfies the following properties:

(D1) (Positive definiteness) For all $x, y \in \mathbb{R}$, $d(x, y) \geq 0$. If $d(x, y) = 0$ then $x = y$.

(D2) (Symmetry) For all $x, y \in \mathbb{R}$, $d(x, y) = d(y, x)$.

(D3) (Triangle inequality) For all $x, y, z \in \mathbb{R}$, $d(x, z) \leq d(x, y) + d(y, z)$.

The reason (D3) is called the triangle inequality is that, for triangles in Euclidean geometry of the plane, we know that the sum of the lengths of two sides of a triangle is at least as much as the length of the third side: so for the points X, Y, Z in a plane forming the three vertices of a triangle: we know that $\ell(XZ) \leq \ell(XY) + \ell(YZ)$; see Figure 1.4. (D3) reminds us of this triangle inequality, and hence the name.

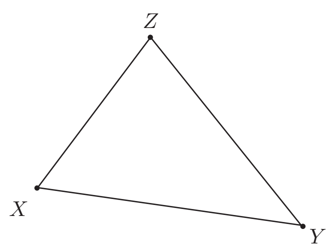

Figure 1.4 How the triangle inequality gets its name.

Exercise 1.21. Prove that if x, y are real numbers, then $||x| - |y|| \leq |x - y|$.

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Exercise 1.22 (When does equality hold in the triangle inequality?).

(1) Show the generalised triangle inequality: if $n \in \mathbb{N}$ and $a_1, \cdots, a_n$ are real numbers, then $|a_1 + \cdots + a_n| \le |a_1| + \cdots + |a_n|$.

(2) $(*)$ We say that the numbers $a_1, \cdots, a_n$ *have the same sign* if either of the following two cases is true:

1° $a_1 \ge 0, \cdots, a_n \ge 0$.
2° $a_1 \le 0, \cdots, a_n \le 0$.

In other words, the numbers have the same sign if on the number line either they all lie on the right of 0 including 0, or they all lie on the left of 0 including 0. Show that equality holds in the generalised triangle inequality if and only if the numbers have the same sign.
Hint: Consider the $n = 2$ case first.

Exercise 1.23. For $a, b \in \mathbb{R}$, show that $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ and $\min\{a, b\} = \frac{a+b-|a-b|}{2}$.

1.8 $(*)$ Remark on the construction of $\mathbb{R}$

Natural numbers

Although we get familiar with the numbers $0, 1, 2, 3, \cdots$ from an early age, we don't learn its abstract construction in elementary school. Such an abstract construction can be given using set theory. One associates

$$\begin{aligned} &0 \text{ with the empty set } \emptyset,\\ &1 \text{ with } \{\emptyset\},\\ &2 \text{ with } \{\emptyset, \{\emptyset\}\},\\ &3 \text{ with } \{\emptyset, \{\emptyset, \{\emptyset\}\}\},\\ &\text{ and so on.} \end{aligned}$$

In this manner, we obtain $0, 1, 2, 3, \cdots$, in other words, the set $\mathbb{N} \bigcup \{0\}$, and one can also define addition via a successor function and establish the usual arithmetic laws of addition (commutativity, associativity etc.).

Integers

We can introduce the integers as pairs (m, n), where $m, n \in \mathbb{N} \bigcup \{0\}$, where (m, n) and (a, b) are considered to be defining the same integer if

$$m + b = n + a.$$

Then $n \in \mathbb{N} \bigcup \{0\}$ can be identified with $(n, 0) \in \mathbb{Z}$ and $(0, n) \in \mathbb{Z}$ is thought of as the non-positive integer $-n$, $n \in \mathbb{N} \bigcup \{0\}$. So -1 is $(0, 1) = (2, 3) = (1975, 1976)$, and so on.

Rational numbers

The rational numbers $\mathbb{Q}$ can be defined using pairs of integers, where the second integer is not zero, and (m,n), (a,b) are considered identical if $mb = na$.

Real numbers

What about the construction of the real number system $\mathbb{R}$?

In this book, we treat the real number system $\mathbb{R}$ as a given. But one might wonder if we can take the existence of real numbers on faith alone. It turns out that a mathematical proof of its existence can be given.

There are several ways of doing this. One is by a method called 'completion of $\mathbb{Q}$', where one considers 'Cauchy sequences' in $\mathbb{Q}$, and defines $\mathbb{R}$ to be 'equivalence classes of Cauchy sequences under a certain equivalence relation'. We refer the interested student to [**S2**, Problem 1, p. 588] or [**R**, Exercises 24, 25, p. 82] for details about this.

Another way, which is more intuitive, is via '(Dedekind) Cuts', where we identify each real number by means of two sets A and B associated with it: A is the set of rationals less than the real number we are defining, and B is set of rational numbers at least as big as the real number we are trying to identify. In other words, if we view the rational numbers lying on the number line, and think of the sets A and B (described above) corresponding to a real number, then this real number is the place along this rational number line where it can be cut, with A lying on the left side of this cut, and B lying on the right side of this cut. See Figure 1.5. More precisely, a *cut* (A,B) in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that $A \bigcup B = \mathbb{Q}$, $A \neq \emptyset$, $B \neq \emptyset$, $A \bigcap B = \emptyset$, if $a \in A$ and $b \in B$ then $a < b$, and A contains no largest element. $\mathbb{R}$ is then taken as the set of all cuts (A,B). Here are two examples of cuts:

$$(A,B) = \Big(\{r \in \mathbb{Q} : r < 0\},\ \{r \in \mathbb{Q} : r \geq 0\}\Big) \quad \text{(giving the real number '0')}$$

$$(A,B) = \Big(\{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\},\ \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \geq 2\}\Big) \quad \text{('}\sqrt{2}\text{')}.$$

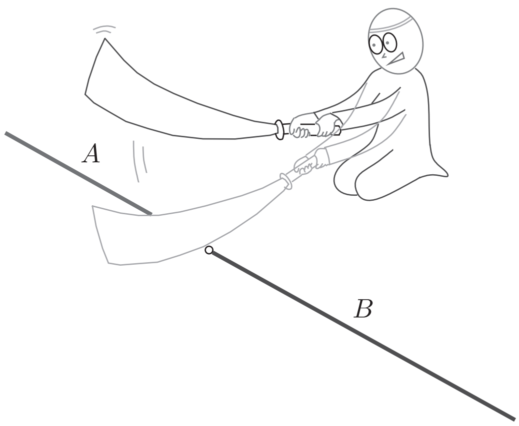

Figure 1.5 Dedekind cut.

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It turns out that $\mathbb{R}$ is a field containing $\mathbb{Q}$, and it can be shown to possess the Least Upper Bound Property. The interested reader is referred to the Appendix to Chapter 1 in the classic textbook by Walter Rudin [**R**].

1.9 Functions

The concept of a 'function' is fundamental in Mathematics and in particular in Calculus. So in this section, we will quickly review:

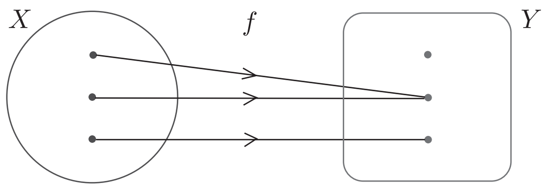
(1) the definition of a function, and standard terminology associated with functions, such as the domain/codomain/range of a function, injective/one-to-one functions, surjective/onto functions, bijective functions/one-to-one correspondences, graph of a function;

(2) Cartesian geometry (which will allow us to visualise functions $f : D\ (\subset \mathbb{R}) \to \mathbb{R}$, by looking at their graphs);

(3) some examples.

Informal view of functions

Let X, Y be sets. A function $f : X \to Y$ is a rule that sends each $x \in X$ to one and only one corresponding point $f(x) \in Y$.

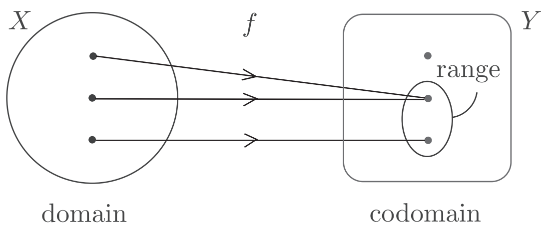


Some terminology:

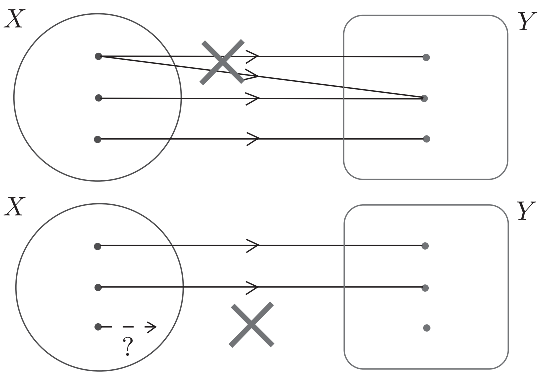
(1) The set X is called the *domain of* f.

(2) The set Y is called the *codomain of* f.

(3) The set $\{y \in Y :$ there exists an $x \in X$ such that $y = f(x)\}$ is called the *range of* f. Note that the range of f is a subset of the codomain.



We disqualify rules that assign multiple points of Y to a point of X, and also those that miss out assigning points of Y to some points of X, from being legitimate functions.



Example 1.16. Let

$$X := \{\text{all students in the classroom}\},$$
$$Y := \mathbb{R},$$

and $f : X \to Y$ be given by $f(x) =$ height in centimeters of student x, $x \in X$. Then f is a function. Indeed, each person has a unique height: one person can't have two heights. Note that there can exist of course two persons having the same height. The domain of this function is the set X of all students in the classroom, and the codomain is the set of *all* real numbers. On the other hand, it is clear that the range of f is a much smaller subset of $\mathbb{R}$: it is the finite set consisting of the heights of the students in the classroom. ♢

Formal definition of a function

Let X, Y be sets. A *function* $f : X \to Y$ is a subset R of

$$X \times Y := \{(x, y) : x \in X,\ y \in Y\}$$

with the following two properties:

(1) For every $x \in X$, there exists a $y \in Y$ such that $(x, y) \in R$.

(2) If (x_1, y_1) and (x_2, y_2) belong to R, and if $x_1 = x_2$, then $y_1 = y_2$.

In plain English, the first requirement above, says that each x in X is sent by f to *some* element of Y (so that no elements of X are 'left out' by the function f), and the second requirement says each element of X is sent to only *one* corresponding element of Y (that is, it is not the case that some element of X is sent to more than one element of Y).

Functions are sometimes also called *maps* or *mappings*. We say for a function $f : X \to Y$ that 'f maps X to Y', and if $x \in X$, then we also say 'f maps x to $f(x)$', written

$$x \mapsto f(x) \ \text{ or } \ x \overset{f}{\mapsto} f(x).$$

In (one variable) Calculus, usually $X, Y \subset \mathbb{R}$.

Exercise 1.24. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = 1 + x^2,$$
$$g(x) = 1 - x^2,$$

$x \in \mathbb{R}$. Compute the following:

(1) $f(3) + g(3)$.
(2) $f(3) - 3 \cdot g(3)$.
(3) $f(3) \cdot g(3)$.
(4) $(f(3))/(g(3))$.
(5) $f(g(3))$.
(6) For $a \in \mathbb{R}$, $f(a) + g(-a)$.
(7) For $t \in \mathbb{R}$, $f(t) \cdot g(-t)$.

Classification of functions

We will now learn about three important classes of functions:

(1) injective or one-to-one functions,
(2) surjective or onto functions, and
(3) bijective functions or one-to-one correspondences.

Let $f : X \to Y$. Then f is called

Injective/ One-to-one	*Surjective/Onto*	*Bijective/ One-to-one correspondence*
if for every x_1 and x_2 in X such that $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.	if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.	if it is both injective and surjective.
'Distinct points have distinct images'.	'Codomain=Range'	

See Figure 1.6.

Example 1.17. The height function $f : X \to \mathbb{R}$ we considered earlier in Example 1.16 will not be one-to-one whenever the set X contains two students having the same height. On the other hand, the height function will be injective if all the students have distinct heights.

Also, the height function is clearly not onto. For example, there is no student whose height is $-399 \in Y = \mathbb{R}$!

As the height function is not onto, it cannot be bijective either. $\diamondsuit$

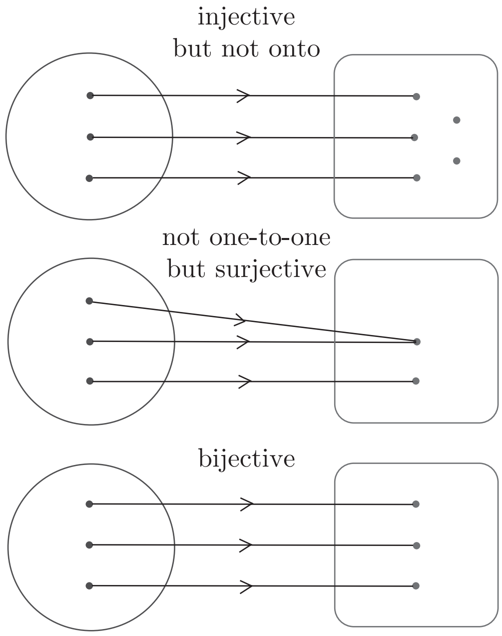


Figure 1.6 Injective/one-to-one functions, surjective/onto functions, bijective functions/one-to-one correspondences.

Example 1.18.

(1) Consider function $f_1 : \mathbb{R} \to [0, \infty)$ given by $f_1(x) = x^2$, $x \in \mathbb{R}$.

Then f_1 is not one-to-one (for example, because $f_1(-1) = 1 = f_1(1)$).
But f_1 is onto, since for every $y \in [0, \infty)$, $x := \sqrt{y} \in \mathbb{R}$ and $f_1(x) = f_1(\sqrt{y}) = (\sqrt{y})^2 = y$.

(2) Consider the function $f_2 : [0, \infty) \to \mathbb{R}$ given by $f_2(x) = \sqrt{x}$, $x \in [0, \infty)$.

Then f_2 is injective, because if $f_2(x_1) = f_2(x_2)$ for some $x_1, x_2 \geq 0$, then $\sqrt{x_1} = \sqrt{x_2}$, and so $x_1 = (\sqrt{x_1})^2 = (\sqrt{x_2})^2 = x_2$.
But f_2 is not surjective, since f_2 never assumes negative values.

(3) The function $f_3 : \mathbb{R} \to \mathbb{R}$ given by $f_3(x) = 2x$, $x \in \mathbb{R}$ is a bijection.

Indeed, f_3 is injective (since if $f_3(x_1) = f_3(x_2)$ for some $x_1, x_2 \in \mathbb{R}$, then $2x_1 = 2x_2$, and so $x_1 = x_2$), and f_3 is surjective (if $y \in \mathbb{R}$, then $f_3(y/2) = 2 \cdot (y/2) = y$). $\diamondsuit$

Exercise 1.25. Show that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x|x|$, $x \in \mathbb{R}$, is a one-to-one correspondence.

Graph of a function. Review of Cartesian geometry

Definition 1.14 (Graph of a function). Let $f : X \to Y$ be a function. Then the *graph of f* is the set $\{(x, f(x)) : x \in X\}$.

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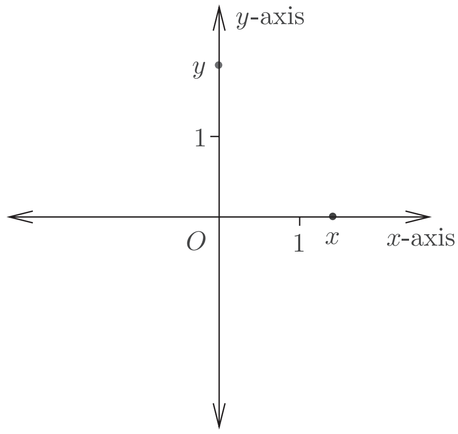
The graph of f is a subset of $X \times Y$. When X and Y are both subsets of $\mathbb{R}$, then we can visualise the function f by 'plotting' its graph in the Cartesian plane $\mathbb{R}^2$. Let us recall how this is done.

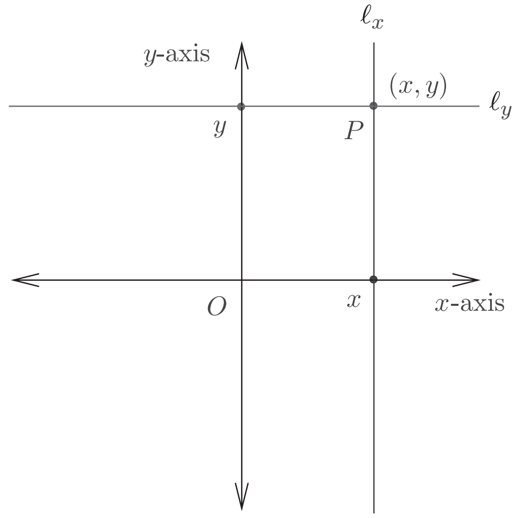
First of all the word 'Cartesian' comes from the name of the mathematician De**cartes** (16th century AD), who described points in the plane with two real numbers, we recall this below.

We first draw two mutually perpendicular lines in the plane, intersecting at a point O called the *origin*. The horizontal line is called the x-axis, and the vertical line is called the *y-axis*.



We choose unit lengths along the x-axis and y-axis, we label the number 1 on the x-axis to the right of the origin, and we label the number 1 on the y-axis above the origin. Thus, any $x \in \mathbb{R}$ is determined on the x-axis, and any $y \in \mathbb{R}$ is determined on the y-axis.

Any point $P = (x, y) \in \mathbb{R} \times \mathbb{R} =: \mathbb{R}^2$ can be depicted in the Cartesian plane by taking it to be the intersection point of the vertical line ℓ_x passing through the point x on the x-axis, and of the horizontal line ℓ_y passing through the point y on the y-axis.



The number x is called the *x-coordinate* of $P = (x, y)$, and the number y is called the *y-coordinate* of $P = (x, y)$.

Exercise 1.26. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a graph as shown in Figure 1.7.
Sketch the graphs of the functions $g_1, g_2, g_3, g_4, g_5, g_6 : \mathbb{R} \to \mathbb{R}$, defined for $x \in \mathbb{R}$ by

$$\begin{aligned} g_1(x) &:= f(x+1), \\ g_2(x) &:= f(x-1), \\ g_3(x) &:= f(2x), \\ g_4(x) &:= f(x/2), \\ g_5(x) &:= f(-x), \\ g_6(x) &:= -f(x). \end{aligned}$$

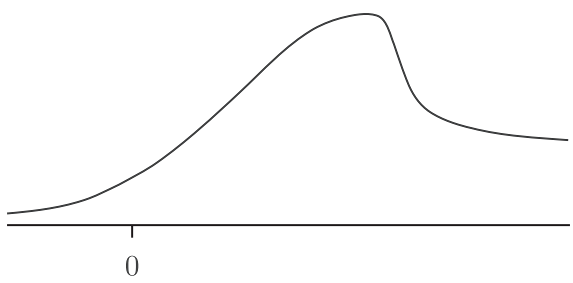
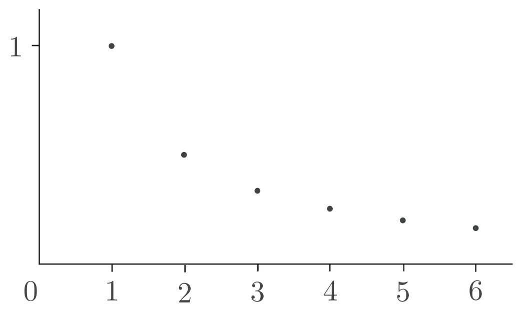


Figure 1.7 The graph of f.

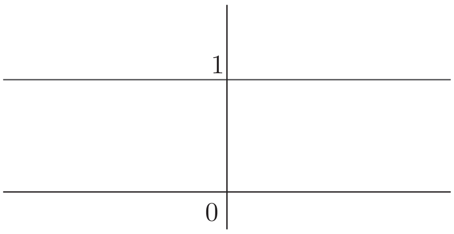
Here are a few examples of functions and their graphs.

Example 1.19.

(1) (Sequence) Let $f_1 : \mathbb{N} \to \mathbb{R}$ be given by $f_1(n) = \dfrac{1}{n}$, $n \in \mathbb{N}$.



(2) (Constant function) Let $f_2 : \mathbb{R} \to \mathbb{R}$ be given by $f_2(x) = c$, $x \in \mathbb{R}$. (Here $c \in \mathbb{R}$ is fixed, say $c = 1$.)



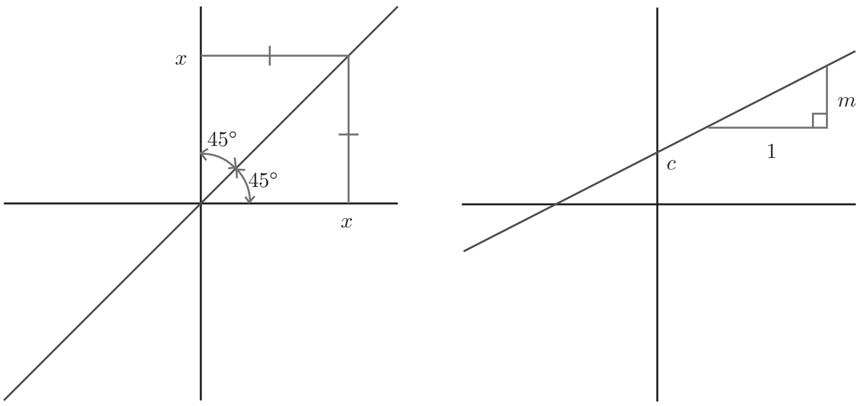
(3) (Identity function) Let $f_3 : \mathbb{R} \to \mathbb{R}$ be given by $f_3(x) = x$, $x \in \mathbb{R}$. See the picture on the left in the following figure. More generally, a *linear function* $L : \mathbb{R} \to \mathbb{R}$ is a function of the form

$$L(x) = mx + c, \quad x \in \mathbb{R},$$

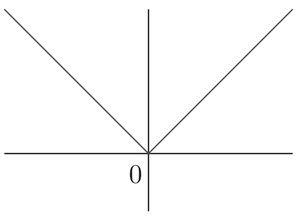
for some constants m and c. The number m is called the *slope of* L, and c is referred to as the *y-axis intercept*. Note that $L(0) = c$, so that the graph of L does intersect the y-axis at the point $(0, c)$ in the Cartesian plane. Also, we note that for all distinct real numbers x_2, x_1,

$$\frac{L(x_2) - L(x_1)}{x_2 - x_1} = \frac{mx_2 + c - (mx_1 + c)}{x_2 - x_1} = m.$$

See the picture on the right below.



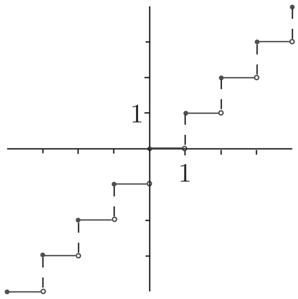
(4) (Absolute value/modulus function) Consider the absolute value/modulus function $|\cdot|$ from $\mathbb{R}$ to $\mathbb{R}$, $x \mapsto |x|$, $x \in \mathbb{R}$. The graph has a ‘corner’ at $x = 0$:

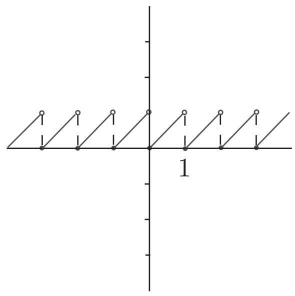


(5) (Integer and fractional part) Consider the greatest integer part function $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{R}$, given by $x \mapsto \lfloor x\rfloor$, $x \in \mathbb{R}$. There are 'jumps' or 'discontinuities' at the integer points. Similarly, one can define the *fractional part* $\{\cdot\} : \mathbb{R} \to \mathbb{R}$ by

$$\{x\} := x - \lfloor x\rfloor, \quad x \in \mathbb{R}.$$

For example, we have $\{-3.05\} = -3.05 - (-4) = 0.95$, $\{-3\} = -3 - (-3) = 0$, and $\{3.05\} = 3.05 - 3 = 0.05$.





$\diamondsuit$

Plotting with `Maple`

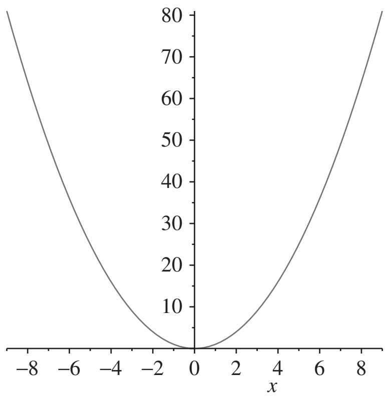
Using a computer package like `Maple` (or `Mathematica`), it is possible to plot the graphs of functions for specified intervals. The basic syntax of the plot command is

```
> plot(f,range);
```

There are many options to the plot command, and the best way to get familiar with this is to experiment with it, and to use the 'help' option. For example,

```
> plot(x^2, x = -9..9);
```

displays the graph of $x \mapsto x^2$ for $x \in [-9, 9]$; see the picture below. In the above command, we indicated the range of x by writing '`x=-9..9`'. Maple automatically chooses a scale on the vertical axis.



Polynomial functions

The simplest functions in Calculus are the constant function $x \mapsto 1$, the identity function given by $x \mapsto x$, and pointwise products of this, namely the *power functions* $x \mapsto x^n$, where $n \in \mathbb{N}$ is fixed. Linear combinations of these are called the *polynomials*, that is, a polynomial $p : \mathbb{R} \to \mathbb{R}$ is a function

$$p(x) = c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3 + \cdots + c_d \cdot x^d, \quad x \in \mathbb{R},$$

where the *coefficients* are the (fixed) numbers $c_0, c_1, c_2, c_3, \cdots, c_d \in \mathbb{R}$, and $c_d \neq 0$. $d \in \{0, 1, 2, 3, \cdots\}$ is then called the *degree of p*. For example,

$$x^6 - 3 \cdot x^4 + 2 \cdot x^2 - \frac{1}{3},$$

is a polynomial of degree 6. If all the coefficients are zeros, then we say that p is the *zero polynomial*, and its degree is taken to be 0.

Exercise 1.27. Use `Maple` (or an equivalent computer program) to plot the graph of the polynomial p, where

$$p(x) = x^6 - 3 \cdot x^4 + 2 \cdot x^2 - \frac{1}{3},$$

for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$. Can you explain the symmetry in the resulting picture?

Rational functions

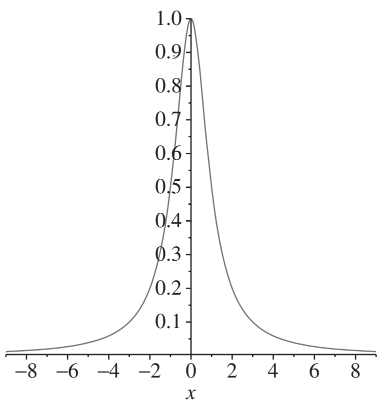
A function $r : D\ (\subset \mathbb{R}) \to \mathbb{R}$ of the form

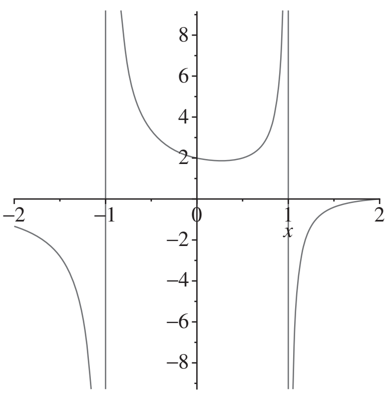
$$r(x) = \frac{n(x)}{d(x)}, \quad x \in D,$$

where n, d are fixed polynomials and $D := \mathbb{R} \backslash \{\zeta \in \mathbb{R} : d(\zeta) = 0\}$, is called a *rational function*. The polynomial n is called the *numerator polynomial of r*, and d is called the *denominator polynomial of r*. For example,

$$x \mapsto \frac{1}{1 + x^2} : \mathbb{R} \to \mathbb{R} \quad \text{and} \quad x \mapsto \frac{2 - x}{1 - x^2} : \mathbb{R} \backslash \{-1, 1\} \to \mathbb{R}$$

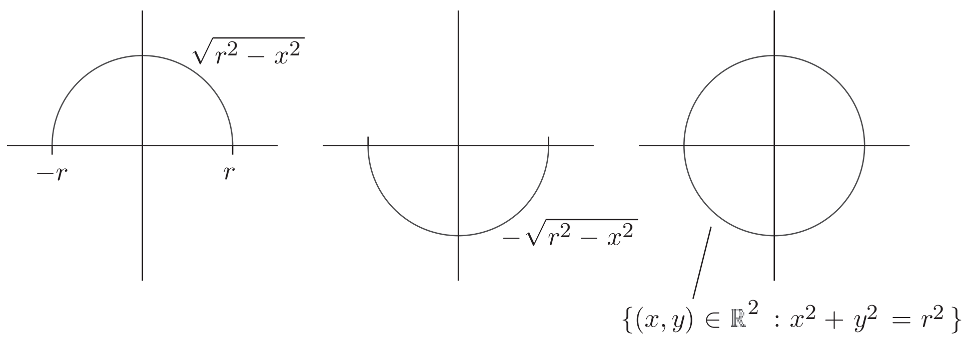
are rational functions. The graphs are displayed below.





Example 1.20 (The circle). Consider all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 = r^2$ for some fixed $r > 0$. Then for $x \in [-r, r]$, $y = \sqrt{r^2 - x^2}$ or $y = -\sqrt{r^2 - x^2}$. (Recall that if $a \geq 0$, then $\sqrt{a}$ denotes the unique positive square root of a. For example, $\sqrt{9} = 3$.) So we can view the circle as made up of the graphs of *two* functions:

$$f_+ : [-r, r] \to \mathbb{R} \text{ given by } f_+(x) = \sqrt{r^2 - x^2}, \quad x \in [-r, r], \text{ and}$$
$$f_- : [-r, r] \to \mathbb{R} \text{ given by } f_-(x) = -\sqrt{r^2 - x^2}, \quad x \in [-r, r]. \qquad \diamondsuit$$



Polynomials and rational functions, and related functions such as the square root function, and their combinations, are loosely called *algebraic functions*. Later on in Chapter 5, we will learn about some other functions that often arise in applications, such as the logarithm, the exponential, and trigonometric functions, and these are examples of 'non-algebraic' functions, or 'transcendental functions'.

Inverse functions

If one has a bijective function f from X to Y, then we can imagine a picture where points from X are taken to points in Y by f. But now if we start from any point y in Y, since the function is surjective, there has to be a point x in X which is sent to $y \in Y$, and moreover, since f is injective, we know that this x is unique. So we can 'reverse the arrow' that takes x to y under f. In this way, we get a new rule/function that takes elements from Y to elements in X, by just reversing all the old arrows of the bijective f (taking elements of X to those in Y). This map is called the 'inverse function of f', denoted by f^{-1}. We summarise this below.

If $f : X \to Y$ is a bijection, then the *inverse function*

$$f^{-1} : Y \to X$$

is defined as follows. Given $y \in Y$, there is *an* $x \in X$ such that $f(x) = y$ (since f is surjective), and moreover, this x is *unique* (since f is injective). So for $y \in Y$, we define $f^{-1}(y) := x$, where x is the *unique* element in X such that $f(x) = y$. It is easy to see that

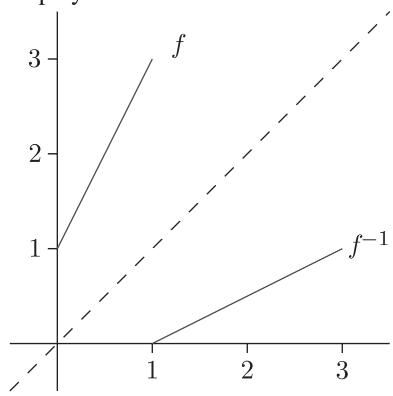
$$f(f^{-1}(y)) = y \text{ for all } y \in Y, \text{ and}$$
$$f^{-1}(f(x)) = x \text{ for all } x \in X.$$

Example 1.21. Let $f : [0, 1] \to [1, 3]$ be given by $f(x) = 2x + 1, 0 \leq x \leq 1$. It can be checked that f is bijective. Then the inverse of f is $f^{-1} : [1, 3] \to [0, 1]$, given by

$$f^{-1}(y) = \frac{y-1}{2}, \quad 1 \leq y \leq 3.$$

◇

The graphs of f and f^{-1} are displayed below.



In the picture above, we notice that the graph of f^{-1} is just the reflection of the graph of f in about the line $y = x$ in the plane. This is no coincidence. The sequence of pictures in Figure 1.8 gives a key step towards explaining this: we look at the two points (a, a), (b, b), and note that

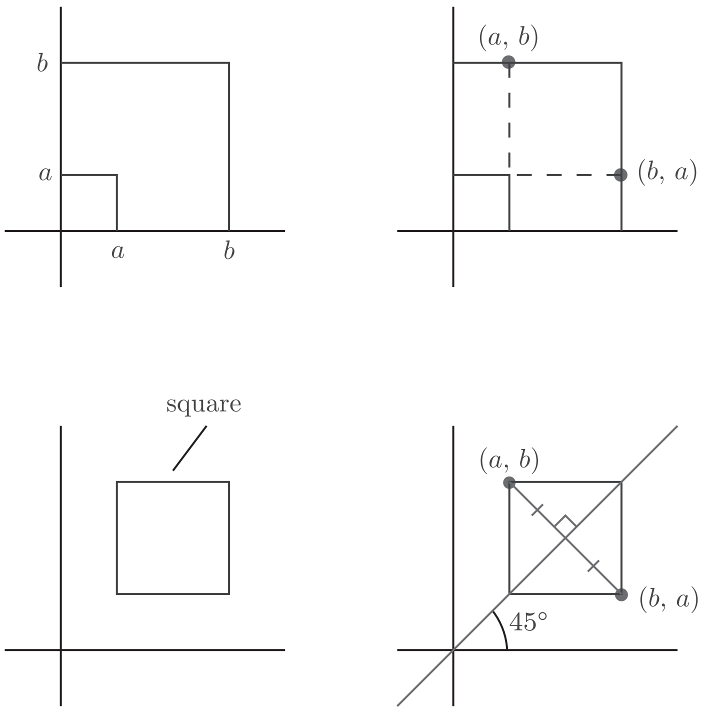


Figure 1.8 The point (b, a) is the reflection of the point (a, b) in the $y = x$ line.

that the line joining (a, b) to (b, a) is the diagonal of a square, the other diagonal of the square being the line $y = x$, and so the point (a, b) is obtained by reflecting the point (b, a) about the line $y = x$. Bearing this fact in mind, we finally note that

$$\begin{gathered}(y, x) \in \text{ graph of } f^{-1} \\ \Updownarrow \\ x = f^{-1}(y) \\ \Updownarrow \\ f(x) = y \\ \Updownarrow \\ (x, y) \in \text{ graph of } f.\end{gathered}$$

And this completes the explanation of the fact that the graph of f^{-1} is just the reflection of the graph of f about the line $y = x$ in the plane.

Example 1.22 (The nth root function $\sqrt[n]{\cdot}$) Let $n \in \mathbb{N}$ be fixed. Let the function $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = x^n$, $x \geq 0$. Then f is one-to-one because if $0 \leq a < b$, then the law of exponents (E3) on page 20 gives $a^n < b^n$. It is also onto by Theorem 1.5. Thus f is bijective, and its inverse is the nth root function $f^{-1} = \sqrt[n]{\cdot} : [0, \infty) \to [0, \infty)$ given by $f^{-1}(x) = \sqrt[n]{x}$, $x \geq 0$. Taking $n = 2$, the graphs of $f := x^2$ and its inverse $f^{-1} = \sqrt{\cdot}$ are shown in Figure 1.9. ♢

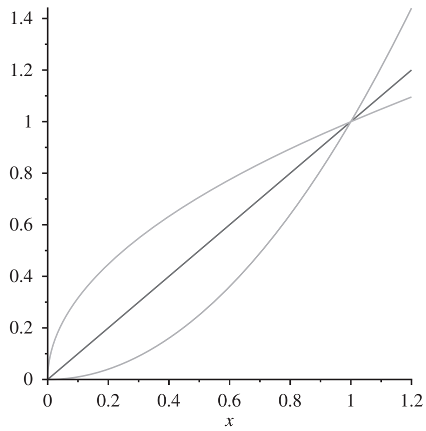


Figure 1.9 The graph of $f := x^2$ and its inverse $f^{-1} = \sqrt{\cdot}$.

Inverse of an injective map. Even if a function f fails to be bijective, but is only injective (and not necessarily surjective), we can define an inverse function from its *range* (and not its codomain) to the domain of f. We explain this now.

Let I be an interval, and $f : I \to \mathbb{R}$ be injective/one-to-one. Let us denote by $f(I)$ the range of f, that is,

$$f(I) := \{f(x) : x \in I\}.$$

We define the *inverse function* $f^{-1} : f(I) \to I$ as follows.

$$f^{-1}(y) := x, \text{ where } x \text{ is the unique element in } I \text{ such that } f(x) = y.$$

Again, since (y, x) belongs to graph of f^{-1} if and only if $f(x) = y$, that is, if and only if (x, y) belongs to graph of f, the graph of f^{-1} is obtained from the graph of f by reflection in the $y = x$ line.

1.10 (∗) Cardinality

This section is independent of the rest of the subject matter of Calculus, and if the reader so desires, it may be skipped.

For finite sets, we can compare sizes by just counting the number of elements, and this is referred to as the *cardinality of the set*: for example, the set $\{A, B, C, \cdots, Z\}$ of alphabet letters in the English language has cardinality 26, while the cardinality of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is 10. Note that finite sets of the same cardinality can be put in a one-to-one correspondence, that is, we can define a bijection between the two sets. Sets that do not have finite cardinality are called *infinite sets*. One can then ask the natural question: can any two infinite sets always also be put in a one-to-one correspondence? For example, we know that the set $\mathbb{N}$ is infinite, and now suppose that we have another infinite set S. Then can we always establish a bijection between the elements of $\mathbb{N}$ and those of S? In other words can we 'list' the elements of S, as the first element of S, the second element of S, and so on? The answer, perhaps surprisingly, is no! For example, such a bijection fails to exist if we take $S = \mathbb{R}$, and this is the content of Theorem 1.11 below. But first, the above discussion motivates the following definition.

Definition 1.15 (Countable set). An infinite set S is said to be *countable* if there is a bijective map from $\mathbb{N}$ onto S.

Example 1.23. Clearly if we consider the identity map $n \mapsto n : \mathbb{N} \to \mathbb{N}$, then we see that $\mathbb{N}$ is countable.

A non-trivial example is that also the set $\mathbb{Z}$ of integers is countable. This is best seen by means of a picture, as shown in Figure 1.10.

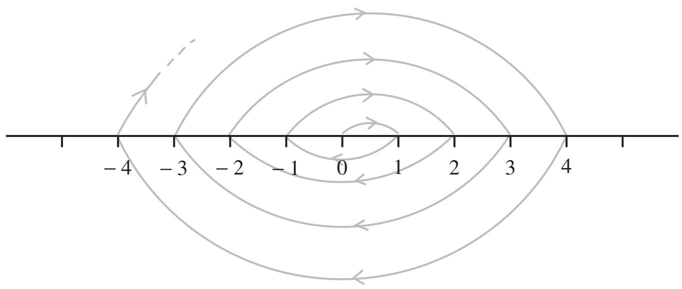


Figure 1.10 Countability of $\mathbb{Z}$.

Clearly the resulting map from $\mathbb{N}$ to $\mathbb{Z}$ is injective (since each integer is crossed by the spiral path *only once* ever—having crossed an integer, the subsequent distance of the path to the origin *increases*), and surjective (since every integer will be crossed by the spiral path *some*time). ♢

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Let us show that the set $\mathbb{Q}$ of rational numbers is countable. To this end, we will need the following two auxiliary results, which are interesting in their own right.

Lemma 1.8. *Every infinite subset of a countable set is countable.*

Proof. First let us show that any infinite subset S of $\mathbb{N}$ is countable.

Let $a_1 := \min S$. If $a_1, \cdots, a_k$ have been constructed, then define

$$a_{k+1} := \min(S \backslash \{a_1, \cdots, a_k\}).$$

Then $a_1 < a_2 < a_3 < \cdots$. Define $\varphi : \mathbb{N} \to S$ by $\varphi(n) = a_n$, $n \in \mathbb{N}$. Then φ is injective (because if $n < m$, then $\varphi(n) < \varphi(m)$). Also, φ is surjective. Indeed, for each element $m \in S$, there are only finitely many natural numbers, and much less elements of S, which are smaller than m. If the number of such elements of S that are smaller than m is n_m, then it is clear that $\varphi(n_m + 1) = m$.

Now let S be countable and let T be an infinite subset of S. Let $\varphi : S \to \mathbb{N}$ be a bijection. There is a bijection from T to the range of[3] $\varphi|_T$. But the range of $\varphi|_T$ is a subset of $\mathbb{N}$, and so it is countable. Hence T is countable too. □

Lemma 1.9. *If A, B are countable, then $A \times B$ is also countable.*

Proof. Since A and B are countable, we can list their elements:

$$\begin{aligned} A &= \{a_1, a_2, a_3, \cdots\}, \\ B &= \{b_1, b_2, b_3, \cdots\}. \end{aligned}$$

Arrange the elements of $A \times B$ in an array and list them by following the path as shown below.

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \cdots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

The resulting map from $\mathbb{N}$ to $A \times B$ is clearly surjective (since every element (a_n, b_m) is hit by the zigzag path *some*time), and it is also injective (since the zigzag path never hits a point after having crossed it because it moves on to a parallel antidiagonal below). □

[3] Here $\varphi|_T$ denotes the restriction of φ to T. In general, if $f : X \to Y$ is a function and S is a subset of X, then the *restriction of f to S* is the function $f|_S : S \to Y$ given by $f|_S(x) = f(x)$ for all $x \in S$.

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We are now ready to show the countability of the rationals.

Theorem 1.10. $\mathbb{Q}$ *is countable.*

Proof. Each $q \in \mathbb{Q}$ can be written uniquely as $q = \frac{n}{d}$, where $n, d \in \mathbb{Z}$, $d > 0$ and the greatest common divisor $\gcd(n, d)$ of n, d is equal that is 1 (that is, n and d have no common factor besides 1, or n, d are coprime/relatively prime). Hence we can consider $\mathbb{Q}$ as a subset of $\mathbb{Z} \times \mathbb{Z}$. But $\mathbb{Z}$ is countable, and so by the previous part, $\mathbb{Z} \times \mathbb{Z}$ is countable. Consequently, $\mathbb{Q}$ is countable. □

Theorem 1.11. $\mathbb{R}$ *is uncountable.*

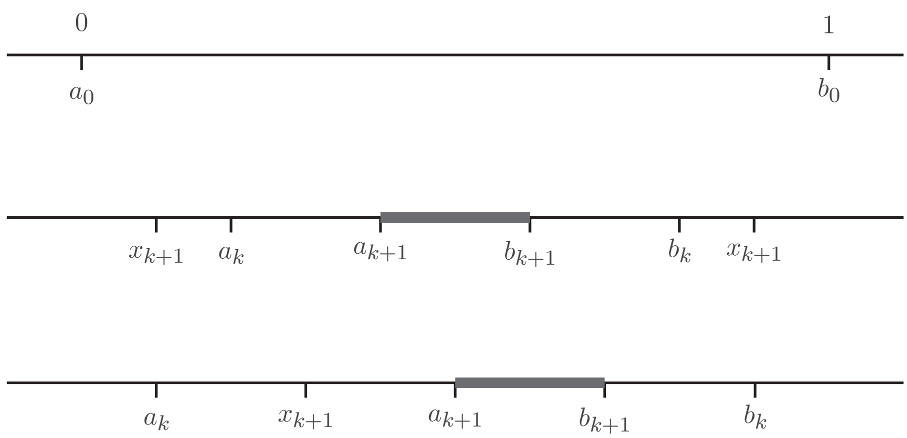
Proof. Using Lemma 1.8, we see that it is enough to show that $[0, 1]$ $(\subset \mathbb{R})$ is uncountable. Suppose, on the contrary, that $[0, 1]$ is countable. Let $x_1, x_2, x_3, \cdots$ be an enumeration of $[0, 1]$. For each $n \in \mathbb{N}$, construct a subinterval $[a_n, b_n]$ of $[0, 1]$, that does not contain x_n, inductively as follows:
Initially, $a_0 := 0$, $b_0 := 1$.
Suppose that for $k \geq 0$, a_k, b_k have been chosen. Choose a_{k+1}, b_{k+1} like this:

If $x_{k+1} \leq a_k$ or $x_{k+1} \geq b_k$, then

$$a_{k+1} := a_k + \frac{b_k - a_k}{3},$$
$$b_{k+1} := a_k + 2 \cdot \frac{b_k - a_k}{3}.$$

If $a_k < x_{k+1} < b_k$, then

$$a_{k+1} := x_{k+1} + \frac{b_k - x_{k+1}}{3},$$
$$b_{k+1} := x_{k+1} + 2 \cdot \frac{b_k - x_{k+1}}{3}.$$



Then for all $n \in \mathbb{N}$, $[a_n, b_n] \neq \emptyset$ and $x_n \notin [a_n, b_n]$.

Moreover, $0 < a_1 < a_2 < a_3 < \cdots < a_n < \cdots < b_n < b_{n-1} < \cdots < b_2 < b_1 < 1$.

Let $a := \sup\limits_{n\in\mathbb{N}} a_n$ and $b := \inf\limits_{n\in\mathbb{N}} b_n$.

Then $a \leq b$, and so $[a, b] \neq \emptyset$. Also, for all $n \in \mathbb{N}$, $[a, b] \subset [a_n, b_n]$ and $x_n \notin [a_n, b_n]$. So for all $n \in \mathbb{N}$, $x_n \notin [a, b]$. Thus the points in $[a, b]$ ($\subset [0, 1]$) are missing from the enumeration, a contradiction! $\square$

Notes

The discussion in Example 1.4.(4) is based on [**J**, Page 10]. The picture in Figure 1.5 is inspired by [**P**, Figure 1.3, page 12]. Exercise 1.24 is based on [**A**, Exercise 1.5.2].