1

A SIGNAL THEORETIC INTRODUCTION TO RANDOM PROCESSES

1.1 INTRODUCTION

Consistent with the nature of the physical universe, the technical world, and the social world and at the usual levels of observation, randomness is ubiquitous. Consistent with this, randomness has received significant attention over human history and pre-history. The appearance of gods, and associated offerings to such deities, in early civilizations was, in part, an attempt to understand the randomness inherent in nature and to have control over this variability. As our understanding of the nature of the physical universe has expanded, interest in random phenomena, and its characterization, has increased significantly. Today, the accumulated knowledge on characterizing randomness is vast. Cohen (2005) provides a good overview of part of the more recent scientific endeavor to understand and characterize random phenomena. Meyer (2009) provides a historical perspective on the mathematical development of stochastic process theory from the 1940s.

Random phenomena occur widely in the physical, social, and technical worlds, and well-known examples include the variation, with time, in measures of the weather (wind velocity, temperature, humidity, etc.), economic activity (inflation rate, stock indexes, currency exchange rates, etc.), an individual's physical state (heart rate, blood pressure, feeling of well-being, etc.), and technical entities (the information flow to a mobile communication device, the fuel economy of a car, etc.). Other technical examples include the lifetime of a set product manufactured by a manufacturer, variations in the queue length of data packets waiting to be forwarded at a given

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network node, the effective information rate on a given communication channel, the voltage variation between the terminals of a conductive medium due to the random movement of electrons, unwanted signal variations at the receiver of a communication system or at the output of a sensor, etc. Random phenomena generally inhibit the performance of a system and, for example, limit the transmission distance for electronic/photonic communication systems, the sensitivity of sensors for monitoring phenomenon in the natural world, the timing accuracy of all timing reference sources, etc. Randomness is not always detrimental to system operation, and the introduction of a random signal component to a system can, in certain circumstances, enhance system performance, and stochastic resonance is a common term associated with such an outcome.

1.2 MOTIVATION

The transformation of the world to the information age has been underpinned by advances in electrical, electronic, and photonic technology. Such technology is based on controlling the random movement of electrons and photons. As electrons have a mass of 9.11×10^{-31} kg, they, potentially, can move with very high velocities (of the order of 10^5 m/s), between collisions, when subject to an electric field and/or due to ambient thermal energy. (Such velocities should not be confused with the average drift velocity of electrons in a conducting media—of the order of 10^{-4}m/s in a copper conductor.) One extraordinary achievement of the modern era has been the development of sophisticated devices and systems with robust well-defined performance based on controlling electrons. Figure 1.1 provides a perspective.

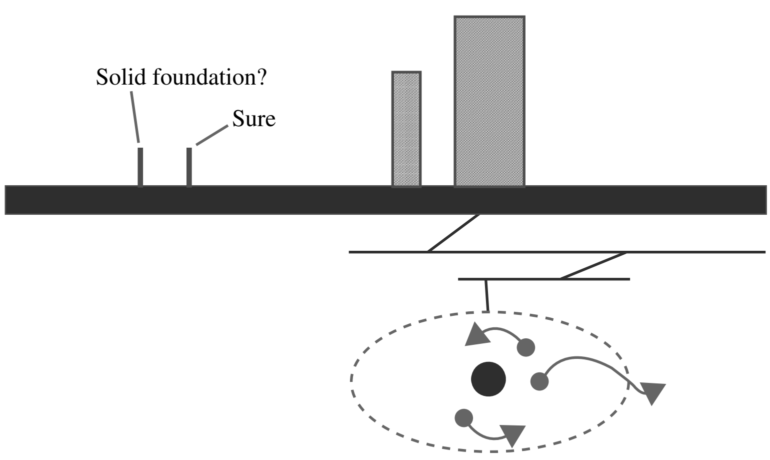

FIGURE 1.1 On the foundation of modern electrical and electronic technology.

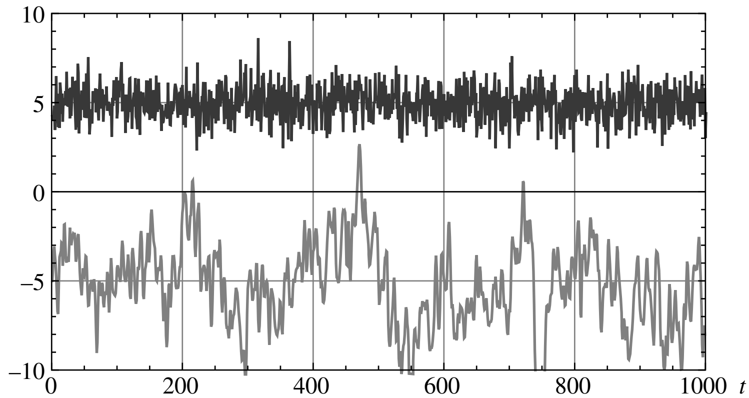

FIGURE 1.2 Upper: white Gaussian noise signal—offset by 5. Lower: 1/*f* noise signal—offset by −5.

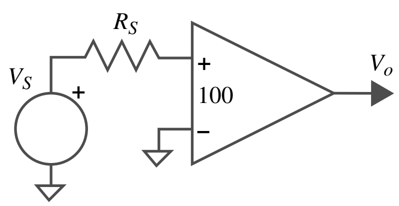

FIGURE 1.3 Schematic diagram of a signal source connected to an amplifier.

The following examples illustrate, in small part, the reality of engineering in the context of the nature of our physical world and the diverse nature of the manifestation of randomness.

First, the voltage at a node in an electrical/electronic system will exhibit randomness consistent, for example, with white Gaussian noise or 1/*f* noise. The nature of such randomness is illustrated in Figure 1.2. Note the *regular* nature of a white Gaussian noise signal in comparison to a 1/*f* noise signal. The irregularity of 1/*f* noise is consistent with the nature of randomness in complex systems with multiple sources of noise, and such noise has been found in signals from a diverse range of systems including graphene, electronic devices, the human heart and brain, the human response to stimuli, phenomena in the natural world, and economic activity. The spectrum of many forms of music is consistent with a 1/*f* noise spectrum, and 1/*f* noise is often a signature of complexity. The introduction, and cited references, in Grigolini et al. (2009) provides a good overview of the diverse sources of such noise.

Second, as an example of the importance of randomness in electrical engineering, consider the random movement of electrons that leads to a signal at the output of an electronic amplifier, illustrated in Figure 1.3, having the form illustrated in Figure 1.4. In a communication context, the noise floor (the output noise signal in the absence of an input signal) of an amplifier limits the distance information can be transmitted and

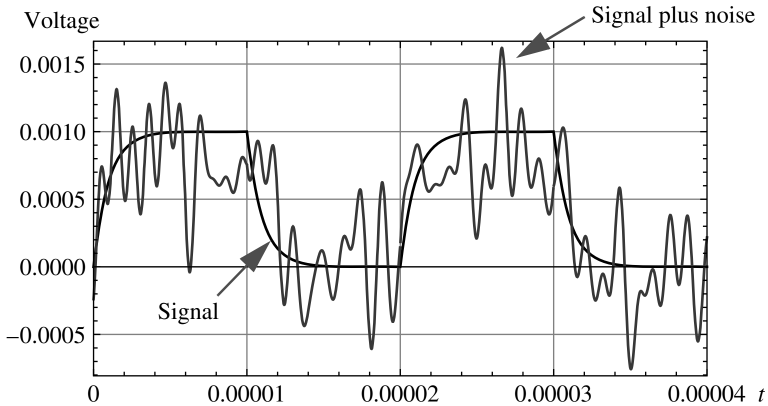

FIGURE 1.4 Example of a signal corrupted by the noise inherent in an amplifier, at the amplifier output, for the case where the maximum signal level is $\sqrt{10}$ times the rms noise level.

recovered. In a sensor context, the amplifier noise limits the sensitivity of the sensor. The effect of the noise on a signal depends on the nature of the noise, the bandwidth of the noise, the noise level, the signal form, and the information required from the signal. Modelling and characterization of noise is fundamental to designing amplifiers to minimize the noise generated and for subsequent processing that, potentially, can limit the effect of the introduced noise.

Third, the randomness of the movement of electrons, and the randomness inherent in the natural environment (e.g., temperature variations), results in the drift of all electronic-based timing references. This necessitates, for example, circuitry, or algorithms, to maintain synchronization and, hence, reliable communication, between the nodes in a communication network. The first-order model of an electronic oscillator is

$$x(t) = A_o \sin(2\pi f_o t + \phi(t)) \tag{1.1}$$

where f_o is the oscillator frequency and $\phi(t)$ is the phase variation due to the influence of random electrical and thermal variations. As a first-order model, the phase $\phi(t)$ variation will exhibit a random walk behavior as illustrated in Figure 1.5.

Noise in a system often leads to the delay or advance of the timing of a signal when compared with the zero noise case. This leads to jitter of the time a signal crosses a set threshold as illustrated in Figure 1.6. When the jitter is associated with a clock signal, which serves as a timing reference for a system, constraints on the switching rate, for reliable system operation, result. In a communication context, jitter in the timing reference at a receiver leads to an increase in the probability of error for the received information.

Fourth, the time a signal first reaches a set threshold level is called the first passage time, and the first passage time is of interest in many areas including the spread of diseases (when does a measure of the disease reach a set level for the first time),

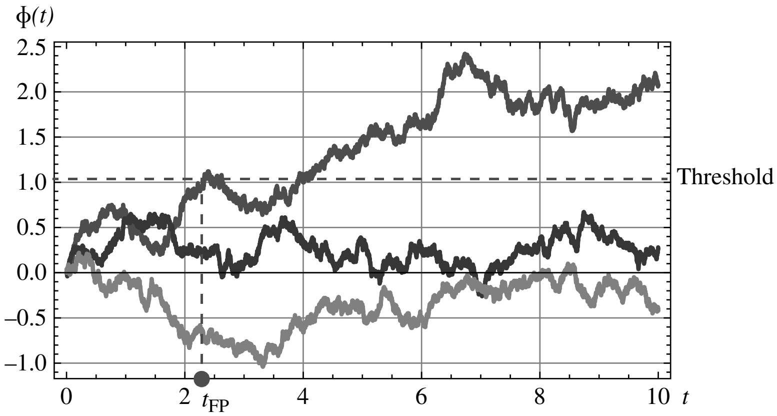

FIGURE 1.5 Three trajectories consistent with the random-walk behavior of the phase of an oscillator (arbitrary units). The time t_{FP} is the time the upper random walk first reaches the threshold level of unity.

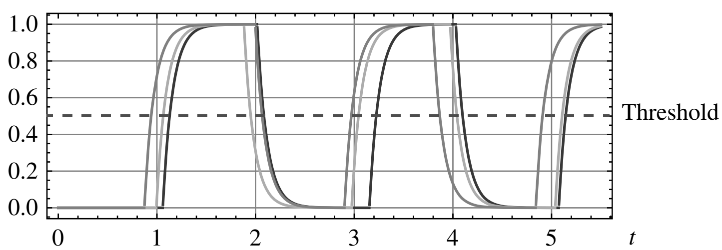

FIGURE 1.6 Three jittered binary signals.

finance (when does a set measure of economic activity reach a set level for the first time), neuronal firing (a neuron fires when the collective inputs reach a set level), etc. The first passage time for a random walk is illustrated in Figure 1.5.

Fifth, in many contexts, our intuition and experience are consistent with events occurring at random and at a regular rate. Examples include the arrival of people at a supermarket queue, the timing of incoming phone calls/emails during a set period of the day, etc. In a technical context, examples include the arrival of photons on a photodetector, the crossing times of electrons in a PN junction, the arrival of data packets at a network node, etc. The simplest model for such timing is the Poisson point process (random timing of points but at a set rate), and examples of times defined by such a process are shown in Figure 1.7. Such point processes underpin, for example, the examples noted above.

Sixth, the efficient and inexpensive conveyance of information underpins, in significant measure, the latest transformation of our world and is based on agreed protocols with respect to encoding of information in signals. Efficient communication requires the use of signals that are spectrally efficient with a unit of information taking

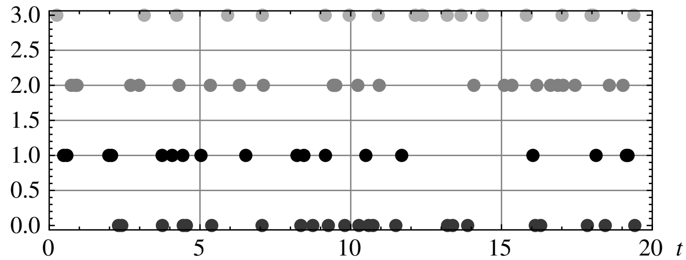

FIGURE 1.7 Four examples of the times defined by a Poisson point random process with a rate of one point per unit time.

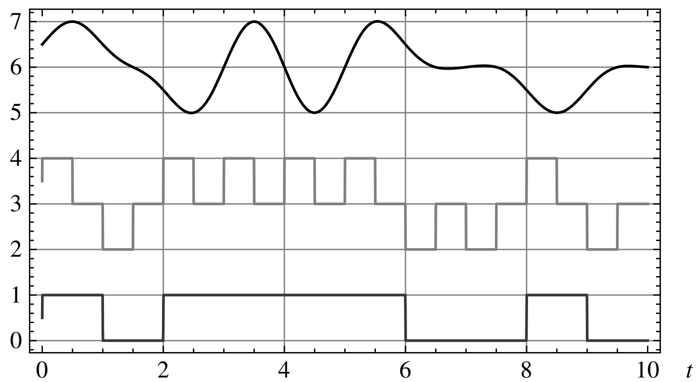

FIGURE 1.8 Three signalling waveforms for encoding the data 1, 0, 1, 1, 1, 1, 0, 0, 1, 0, at a rate of 1 bit per second. Bottom: signalling with rectangular pulses—reference level of zero. Middle: signalling with return to zero pulses and polar coding—reference level of three. Top: signalling with raised cosine pulses using bipolar coding—reference level of six.

the smallest possible amount of the available capacity of a communication channel. Three different signals, each with different spectral efficiency but encoding the same information, are shown in Figure 1.8. Further, the nature of the signals used has implications for the ability of the receiver to recover the sent information in the context of noise and interference. Signal theory, random process theory, and methods of characterizing the spectral content of communication signals, underpin modern communication.

1.2.1 Usefulness of Randomness

Randomness is not always detrimental to the outcomes of a system. One example is illustrated in Figure 1.9 where the irregularity of the main traffic flow facilitates the movement of traffic from the right.

Another application where noise is useful is illustrated in Figures 1.10 and 1.11 and arises when an input level is to be estimated from samples provided by a device, for

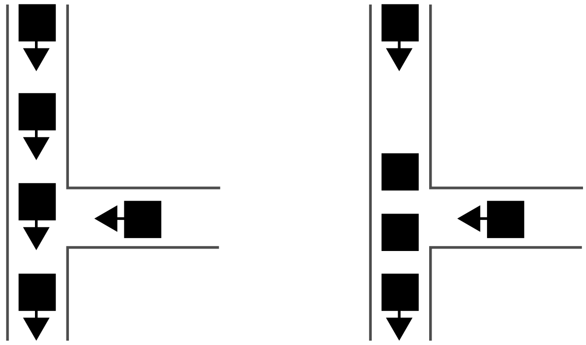

FIGURE 1.9 Regular and irregular traffic flow.

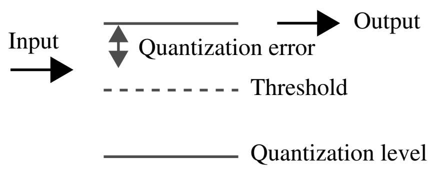

FIGURE 1.10 Illustration of the quantization error arising from sampling a signal with finite amplitude resolution.

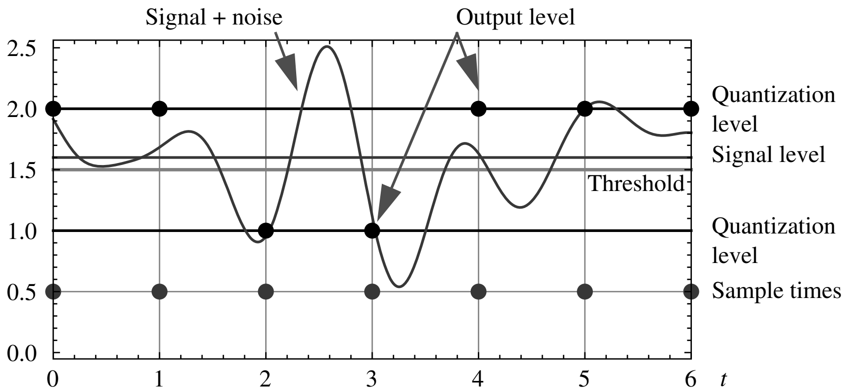

FIGURE 1.11 Illustration of the output values arising from sampling a constant signal level plus additive noise. The sampling rate is unity.

example, an analogue to digital converter, that produces quantized levels. In the absence of noise, the output level will differ from the input level by the quantization error. When noise, with a variation greater than the quantization resolution, is present, it is possible to recover the input level accurately by averaging output values taken at times greater than the correlation time of the noise. The resolution can be improved according to $1/\sqrt{N}$ where N is the number of data values averaged.

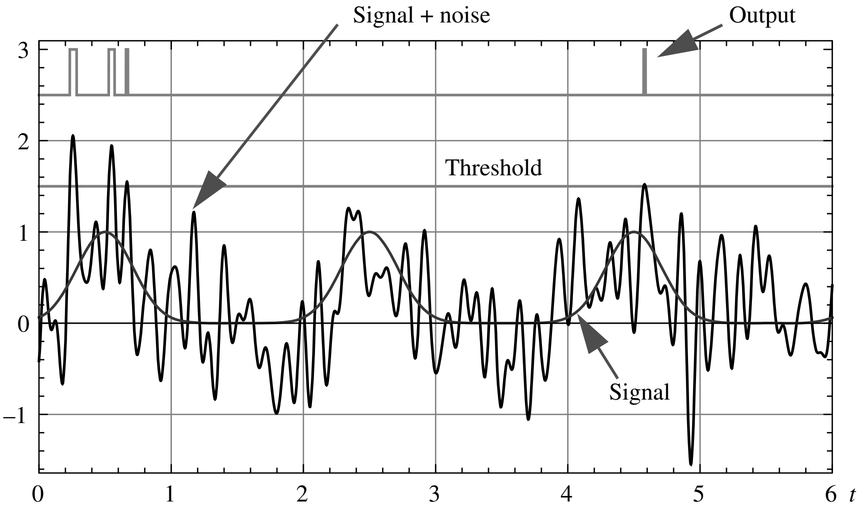

FIGURE 1.12 Illustration of the beneficial effect of additive noise on the detection of a pulse train signal that is below a threshold level. The output level (offset) is a pulse when the signal plus noise is above the threshold.

Additive noise, or the noise inherent in a system, can aid subthreshold detection of a periodic signal as illustrated in Figure 1.12. Such detection occurs, for example, in neuronal networks. The phrase *stochastic resonance* is widely used when noise aids the detection of a signal.

1.2.2 Engineering

The first requirement for engineering, in the context of random phenomena, is the modelling of the random phenomena. The second requirement is the characterization of the random phenomena such that measures of system performance can be ascertained and the effects of the randomness minimized or utilized as appropriate. The goal of this book is to provide a theoretical basis that leads to the facilitation of both the modelling and the characterization of the prototypical random phenomena encountered in electrical engineering.

1.3 BOOK OVERVIEW

The theory for modelling and characterizing random phenomena is vast, and a single book can, at best, provide a modest introduction in a specific area. This book, and the approach taken for introducing and characterizing random phenomena, has arisen out of long-term research in the electronics and communications field. The theory and examples included are consistent with the broad electrical/electronic/communications engineering discipline and include the prototypical random phenomena of these disciplines: the random walk, Brownian motion, the random telegraph signal, the Poisson point process, the Poisson counting process, shot noise, white noise, 1/*f* noise, signalling random processes (which underpin most forms of communication), jitter, random clustering, and birth–death random processes.

The rationale for the book is threefold: first, random process theory should be grounded in signal theory as well as probability theory. Second, results for random processes should be established on the finite interval and results for the infinite interval obtained by taking an appropriate limit. Third, attention to mathematical rigor provides clarity and facilitates understanding, and such rigor is well suited to the modelling and characterization of random phenomena. Detailed proofs of results have been included to provide a comprehensive treatment of material at a moderate mathematical level. The book assumes a prior introduction to probability theory and random variable theory.

The approach taken, with a strong mathematical and signal theory basis, provides a foundation for random process theory that facilitates the continued development of random process theory in the context of many unsolved problems and the increasing importance of modelling and characterizing random phenomena. The modelling of random processes on the finite interval allows transient as well as steady-state results to be obtained. Importantly, it allows the use of finite-dimensional functions for characterizing random phenomena, and this facilitates the development of random process theory. The use of a signal theory basis provides a general framework for defining functions used for characterizing random phenomena including the autocorrelation function and the power spectral density. This has pedagogical value as the definitions have a signal basis rather than a random process basis and is less problematic for students. The use of a signal basis set approach for defining the power spectral density, which is the most widely used measure for characterizing random phenomena, provides a simple and natural interpretation of this function for the general case and for the usual case where a sinusoidal basis set is assumed.

Chapters 2–5 provide the necessary mathematical theory, background signal theory, random variable theory, and random process theory for subsequent discussion of random processes. Chapter 6 details the prototypical random processes that are fundamental to electrical, electronic, and communication engineering. Chapters 7–10 provide a basis for the characterization of random phenomena: Chapter 7 provides a general overview, Chapter 8 details probability mass function/probability density function evolution, Chapter 9 details the autocorrelation function, and Chapter 10 details the power spectral density. Chapter 11 provides an introduction to order statistics, and this provides the background for a discussion of the Poisson point random process in Chapter 12. Chapter 13 provides an introduction to birth–death random processes, while Chapter 14 provides an introduction to first passage time theory.