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A Crash Course on Lagrangian Dynamics

1.1 Objectives

This chapter presents the fundamental concepts in Lagrangian dynamics and outlines a procedure for deriving the equation(s) of motion for holonomic mechanical systems using Lagrange's equation. Extensive examples are presented to cover a large variety of mechanical systems containing particles and rigid bodies. Finally, the procedures for finding the equilibrium position(s) and linearizing the equation(s) of motion in preparation for vibration analysis are also presented.

1.2 Concept of "Equation of Motion"

An *equation of motion* for a mechanical system is a differential equation that governs the changes in positions of components in the system with respect to time.

There are three key phrases in the above definition. Phrases *differential equation* and *with respect to time* specify a particular mathematical form for the equation and distinguish the equation of motion from its solution. That is, an equation of motion is a differential equation involving time derivatives. The phrase *positions* specifies a particular kinematic quantity, the most fundamental one from which other measures of motion, namely, displacements, velocities, and accelerations, can be obtained.

The concept of the *equation of motion* has suffered a degree of misuse in some dynamics textbooks. In these textbooks, Newton's second law, which many of us conveniently recite as $\boldsymbol{F} = m\boldsymbol{a}$, is called the equation of motion. In fact, Newton's second law is the fundamental physical law that governs the motion of any physical system. It is often a crucial tool to use in obtaining the equation(s) of motion for a system. But it is too primitive a form to be called an equation of motion. In our daily lives, we do not call a foundation as a structure. The same goes here.

Fundamentals of Mechanical Vibrations, First Edition. Liang-Wu Cai.

Companion Website: www.wiley.com/go/cai/fundamentals_mechanical_vibrations

In the mean time, we must keep our minds open to the notion that there are other ways to establishing the equation of motion for a system. *Lagrangian dynamics* is one such alternative.

Before jumping into the details of Lagrangian dynamics, let us look at the way the equation of motion for a system can be obtained by using Newton's second law.

■ Example 1.1: Simple Mass–Spring–Dashpot System

Derive the equation of motion for the mass–spring–dashpot system as shown in Fig. 1.1.

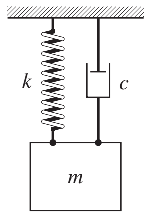

Figure 1.1 Mass–spring–dashpot system

□ Solution 1:

- Define x as the downward displacement of the mass, measured from its position when the spring is unstretched.
- *Kinematics of Mass m*: This example is simple enough to allow us to write directly:

$$\text{velocity} = \dot{x} \qquad \text{and} \qquad \text{acceleration} = \ddot{x} \tag{a}$$

  where, as in dynamics, an overhead dot represents the time derivative, and overhead double-dot represents the second derivative with respect to time t.
- *Kinetics of Mass m*: A free-body diagram can be drawn, as in Fig. 1.2, to show all the forces acting on the mass in a generic instant in time when the system is in motion. In Fig. 1.2, F_s and F_d are the forces exerted by the spring and the dashpot, respectively. Applying Newton's second law based on the free-body diagram in Fig. 1.2 gives

$$mg - F_s - F_d = m\ddot{x} \tag{b}$$

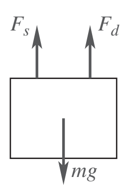

Figure 1.2 Free-body diagram for mass m at a generic time instant

- *Constitutive Relations*: Besides the mass, the system contains a spring and a dashpot, whose forces are proportional to the deflection and the velocity, respectively. That is,

$$F_s = kx \qquad \text{and} \qquad F_d = c\dot{x} \tag{c}$$

- Substituting eqns. (c) into eqn. (b) gives, with a slight rearrangement,

$$m\ddot{x} + c\dot{x} + kx = mg \tag{d}$$

which is the equation of motion for this system.

□ **Solution 2:**

- Define y as the downward displacement of the mass, measured from its static equilibrium position.
- *Kinematics of Mass m*: Kinematics of mass m is unchanged, except replacing x by y.
- *Kinetic of Mass m*: The free-body diagram remains the same as in Fig. 1.2; and hence Newton's second law gives

$$mg - F_s - F_d = m\ddot{y} \tag{e}$$

- *Constitutive Relations* for system components are

$$F_s = k(y + \Delta) \qquad \text{and} \qquad F_d = c\dot{y} \tag{f}$$

where we note that the spring is already stretched at equilibrium, denoted as Δ, and that the spring force is proportional to the total amount of deformation in the spring.
- Substituting eqn. (f) into eqn. (e) gives

$$mg - ky - k\Delta - c\dot{y} = m\ddot{y} \tag{g}$$

- To eliminate Δ from eqn. (g), we look at the static equilibrium: the dashpot does not exert any force to the mass. The free-body diagram for the mass in its static equilibrium state is shown in Fig. 1.3. The equilibrium requires

$$k\Delta = mg \tag{h}$$

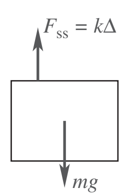

Figure 1.3 Free-body diagram for mass m at the static equilibrium

- Combining eqns. (g) and (h) gives, after a slight rearrangement,

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{i}$$

which is the equation of motion for the system.

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This example illustrates a typical process for deriving the equation(s) of motion for a mechanical system by using Newton's second law. Through this example, we can make the following observations:

- Before we can proceed to deriving the equation, some variables (such as x and y) must be defined so that the locations of the system's components can be described. In the end, the equations of motion for the system are differential equations of these variables. Such variables are called the *generalized coordinates*. We shall study this concept more thoroughly and rigorously in the next section.
- Different definitions of the generalized coordinates result in different equations of motion for the same system. That is, the equations of motion are not unique, depending on the choice of the generalized coordinates by which the system is described.
- The following three pieces of information are generally needed: (1) kinematics of the system components at a generic time instant during the motion; (2) the properties of system components, which are called the *constitutive relations*; and (3) Newton's second law that threads all these pieces of information together.

However, there are some shortcomings in this approach of deriving the equations of motion for a system. For example, how do we know how many equations should be obtained? For a complex system, which part or parts of the system should be isolated to draw the free-body diagrams? Finding the answer to these questions is *ad hoc*: we have to look into individual systems case by case. This means that we may be able to solve one problem effortlessly, but we might stumble on the next. We all have experienced the situations in which drawing a free-body diagram reveals more unknowns than desired and calls for drawing more free-body diagrams and writing out more equations. We may also recall that, in kinematics, finding the acceleration for a particle or a point in a rigid body involves substantially more work than finding the velocity.

Lagrangian dynamics avoids most of these issues and provides a structured way to analyze a system and to subsequently obtain its equation(s) of motion. It works in a way similar to the energy method: only positions and velocities are required in the formulation. The procedure is unchanged regardless of the system's complexity. Furthermore, it can be extended to handle systems in other physical domains, such as electrical, electromagnetic, and electromechanical systems.

Deriving the equation of motion is the first step toward understanding a system. Having the equation(s) of motion in hand, the first and foremost, we can find and subsequently analyze the solution to the equation(s) of motion. This plainly stated activity, in fact, encompasses almost all subjects of study in mechanical engineering, including vibration analyses. As we shall see, vibration analyses are to find, analyze, and study the solutions to a particular class of equations of motion.

Having the equation of motion, before putting all the efforts into finding the solution, a few less ambitious things having great engineering interests can be done:

- Determine the system's equilibrium configuration(s). In the equilibrium state, the system does not move at all. For the above example, setting the velocity and acceleration to zero, the equation of motion derived in *Solution 1* would become

$$kx_{\text{eq}} = mg \qquad \text{or} \qquad x_{\text{eq}} = \frac{mg}{k}$$

  where x_{eq} denotes the position of the mass at equilibrium. The equation of motion derived in *Solution 2* would give $y_{\text{eq}} = 0$, as expected.
- Analyze the dynamic stability and other associated behaviors of the system.

- Introduce approximations to simplify the equation(s) of motion or to obtain approximate solutions.

We discuss these topics separately later in this chapter. In fact, these things must be explored before we can proceed with vibration analyses of a system.

1.3 Generalized Coordinates

Coordinates usually are associated with a particular coordinate system and usually appear in a set. For instance, in a Cartesian coordinate system, the coordinates for a point are (x, y, z). In a polar coordinate system, the coordinates are (r, θ). Adding the word *generalized* frees us from abiding to any particular coordinate system so that we can choose whatever parameter that is convenient to describe the position of a point in a system. Hence,

A *generalized coordinate* is a parameter that is used to locate a part of a system. A generalized coordinate is a scalar quantity.

A group of parameters that is used to locate a system is a *set* of generalized coordinates. To denote a set of generalized coordinates, we follow the mathematical notation for explicitly defining a set by listing all elements contained in the set. We write, for example, $\{x_1, x_2\}$.

A *complete set* of generalized coordinates is a set of generalized coordinates that completely locates all parts of a system in all geometrically admissible configurations. A *geometrically admissible configuration* is a configuration that is allowed by the geometrical constraints in the system.

An *independent set* of generalized coordinates is a set of generalized coordinates in which if all but one of them are fixed, there still exists a continuous range of values for that unfixed generalized coordinate.

A *complete and independent set* of generalized coordinates is, literally, a set of generalized coordinates that is both complete and independent.

Because generalized coordinates can be chosen or defined almost at will, generalized coordinates for a system are not unique. For this same reason, it is utterly important to define them clearly, aided by a schematic depiction if necessary. When defining a generalized coordinate, usually the following four vital aspects should be clearly indicated for each generalized coordinate: the exact physical meaning, where it is measured from, and relative to whom, and which direction is positive.

■ Example 1.2: Particle Moves Freely in Three-Dimensional Space

A particle can move freely in a three-dimensional space. Define a set of generalized coordinates and subsequently judge whether it is a complete and independent set of generalized coordinates.

□ Solution:

Before we start, as a preparatory setup, we define a Cartesian coordinate system to facilitate the discussions that follow. The Cartesian coordinate system is fixed in space.

We define $\{x, y, z\}$ as a set of generalized coordinates, where x, y, and z are the Cartesian coordinates of the particle. Note that the positive directions for the coordinates have already been defined in the Cartesian coordinate system.

- *Is It a Complete Set?* YES, because there is only one particle in the three-dimensional space, the three coordinates completely specified a point in the three-dimensional space.
- *Is It an Independent Set?* To answer this question, we need to conduct a series of tests: if all but one of the generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to change. In this problem, if x and y are fixed, z can still be freely changed along a line parallel to the z-axis. Similar conclusion can be drawn for leaving any other coordinate unfixed. So, the answer to this question is also YES.

Therefore, we conclude that $\{x, y, z\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion: Using a Coordinate System

When using coordinates in a well-established coordinate system, such as the Cartesian coordinate system in this example, as the generalized coordinates, in general, the coordinate system has already included the specifications for the directions of positive coordinates. An important task in defining a coordinate system is to specify how its origin moves, such as relative to whom, and how the coordinate axes are oriented.

■ Example 1.3: Simple Planar Pendulum

A simple pendulum is made up by a particle hung to a pivot point through a massless string. A planar pendulum is a simple pendulum whose motion is restricted to within a plane, typically the plane of paper, as shown in Fig. 1.4. Assume that the string of length l remains taut at all times. Define a set of generalized coordinates for this pendulum and subsequently judge whether it is a complete and independent set of generalized coordinates.

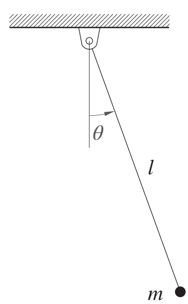

Figure 1.4 Simple planar pendulum

□ Solution 1: Angular Displacement

We define $\{\theta\}$ as a set of generalized coordinates, where θ is the angular displacement of the pendulum, measured in the counterclockwise direction from the vertical, as shown in Fig. 1.4. Note that, in the sketch, the arrow for θ goes only in one direction, indicating its positive direction.

- *Is It a Complete Set?* YES, because there is only one particle in the system and it can only move around a circle of radius l centered at the pivot point. Once the angle θ is determined, the location of the particle is uniquely determined.

- *Is It an Independent Set?* To answer this question, we need to conduct a test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? In this case, there is only one generalized coordinate. When "all but one are fixed," actually nothing is fixed, and the unfixed one is θ. As the particle is allowed to move along the circle of radius l, θ indeed has a continuous range to vary. So, the answer to this question is also YES.

Therefore, we conclude that $\{\theta\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion: Geometrically Admissible Configurations

The definition for the completeness of a set of generalized coordinate includes a qualifying term "geometrically admissible configurations." However, this phrase is not mentioned in the preceding discussions. A geometrically admissible configuration requires the pendulum remain in the plane of paper and the string being taut. These constraints have been implicitly invoked when we say "the pendulum can only move around a circle."

□ Solution 2: Cartesian Coordinates

In the preparatory setup, we define a Cartesian coordinate system Oxy such that its origin O is fixed at the pivot point, the x-axis points horizontally to the right, and the y-axis points vertically upward.

We then define $\{x, y\}$ as a set of generalized coordinates, where x and y are Cartesian coordinates of the mass of the pendulum.

- *Is It a Complete Set?* YES, because there is only one particle that can only move within the plane of paper: once the Cartesian coordinates x and y are specified, the location of the mass is completely determined.
- *Is It an Independent Set?* To answer this question, we need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? If we fix x, since the particle can only move along the dotted circle shown in Fig. 1.5, y can be located at two possible positions as indicated: at the intersections of the vertical line and the circle. However, they are two *discrete* positions, and do not constitute a *continuous range*. This suffices to give NO as the answer to this question without conducting any further test.

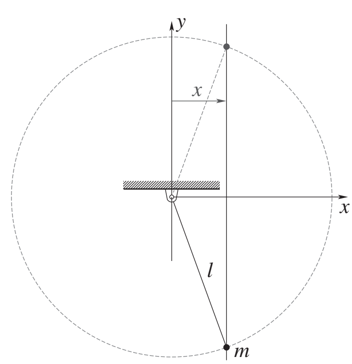

Figure 1.5 Two possible positions for the pendulum when x is fixed

Therefore, we conclude that $\{x, y\}$ as defined is not a complete and independent set of generalized coordinates. Specifically, it is a complete set but not an independent set.

□ Solution 3: Cartesian Coordinates, Remedied

Although the problem statement does not ask us to define a complete and independent set of generalized coordinates, we are still curious as how to remedy the set we just defined to make it a complete and independent set.

We define $\{x\}$ as a set of generalized coordinates, where x is the x-coordinate of the mass of the pendulum in the Cartesian coordinate system as defined in *Solution 2*.

- *Is It a Complete Set?* When x is fixed, as discussed earlier, there are two possible positions for the pendulum. In the Cartesian coordinate system, the y-coordinate can be found from the following relation:

$$x^2 + y^2 = l^2 \qquad \text{or} \qquad y = \pm\sqrt{l^2 - x^2} \tag{a}$$

  This identifies two possible y values in Fig. 1.5. The exact locations of the two possible positions are completely determined. We consider the answer to this question is YES.
- *Is It an Independent Set?* To answer this question, we need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? In this case, if we fix all but x, nothing is actually fixed, and x can vary in the range $-l \le x \le l$. Thus, the answer to this question is YES.

Therefore, we conclude that $\{x\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion

The condition such as the one in eqn. (a) is called a *geometric constraint* of the system. The question about which of the two positions the particle is located at a given time will be answered or become apparent by other information of the system under consideration, such as the initial conditions of the system. A mechanical system moves continuously in space from one location into an adjacent one.

■ Example 1.4: Rigid Slender Rod Moves in a Plane

A rigid slender rod of length L is allowed to move freely on the plane of paper. Define a set of generalized coordinates and subsequently judge whether it is a complete and independent set of generalized coordinates.

□ Solution:

In the preparatory setup, we define a Cartesian coordinate system Oxy that is fixed in space, as shown in Fig. 1.6.

We then define $\{x_1, x_2, y_1, y_2\}$ as a set of generalized coordinate, where x_1 and y_1 are the Cartesian coordinates of the left end of the rod and x_2 and y_2 are the Cartesian coordinates of the other end of the rod.

- *Is It a Complete Set?* YES, because once the two ends of the rod are fixed, every point on the rod can be located.

- *Is It an Independent Set?* To answer this question, we need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? We first fix x_1, y_1, and x_2 and leave y_2 unfixed. As a rigid body, the length of the rod, L, is fixed. This geometrical constraint gives y_2 as

$$y_2 = y_1 \pm \sqrt{L^2 - (x_1 - x_2)^2} \tag{a}$$

There are two possible values for y_2, as indicated by the ± sign, as shown in Fig. 1.7. They are two discrete values, but do not constitute a continuous range of values. Thus, the answer to this question is NO.

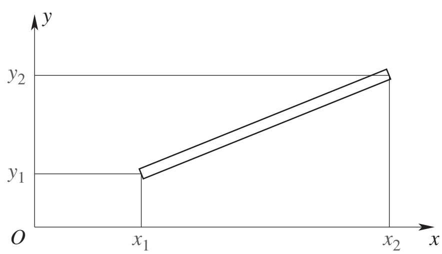

Figure 1.6 Rigid slender rod moves on the plane

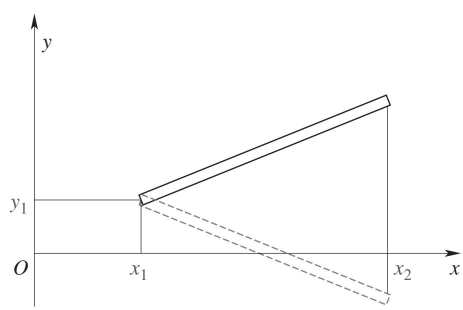

Figure 1.7 Two possible locations of the rod when x_1, y_1, and x_2 are fixed

Therefore, we conclude that $\{x_1, x_2, y_1, y_2\}$ as defined is not a complete and independent set of generalized coordinates. Specifically, it is a complete set but not an independent set.

□ **Remedied Solution:**

To define a set of complete and independent set, we only need to pick out any three of the four generalized coordinates, knowing that the fourth can be obtained from the geometrical constraint in eqn. (a).

In this solution, we pick out $\{x_1, y_1, x_2\}$ as a set of generalized coordinates. Their definitions are exactly the same as before and would not be repeated here.

- *Is It a Complete Set?* YES. As analyzed before, when needed, the remaining coordinate y_2 can be found from eqn. (a). Then, the exact location of the rod is completely determined.

- *Is It an Independent Set?* We cannot take for granted that such a choice would automatically be a complete and independent set. We still need to conduct a series of tests.
  - First, we fix x_1 and y_1 and leave x_2 unfixed. The possible scenario is that the rod can rotate about its left end, while its right end moves along a circle, as shown in Fig. 1.8. Along this circle, x_2 certainly has a continuous range to vary. Furthermore, the range is between the extreme ends of the circle, that is, $x_1 - L \leq x_2 \leq x_1 + L$.

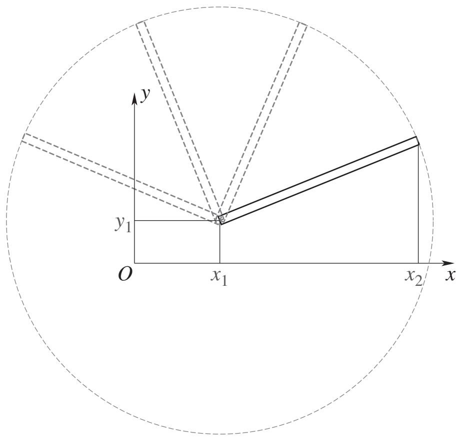

Figure 1.8 Possible locations of the rod when x_1 and y_1 are fixed

  - Next, we fix x_1 and x_2 and leave y_1 unfixed. This allows the rod to slide up and down along a vertical strip confined by x_1 and x_2, as shown in Fig. 1.9. Thus, y_2 has a continuous range to vary. Furthermore, the range is unlimited: $-\infty < y_1 < \infty$.

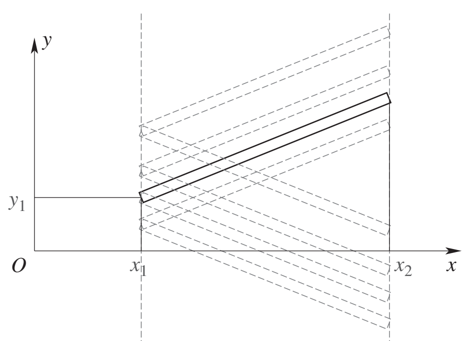

Figure 1.9 Possible locations of the rod when x_1 and x_2 are fixed

  - Lastly, we fix y_1 and x_2 and leave x_1 unfixed. The scenario is similar to a ladder sliding along a wall–floor corner while remaining in simultaneous contacts with both the wall and the floor. In this analogy, the "wall" is defined by the vertical lines $x = x_2$; while the "floor" is defined by the horizontal line $y = y_1$. The difference is that the image mirrored either by the "wall" or by the "floor" is also valid, as sketched in Fig. 1.10. Thus, x_1 has a continuous range to vary. Furthermore, the range is $x_2 - L \leq x_1 \leq x_2 + L$.

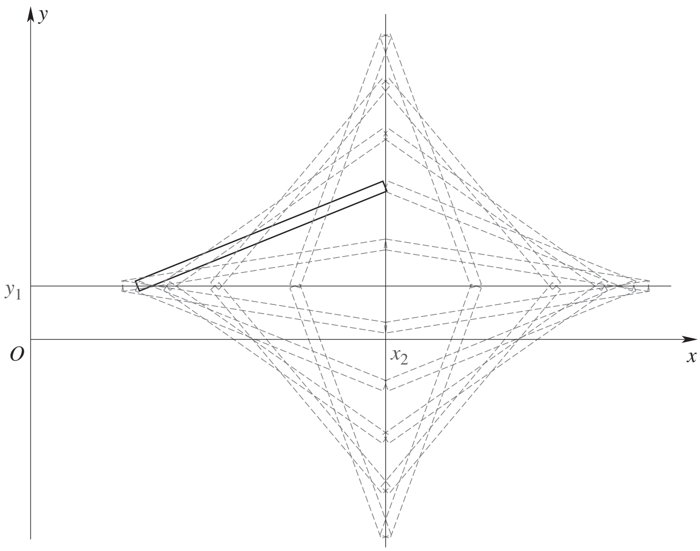

Figure 1.10 Possible locations of the rod when y_1 and x_2 are fixed

This completes the independence test. In all three tests, the unfixed coordinate always has a continuous range of values to vary. Thus, the answer to this question is YES.

We conclude that $\{x_1, y_1, x_2\}$ as defined is a complete and independent set of generalized coordinates.

■ Example 1.5: Particle Moves Along a Wire of Known Shape

A particle moves along a wire on the xy-plane of a known shape given by

$$y = a + bx^2$$

as shown in Fig. 1.11. Define a complete and independent set of generalized coordinates.

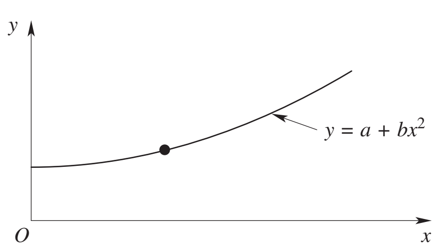

Figure 1.11 Particle moves on a wire of known shape

□ **Solution:**

Since a Cartesian coordinate system has already been defined, it is convenient to choose the Cartesian coordinates x and y of the particle as the generalized coordinates. However, Example 1.4 gives us sufficient reason to pause before jumping into making the declaration. A careful examination suggests that we can only choose one of the coordinates, and the function describing the wire shape provides the other coordinate.

Therefore, we define $\{x\}$ as a set of generalized coordinates, where x is the x-coordinate of the particle in the Cartesian coordinate system Oxy.

- *Is It a Complete Set?* YES, as we have just analyzed above.
- *Is It an Independent Set?* We still need to conduct the test for the independence. If all but one fixed, the only possibility is to leave x unfixed, and nothing is fixed. The particle is still free to move along the wire.

Therefore, we conclude that $\{x\}$ as defined is a complete and independent set of generalized coordinates.

■ **Example 1.6: Disk Rolls Without Slip on Ground**

A circular disk of radius R rolls without slip on a horizontal ground. Define a complete and independent set of generalized coordinates.

□ **Solution:**

In the preparatory setup, we paint a radius on the disk between its center and the contact point with ground in a reference configuration. We also define a Cartesian coordinate system Oxy such that its origin is fixed at the contact point in the reference configuration and its x-axis lies on the ground. Figure 1.12 shows both the reference and the displaced configurations of the disk. In the displaced configuration, C is the center of the disk, A is the contact point, and B is the contact at the reference configuration. Angle θ denotes the angular displacement of the painted radius.

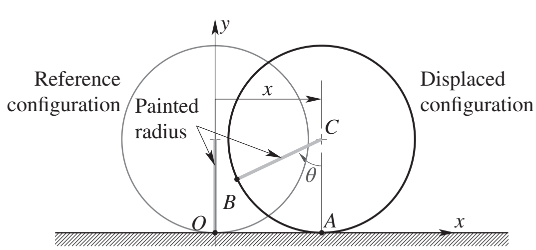

Figure 1.12 Disk rolls without slip on horizontal ground, showing reference and displaced configurations

We can now define the x-coordinate of the center C as a generalized coordinate. Once x is fixed, the location of the center C is completely determined. As a rigid body, locating the center is not sufficient. We need to fix its orientation. The angle θ seems to be a perfect candidate for another generalized coordinate. However, since the disk rolls without slip, it requires that the length OA equals to the arc length $\widehat{AB}$. Since length OA equals x, given the radius R of the disk, θ can be uniquely determined.

Therefore, we define $\{x\}$ as a set of generalized coordinate, where x is the x-coordinate of the center of the disk in the Cartesian coordinate system Oxy.

- *Is It a Complete Set?* YES. As we have just analyzed, this is a complete set.
- *Is It an Independent Set?* We need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary. If we fix all but x, actually nothing is fixed, the disk can roll on the ground, and hence x has a continuous range to vary. Thus, the answer to this question is YES.

Finally, we conclude that $\{x\}$ as defined is a complete and independent set of generalized coordinates.

■ Example 1.7: Disk Rolls on Ground, Slipping Allowed

A circular disk of radius R rolls on a horizontal ground. The disk is allowed to slip while rolling. Define a complete and independent set of generalized coordinates.

□ Solution:

The preparatory setup is the same as in Example 1.6.

We define $\{x, \theta\}$ as a set of generalized coordinate, where x is the x-coordinate of the center of the disk and θ is the clockwise angular displacement of the painted radius.

- *Is It a Complete Set?* Based on the analysis in Example 1.6, we can readily conclude that, YES, it is a complete set.
- *Is It an Independent Set?* We need to conduct the following series of tests: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary. We first fix x and allow θ to vary. Since slipping is allowed, when the center is fixed, the disk can still rotate just like a spinning wheel of a car on a jack stand. If we fix θ and allow x to vary, because slipping is allowed, the disk can slide without rotation, just like a locked wheel skidding on ice. So, YES, it is an independent set.

Therefore, we conclude that $\{x, \theta\}$ is a complete and independent set of generalized coordinates.

1.4 Admissible Variations

A *variation* is a hypothetical small change in a generalized coordinate.

A variation is a close cousin of a *virtual displacement*. Let us discuss the virtual displacement first. A virtual displacement differs from a real displacement in two aspects: (1) It occurs instantaneously without advancing the time. Or, put it differently: it is the difference between the real position and an alternative position, as if we say to ourselves: “what if at this particular moment this point is located there instead of here.” (2) It is *small* in the sense of a differential change.

Now let us look at the relation between a virtual displacement and a variation. A real change in position is a *displacement* (vector). Since the system is described by the chosen complete and independent set of generalized coordinates, this displacement is expressible in terms of this set of generalized coordinates (scalars). A virtual displacement is a *hypothetical* and *small* displacement. It is expressible in *hypothetical* and *small* changes in generalized coordinates, which are defined as *variations*. Normally, displacements are vectors, so are the virtual displacements. Generalized coordinates are scalars, so are the variations.

An *admissible variation* is a hypothetical small change in a generalized coordinate that is allowed by the geometrical constraints of the system.

Since the admissible variations are associated with the generalized coordinates, once a set of generalized coordinates has been defined, a set of admissible variations can be naturally derived. In other words, it does not need to be defined. For a set of generalized coordinates, we usually write, for example, $\{\delta x_1, \delta x_2\}$ as the associated set of admissible variations. Mathematically, the variation operator δ follows the same rules as the differential operator d. But, what is important is to conduct similar tests to see whether the set of admissible variations is *complete and independent*. Passing both completeness and independence tests makes the set a *complete and independent set* of admissible variations.

■ Example 1.8: Rigid Slender Rod Moves on Plane, Variations

Determine whether the set of variations associated with the set of generalized coordinates defined in Example 1.4 is a complete and independent set of admissible variations.

□ Solution:

In Example 1.4, a complete and independent set of generalized coordinate has been defined as $\{x_1, y_1, x_2\}$, where x_1, x_2, and y_1 are all Cartesian coordinates. The corresponding set of admissible variations is $\{\delta x_1, \delta y_1, \delta x_2\}$. Being "admissible" means that the varied configuration remains entirely on the Oxy plane.

Assume a varied configuration of the system, shown as the dashed configuration in Fig. 1.13. The left end is shown in an enlarged view, in which $\delta \boldsymbol{r}_A$ is its virtual displacement; δx_1 and δy_1 are the admissible variations associated with x_1 and y_1, respectively. They, along with x_1 and y_1, locate the left end of rod in the varied configuration. Similarly, δx_2 and x_2 locate the x-coordinate of the right end of the rod in the varied configuration. With the geometrical constraint of a fixed length L, the right end of the rod in the varied configuration is thus located. Consequently, every point in the rod in the varied configuration can be located. Thus, this set of admissible variations is complete.

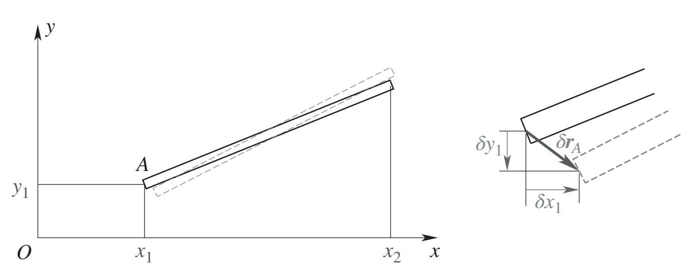

Figure 1.13 Varied configuration of slender rod on plane. The enlarged view of left end A on the right shows virtual displacement $\delta \boldsymbol{r}_A$ being represented by variations δx_1 and δy_1

Furthermore, we need to test when all but one admissible variations are fixed, whether the remaining unfixed one has a continuous range of values to vary. If δx_1 and δy_1 are fixed, the rod is allowed to rotate about the varied position of the left end while the other end moves around the circle of radius L, similar to Fig. 1.8. So, any small variation within the circle is the range in which δx_2 can vary. We can run through the other two tests and find that the situations are similar to the respective tests for the generalized coordinates in Example 1.4. In all tests, the unfixed one has a continuous range of values to vary.

Therefore, we conclude that $\{\delta x_1, \delta y_1, \delta x_2\}$ is a set of complete and independent admissible variations.

■ Example 1.9: Particle Moves Along a Wire of Known Shape, Variations

Determine whether the set of variations associated the set of generalized coordinates defined in Example 1.5 is a complete and independent set of admissible variations.

□ Solution:

In Example 1.5, a complete and independent set of generalized coordinates has been defined as $\{x\}$, where x is the Cartesian coordinates of the particle. The associated set of admissible variations is $\{\delta x\}$. We verify that a varied location can be completely located by δx along with x and that δx has a continuous range of values to vary. Therefore, we conclude that $\{\delta x\}$ is a set of complete and independent admissible variations for this set of generalized coordinates.

■ Example 1.10: Disk Rolls without Slip on Ground, Variations

Determine whether the set of variations associated the set of generalized coordinates defined in Example 1.6 is a complete and independent set of admissible variations.

□ Solution:

In Example 1.6, a complete and independent set of generalized coordinates has been defined as $\{x\}$, where x is the position of the center of the disk from its initial position and positive in the right direction. The associated set of admissible variations is $\{\delta x\}$. We verify that a varied location can be completely located by δx along with x; and that δx has a continuous range of values to vary. Therefore, we conclude that $\{\delta x\}$ is a set of complete and independent admissible variations for this set of generalized coordinates.

■ Example 1.11: Disk Rolls on Ground, Slipping Allowed, Variations

Determine whether the set of variations for the set of generalized coordinates defined in Example 1.7 is a set of complete and independent admissible variations.

□ Solution:

In Example 1.7, a complete and independent set of generalized coordinates has been defined as $\{x, \theta\}$, where x is the Cartesian coordinate of the particle and θ is the angle of the vertical makes with the line that is initially painted as vertical. The associated set of admissible variations is $\{\delta x, \delta\theta\}$. We verify that a varied location can be completely located by δx and $\delta\theta$, along with x and θ; and that when either one is fixed, the unfixed one has a continuous range of values to vary. Therefore, we conclude that $\{\delta x, \delta\theta\}$ is a set of complete and independent admissible variations for this set of generalized coordinates.

1.5 Degrees of Freedom

The number of variations in a set of complete and independent admissible variations for a system is called the number of *degrees of freedom* of the system.

A system is said to be *holonomic* if the number of degrees of freedom equals to the number of generalized coordinates in a set of complete and independent generalized coordinates. Otherwise, the system is *nonholonomic*. Most of the mechanical systems are holonomic and we will only discuss holonomic systems in this book. Treatments for nonholonomic system are beyond the scope of this book.

The number of degrees of freedom is a property of a system. No matter how we define the generalized coordinates, and there is probably an infinite number of ways of doing so, the number of degrees of freedom is always the same.

■ Example 1.12: Revisiting Earlier Examples

Determine the number of degrees of freedom and the holonomicity for systems in Examples 1.4 through 1.7.

□ Solution:

- In Example 1.4: The number of degrees of freedom is 3. The system is holonomic.
- In Example 1.5: The number of degrees of freedom is 1. The system is holonomic.
- In Example 1.6: The number of degrees of freedom is 1. The system is holonomic.
- In Example 1.7: The number of degrees of freedom is 2. The system is holonomic.

■ Example 1.13: Specified Motion

The system shown in Fig. 1.14 consists of a cart of mass m_1 that moves on the horizontal rail and a pendulum of length l and mass m_2. The pendulum is pinned to the cart. The cart moves in accordance with $x = x_0(t)$, driven by an external mechanism (not shown), where x is measured from a fixed reference position. Determine the number of degrees of freedom of the system.

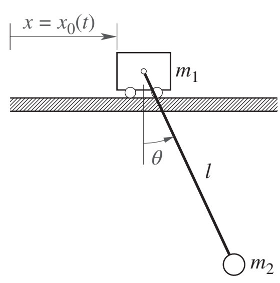

Figure 1.14 Mass–pendulum system with a specified motion

□ Solution:

In this example, by intuition we are inclined to define x and θ as a set of generalized coordinates, where x is as defined in the problem statement and θ is the counterclockwise angular displacement of the pendulum with respect to the vertical. However, we must recognize that such a set is not independent, because $x(t)$ is known for all times, as a part of the geometrical constraint of the system. More importantly, it cannot be varied. In such a case, the cart is said to be undergoing a *specified motion*, and the condition $x = x_0(t)$ is an *auxiliary condition*, which can be considered as a generalized coordinate (but not included in the *complete and independent* set) when needed in deriving the equations of motion for the system.

We define $\{\theta\}$ as a complete and independent set of generalized coordinates for the system; and $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates. The system is holonomic and has one degree of freedom.

—————— · ——————

So far, all examples include only holonomic systems. A curious reader might wonder: how a nonholonomic system looks like, or whether nonholonomic systems even exit. In fact, we deal with nonholonomic system almost daily. When we parallel park a car, if we park it too far from the curb in the first attempt and decide to “correct” it instead of starting over, we have to move the car back and forth in order to move the car laterally just a little bit. That little bit cannot be achieved directly by small variations. A careful reader shall examine this situation as an exercise, by modeling a wheel of the car as a disk that rolls without slip on the ground, and also examine the difference from the situation described in Example 1.6.

In general, a nonholonomic system contains nonholonomic constraints. A nonholonomic constraint is one that cannot be explicitly expressed as an equation involving only the generalized coordinates and time. The analysis in the remainder of this chapter is not applicable to nonholonomic systems.

1.6 Virtual Work and Generalized Forces

Loosely speaking, a *virtual work* is the work done by a force over a virtual displacement. However, in Lagrangian dynamics, we would like to give it a more articulate definition: the virtual work is the *total work done by all nonconservative forces acting on the system over a variation in the admissible configuration*.

There are two key points in this definition. First, the virtual work accounts only for work done by nonconservative forces. Typical nonconservative forces include those that introduce energy into the system, such as externally applied forces, or those that cause energy losses, such as dashpot forces and frictional forces. Second, the variation in the admissible configuration must be general enough to cause a virtual displacement at every location wherever a force is acting upon.

This definition of virtual for the virtual work results in the following expression:

$$\delta W^{\text{n.c.}} = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{R}_i \tag{1.1}$$

where N is the number of nonconservative forces acting on the system, $\boldsymbol{F}_i$'s are the individual forces, and $\delta\boldsymbol{R}_i$'s are the virtual displacements at the locations where the forces are acting.

On the other hand, having selected a complete and independent set of generalized coordinates for a holonomic system guarantees that any virtual displacement can be expressed in terms of admissible variations. Therefore, the virtual work can be expressed as

$$\delta W^{\text{n.c.}} = \sum_{j=1}^{M} \Xi_j \delta\xi_j \tag{1.2}$$

where M is the number of admissible variations in a complete and independent set, which is also the number of degrees of freedom of the system and Ξ_j is called the *generalized force* associated with the generalized coordinate ξ_j.

Equation (1.1) is used to calculate the virtual work; and eqn. (1.2) is used to identify the generalized forces. In calculating the virtual work, there are two methods that can be deployed. One method is to identify a varied configuration and determine the virtual displacement at every location where a force is applied, and use the vector dot product to calculate the virtual work. Another method is to vary one generalized coordinate at a time, which is mapped into a set of corresponding virtual displacements, and calculate the resulting virtual work. After we have varied all the generalized coordinates, summing the resulting virtual work gives the total virtual work. In practice, we need to keep the following in mind:

- Conservative forces should not be included in calculating the generalized forces. Common conservative forces include weights and spring forces. They are included in the system's potential energy.
- A generalized force is a scalar quantity.
- The dimension (unit) of a generalized force is not always the force. It depends on the corresponding generalized coordinate. But the product of a generalized force and the corresponding generalized coordinate always has a unit of work or energy. When a generalized coordinate is a displacement or a position measured in a linear length, the generalized force has the dimension of a force. When a generalized coordinate is an angular displacement or a position measured by an angle, the generalized force has the dimension of a moment or a torque.

■ Example 1.14: Generalized Forces for Externally Applied Force

Find the generalized forces for the force $\boldsymbol{F}(t)$ acting on a particle moving in the plane, as shown in Fig. 1.15. The force forms an angle α with the x-axis.

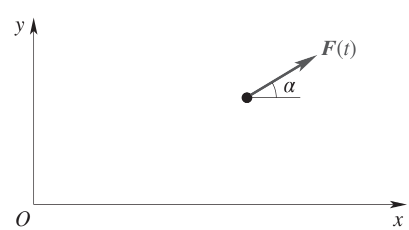

Figure 1.15 Force acting on a particle

□ Solution 1: Using Cartesian Coordinates

- *Generalized Coordinates*: We defined $\{x, y\}$ as a complete and independent set of generalized coordinates, where x and y are the Cartesian coordinates of the particle.
- *Admissible Variations*: We verify that $\{\delta x, \delta y\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Virtual Work*: We use both methods to calculate the virtual work.

Method 1: We introduce a virtual displacement $\delta \boldsymbol{r}$ to the particle, as shown in Fig. 1.16a. The virtual work done by the force over this virtual displacement is, according to eqn. (1.1),

$$\delta W^{\text{n.c.}} = \boldsymbol{F}(t) \cdot \delta \boldsymbol{r} \tag{a}$$

To carry out the dot product, we decompose both $\delta \boldsymbol{r}$ and $\boldsymbol{F}(t)$ into the x- and y-components, as shown in Fig. 1.16b. Then,

$$\begin{aligned}\delta W^{\text{n.c.}} &= \left[F(t)\cos\,\alpha \boldsymbol{i} + F(t)\sin\,\alpha \boldsymbol{j}\right] \cdot (\delta x \boldsymbol{i} + \delta y \boldsymbol{j}) \\ &= F(t)\cos\,\alpha\delta x + F(t)\sin\,\alpha\delta y \end{aligned}\tag{b}$$

where we have used $\boldsymbol{i}$ and $\boldsymbol{j}$ to denote the unit vectors for the x- and y-axes, respectively.

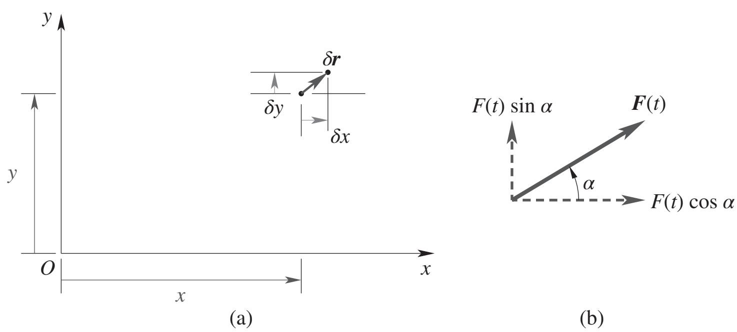

Figure 1.16 (a) Virtual displacement $\delta \boldsymbol{r}$ is decomposed into admissible variations δx and δy. (b) $\boldsymbol{F}$ is decomposed into the x- and y-components

Method 2: We first vary x by δx: the particle moves along the x-axis at a distance δx, as shown in Fig. 1.16a. Only the x-component of the force, $F(t)\cos\,\alpha$, does work over this virtual displacement, and the virtual work is $F(t)\cos\,\alpha\delta x$. We then vary y by δy. Similarly, the work done over this virtual displacement is $F(t)\sin\,\alpha\delta y$. Summing up, the total virtual work due to force $\boldsymbol{F}(t)$ is

$$\delta W^{\text{n.c.}} = F(t)\cos\,\alpha\delta x + F(t)\sin\,\alpha\delta y \tag{c}$$

which is the same as using the first method, as in eqn. (b).

- *Generalized Forces*: According to eqn. (1.2), the generalized forces are

$$\Xi_x = F(t)\cos\,\alpha \tag{d}$$

$$\Xi_y = F(t)\sin\,\alpha. \tag{e}$$

□ Solution 2: Using Polar Coordinates

- *Generalized Coordinates*: We alternatively defined $\{r, \theta\}$ as a complete and independent set of generalized coordinates, where r and θ are the polar coordinates of the particle.
- *Admissible Variations*: We verify that $\{\delta r, \delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.

- *Virtual Work*: To find the virtual work, we first vary r by δr. In the meantime, the force is decomposed into radial and tangential (azimuthal) components, as shown in Fig. 1.17. Note that the angle between F and the radial direction is $\alpha - \theta$. The virtual work due to this variation is $F(t)\cos(\alpha - \theta)\delta r$. We then vary θ by $\delta\theta$. At the location where the force is applied, this variation results in a linear virtual displacement of $r\delta\theta$, also shown in Fig. 1.17. Hence, the virtual work due to this variation is $F(t)\sin(\alpha - \theta)r\delta\theta$. Thus, the virtual work done by the force $\boldsymbol{F}(t)$ is

$$\delta W^{\text{n.c.}} = F(t)\cos(\alpha - \theta)\delta r + F(t)\sin(\alpha - \theta)r\delta\theta \tag{f}$$

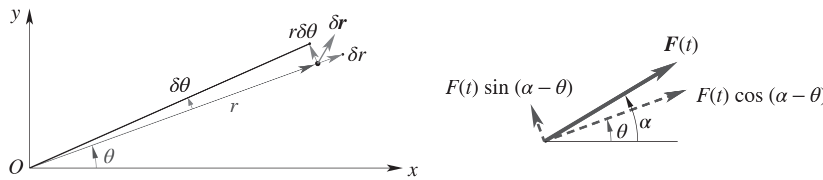

Figure 1.17 Variations in generalized coordinates and decomposition of force $\boldsymbol{F}(t)$ into radial and tangential (azimuthal) components

- *Generalized Forces*: According to eqn. (1.2), the generalized forces are

$$\Xi_r = F(t)\cos(\alpha - \theta) \tag{g}$$

$$\Xi_\theta = F(t)r\sin(\alpha - \theta). \tag{h}$$

- Note that the generalized force Ξ_θ has the dimension of a moment or a torque.

■ Example 1.15: Buoyant Force and Drag on Pendulum

A planar pendulum of mass m and length l is submerged in water, which exerts a constant buoyant force F_0 on the bob vertically upward, as shown in Fig. 1.18. The water also exerts a drag force whose magnitude is proportional to the speed, cv, where c is a constant and v is the speed of the bob, and in the opposite direction of the velocity. The bob remains submerged during the motion. Find the generalized force for the system.

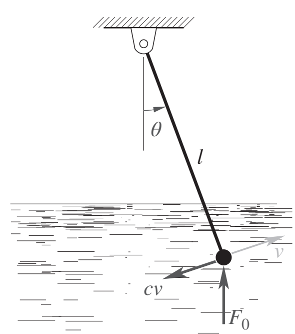

Figure 1.18 Planar pendulum with its bob submerged in water

□ **Solution:**

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the angle the pendulum makes with the vertical, which is positive in the counterclockwise direction.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Virtual Work*: The buoyant force is a constant value, which is very similar to the weight: the work it does depends only on the change in vertical elevation and is independent of the path. This is the signature for a conservative force. Thus, we do not include the buoyant force in the virtual work. Instead, we consider it in the potential energy.

  For the drag, its magnitude is $cl\dot{\theta}$. If we vary θ by $\delta\theta$, this variation causes the bob to travel a virtual displacement of an arc length $l\delta\theta$. Then, the virtual work done is

$$\delta W^{\text{n.c.}} = -(cl\dot{\theta})(l\delta\theta) = -cl^2\dot{\theta}\delta\theta \tag{a}$$

- *Generalized Forces*: According to eqn. (1.2), the generalized force is

$$\Xi_\theta = -cl^2\dot{\theta} \tag{b}$$

- *Potential Energy for Buoyant Force*: The potential energy due to the buoyant force is its hydrostatic pressure. If the mass density of the water is ρ_0, the potential energy can be written as

$$V_{\text{buoyant}} = \rho_0 V_{\text{obj}} g d \tag{c}$$

  where V_{obj} is the volume of the object and d is the depth, which is positive in the downward direction, of its geometric center. Furthermore, $\rho_0 V_{\text{obj}} g$ gives the buoyant force F_0. Using the pivot point as the datum, the bob is located below the pivot point and the potential energy for the buoyant force is

$$V_{\text{buoyant}} = F_0 l \cos\theta \tag{d}$$

■ **Example 1.16: Generalized Force for Dashpot**

A dashpot of dashpot constant c has one of its ends connected to a wall while the other end can move freely, as shown in Fig. 1.19. Find the generalized force for the dashpot.

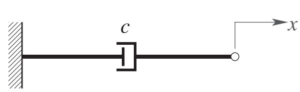

Figure 1.19 Dashpot with one end fixed to the wall

□ **Solution:**

- *Generalized Coordinates*: We define $\{x\}$ as a complete and independent set of generalized coordinates, where x is the rightward displacement of the movable end of the dashpot measured from a fixed reference position.
- *Admissible Variations*: We verify that $\{\delta x\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.

- *Virtual Work*: The dashpot is a nonconservative element that produces a nonconservative force. To calculate the virtual work, we vary x by δx. Although the displacement is varied, the velocity $\dot{x}$ remains constant during the variation. In symbolic analysis, we only need to focus on the positive senses of parameters. Since x is positive when it moves to the right, both the velocity $\dot{x}$ and the variation δx are moving to the right. The force produced by the dashpot equals to $c\dot{x}$, in the direction opposite to $\dot{x}$ and δx. Thus, the virtual work done by the dashpot due to this variation is negative and is expressible as

$$\delta W^{\text{n.c.}} = -c\dot{x}\delta x \tag{a}$$

- *Generalized Forces*: According to eqn. (1.2), the generalized force corresponding to x is

$$\Xi_x = -c\dot{x} \tag{b}$$

■ Example 1.17: Generalized Forces for "Floating Dashpot"

A "floating dashpot" is a dashpot whose two ends move independently, such as the one shown in Fig. 1.20. Find the generalized forces for the "floating dashpot" in Fig. 1.20.

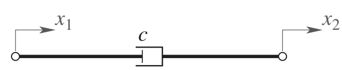

Figure 1.20 "Floating" dashpot: dashpot with independently moveable ends

□ Solution 1: Absolute Displacements as Generalized Coordinates

- *Generalized Coordinates*: We define $\{x_1, x_2\}$ as a complete and independent set of generalized coordinates, where x_1 and x_2 are the displacements of the left and the right ends, respectively, of the dashpot, with respect to the ground, measured to the right from their respective positions in a reference configuration (such as when $t = 0$).
- *Admissible Variations*: We verify that $\{\delta x_1, \delta x_2\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Virtual Work*: At any instant, two ends of the dashpot have speeds of $\dot{x}_1$ and $\dot{x}_2$. They both are positive to the right. There are two possible scenarios, discussed separately below.

  Case 1: If $\dot{x}_1 > \dot{x}_2$, the dashpot is being compressed. The dashpot forces acting on the ends, which have the same magnitude and are denoted as $F_d = c(\dot{x}_1 - \dot{x}_2)$, are against the compression. Thus, they are in directions of stretching the dashpot, as shown in Fig. 1.21. If we vary x_1 by δx_1 and x_2 by δx_2, the dashpot force on the left end is in the opposite direction of δx_1; while the dashpot force on the right end is in the same direction as δx_2. Thus, the total virtual work in this scenario is

$$\delta W^{\text{n.c.}} = -c(\dot{x}_1 - \dot{x}_2)\delta x_1 + c(\dot{x}_1 - \dot{x}_2)\delta x_2 \tag{a}$$

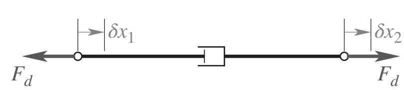

Figure 1.21 Dashpot force when the floating dashpot is compressed

Case 2: If $\dot{x}_1 < \dot{x}_2$, the dashpot is being stretched. The dashpot forces acting on the two ends, expressible as $F_d = c(\dot{x}_2 - \dot{x}_1)$, are against the stretching. Thus, they are in the directions of compressing the dashpot, as shown in Fig. 1.22.

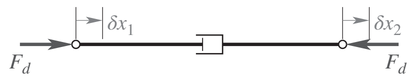

Figure 1.22 Dashpot force when the floating dashpot is stretched

If we vary x_1 by δx_1 and x_2 by δx_2, the dashpot force on the left end is in the same direction of δx_1 while the dashpot force on the right end is in the opposite direction as δx_2. Thus, the total virtual work in this scenario is

$$\delta W^{\text{n.c.}} = c(\dot{x}_2 - \dot{x}_1)\delta x_1 - c(\dot{x}_2 - \dot{x}_1)\delta x_2 \tag{b}$$

Conclusion: Equations (a) and (b) are in fact identical and can be uniformly written as

$$\delta W^{\text{n.c.}} = -c(\dot{x}_1 - \dot{x}_2)(\delta x_1 - \delta x_2) = -c(\dot{x}_2 - \dot{x}_1)(\delta x_2 - \delta x_1) \tag{c}$$

The ingredients in these two expressions are: the negative sign, the dashpot constant, the difference in the velocities, and the difference in the variations. The key is that the order for the subscripts in these two pairs of parentheses is the same. Also note that the two cases discussed earlier do not include the case $\dot{x}_1 = \dot{x}_2$. This is a special case where $\delta W^{\text{n.c.}} = 0$ and is included in the uniform expression in eqn. (c).

- *Generalized Forces*: Therefore, according to eqn. (1.2), the generalized forces are

$$\Xi_{x_1} = -c(\dot{x}_1 - \dot{x}_2) \tag{d}$$

$$\Xi_{x_2} = -c(\dot{x}_2 - \dot{x}_1) \tag{e}$$

□ Solution 2: Relative Displacement as Generalized Coordinate

- *Generalized Coordinates*: We define $\{x_1, y_2\}$ as a complete and independent set of generalized coordinates, where x_1 is the absolute displacement of m_1 and y_2 is the displacement of m_2 relative to m_1, both are positive to the right, measured from their respective positions in a reference configuration.
- *Admissible Variations*: We verify that $\{\delta x_1, \delta y_2\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Virtual Work*: The dashpot force is determined by the relative velocities of its two ends, that is, $c\dot{y}_2$. If x_1 is varied by δx_1 but y_2 is fixed, both masses have the same virtual displacement of δx_1. But the dashpot forces on the two masses are in opposite directions. This means the virtual works done by the dashpot forces cancel each other out; hence, the total virtual work is zero. If x_1 is fixed while y_2 is varied by δy_2, there is no virtual displacement at m_1, but m_2 has a virtual displacement of δy_2. Hence, the virtual work is $-c\dot{y}_2\delta y_2$. Summing up, the total virtual work done by the dashpot is

$$\delta W^{\text{n.c.}} = -c\dot{y}_2\delta y_2 \tag{f}$$

This is similar to the case when one end of the dashpot is fixed, as in Example 1.16.

- *Generalized Forces*: According to eqn. (1.2), the generalized forces are

$$\Xi_{x_1} = 0 \tag{g}$$

$$\Xi_{y_2} = -c\dot{y}_2 \tag{h}$$

1.7 Lagrangian

The *Lagrangian* of a system is given by

$$\mathcal{L} = T - V \tag{1.3}$$

where T is the total kinetic energy[1] of the system, and V is the total potential energy of the system. Since the system is completely described by the chosen set of generalized coordinates, denoted as ξ_j where $j = 1, 2, \ldots, M$, the system's potential energy, which involves positions, is expressible in terms of ξ_j; and the system's kinetic energy, which involves velocities and sometimes the positions as well, is expressible in terms of ξ_j and $\dot{\xi}_j$. Therefore, the Lagrangian, in general, is a function of both ξ_j and $\dot{\xi}_j$, that is, $\mathcal{L} = \mathcal{L}(\xi_j, \dot{\xi}_j)$.

1.8 Lagrange's Equation

For every generalized coordinate ξ_j, the *Lagrange's equation* is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_j}\right) - \frac{\partial \mathcal{L}}{\partial \xi_j} = \Xi_j \tag{1.4}$$

This comprises the set of *equations of motion* for the system. There is exactly one equation of motion corresponding to every generalized coordinate. The number of equations of motion for a system equals to the number of generalized coordinates, which, in turn, equals to the number of degrees of freedom of the system.

Lagrange's equation is derived from *Hamilton's principle*, which is also known as the *principle of virtual work*. Historically, there are claims that Hamilton's principle is the most fundamental principle for dynamics from which Newton's second law can be derived. Not surprisingly, there are counter-claims that Newton's second law is more fundamental from which Hamilton's principle can be derived. No matter how science historians will settle their claims, it is safe to say that using Lagrange's equation is a valid alternative to Newton's second law.

1.9 Procedure for Deriving Equation(s) of Motion

Here, we outline the procedure for deriving the equation(s) of motion for a mechanical system. Note that, in the process, Newton's second law is never invoked.

[1] In a strict sense, the Lagrangian is defined as $\mathcal{L} = T^* - V$, where T^* is called the *kinetic coenergy*. For Newtonian particles and rigid bodies that move at speeds significantly below the speed of light, $T = T^*$. So here we use what most other textbooks use. Interested readers are referred to an excellent textbook on Lagrangian dynamics by J. H. Williams, Jr., *Fundamentals of Applied Dynamics* (John Wiley & Sons, 1995, New York) and a classic textbook by Stephen H. Crandall, D. C. Karnopp, E. F. Kurtz & D. C. Pridmore-Brown, *Dynamics of Mechanical and Electromechanical Systems* (McGraw-Hill, 1968, New York) for more comprehensive and rigorous discussions.

- Preparatory setup: This optional step establishes aids, such as coordinate systems, that facilitate mathematical description of the geometry of the problem.
- Define a set of generalized coordinates and verify its completeness and independence. The definition of a generalized coordinate should include four vital aspects: what, where, relative to whom, and in which direction. The verification can be a mental experiment, without being written out.
- Verify that the associate set of admissible variations is complete and independent.
- Check the holonomicity of the system. If holonomic, identify the number of degrees of freedom of the system. If not, Lagrange's equation is not applicable.
- Derive the expression for the total virtual work according to eqn. (1.1) and subsequently obtain the generalized forces through virtual work, using eqn. (1.2).
- Derive the Lagrangian for the system. This can be separated into three substeps: the kinetic energy, the potential energy, and then the Lagrangian via eqn. (1.3).
- Use Lagrange's equation in eqn. (1.4) to obtain the equation(s) of motion for the system: one equation for each generalized coordinate.

1.10 Worked Examples

In this section, we present a carefully selected collection of worked examples of deriving the equation(s) of motion for the systems. In many of these examples, equation(s) of motion are also derived for additional sets of generalized coordinates. The first several examples focus on systems consisting of only particles and then followed by examples on systems consisting of rigid bodies.

1.10.1 Systems Containing Only Particles

■ Example 1.18: Simple Mass–Spring–Dashpot System, Revisited

Use Lagrangian dynamics to rederive the equation of motion for the mass–spring–dashpot system as shown in Fig. 1.1.

□ Solution 1: Generalized Coordinate Defined from Spring's Unstretched Position

- *Generalized Coordinates*: We define $\{x\}$ as a complete and independent set of generalized coordinates, where x is the downward displacement of the mass, measured from its position when the spring is unstretched, as sketched in Fig. 1.23.

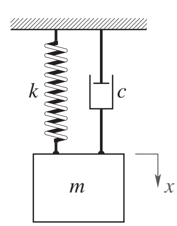

Figure 1.23 Mass–spring–dashpot system

- *Admissible Variations*: We verify that $\{\delta x\}$ is a complete and independent set of admissible variation in this set of generalized coordinates.
- *Holonomicity*: We observe that the number of generalized coordinate (one) equals to the number of admissible variation (one). Thus, we conclude that the system is holonomic. Furthermore, the system has one degree of freedom.
- *Generalized Forces*: The dashpot connects the mass to the wall. We can directly use the results from Example 1.16 for the virtual work done by the dashpot, as

$$\delta W^{\text{n.c.}} = -c\dot{x}\delta x \tag{a}$$

Thus, according to eqn. (1.2), the generalized force associated with x is

$$\Xi_x = -c\dot{x} \tag{b}$$

- *Kinetic Energy*: The velocity of the mass is $\dot{x}$. Thus,

$$T = \frac{1}{2}m\dot{x}^2 \tag{c}$$

- *Potential Energy*: Two elements in the system contribute to the potential energy: the mass and the spring. Note that the generalized coordinate is defined as positive downward and the gravitational energy decreases as x increases.

$$V = -mgx + \frac{1}{2}kx^2 \tag{d}$$

- *Lagrangian*: Combining eqns. (c) and (d) gives, according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 + mgx - \frac{1}{2}kx^2 \tag{e}$$

- *Lagrange's Equation*: The Lagrange's equation for x is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{dx} = \Xi_x \tag{f}$$

When taking the partial derivative of $\mathcal{L}$ with respect to x or $\dot{x}$, we should treat x and $\dot{x}$ as independent variables. Therefore,

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial x} = mg - kx \tag{g}$$

Substituting eqns. (g) and (a) into eqn. (f) gives

$$m\ddot{x} - (mg - kx) = -c\dot{x}$$

or, moving the constant term mg to the right-hand side and $c\dot{x}$ term to the left-hand side, which are customarily done for differential equations, gives

$$m\ddot{x} + c\dot{x} + kx = mg \tag{h}$$

Equation (h) is the equation of motion for the system.

□ Solution 2: Generalized Coordinate Defined from Equilibrium Position

- *Generalized Coordinates*: We defined $\{y\}$ as a complete and independent set of generalized coordinates, where y is the downward displacement of the mass, measured from its static equilibrium position.
- *Admissible Variations*: We verify that $\{\delta y\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: With this new definition of the generalized coordinate, the relation between the dashpot force and the virtual displacement remains the same. That is,

$$\delta W^{\text{n.c.}} = -c\dot{y}\delta y \tag{i}$$

Thus, according to eqn. (1.2), the generalized force associated with y is

$$\Xi_y = -c\dot{y} \tag{j}$$

- *Kinetic Energy*: The velocity of the mass is $\dot{y}$. Thus,

$$T = \frac{1}{2}m\dot{y}^2 \tag{k}$$

- *Potential Energy*: At equilibrium, the spring is already stretched, and the amount is denoted as Δ. In calculating the potential energy stored in the spring, we need to add Δ into displacement y to account for the *total amount of stretch* in the spring. Thus,

$$V = -mgy + \frac{1}{2}k(y+\Delta)^2 \tag{l}$$

- *Lagrangian*: Combining eqns. (k) and (l) gives, according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{y}^2 + mgy - \frac{1}{2}k(y+\Delta)^2 \tag{m}$$

- *Lagrange's equation*: The Lagrange's equation for y, according to eqn. (1.4), is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) - \frac{\partial\mathcal{L}}{dy} = \Xi_y \tag{n}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{y}} = m\dot{y} \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial y} = mg - k(y+\Delta) \tag{o}$$

substituting eqns. (o) and (j) into eqn. (n) gives

$$m\ddot{y} - [mg - k(y+\Delta)] = -c\dot{y} \tag{p}$$

Moving the constant terms to the right-hand side,

$$m\ddot{y} + c\dot{y} + ky = mg - k\Delta \tag{q}$$

Equation (q) is the equation of motion for the system. But it is not the final form, since Δ is not a parameter given in the problem statement. Its expression needs to be determined. Recall that the equation of motion can be used to determine the equilibrium positions of the system. In the above solution process, eqn. (q) is already the equation of motion for the system. At the static equilibrium, $\ddot{y} = \dot{y} = 0$; and $y = 0$ since y is measured from the equilibrium position. Equation (q) itself can be used to find that, at equilibrium, $mg = k\Delta$. Substituting this relation into eqn. (q) gives

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{r}$$

Equation (r) is the equation of motion for the system.

□ Discussion: Equilibrium Condition:

According to the analysis of the static equilibrium of the system in eqn. (h) in *Solution 2* of Example 1.1, $k\Delta = mg$. This is, not surprisingly, identical as the one found in Example 1.1.

In general, if the generalized coordinates are defined from the equilibrium configuration, we do not need to perform a separate analysis of the equilibrium state. This is especially convenient if we do not need to know the exact expression for Δ. We can simply state that, because $\ddot{y} = \dot{y} = y = 0$ at equilibrium, the constant terms (such as those gathered on the right-hand side of eqn. (q)) in the equation sum to zero as the *equilibrium condition*.

■ Example 1.19: Submerged Simple Pendulum

Derive the equation of motion for the submerged planar pendulum in Example 1.15.

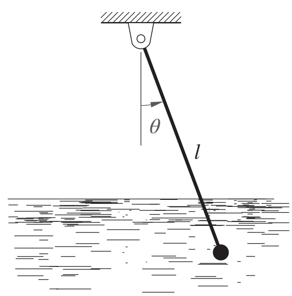

Figure 1.24 Simple pendulum submerged in water

□ Solution:

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the angle the pendulum makes with the vertical, which is positive in the counterclockwise direction. This is the same definition used in Example 1.15.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: The generalized force has been found in Example 1.15 as

$$\Xi_\theta = -cl^2\dot{\theta} \tag{a}$$

- *Kinetic Energy*: The bob in the pendulum has a velocity of $l\dot{\theta}$. Thus,

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{b}$$

- *Potential Energy*: We choose the pivot point as the datum for both gravitational and buoyant potential energies. The bob is located below the datum by a distance of $l\ \cos\ \theta$. The potential energy for the buoyant force has been found in Example 1.15. Hence,

$$V = -mgl\ \cos\ \theta + F_0 l\ \cos\ \theta = (F_0 - mg)l\ \cos\ \theta \tag{c}$$

- *Lagrangian*: Combining eqns. (b) and (c) gives, according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - (F_0 - mg)l\ \cos\ \theta \tag{d}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{d\theta} = \Xi_\theta \tag{e}$$

  Since

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = (F_0 - mg)l\ \sin\ \theta \tag{f}$$

  substituting eqns. (f) and (a) into eqn. (e) gives

$$ml^2\ddot{\theta} - (F_0 - mg)l\ \sin\ \theta = -cl^2\dot{\theta} \tag{g}$$

  Rearranging and canceling an l give

$$ml\ddot{\theta} + cl\dot{\theta} + (mg - F_0)\sin\ \theta = 0 \tag{h}$$

  Equation (h) is the equation of motion for the system.

□ Discussion: Small Motions

A pendulum is a common device for studying vibration in the introductory engineering physics course. In most cases, we assume "small motions" and use the approximation $\sin\ \theta \approx \theta$ to simplify the equation. For the present system, doing so gives

$$ml\ddot{\theta} + cl\dot{\theta} + (mg - F_0)\theta = 0 \tag{i}$$

where the left-hand side consists only of linear terms of $\ddot{\theta}$, $\dot{\theta}$, and θ. This process is known as linearization, in which we keep only up to the first-order (linear) small terms.

We can employ this "small motion" assumption much earlier in order to reduce the complexity in some expressions. We should keep in mind that, when using Lagrange's equation, both energies will be taken partial derivatives. This means that, when writing the expressions for energies, we need to keep up to the second-order small terms if we want to keep the first-order small terms in the resulting equation(s) of motion.

□ Solution 2: Early Adoption of "Small Motion" Assumption

We redo this example by adopting the small motion assumption early. We continue to use the same generalized coordinate. Hence, we start from the step of generalized forces.

- *Generalized Forces*: The generalized force is unchanged as in eqn. (a).
- *Kinetic Energy*: This is unchanged as in eqn. (b).

- *Potential Energy*: Using the "small motion" assumption: when θ is small, keeping up to the second order of θ gives

$$\sin\,\theta \approx \theta \qquad \text{and} \qquad \cos\,\theta \approx 1 - \frac{1}{2}\theta^2 \tag{j}$$

The potential energy in eqn. (c) becomes

$$V = (F_0 - mg)l\left(1 - \frac{1}{2}\theta^2\right) \tag{k}$$

- *Lagrangian*: Combining eqns. (b) and (k), according to eqn. (1.4), gives

$$\mathcal{L} = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - (F_0 - mg)l\left(1 - \frac{1}{2}\theta^2\right) \tag{l}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{d\theta} = \Xi_\theta \tag{m}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = (F_0 - mg)l\theta \tag{n}$$

substituting eqns. (n) and (a) into eqn. (m) gives

$$ml^2\ddot{\theta} - (F_0 - mg)l\theta = -cl^2\dot{\theta} \tag{o}$$

Rearranging and canceling out an l give

$$ml\ddot{\theta} + cl\dot{\theta} + (mg - F_0)\theta = 0 \tag{p}$$

Equation (p) is the equation of motion for the system under the "small motion" assumption, which is the same as eqn. (i) discussed earlier.

■ Example 1.20: Pendulum with Specified Base Motion

The system shown in Fig. 1.25 consists of a cart of mass m_1 that moves on the horizontal rail, and a pendulum of mass m_2 and length l is pinned to the cart. The pendulum swings freely without friction. The cart moves in accordance with $x = x_0(t)$, driven by an external mechanism (not shown), where x is the location of the cart measured from a fixed reference position. Derive the equation(s) of motion for the system.

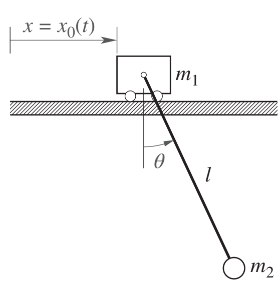

Figure 1.25 Pendulum with specified motion at its base

□ **Solution:**

- *Preparatory Setup*: We have discussed in Example 1.13 that $x_0(t)$ is a specified motion and shall not be considered as a generalized coordinate.
- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the counterclockwise angular displacement of the pendulum, measured from the vertical.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: Although there must be an external force acting on the cart to make it move, this force is specified through a specified motion and will be accounted for by the kinematics of the system. Besides this force, there is no other nonconservative force in the system. Thus, $\delta W^{\text{n.c.}} = 0$, and, according to eqn. (1.2),

$$\Xi_\theta = 0 \tag{a}$$

- *Kinetic Energy*: The pendulum swings relative to the translating cart. Recall the relative motions in the particle kinematics in eqn. (A.12),

$$\boldsymbol{v}_{\text{bob}} = \boldsymbol{v}_{\text{cart}} + \boldsymbol{v}_{\text{bob/cart}} \tag{b}$$

  where $\boldsymbol{v}_{\text{bob}}$ and $\boldsymbol{v}_{\text{cart}}$ are the absolute velocities of the pendulum bob and the cart, respectively, and $\boldsymbol{v}_{\text{bob/cart}}$ is the relative velocity of the bob with respect to the cart. The word "absolute" is added to emphasize its difference from a relative velocity. This vector summation is sketched in Fig. 1.26, where $l\dot{\theta}$ is $\boldsymbol{v}_{\text{bob/cart}}$ and $\dot{x}$ is $\boldsymbol{v}_{\text{cart}}$.

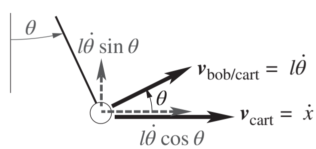

Figure 1.26 Velocity composition of the pendulum pivoted on moving cart

To calculate the kinetic energy, the velocity due to the swinging of the pendulum $\boldsymbol{v}_{\text{bob/cart}}$ is decomposed into horizontal and vertical components, as shown in the dashed vectors in Fig. 1.26. Thus,

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2\right] \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right) \end{aligned} \tag{c}$$

- *Potential Energy*: We choose the pivot point of the pendulum as the datum for gravitational potential energy. Although this is a moving point, its vertical position remains fixed and hence can be used as a datum point. Thus,

$$V = -m_2 gl\cos\theta \tag{d}$$

- *Lagrangian*: Combining eqns. (c) and (d), according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right) + m_2 gl\cos\theta \tag{e}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{d\theta} = \Xi_\theta \tag{f}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta \tag{g}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = -m_2 gl\sin\theta - m_2\dot{x}l\dot{\theta}\sin\theta \tag{h}$$

substituting eqns. (g), (h), and (a) into eqn. (f) gives

$$\frac{d}{dt}\left(m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta\right) + m_2 gl\sin\theta + m_2\dot{x}l\dot{\theta}\sin\theta = 0$$

or, expanding the derivative and canceling the common factor $m_2 l$,

$$l\ddot{\theta} + \ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta + g\sin\theta + \dot{x}\dot{\theta}\sin\theta = 0$$

Canceling out the two $\dot{x}\dot{\theta}\sin\theta$ terms gives

$$l\ddot{\theta} + \ddot{x}_0(t)\cos\theta + g\sin\theta = 0 \tag{i}$$

Equation (i) is the equation of motion for the system.

■ Example 1.21: Damped Two-Mass System

Two masses m_1 and m_2 are constrained between two walls by three springs k_1, k_2, and k_3 and three dashpots c_1, c_2, and c_3, as shown in Fig. 1.27. Assume masses move frictionlessly on the horizontal floor. Derive the equation(s) of motion for the system.

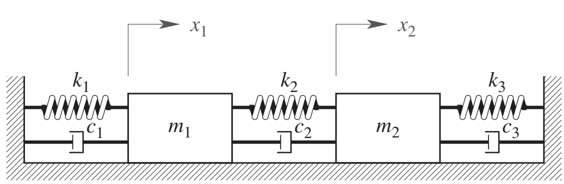

Figure 1.27 Two-mass mass–spring–dashpot system

□ Solution 1: Absolute Displacements as Generalized Coordinates

- *Generalized Coordinates*: We define $\{x_1, x_2\}$ as a complete and independent set of generalized coordinates, where x_1 and x_2 are the displacement of masses m_1 and m_2, respectively, with respect to the ground, measured to the right from their respective positions in equilibrium.

- *Admissible Variations*: We verify that $\{\delta x_1, \delta x_2\}$ is a set of complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has two degrees of freedom.
- *Generalized Forces*: There are three nonconservative force-producing dashpots in the system. Among them, c_1 and c_3 are simple dashpots as analyzed in Example 1.16, and c_2 is a "floating dashpot" as analyzed in Example 1.17. Thus, the total virtual work done by these dashpots is

$$\begin{aligned}\delta W^{\text{n.c.}} &= -c_1\dot{x}_1\delta x_1 - c_2(\dot{x}_2 - \dot{x}_1)(\delta x_2 - \delta x_1) - c_3\dot{x}_2\delta x_2 \\ &= \left[-c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1)\right]\delta x_1 + \left[-c_2(\dot{x}_2 - \dot{x}_1) - c_3\dot{x}_2\right]\delta x_2\end{aligned}$$

  Thus, according to eqn. (1.2),

$$\Xi_{x_1} = -(c_1 + c_2)\dot{x}_1 + c_2\dot{x}_2 \tag{a}$$

$$\Xi_{x_2} = c_2\dot{x}_1 - (c_2 + c_3)\dot{x}_2 \tag{b}$$

- *Kinetic Energy*: The velocities of the masses are $\dot{x}_1$ and $\dot{x}_2$. Thus,

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

- *Potential Energy*: There is no change in the height of the masses, so the gravitational potential energies for both masses do not change. The springs may have already been stretched at equilibrium, denoted as Δ_1, Δ_2, and Δ_3, respectively. These are in addition to stretches due to displacements, which are, x_1, $x_2 - x_1$, and $-x_2$, respectively. Thus,

$$V = \frac{1}{2}k_1(x_1 + \Delta_1)^2 + \frac{1}{2}k_2(x_2 - x_1 + \Delta_2)^2 + \frac{1}{2}k_3(-x_2 + \Delta_3)^2$$

- *Lagrangian*: According to eqn. (1.3)

$$\begin{aligned}\mathcal{L} = T - V = &\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1(x_1 + \Delta_1)^2 \\ &- \frac{1}{2}k_2(x_2 - x_1 + \Delta_2)^2 - \frac{1}{2}k_3(-x_2 + \Delta_3)^2\end{aligned} \tag{c}$$

- *Lagrange's Equation*: The system has two equations of motion.
  – *The x_1-Equation*: The Lagrange's equation for x_1 is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{dx_1} = \Xi_{x_1} \tag{d}$$

  Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1\dot{x}_1 \tag{e}$$

  and

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k_1\left(x_1 + \Delta_1\right) + k_2\left(x_2 - x_1 + \Delta_2\right) \tag{f}$$

  substituting eqns. (e), (f), and (a) into eqn. (d) gives

$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = -k_1\Delta_1 + k_2\Delta_2 \tag{g}$$

– *The x_2-Equation*: The Lagrange's equation for x_2 is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{dx_2} = \Xi_{x_2} \tag{h}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2 \tag{i}$$

and

$$\frac{\partial \mathcal{L}}{\partial x_2} = -k_2\left(x_2 - x_1 + \Delta_2\right) + k_3\left(-x_2 + \Delta_3\right) \tag{j}$$

substituting eqns. (i), (j), and (b) into eqn. (h) gives

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + \left(c_2 + c_3\right)\dot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 = -k_2\Delta_2 + k_3\Delta_3 \tag{k}$$

– *Equilibrium Conditions*: Since the generalized coordinates are defined from the respective equilibrium positions, we can simply require all constant terms in both equations to vanish as the equilibrium conditions. That is,

$$-k_1\Delta_1 + k_2\Delta_2 = 0$$
$$-k_2\Delta_2 + k_3\Delta_3 = 0$$

We have only two equations to solve for three Δ's: the system is statically indeterminate. However, if we do not need to know the explicit expressions for these "equilibrium stretches," they suffice to remove these Δ's from the equations of motion for the system, in eqns. (g) and (k), to give

$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0 \tag{l}$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + \left(c_2 + c_3\right)\dot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 = 0 \tag{m}$$

Equations (l) and (m) comprise the equations of motion for the system.

□ Solution 2: Relative Displacement as Generalize Coordinate

- *Generalized Coordinates*: We define $\{x_1, y_2\}$ as a complete and independent set of generalized coordinates, where x_1 is the displacement of mass m_1 with respect to the ground, measured from its equilibrium position, and y_2 is the relative displacement of mass m_2 with respect to mass m_1, measured from its equilibrium position.
- *Admissible Variations*: We verify that $\{\delta x_1, \delta y_2\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has two degrees of freedom.
- *Generalized Forces*: There are three nonconservative elements. Dashpot c_1 is a simple dashpot as analyzed in Example 1.16, which gives the virtual work as $-c_1\dot{x}_1\delta x_1$. Dashpot c_2 is a "floating dashpot" as analyzed in *Solution 2* in Example 1.17, which gives the virtual work as $-c_2\dot{y}_2\delta y_2$.

For dashpot c_3, since the (absolute) velocity of m_2 is $\dot{x}+\dot{y}$, the dashpot force is $c_3(\dot{x}_1+\dot{y}_2)$. If x_1 is varied by δx_1 but y_2 is fixed, m_2 has a virtual displacement of δx_1; the virtual work is $-c_3(\dot{x}_1+\dot{y}_2)\delta x_1$. If x_1 is fixed but y_2 is varied by δy_2, m_2 has a virtual displacement of δy_2, and the virtual work is $-c_3(\dot{x}_1+\dot{y}_2)\delta y_2$. Thus, the total virtual work done by dashpot c_3 is $-c_3(\dot{x}_1+\dot{y}_2)(\delta x_1+\delta y_2)$.

Summing up, the total virtual work done by all three dashpots is

$$\begin{aligned}\delta W^{\text{n.c.}} &= -c_1\dot{x}_1\delta x_1 - c_2\dot{y}_2\delta y_2 - c_3(\dot{x}_1+\dot{y}_2)(\delta x_1+\delta y_2)\\ &= \left[-c_1\dot{x}_1 - c_3(\dot{x}_1+\dot{y}_2)\right]\delta x_1 + \left[-c_2\dot{y}_2 - c_3(\dot{y}_2+\dot{x}_1)\right]\delta y_2\end{aligned}$$

Then, according to eqn. (1.2),

$$\Xi_{x_1} = -(c_1+c_3)\dot{x}_1 - c_3\dot{y}_2 \tag{n}$$

$$\Xi_{y_2} = -c_3\dot{x}_1 - (c_2+c_3)\dot{y}_2 \tag{o}$$

- *Kinetic Energy*: The velocity of m_1 is $\dot{x}_1$, and the (absolute) velocity of m_2 is $(\dot{x}_1+\dot{y}_2)$. Thus,

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_1+\dot{y}_2)^2$$

- *Potential Energy*: Denote the amount of stretches in the springs at equilibrium as Δ_1, Δ_2, and Δ_3. The total amount of stretch in three springs are $x_1+\Delta_1$, $y_2+\Delta_2$, and $-x_1-y_2+\Delta_3$, respectively.

$$V = \frac{1}{2}k_1(x_1+\Delta_1)^2 + \frac{1}{2}k_2(y_2+\Delta_2)^2 + \frac{1}{2}k_3(-x_1-y_2+\Delta_3)^2$$

- *Lagrangian*: According to eqn. (1.3),

$$\begin{aligned}\mathcal{L} = T - V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_1+\dot{y}_2)^2 - \frac{1}{2}k_1(x_1+\Delta_1)^2\\ - \frac{1}{2}k_2(y_2+\Delta_2)^2 - \frac{1}{2}k_3(-x_1-y_2+\Delta_3)^2\end{aligned} \tag{p}$$

- *Lagrange's Equation*: The system has two equations of motion.
  – *The x_1-Equation*: The Lagrange's equation for x_1 is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_1}\right) - \frac{\partial\mathcal{L}}{dx_1} = \Xi_{x_1} \tag{q}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{x}_1} = m_1\dot{x}_1 + m_2(\dot{x}_1+\dot{y}_2) = (m_1+m_2)\dot{x}_1 + m_2\dot{y}_2 \tag{r}$$

and

$$\frac{\partial\mathcal{L}}{\partial x_1} = -k_1(x_1+\Delta_1) - k_3(x_1+y_2-\Delta_3) \tag{s}$$

substituting eqns. (r), (s), and (n) into eqn. (q) gives

$$(m_1+m_2)\ddot{x}_1 + m_2\ddot{y}_2 + (c_1+c_3)\dot{x}_1 + c_3\dot{y}_2 + (k_1+k_3)x_1 + k_3y_2 = -k_1\Delta_1 + k_3\Delta_3 \tag{t}$$

– *The y_2-Equation*: The Lagrange's equation for y_2 is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_2}\right) - \frac{\partial \mathcal{L}}{d y_2} = \Xi_{y_2} \tag{u}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{y}_2} = m_2(\dot{y}_2 + \dot{x}_1) \tag{v}$$

and

$$\frac{\partial \mathcal{L}}{\partial y_2} = -k_2(y_2 + \Delta_2) - k_3(x_1 + y_2 - \Delta_3) \tag{w}$$

substituting eqns. (v), (w), and (o) into eqn. (u) gives

$$m_2\ddot{x}_1 + m_2\ddot{y}_2 + c_3\dot{x}_1 + (c_2 + c_3)\dot{y}_2 + k_3x_1 + (k_2 + k_3)y_2 = -k_2\Delta_2 + k_3\Delta_3 \tag{x}$$

The constant terms in eqns. (t) and (x) have been moved to the right-hand sides. They vanish under the "equilibrium conditions." Thus,

$$(m_1 + m_2)\ddot{x}_1 + m_2\ddot{y}_2 + (c_1 + c_3)\dot{x}_1 + c_3\dot{y}_2 + (k_1 + k_3)x_1 + k_3y_2 = 0 \tag{y}$$

$$m_2\ddot{x}_1 + m_2\ddot{y}_2 + c_3\dot{x}_1 + (c_2 + c_3)\dot{y}_2 + k_3x_1 + (k_2 + k_3)y_2 = 0 \tag{z}$$

Equations (y) and (z) comprise the equations of motion for the system.

■ Example 1.22: Double Pendulum

A planar pendulum of mass m and length l is attached to another identical planar pendulum, which is connected to the ceiling, as shown in Fig. 1.28. Derive the equation(s) of the motion for the system.

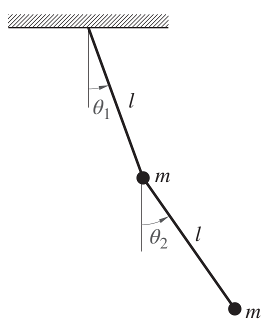

Figure 1.28 Double pendulum

□ Solution:

- *Generalized Coordinates*: We define $\{\theta_1, \theta_2\}$ as a set of complete and independent generalized coordinates, where θ_1 and θ_2 are the counterclockwise angular displacements of the two pendulums with respect to the vertical.

- *Admissible Variations*: We verify that $\{\delta\theta_1, \delta\theta_2\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has two degrees of freedom.
- *Generalized Forces*: There is no nonconservative element in the system. Thus, $\delta W^{\text{n.c.}} = 0$. Then, according to eqn. (1.2),

$$\Xi_{\theta_1} = 0 \quad \Xi_{\theta_2} = 0 \tag{a}$$

- *Kinetic Energy*: The mass of pendulum 1 has a velocity of $l\dot{\theta}_1$, which can be decomposed into the x- and y-components, as

$$\boldsymbol{v}_1 = l\dot{\theta}_1(\cos\,\theta_1\boldsymbol{i} + \sin\,\theta_1\boldsymbol{j}) \tag{b}$$

The mass of pendulum 2 has a velocity of $l\dot{\theta}_2$ relative to a translating reference frame attached to the mass of pendulum 1 (but does not rotate with the pendulum), that is,

$$\boldsymbol{v}_{2/1} = l\dot{\theta}_2(\cos\,\theta_2\boldsymbol{i} + \sin\,\theta_2\boldsymbol{j}) \tag{c}$$

Recall the relation for relative motions in eqn. (A.12), the absolute velocity of pendulum 2 is

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_{2/1} = l\left(\dot{\theta}_1\cos\,\theta_1 + \dot{\theta}_2\cos\,\theta_2\right)\boldsymbol{i} + l\left(\dot{\theta}_1\sin\,\theta_1 + \dot{\theta}_2\sin\,\theta_2\right)\boldsymbol{j} \tag{d}$$

Hence,

$$T = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\,\cos(\theta_2 - \theta_1)\right] \tag{e}$$

- *Potential Energy*: We choose the pivot point as the datum for the gravitational potential energy. Both masses are located below the datum. Thus,

$$\begin{aligned} V &= mg(-l\,\cos\,\theta_1) + mg(-l\,\cos\,\theta_1 - l\,\cos\,\theta_2) \\ &= -mgl(2\,\cos\,\theta_1 + \cos\,\theta_2) \end{aligned} \tag{f}$$

- *Lagrangian*: Combining eqns. (e) and (f) gives, according to eqn. (1.3),

$$\begin{aligned} \mathcal{L} &= T - V \\ &= ml^2\left[\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\,\cos(\theta_2 - \theta_1)\right] + mgl(2\,\cos\,\theta_1 + \cos\,\theta_2) \end{aligned} \tag{g}$$

- *Lagrange's Equations*: The system has two equations of motion.
  – *The θ_1-Equation*: Lagrange's equation for θ_1 is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = \Xi_{\theta_1} \tag{h}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = 2ml^2\dot{\theta}_1 + ml^2\dot{\theta}_2\cos(\theta_2 - \theta_1) \tag{i}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) - 2mgl\,\sin\theta_1 \tag{j}$$

substituting eqns. (i), (j), and (a) into eqn. (h) gives

$$\begin{aligned}2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2\cos(\theta_2-\theta_1) - ml^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)\\ -ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + 2mgl\,\sin\theta_1 = 0\end{aligned} \tag{k}$$

Canceling out the two terms of $ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)$ gives

$$2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2\cos(\theta_2-\theta_1) - ml^2\dot{\theta}_2^2\sin(\theta_2-\theta_1) + 2mgl\,\sin\theta_1 = 0 \tag{l}$$

– *The θ_2-Equation*: Lagrange's equation for θ_2 is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) - \frac{\partial\mathcal{L}}{\partial\theta_2} = \Xi_{\theta_2} \tag{m}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = ml^2\dot{\theta}_2 + ml^2\dot{\theta}_1\cos(\theta_2-\theta_1) \tag{n}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = -ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - mgl\,\sin\theta_2 \tag{o}$$

substituting eqns. (n), (o), and (a) into eqn. (m) gives

$$\begin{aligned}ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1\cos(\theta_2-\theta_1) - ml^2\dot{\theta}_1(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)\\ +ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + mgl\,\sin\theta_2 = 0\end{aligned} \tag{p}$$

Canceling out the two terms of $ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)$ gives

$$ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1\cos(\theta_2-\theta_1) - ml^2\dot{\theta}_1^2\sin(\theta_2-\theta_1) + mgl\,\sin\theta_2 = 0 \tag{q}$$

Equations (l) and (q) are the equations of motion for the system.

1.10.2 Systems Containing Rigid Bodies

In this section, the systems under consideration contain rigid bodies. The expressions for kinetic and potential energies of a rigid body are slightly more complicated. The readers are advised to review Appendix A before proceeding. But the procedure for deriving the equation(s) of motion remains the same.

■ Example 1.23: Rigid-Link Pendulum

A uniform rigid slender rod of mass m and length L is pivoted at its top end and is allowed to move within the plane of paper, as shown in Fig. 1.29. Assume the pivot is frictionless. Derive the equation(s) of motion for the system.

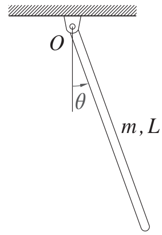

Figure 1.29 Rigid-link planar pendulum

□ **Solution:**

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the counterclockwise angular displacement of the rod measured from the vertical.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: There is no nonconservative force in the system. Thus, $\delta W^{\text{n.c.}} = 0$. According to eqn. (1.2),

$$\Xi_\theta = 0 \tag{a}$$

- *Kinetic Energy*: The rod is a rigid body rotating about a fixed point O. We can use a simpler version of the equation for the kinetic energy of the rigid body:

$$T = \frac{1}{2} I_O \omega^2$$

where I_O is the moment of inertia of the rigid body about the fixed point O and ω is its angular velocity. Recall the moment of inertia for a uniform slender rod about its centroid

$$I_C = \frac{1}{12} mL^2$$

Using the parallel axes theorem (see eqn. (A.21)), the moment of inertia for the rigid slender rod about the pivot point O is

$$I_O = I_C + m\left(\frac{L}{2}\right)^2 = \frac{1}{12} mL^2 + \frac{1}{4} mL^2 = \frac{1}{3} mL^2$$

The angular velocity of the rod is $\dot{\theta}$. Thus,

$$T = \frac{1}{2}\left(\frac{1}{3} mL^2\right)\dot{\theta}^2 = \frac{1}{6} mL^2 \dot{\theta}^2 \tag{b}$$

- *Potential Energy*: We choose the pivot point as the datum for gravitational potential energy. The centroid is located below the datum.

$$V = -mg\frac{L}{2}\cos\theta = -\frac{1}{2} mgL\cos\theta \tag{c}$$

- *Lagrangian*: Combining eqns. (b) and (c) gives, according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{1}{6}mL^2\dot{\theta}^2 + \frac{1}{2}mgL\cos\theta \tag{d}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{d\theta} = \Xi_\theta \tag{e}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{1}{3}mL^2\dot{\theta} \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial\theta} = -\frac{1}{2}mgL\sin\theta \tag{f}$$

substituting eqns. (f) and (a) into eqn. (e) gives

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mgL\sin\theta = 0 \tag{g}$$

Equation (g) is the equation of motion for the system.

■ Example 1.24: Inverted Rigid-Link Pendulum

A uniform rigid slender rod of mass m and length L is pivoted at its bottom end and is allowed to rotate within the plane of paper. Its top end is constrained by two identical linear springs, whose pivot points are a distance L from the pivot for the rod, as shown in Fig. 1.30. The springs are unstretched when the rod is vertical. Derive the equation(s) of motion for the system.

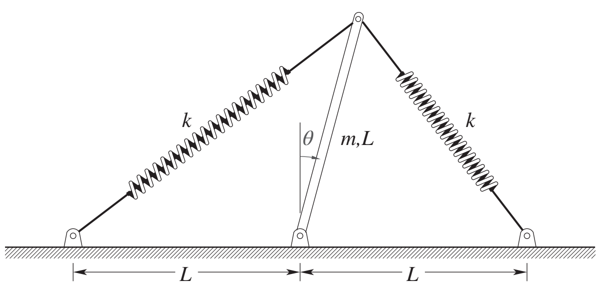

Figure 1.30 Inverted rigid-link planar pendulum

□ Solution:

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is clockwise angular displacement of the rod measured from the vertical.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: Since there is no nonconservative force in the system, $\delta W^{\text{n.c.}} = 0$. Thus,

$$\Xi_\theta = 0 \tag{a}$$

- *Kinetic Energy*: The rod is rotating about a fixed point at an angular velocity of $\dot{\theta}$. The moment of inertia about the pivot point, as calculated in Example 1.23, is $I_O = \frac{1}{3}mL^2$. Thus,

$$T = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 = \frac{1}{6}mL^2\dot{\theta}^2 \tag{b}$$

- *Potential Energy*: We choose the pivot point as the datum for the gravitational potential energy. The gravitational potential energy is

$$V_{\text{gravity}} = mg\frac{L}{2}\cos\,\theta = \frac{1}{2}mgL\,\cos\,\theta \tag{c}$$

For the potential energy stored in the springs, consider first the spring on the left. The current length of this spring, L_L, can be found by the law of cosine as

$$L_L^2 = L^2 + L^2 - 2L^2\cos\left(\frac{\pi}{2}+\theta\right) = 2L^2(1+\sin\theta) \tag{d}$$

Similarly, for the spring on the right,

$$L_R^2 = L^2 + L^2 - 2L^2\cos\left(\frac{\pi}{2}-\theta\right) = 2L^2(1-\sin\theta) \tag{e}$$

The unstretched length of both springs is $\sqrt{2}L$. Thus, the potential energy stored in the two springs is

$$\begin{aligned} V_{\text{springs}} &= \frac{1}{2}k\left(\sqrt{2}L\sqrt{1-\sin\theta} - \sqrt{2}L\right)^2 + \frac{1}{2}k\left(\sqrt{2}L\sqrt{1+\sin\theta} - \sqrt{2}L\right)^2 \\ &= kL^2\left[\left(\sqrt{1-\sin\theta}-1\right)^2 + \left(\sqrt{1+\sin\theta}-1\right)^2\right] \\ &= 2kL^2\left[2-\left(\sqrt{1-\sin\theta}+\sqrt{1+\sin\theta}\right)\right] \end{aligned} \tag{f}$$

- *Lagrangian*: Combining eqns. (b) and (f), according to eqn. (1.3), gives

$$\mathcal{L} = T - V = \frac{1}{6}mL^2\dot{\theta}^2 - \frac{1}{2}mgL\,\cos\,\theta - 2kL^2\left[2-\left(\sqrt{1-\sin\theta}+\sqrt{1+\sin\theta}\right)\right] \tag{g}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{d\theta} = \Xi_\theta \tag{h}$$

Note that

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{1}{3}mL^2\dot{\theta} \tag{i}$$

and

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\theta} &= \frac{1}{2}mgL\,\sin\theta + 2kL^2\left(\frac{1}{2}\frac{-\cos\,\theta}{\sqrt{1-\sin\theta}} + \frac{1}{2}\frac{\cos\,\theta}{\sqrt{1+\sin\theta}}\right) \\ &= \frac{1}{2}mgL\,\sin\theta - kL^2\left(\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}\right) \end{aligned} \tag{j}$$

where, considering the geometrically admissible range of $-\pi/2 \le \theta \le \pi/2$, the following relation has been used:

$$\sqrt{(1-\sin\theta)(1+\sin\theta)} = \cos\ \theta$$

Substituting eqns. (i), (j), and (a) into eqn. (h) gives

$$\frac{1}{3}mL^2\ddot{\theta} - \frac{1}{2}mgL\ \sin\ \theta + kL^2\left(\sqrt{1+\sin\ \theta} - \sqrt{1-\sin\ \theta}\right) = 0 \qquad \text{(k)}$$

Equation (k) is the equation of motion for the system. Noting that

$$\left(\sqrt{1+\sin\ \theta} - \sqrt{1-\sin\ \theta}\right)^2 = 2 - 2\sqrt{(1-\sin\ \theta)(1+\sin\ \theta)} = 2\,(1-\cos\ \theta) \qquad \text{(l)}$$

Equation (k) can be further simplified into

$$\frac{1}{3}mL^2\ddot{\theta} - \frac{1}{2}mgL\ \sin\ \theta + \sqrt{2}kL^2\sqrt{1-\cos\ \theta} = 0 \qquad \text{(m)}$$

■ Example 1.25: Restrained Double Pulley

A double pulley comprises two pulleys, modeled as uniform circular disks, wielded together such that they rotate about their common center as one piece. The larger pulley has mass m_1 and radius R_1, and the smaller pulley has mass m_2 and radius R_2. A mass m_0 is attached to the double pulley via a massless inextensible rope that is wrapped around the perimeter of the smaller pulley without slippage. The larger pulley is restrained by a spring k and a dashpot c, as shown in Fig. 1.31. Derive the equation(s) of motion for the system.

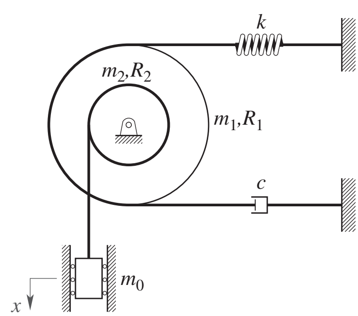

Figure 1.31 A double pulley constrained by spring and dashpot

□ Solution 1: Linear Displacement as Generalized Coordinate

- *Generalized Coordinates*: We define $\{x\}$ as a complete and independent set of generalized coordinates, where x is the downward displacement of mass m_0 with respect to the ground measured from the equilibrium position.
- *Admissible Variations*: We verify that $\{\delta x\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.

- *Generalized Forces*: The dashpot is the nonconservative element in the system. A velocity of $\dot{x}$ at the perimeter of the smaller pulley of radius R_2 produces an angular velocity of $\dot{x}/R_2$ in the double pulley and, in turn, a velocity of $\dot{x}R_1/R_2$ at the dashpot. Similarly, a variation in x of δx at radius R_2 produces a virtual angular displacement of $\delta x/R_2$ in the double pulley and a virtual displacement of $\delta x R_1/R_2$ at the dashpot. Thus,

$$\delta W^{\text{n.c}} = -c\left(\dot{x}\frac{R_1}{R_2}\right)\left(\delta x\frac{R_1}{R_2}\right) = -c\left(\frac{R_1}{R_2}\right)^2 \dot{x}\delta x \tag{a}$$

  According to eqn. (1.2),

$$\Xi_x = -c\left(\frac{R_1}{R_2}\right)^2 \dot{x} \tag{b}$$

- *Kinetic Energy*: As mass m_0 moves at a speed of $\dot{x}$, the double pulley rotates at an angular velocity of $\dot{x}/R_2$. Recall that the moment of inertia for a circular disk of mass m and radius R about its centroid is $\frac{1}{2}mR^2$. Hence,

$$\begin{aligned} T &= \frac{1}{2}m_0\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}m_1R_1^2\right)\left(\frac{\dot{x}}{R_2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}m_2R_2^2\right)\left(\frac{\dot{x}}{R_2}\right)^2 \\ &= \frac{1}{2}\left(m_0 + \frac{R_1^2}{2R_2^2}m_1 + \frac{1}{2}m_2\right)\dot{x}^2 \end{aligned} \tag{c}$$

- *Potential Energy*: When mass m_0 moves a distance x, the pulley rotates by an angle of x/R_2, and, on the outer rim, the spring is stretched by an additional xR_1/R_2. Thus,

$$V = -mgx + \frac{1}{2}k\left(x\frac{R_1}{R_2} + \Delta\right)^2 \tag{d}$$

  where Δ is the amount of stretch in the spring at equilibrium.
- *Lagrangian*: Combining eqns. (c) and (d) gives, according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{1}{2}\left(m_0 + \frac{R_1^2}{2R_2^2}m_1 + \frac{1}{2}m_2\right)\dot{x}^2 + mgx - \frac{1}{2}k\left(x\frac{R_1}{R_2} + \Delta\right)^2 \tag{e}$$

- *Lagrange's Equation*: The Lagrange's equation for x is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{dx} = \Xi_x \tag{f}$$

  Since

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = \left(m_0 + \frac{R_1^2}{2R_2^2}m_1 + \frac{1}{2}m_2\right)\dot{x} \tag{g}$$

  and

$$\frac{\partial\mathcal{L}}{\partial x} = mg - k\left(x\frac{R_1}{R_2} + \Delta\right)\frac{R_1}{R_2} \tag{h}$$

substituting eqns. (g), (h), and (b) into eqn. (f) gives

$$\left(m_0 + \frac{R_1^2}{2R_2^2}m_1 + \frac{1}{2}m_2\right)\ddot{x} - mg + k\left(x\frac{R_1}{R_2} + \Delta\right)\frac{R_1}{R_2} = -c\left(\frac{R_1}{R_2}\right)^2\dot{x}$$

or

$$\left(m_0 + \frac{R_1^2}{2R_2^2}m_1 + \frac{1}{2}m_2\right)\ddot{x} + c\left(\frac{R_1}{R_2}\right)^2\dot{x} + k\left(\frac{R_1}{R_2}\right)^2 x + k\frac{R_1}{R_2}\Delta - mg = 0 \tag{i}$$

At equilibrium, all constant terms (Δ and mg) cancel out as the equilibrium condition, and eqn. (i) becomes

$$\left[(2m_0 + m_2)R_2^2 + m_1R_1^2\right]\ddot{x} + 2cR_1^2\dot{x} + 2kR_1^2 x = 0 \tag{j}$$

Equation (j) is the equation of motion for the system.

□ Solution 2: Angular Displacement as Generalized Coordinate

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the counterclockwise angular displacement of the pulleys measured from the equilibrium configuration, as shown in Fig. 1.32.

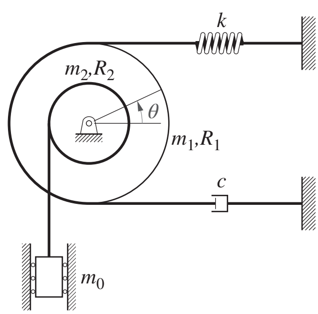

Figure 1.32 Alternative generalized coordinate for the double pulley constrained by spring and dashpot

- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: The dashpot is the nonconservative element in the system. An angular velocity of $\dot{\theta}$ in the pulleys produces a velocity of $\dot{\theta}R_1$ at the dashpot. A variation $\delta\theta$ produces a virtual displacement of $R_1\delta\theta$ at the end of the dashpot. Then,

$$\delta W^{\text{n.c}} = -c\left(R_1\dot{\theta}\right)\left(R_1\delta\theta\right) = -cR_1^2\dot{\theta}\delta\theta \tag{k}$$

Thus, according to eqn. (1.2),

$$\Xi_\theta = -cR_1^2\dot{\theta} \tag{l}$$

- *Kinetic Energy*: When the pulley rotates at an angular velocity $\dot{\theta}$, mass m_0 moves at the same velocity as the outer rim of the smaller pulley, at $\dot{\theta}R_2$. Hence,

$$\begin{aligned} T &= \frac{1}{2}m_0\left(\dot{\theta}R_2\right)^2 + \frac{1}{2}\left(\frac{1}{2}m_1R_1^2\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{2}m_2R_2^2\right)\dot{\theta}^2 \\ &= \frac{1}{2}\left(m_0R_2^2 + \frac{1}{2}m_1R_1^2 + \frac{1}{2}m_2R_2^2\right)\dot{\theta}^2 \end{aligned} \tag{m}$$

- *Potential Energy*: When the pulley rotates by an angle θ, the spring is additionally stretched by θR_1 while mass m_0 is lowered by θR_2. Thus,

$$V = -mgR_2\theta + \frac{1}{2}k\left(\theta R_1 + \Delta\right)^2 \tag{n}$$

  where Δ is the amount of stretch in the spring at equilibrium, and the equilibrium position of the mass is used as the datum for its gravitational potential energy.
- *Lagrangian*: Combining eqns. (m) and (n) gives, according to eqn. (1.3)

$$\mathcal{L} = T - V = \frac{1}{2}\left(m_0R_2^2 + \frac{1}{2}m_1R_1^2 + \frac{1}{2}m_2R_2^2\right)\dot{\theta}^2 + mgR_2\theta - \frac{1}{2}k\left(\theta R_1 + \Delta\right)^2 \tag{o}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{d\theta} = \Xi_\theta \tag{p}$$

  Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \left(m_0R_2^2 + \frac{1}{2}m_1R_1^2 + \frac{1}{2}m_2R_2^2\right)\dot{\theta} \tag{q}$$

  and

$$\frac{\partial\mathcal{L}}{\partial\theta} = mgR_2 - kR_1\left(\theta R_1 + \Delta\right) \tag{r}$$

  substituting eqns. (q), (r), and (l) into eqn. (p) gives

$$\left(m_0R_2^2 + \frac{1}{2}m_1R_1^2 + \frac{1}{2}m_2R_2^2\right)\ddot{\theta} - mgR_2 + kR_1\left(\theta R_1 + \Delta\right) = -cR_1^2\dot{\theta} \tag{s}$$

  At equilibrium, constant terms (Δ and mg) form the equilibrium condition and cancel out, and eqn. (s) becomes

$$\left(m_0R_2^2 + \frac{1}{2}m_1R_1^2 + \frac{1}{2}m_2R_2^2\right)\ddot{\theta} + cR_1^2\dot{\theta} + kR_1^2\theta = 0 \tag{t}$$

  Equation (t) is the equation of motion for the system.

■ Example 1.26: Platform on Two Supports

A platform, which is modeled as a uniform rigid slender rod of a mass m and length L, is supported by two springs k_1 and k_2, and two dashpots c_1 and c_2 at its two ends, as shown in Fig. 1.33. Assume small motions. Derive the equation(s) of motion for the system.

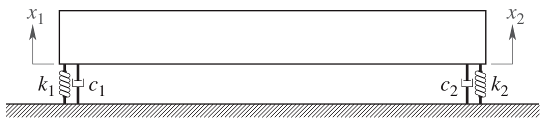

Figure 1.33 Platform supported by springs and dashpots at two ends

□ Solution 1: Using Two End Displacements as Generalized Coordinates

- *Generalized Coordinates*: We define $\{x_1, x_2\}$ as a complete and independent set of generalized coordinates, where x_1 and x_2 are the upward displacement of the left and right ends, respectively, of the platform with respect to ground and measured from their respective equilibrium positions.
- *Admissible Variations*: We verify that $\{\delta x_1, \delta x_2\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has two degrees of freedom.
- *Generalized Forces*: There are two nonconservative elements (dashpots) in the system. Each dashpot is connected in the same way as the one analyzed in Example 1.16. Thus,

$$\delta W^{\text{n.c.}} = -c_1\dot{x}_1\delta x_1 - c_2\dot{x}_2\delta x_2$$

Thus, according to eqn. (1.2),

$$\Xi_{x_1} = -c_1\dot{x}_1 \tag{a}$$

$$\Xi_{x_2} = -c_2\dot{x}_2 \tag{b}$$

- *Kinetic Energy*: There is no identifiable fixed point in the rigid body, we use the general expression for the kinetic energy in eqn. (A.27) and recall that the centroidal moment of inertia for a slender rod is $I_C = \frac{1}{12}mL^2$.

  To find the velocity at the center of the platform, we need to analyze the geometry, as sketched in Fig. 1.34. In Fig. 1.34, the bottom horizontal line represents the ground; l's are the unstretched length of the springs, Δ's are the compressions in the springs at equilibrium, and x's are the current positions. From the geometry, the y-coordinate of the centroid is expressible as

$$y_C = \frac{(l_1 + x_1 - \Delta_1) + (l_2 + x_2 - \Delta_2)}{2} \tag{c}$$

and the inclined angle of the platform, denoted as θ, is expressible as

$$\theta \approx \sin\theta = \frac{(l_1 + x_1 - \Delta_1) - (l_2 + x_2 - \Delta_2)}{L} \tag{d}$$

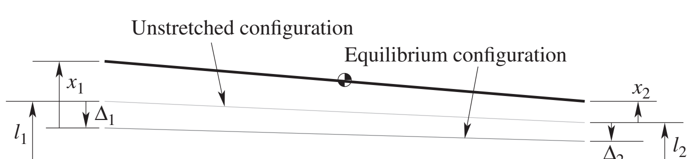

Figure 1.34 Geometry of the platform at a generic moment

In writing these equations, we have used the small motion assumption such that all displacements are in vertical direction, and the horizontal distance between two ends of the platform remains as L.

Taking a time derivative, the velocity of the centroid and the angular velocity of the platform are

$$v_C = \frac{\dot{x}_1 + \dot{x}_2}{2} \qquad \text{and} \qquad \dot{\theta} = \frac{\dot{x}_2 - \dot{x}_1}{L} \tag{e}$$

Thus, the kinetic energy of the platform is

$$\begin{aligned} T &= \frac{1}{2}v_C^2 + \frac{1}{2}I_C\dot{\theta}^2 \\ &= \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2 \\ &= \frac{1}{6}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1\dot{x}_2\right) \end{aligned} \tag{f}$$

- *Potential Energy*: Again, referring to the geometry in Fig. 1.34, the amounts of stretches in springs are $x_1 - \Delta_1$ and $x_2 - \Delta_2$, respectively. For the gravitational potential energy, we can use the y-coordinate for the centroid y_C in eqn. (c). Thus,

$$V = \frac{1}{2}k_1(x_1 - \Delta_1)^2 + \frac{1}{2}k_2(x_2 - \Delta_2)^2 + \frac{1}{2}mg\left[(l_1 + x_1 - \Delta_1) + (l_2 + x_2 - \Delta_2)\right] \tag{g}$$

- *Lagrangian*: Combining eqns. (f) and (g) gives, according to eqn. (1.3),

$$\begin{aligned} \mathcal{L} = T - V = {} & \frac{1}{6}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1\dot{x}_2\right) - \frac{1}{2}k_1(x_1 - \Delta_1)^2 - \frac{1}{2}k_2(x_2 - \Delta_2)^2 \\ & - \frac{1}{2}mg\left[(l_1 + x_1 - \Delta_1) + (l_2 + x_2 - \Delta_2)\right] \end{aligned} \tag{h}$$

- *Lagrange's Equation*: The system has two equations of motion.
  - The x_1-Equation: The Lagrange's equation for x_1 is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{dx_1} = \Xi_{x_1} \tag{i}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{1}{3}m\dot{x}_1 + \frac{1}{6}m\dot{x}_2 \tag{j}$$

and

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k_1(x_1 - \Delta_1) - \frac{1}{2}mg \tag{k}$$

substituting eqns. (j), (k), and (a) into eqn. (i) gives

$$\frac{1}{3}m\ddot{x}_1 + \frac{1}{6}m\ddot{x}_2 + k_1(x_1 - \Delta_1) + \frac{1}{2}mg = -c_1\dot{x}_1$$

Rearranging and canceling the constant terms using the equilibrium condition give

$$\frac{1}{3}m\ddot{x}_1 + \frac{1}{6}m\ddot{x}_2 + c_1\dot{x}_1 + k_1x_1 = 0 \tag{l}$$

– *The x_2-Equation*: The Lagrange's equation for x_2 is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{dx_2} = \Xi_{x_2} \tag{m}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{1}{6}m\dot{x}_1 + \frac{1}{3}m\dot{x}_2 \tag{n}$$

and

$$\frac{\partial \mathcal{L}}{\partial x_2} = -k_2(x_2 - \Delta_2) - \frac{1}{2}mg \tag{o}$$

substituting eqns. (n), (o), and (b) into eqn. (m) gives

$$\frac{1}{6}m\ddot{x}_1 + \frac{1}{3}m\ddot{x}_2 + k_2(x_2 - \Delta_2) + \frac{1}{2}mg = -c_2\dot{x}_2$$

Rearranging and canceling the constant terms using the equilibrium condition give

$$\frac{1}{6}m\ddot{x}_1 + \frac{1}{3}m\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = 0 \tag{p}$$

Equations (l) and (p) comprise the equations of motion for the system.

□ Solution 2: Using Displacement and Rotation as Generalized Coordinates

- *Generalized Coordinates*: We define $\{y, \theta\}$ as a complete and independent set of generalized coordinates, where y is the upward displacement of the centroid of the platform with respect to the ground and θ is the counterclockwise angular displacement of the platform; both are measured from the equilibrium configuration, as shown in Fig. 1.35.

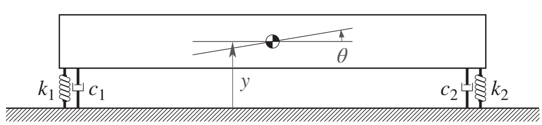

Figure 1.35 Linear and angular displacements as generalized coordinates for platform

- *Admissible Variations*: We verify that $\{\delta y, \delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has two degrees of freedom.
- *Generalized Forces*: There are two nonconservative elements (dashpots). In determining the virtual work, we look at the velocities first. The platform is rotating about the centroid at angular velocity of $\dot{\theta}$ while being carried at the velocity of the centroid $\dot{y}$. Assuming small motions, the relative velocities at both ends due to the rotation are in the vertical direction. Thus, the left end has a velocity of $\dot{y} - \frac{1}{2}L\dot{\theta}$, and the right end has a velocity of $\dot{y} + \frac{1}{2}L\dot{\theta}$.

  Now consider the virtual displacements. If y is varied by δy but θ is fixed, both ends have an upward virtual displacement of δy. If y is fixed but θ is varied by $\delta\theta$, the left end has a downward virtual displacement of $\frac{1}{2}L\delta\theta$ and the right end has an upward virtual displacement of the same amount, as sketched in Fig. 1.36. Thus, the combined total virtual displacement at the left end is $\delta y - \frac{1}{2}L\delta\theta$ and at the right end is $\delta y + \frac{1}{2}L\delta\theta$.

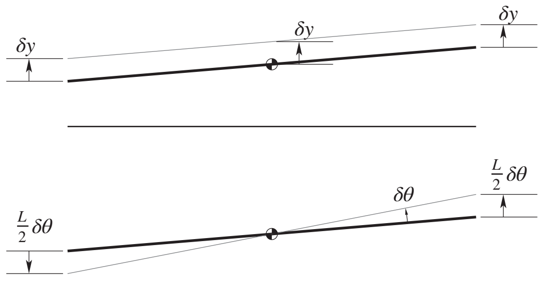

Figure 1.36 Variations and virtual displacements in the platform

Thus, the total virtual work is

$$\begin{aligned}\delta W^{\text{n.c.}} &= -c_1\left(\dot{y} - \frac{1}{2}L\dot{\theta}\right)\left(\delta y - \frac{1}{2}L\delta\theta\right) - c_2\left(\dot{y} + \frac{1}{2}L\dot{\theta}\right)\left(\delta y + \frac{1}{2}L\delta\theta\right)\\ &= -\left[(c_1 + c_2)\dot{y} + \frac{1}{2}(c_2 - c_1)L\dot{\theta}\right]\delta y + \frac{1}{2}L\left[(c_1 - c_2)\dot{y} - \frac{1}{2}(c_1 + c_2)L\dot{\theta}\right]\delta\theta\end{aligned}$$

Then, according to eqn. (1.2),

$$\Xi_y = -(c_1 + c_2)\dot{y} - \frac{1}{2}(c_2 - c_1)L\dot{\theta} \tag{q}$$

$$\Xi_\theta = \frac{1}{2}L\left[(c_1 - c_2)\dot{y} - \frac{1}{2}(c_1 + c_2)L\dot{\theta}\right] \tag{r}$$

- *Kinetic Energy*: The centroid of the platform has a velocity of $\dot{y}$ and the platform has an angular velocity of $\dot{\theta}$. Thus,

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{24}mL^2\dot{\theta}^2 \tag{s}$$

- *Potential Energy*: Assume that the spring compressions at equilibrium are Δ_1 and Δ_2, respectively. The displacements at the two ends are $y \pm \frac{1}{2}L\dot{\theta}$. For the gravitational potential energy, we choose the location of the centroid at equilibrium as the datum. Then,

$$V = \frac{1}{2}k_1\left(y - \frac{1}{2}\theta L - \Delta_1\right)^2 + \frac{1}{2}k_2\left(y + \frac{1}{2}\theta L - \Delta_2\right)^2 + mgy \tag{t}$$

- *Lagrangian*: Combining eqns. (s) and (t) gives, according to eqn. (1.3),

$$\begin{aligned}\mathcal{L} = \frac{1}{2}m\dot{y}^2 &+ \frac{1}{24}mL^2\dot{\theta}^2 - \frac{1}{2}k_1\left(y - \frac{1}{2}\theta L - \Delta_1\right)^2\\ &- \frac{1}{2}k_2\left(y + \frac{1}{2}\theta L - \Delta_2\right)^2 - mgy\end{aligned} \tag{u}$$

- *Lagrange's Equation*: The system has two equations of motion.

– *The y-Equation*: The Lagrange's equation for y is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{dy} = \Xi_y \tag{v}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} \tag{w}$$

and

$$\frac{\partial \mathcal{L}}{\partial y} = -k_1\left(y - \frac{1}{2}\theta L - \Delta_1\right) - k_2\left(y + \frac{1}{2}\theta L - \Delta_2\right) - mg \tag{x}$$

substituting eqns. (w), (x), and (q) into eqn. (v) gives

$$m\ddot{y} + k_1\left(y - \frac{1}{2}\theta L - \Delta_1\right) + k_2\left(y + \frac{1}{2}\theta L + \Delta_2\right) - mg$$
$$= -(c_1 + c_2)\dot{y} - \frac{1}{2}(c_2 - c_1)L\dot{\theta}$$

Rearranging and canceling out the constant terms using the equilibrium condition give

$$m\ddot{y} + (c_1 + c_2)\dot{y} - \frac{1}{2}(c_1 - c_2)L\dot{\theta} + (k_1 + k_2)y + \frac{1}{2}(k_2 - k_1)L\theta = 0 \tag{y}$$

– *The θ-Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{d\theta} = \Xi_\theta \tag{z}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{12}mL^2\dot{\theta} \tag{aa}$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = k_1\left(y - \frac{1}{2}\theta L - \Delta_1\right)\frac{1}{2}L - k_2\left(y + \frac{1}{2}\theta L - \Delta_2\right)\frac{1}{2}L$$
$$= -\frac{L}{2}(k_2 - k_1)y - \frac{L^2}{4}(k_1 + k_2)\theta - \frac{L}{2}(k_1\Delta_1 - k_2\Delta_2) \tag{ab}$$

substituting eqns. (aa), (ab), and (r) into eqn. (z) gives

$$\frac{1}{12}mL^2\ddot{\theta} + \frac{L}{2}(k_2 - k_1)y + \frac{L^2}{4}(k_1 + k_2)\theta + \frac{L}{2}(k_1\Delta_1 - k_2\Delta_2)$$
$$= \frac{L}{2}\left[(c_1 - c_2)\dot{y} - (c_1 + c_2)\frac{L}{2}\dot{\theta}\right]$$

Rearranging and canceling out the constant terms using the equilibrium condition give

$$mL\ddot{\theta} + 6(c_2 - c_1)\dot{y} + 3(c_1 + c_2)L\dot{\theta} + 6(k_2 - k_1)y + 3(k_1 + k_2)L\theta = 0 \tag{ac}$$

Equations (y) and (ac) comprise the equations of motion for the system.

■ **Example 1.27: Washing Machine with Load Imbalance**

A front-loading washing machine is confined to move only in the vertical direction. Its rubber feet are modeled as two springs of $\frac{1}{2}k$ and a dashpot c, as sketched in Fig. 1.37. The drum and the clothes are modeled together as one piece of mass m and centroidal moment of inertia I_C, and its centroid is located at a distance e from its geometric center. The machine body has mass M. Assume the drum rotates at a constant angular velocity of Ω. Derive the equation(s) of motion for the system.

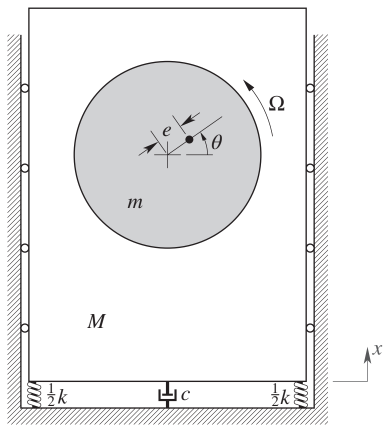

Figure 1.37 Model for front-loading washing machine

□ **Solution:**

- *Preparatory Setup*: The drum is undergoing a specified motion. We define θ as the angle the line connecting the mass center to geometric center makes with the horizontal. This angle can be described as

$$\theta = \Omega t + \theta_0 \tag{a}$$

where θ_0 is the angle at $t = 0$. We shall keep in mind that θ is completely specified at all times, and hence shall not be included as a generalized coordinate.
- *Generalized Coordinates*: We define $\{x\}$ as a set of complete and independent generalized coordinates, where x is the upward displacement of the machine body measured from its equilibrium position with respect to the ground.
- *Admissible Variations*: We verify that $\{\delta x\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: There is one nonconservative element: a simple dashpot. Thus,

$$\delta W^{\text{n.c.}} = -c\dot{x}\delta x \tag{b}$$

Then, according to eqn. (1.2),

$$\Xi_x = -c\dot{x} \tag{c}$$

- *Kinetic Energy*: The machine body simply moves up and down at a velocity of $\dot{x}$. The drum rotates relative to the moving body. The velocity composition at the mass center of the drum is sketched in Fig. 1.38. Thus, the kinetic energy of the washing machine is

$$
\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + e\Omega\cos\ \theta)^2 + (e\Omega\sin\theta)^2] + \frac{1}{2}I_C\Omega^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + (e\Omega)^2 + 2e\dot{x}\Omega\cos\ \theta\right] + \frac{1}{2}I_C\Omega^2 \qquad \text{(d)}
\end{aligned}
$$

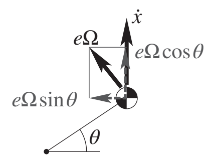

Figure 1.38 Composition and decomposition of velocity at the mass center of the drum

- *Potential Energy*: The machine body can only move up and down. The two springs move in unison and effectively become one. We use the ground as the datum for gravitational potential energy. Then,

$$V = \frac{1}{2}k(x - \Delta)^2 + Mgx + mg(x + e\sin\theta) \qquad \text{(e)}$$

where Δ is the amount of compression in the spring at equilibrium.
- *Lagrangian*: Combining eqns. (d) and (e) gives, according to eqn. (1.3),

$$
\begin{aligned}
\mathcal{L} = T - V = &\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + (e\Omega)^2 + 2e\dot{x}\Omega\cos\ \theta\right] + \frac{1}{2}I_C\Omega^2 \\
&- \frac{1}{2}k(x - \Delta)^2 - Mgx - mg(x + e\sin\theta) \qquad \text{(f)}
\end{aligned}
$$

- *Lagrange's Equation*: Lagrange's equation for x is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} = \Xi_x \qquad \text{(g)}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = M\dot{x} + m(\dot{x} + e\Omega\cos\ \theta) \qquad \text{(h)}$$

and

$$\frac{\partial\mathcal{L}}{\partial x} = -k(x - \Delta) - Mg - mg \qquad \text{(i)}$$

Substituting eqns. (h), (i), and (c) into eqn. (g) gives

$$(M + m)\ddot{x} - me\Omega^2\sin\theta + kx - k\Delta + (M + m)g = -c\dot{x} \qquad \text{(j)}$$

Using the equilibrium condition to eliminate the constant terms gives the following equation of motion for the system:

$$(M + m)\ddot{x} + c\dot{x} + kx = me\Omega^2\sin(\Omega t + \theta_0) \qquad \text{(k)}$$

where eqn. (a) has been used.

■ Example 1.28: Disk Rolling Inside Circular Track

A circular disk of mass m and radius r can roll without slip inside a fixed circular track of radius R ($R > r$), as shown in Fig. 1.39. Derive the equation(s) of motion for the disk.

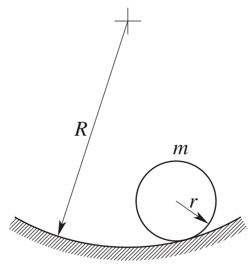

Figure 1.39 Small disk rolling without slip on a large circular track

□ Solution:

- *Preparatory Setup*: We paint a radius on the disk that connects its center to the contact point when the disk is located at its lowest position, which is used as a reference configuration. The geometry of the disk in a displaced configuration is depicted in Fig. 1.40. In this figure, point B on the disk is in contact with point A on the track in the reference configuration. BC is the painted radius and CD is the radius that connects the contact point D to the center of the disk C in the displaced configuration, whose extension passes through the center of the track O.

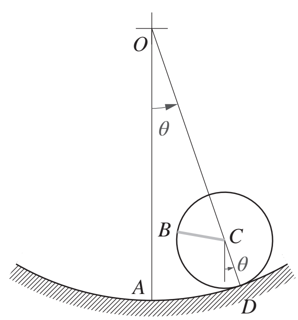

Figure 1.40 Geometry of disk rolling on a circular track

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the angle that the line connecting the two centers forms with the vertical, which is positive in the counterclockwise direction, as shown in Fig. 1.40.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: There is no externally applied force, nor nonconservative elements. Thus, $\delta W^{\text{n.c.}} = 0$. According to eqn. (1.2),

$$\Xi_\theta = 0 \tag{a}$$

- *Kinetic Energy*: There is no apparent fixed point in the disk. Thus, we use the general formula in eqn. (A.27) for the kinetic energy.

  For the velocity of the center of the disk v_C, note that line OC rotates at $\dot{\theta}$,

$$v_C = (R - r)\dot{\theta}$$

For the angular velocity of the disk, we look at the angular displacement first. Because of the nonslip condition, the arc length of $\widehat{AD}$ (of radius R) must equal to the arc length of $\widehat{BD}$ (of radius r). On the other hand, the angular displacement of the disk is the angle between BC line and the vertical, which is

$$\text{Angular displacement of disk} = \frac{\widehat{BD}}{r} - \theta = \frac{\widehat{AD}}{r} - \theta = \left(\frac{R}{r} - 1\right)\theta$$

Taking a time derivative of the above relation gives the angular velocity of the disk as

$$\omega_{\text{disk}} = \left(\frac{R}{r} - 1\right)\dot{\theta}$$

Therefore, the kinetic energy of the disk is

$$\begin{aligned} T &= \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega_{\text{disk}}^2 \\ &= \frac{1}{2}m\left[(R-r)\dot{\theta}\right]^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left[\left(\frac{R}{r} - 1\right)\dot{\theta}\right]^2 = \frac{3}{4}m(R-r)^2\dot{\theta}^2 \end{aligned} \tag{b}$$

- *Potential Energy*: We use the center of the track O as the datum for the gravitational potential energy. Then,

$$V = -mg(R - r)\cos\,\theta \tag{c}$$

- *Lagrangian*: Combining eqns. (b) and (c) gives, according to eqn. (1.3),

$$\mathcal{L} = T - V = \frac{3}{4}m(R-r)^2\dot{\theta}^2 + mg(R-r)\cos\,\theta \tag{d}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{d\theta} = \Xi_\theta \tag{e}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{3}{2}m(R-r)^2\dot{\theta} \tag{f}$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mg(R - r)\sin\theta \tag{g}$$

substituting eqns. (f), (g), and (a) into eqn. (e) gives

$$\frac{3}{2}m(R-r)^2\ddot{\theta} + mg(R-r)\sin\theta = 0$$

Canceling a common factor $m(R - r)$ gives

$$3(R - r)\ddot{\theta} + 2g\sin\theta = 0 \tag{h}$$

Equation (h) is the equation of motion for the system.

■ Example 1.29: Rocking Cylinder Segment

A segment of a circular cylinder of radius R and mass m can roll without slip on a horizontal ground. The total height of the segment is H, and when it rests in upward equilibrium configuration, as shown in Fig. 1.41, the center of mass is located at a distance h above the ground ($h < H$). A force $F(t)$ acts on the edge of the segment and maintains in the horizontal direction at all times, and yet not large enough to overcome the static friction between the cylinder and the ground. The centroidal moment of inertia of the cylinder segment is I. Find the equation(s) of motion for the cylinder segment.

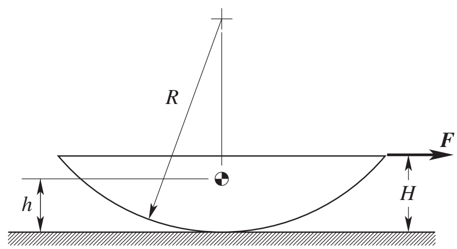

Figure 1.41 Rocking cylinder segment and its geometry

□ Solution:

- *Preparatory Setup*: Fig. 1.42 illustrates the geometry of the cylinder segment in a displaced configuration. In this figure, O is the center of the cylindrical surface, A is the contact point on the ground, A' is the matching contact point on the cylinder, B is the contact point on the ground in the equilibrium configuration, B' is the matching contact point on the cylinder, C is the centroid, D is the edge of the segment where the external force acts, CE and GD are horizontal projection lines, and θ_0 is constant angle describing the angular size of the segment, as

$$\cos\theta_0 = \frac{R-H}{R} \qquad \text{or} \qquad \theta_0 = \cos^{-1}\left(1 - \frac{H}{R}\right) \tag{a}$$

We also define a Cartesian coordinate system Bxy such that the x-axis lies on the ground.

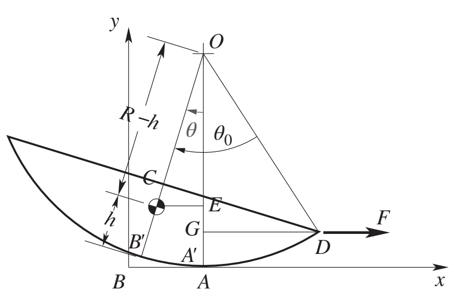

Figure 1.42 Generalized coordinate and geometry of a cylinder segment

- *Generalized Coordinates*: We define $\{\theta\}$ as a complete and independent set of generalized coordinates, where θ is the angle measured from the vertical to line OC, as shown in Fig. 1.42, positive in the clockwise direction.
- *Admissible Variations*: We verify that $\{\delta\theta\}$ is a complete and independent set of admissible variations in this set of generalized coordinates.
- *Holonomicity*: We conclude that the system is holonomic and has one degree of freedom.
- *Generalized Forces*: The virtual work by the applied force is expressible as

$$\delta W^{\text{n.c.}} = F\delta x_D \tag{b}$$

where δx_D is the horizontal virtual displacement at point D. To find δx_D, we first write the x-coordinate of point D in terms of the generalized coordinate θ. From the geometry in Fig. 1.42, because of the nonslip condition, $\overline{BA} = \widehat{B'A} = R\theta$. Then,

$$x_D = \overline{BA} + \overline{GD} = R\theta + R\sin(\theta_0 - \theta) \tag{c}$$

where θ_0 is given in eqn. (a). Then, δx_D can be determined by treating δ like a differential operator as

$$\delta x_D = R\delta\theta - R\cos(\theta_0 - \theta)\delta\theta = R\left[1 - \cos(\theta_0 - \theta)\right]\delta\theta \tag{d}$$

and, substituting eqn. (d) into eqn. (b) gives

$$\delta W^{\text{n.c.}} = FR\left[1 - \cos(\theta_0 - \theta)\right]\delta\theta \tag{e}$$

Then, according to eqn. (1.2),

$$\Xi_\theta = FR\left[1 - \cos(\theta_0 - \theta)\right] \tag{f}$$

- *Kinetic Energy*: There is no apparent fixed point in the cylinder segment, we will use the general expression for the kinetic energy as in eqn. (A.27). The angular velocity of the cylinder segment is $\dot{\theta}$. To find the velocity of the centroid C, a convenient approach is to find the coordinates for the centroid C first. According to Fig. 1.42,

$$x_C = \overline{BA} - \overline{EC} = R\theta - (R - h)\sin\theta \tag{g}$$

$$y_C = \overline{AE} = R - \overline{OE} = R - (R - h)\cos\,\theta \tag{h}$$

Taking a time derivative of the above expressions gives

$$\boldsymbol{v}_C = [R\dot{\theta} - (R - h)\cos\,\theta\dot{\theta}]\boldsymbol{i} + (R - h)\sin\theta\dot{\theta}\boldsymbol{j} \tag{i}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors for the Bxy coordinate system. Then, the kinetic energy for the cylinder segment is

$$\begin{aligned} T &= \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2 \\ &= \frac{1}{2}m\left\{\left[R\dot{\theta} - (R - h)\cos\,\theta\dot{\theta}\right]^2 + \left[(R - h)\sin\theta\dot{\theta}\right]^2\right\} + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}\left\{m\left[R^2 + (R - h)^2 - 2R(R - h)\cos\,\theta\right] + I\right\}\dot{\theta}^2 \end{aligned} \tag{j}$$

- *Potential Energy*: Since we already have the expression for y_C, we can simply choose the ground as the datum for the gravitational potential energy, and

$$V = mgy_C = mg\left[R - (R-h)\cos\,\theta\right] \tag{k}$$

- *Lagrangian*: Combining eqns. (j) and (k) gives, according to eqn. (1.3),

$$\mathcal{L} = \frac{1}{2}\left\{m\left[R^2 + (R-h)^2 - 2R(R-h)\cos\,\theta\right] + I\right\}\dot{\theta}^2 - mg\left[R - (R-h)\cos\,\theta\right] \tag{l}$$

- *Lagrange's Equation*: The Lagrange's equation for θ is, according to eqn. (1.4),

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{d\theta} = \Xi_\theta \tag{m}$$

  Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \left\{m\left[R^2 + (R-h)^2 - 2R(R-h)\cos\,\theta\right] + I\right\}\dot{\theta} \tag{n}$$

  and

$$\frac{\partial\mathcal{L}}{\partial\theta} = -mg(R-h)\sin\theta, \tag{o}$$

  substituting eqns. (n), (o), and (f) into eqn. (m) gives

$$\begin{aligned}&\left\{m\left[R^2 + (R-h)^2 - 2R(R-h)\cos\,\theta\right] + I\right\}\ddot{\theta}\\ &\qquad + mg(R-h)\sin\theta = FR[1 - \cos(\theta_0 - \theta)]\end{aligned} \tag{p}$$

  Equation (p) is the equation of motion for the system.

1.11 Linearization of Equations of Motion

Mechanical vibration studies the small motions of a system around its equilibrium configuration. Meanwhile we have observed that many equations of motion for different systems are often nonlinear; and solving such nonlinear equations could be extremely difficult, if possible at all. Fortunately, the key qualifying term *small motions* saves our day. By assuming small motions, we could reduce a nonlinear equation into a linear one, for which we could solve rather comfortably. Another key qualifying term *equilibrium configuration* indicates that the small motion assumption should be applied to the system near an equilibrium position. Before linearizing an equation of motion, we need to find the equilibrium position(s) first.

At the beginning of this chapter, we have briefly discussed the equilibrium position and the stability of a simple mass–spring–dashpot system in Example 1.1. In this section, we use another example to illustrate the procedure of finding the equilibrium position(s) and then linearizing the equation(s) of motion.

Consider the following example of an inverted rigid-link pendulum, constrained by two springs as shown in Fig. 1.43. The equation of motion for this system has been derived in Example 1.24.

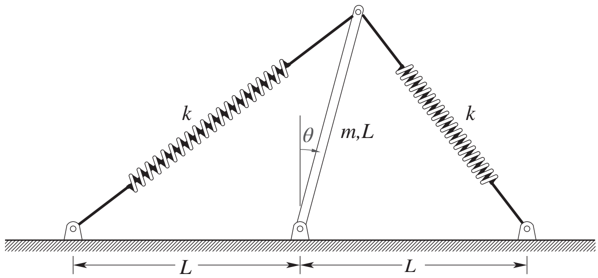

Figure 1.43 Inverted rigid-link planar pendulum

1.11.1 Equilibrium Position(s)

In the equilibrium state, the system does not move, and hence all velocities (including angular velocities) and accelerations (including angular accelerations) vanish. To find the equilibrium position, we set all time derivatives to zero. The resulting equation is called the *equilibrium equation*. Roots of this equation give the *equilibrium positions* of the system.

■ **Example 1.30: Equilibrium Positions of Inverted Rigid-Link Pendulum**

Find all the equilibrium positions for the inverted rigid-link pendulum in Fig. 1.43.

□ **Solution:**

Using θ, the clockwise angular displacement of the pendulum measured from the vertical, as the generalized coordinate, the equation of motion for the inverted rigid-link pendulum, has been found in Example 1.24 as

$$\frac{1}{3}mL\ddot{\theta} - \frac{1}{2}mg\sin\theta + \sqrt{2}kL\sqrt{1-\cos\theta} = 0 \tag{a}$$

Setting $\ddot{\theta} = 0$ (and $\dot{\theta} = 0$ if this term also presents), eqn. (a) becomes the following *equilibrium equation*:

$$-mg\sin\theta + 2\sqrt{2}kL\sqrt{1-\cos\theta} = 0 \tag{b}$$

or

$$\sin\theta = \frac{2\sqrt{2}kL}{mg}\sqrt{1-\cos\theta} \tag{c}$$

To find the root(s) of this equation, we square both sides of eqn. (c) and make use of the following trigonometry identity $\sin^2\theta = 1 - \cos^2\theta = (1+\cos\theta)(1-\cos\theta)$,

$$(1+\cos\theta)(1-\cos\theta) = 8\left(\frac{kL}{mg}\right)^2(1-\cos\theta) \tag{d}$$

We shall exercise caution not to hastily cancel out the common factor $(1-\cos\theta)$ without a reason. Instead, we rewrite eqn. (d) as

$$(1-\cos\theta)\left[1+\cos\theta - 8\left(\frac{kL}{mg}\right)^2\right] = 0 \tag{e}$$

A root to this equation would make either of the two factors to vanish. That is,

$$\cos\,\theta = 1 \qquad \text{or} \qquad \cos\,\theta = 8\left(\frac{kL}{mg}\right)^2 - 1$$

Furthermore, when taking the inverse of the cosine function, we need to be cautious again not to miss the negative solution. In all, the system has three equilibrium positions:

$$\theta_1^{\text{eq}} = 0 \qquad \text{and} \qquad \theta_{2,3}^{\text{eq}} = \pm\cos^{-1}\left[8\left(\frac{KL}{mg}\right)^2 - 1\right] \tag{f}$$

□ **Discussion:**

We now can see that, had we carelessly cancel the common factor $(1 - \cos\,\theta)$, we would have missed one important (and actually obvious) equilibrium position $\theta = 0$.

The equilibrium positions 2 and 3 exist only when

$$0 \leq 8\left(\frac{kL}{mg}\right)^2 - 1 \leq 1 \tag{g}$$

This requirement comes from the geometrically admissible configuration of $-\pi/2 \leq \theta \leq \pi/2$, which gives $0 \leq \cos\,\theta \leq 1$. Rearranged, the above condition can be written as

$$\frac{\sqrt{2}}{2} \leq \frac{2kL}{mg} \leq 1 \tag{h}$$

Physically, this condition means: if the spring is either too strong or too weak, the system has only one equilibrium position that is vertically inverted. We will gain further insight as what would happen if the spring is too weak or too strong once we have obtained a linearized the equation of motion.

1.11.2 Linearization

Example 1.30 shows that a system may have multiple equilibrium positions. In such cases, linearization must be performed for a specific equilibrium position, and the resulting linearized equations of motion are generally different for different equilibrium positions. In the following, we linearize the equation of motion for each equilibrium position.

When we consider small motions around an equilibrium position, we can make the following variable substitution:

$$x = x^{\text{eq}} + y \tag{1.5}$$

where x is the generalized coordinate used in deriving the equation of motion, x^{eq} is the equilibrium position, and y represents the small motions. From eqn. (1.5), since x^{eq} is a constant, it is obvious that

$$\dot{x} = \dot{y} \qquad \ddot{x} = \ddot{y} \tag{1.6}$$

"Small" is a comparative term that should be restricted to nondimensional or normalized parameters. For example, a linear displacement can be normalized by a representative length in the system. This way, the displacement is said to be small compared to that length.

Mathematically, y being small is written as $y \ll 1$. It is generally also implied that the associated velocity and the acceleration are small, that is, $\dot{y} \ll 1$ and $\ddot{y} \ll 1$.

The fundamental idea in a linearization process is to keep only up to the first order small terms. Terms that are explicitly higher order small, such as y^2 or $y\dot{y}$, are simply dropped; and terms that are nonlinear but not explicitly higher order small are expanded in power series using Taylor expansions first and then keep only up to the first order small terms. Recall that the Taylor expansion for a general function $f(x)$ near the point $x = x^{\text{eq}}$ can be written as

$$f(x) = f(x^{\text{eq}}) + f'(x)\big|_{x=x^{\text{eq}}} y + \cdots \tag{1.7}$$

In the following, we illustrate the linearization process with two examples.

■ Example 1.31: Linearize the Equation of Motion for Equilibrium Position 1

Linearize the equation of motion for the inverted rigid-link pendulum for the equilibrium position $\theta_1^{\text{eq}} = 0$.

□ Solution:

We use ϕ to represent the "small motions" near this equilibrium position. The generalized coordinate θ is an angle, thus "small" means a small angle, which is a nondimensional parameter of the unit of radians. The change of variable is as the following:

$$\theta = \theta_1^{\text{eq}} + \phi \tag{a}$$

where ϕ is small. Since $\theta_1^{\text{eq}} = 0$, this quickly leads to $\phi = \theta$, $\dot{\phi} = \dot{\theta}$, and $\ddot{\phi} = \ddot{\theta}$.

To linearize the equation of motion in eqn. (a) in Example 1.30, we use Taylor expansion to expand the nonlinear terms in the equation of motion near $\theta = 0$. These terms include $\sin\theta$ and $\sqrt{1-\cos\theta}$. We write their Taylor expansions as

$$\sin\theta \approx \sin\theta|_{\theta=0} + \cos\theta|_{\theta=0}\,\phi = \phi$$

$$\sqrt{1-\cos\theta} \approx \sqrt{1-\cos\theta}\Big|_{\theta=0} + \left(\sqrt{1-\cos\theta}\right)'\Big|_{\theta=0}\phi = \frac{1}{2}\left.\frac{\sin\theta}{\sqrt{1-\cos\theta}}\right|_{\theta=0}\phi$$

Note that $\sin\theta/\sqrt{1-\cos\theta}$ would become $0/0$ at $\theta = 0$. This can be evaluated using the L'Hôpital's rule or replacing $\sin\theta$ by $\sqrt{(1-\cos\theta)(1+\cos\theta)}$. Using the latter,

$$\sqrt{1-\cos\theta} \approx \frac{1}{2}\sqrt{1+\cos\theta}\Big|_{\theta=0}\phi = \frac{\sqrt{2}}{2}\phi$$

Then, substituting the above expansions into the equation of motion in eqn. (a) in Example 1.30 while keeping up to the first-order terms gives

$$2mL\ddot{\phi} - 3mg\phi + 6\sqrt{2}kL\frac{\sqrt{2}}{2}\phi = 0 \tag{b}$$

or

$$\ddot{\phi} + \frac{3g}{2L}\left(\frac{2kL}{mg} - 1\right)\phi = 0 \tag{c}$$

which is the linearized equation of motion for small motions of the inverted rigid-link pendulum near its vertical equilibrium position.

□ **Discussion:**

In the cases where $\theta^{\text{eq}} = 0$, we can directly assume θ is small and proceed with the analysis, without introducing a new variable ϕ. In fact, this is a very common scenario as we often define the generalized coordinates to be measure from an equilibrium position.

■ **Example 1.32: Linearize Equation of Motion for Equilibrium Positions 2 and 3**

Linearize the equation of motion for the inverted rigid-link pendulum about its equilibrium positions at

$$\theta_{2,3}^{\text{eq}} = \pm\cos^{-1}\left[8\left(\frac{kL}{mg}\right)^2 - 1\right]$$

□ **Solution:**

We again use ϕ to represent the small motions near the equilibrium position, as

$$\theta = \theta^{\text{eq}} + \phi \tag{a}$$

where ϕ is small and θ^{eq} denotes either θ_2^{eq} or θ_3^{eq}, and

$$\cos\,\theta^{\text{eq}} = 8\left(\frac{kL}{mg}\right)^2 - 1 \tag{b}$$

Taylor expansions for $\sin\theta$ and $\sqrt{1-\cos\,\theta}$ about θ^{eq} are

$$\sin\theta \approx \sin\,\theta^{\text{eq}} + \cos\,\theta^{\text{eq}}\phi \tag{c}$$

$$\sqrt{1-\cos\,\theta} \approx \sqrt{1-\cos\,\theta^{\text{eq}}} + \frac{1}{2}\sqrt{1+\cos\,\theta^{\text{eq}}}\phi \tag{d}$$

Substituting eqns. (c) and (d) into the equation of motion in eqn. (a) in Example 1.30 gives

$$2mL\ddot{\phi} - 3mg\,(\sin\,\theta^{\text{eq}} + \cos\,\theta^{\text{eq}}\phi) + 6\sqrt{2}kL\left[\sqrt{1-\cos\,\theta^{\text{eq}}} + \frac{1}{2}\sqrt{1+\cos\,\theta^{\text{eq}}}\phi\right] = 0 \tag{e}$$

Since the constant terms (the first terms in the Taylor expansions in eqns. (c) and (d)) cancel out according to the equilibrium equation as given in eqn. (b) in Example 1.30, eqn. (e) simplifies to

$$2mL\ddot{\phi} + \left[-3mg\cos\,\theta^{\text{eq}} + 6\sqrt{2}kL\frac{1}{2}\sqrt{1+\cos\,\theta^{\text{eq}}}\right]\phi = 0 \tag{f}$$

Substituting the expression for $\cos\,\theta^{\text{eq}}$ in eqn. (b) into eqn. (f) gives

$$2mL\ddot{\phi} + \left\{-3mg\left[8\left(\frac{kL}{mg}\right)^2 - 1\right] + 3\sqrt{2}kL\sqrt{8}\frac{kL}{mg}\right\}\phi = 0 \tag{g}$$

or

$$\ddot{\phi} + \frac{3g}{2L}\left[1 - \left(\frac{2kL}{mg}\right)^2\right]\phi = 0 \tag{h}$$

which is the linearized equation of motion for small motions of the inverted rigid-link pendulum near its two slanted equilibrium positions. Note that this linearized equation is different from the one for the vertically inverted equilibrium position in Example 1.31.

1.11.3 Observations and Further Discussions

Through this series of examples, we make the following observations:

- A system may have multiple equilibrium positions, and the linearized equation of motion is generally different for each equilibrium position. When finding the equilibrium position(s) for a system, make sure to find all possible roots to the equilibrium equation.
- The key for the linearization is using Taylor expansions while keeping only up to linear terms to approximate nonlinear terms. All higher-order terms are omitted.
- When the Taylor expansions are substituted into the nonlinear equation of motion, the zeroth-order terms satisfy the equilibrium equation and cancel out.
- The word *small* in "small motions" is a relative term. In order to assume a parameter being small, that parameter must be nondimensionalized or normalized.

1.11.3.1 Stability of an Equilibrium Position

At each equilibrium position, if the system is slightly disturbed, will the system come back to the equilibrium? This is the test for the stability of the equilibrium. If it comes back, the equilibrium is said to be *stable* and a vibration ensues, which is what we will study. If it does not come back, the equilibrium is said to be *unstable*.

A linearized equation of motion presents a clear indicator about the stability of the equilibrium. For many systems, like Examples 1.31 and 1.32, the linearized equation of motion is of the following form:

$$\ddot{x} + \lambda x = 0 \tag{1.8}$$

From differential equation theories, the solution for eqn. (1.8) is, in complex parameters, $e^{\pm i\sqrt{\lambda}t}$. If λ is positive, the solution can be alternatively written as $\sin\sqrt{\lambda}t$ or $\cos\sqrt{\lambda}t$: such a solution is oscillatory. However, if $\lambda < 0$, the solution would be written as $e^{\pm\sqrt{|\lambda|}t}$. In such a case, the component of $e^{-\sqrt{|\lambda|}t}$ will decade with time, but the component of $e^{\sqrt{|\lambda|}t}$ will grow exponentially with time. This means that the equilibrium is unstable. Vibration only occurs around a stable equilibrium position. Thus, we can conclude that if the resulting linearized equation of motion is of the form in eqn. (1.8), the *equilibrium is stable only when* $\lambda > 0$.

In a more general case, a linear second-order differential equation is of the form:

$$\ddot{x} + \gamma\dot{x} + \lambda x = 0 \tag{1.9}$$

It can be shown that the equilibrium is stable if and only if $\gamma \geq 0$ and $\lambda > 0$.

■ Example 1.33: Stability of Inverted Rigid-Link Pendulum

Determine the stabilities of the three equilibrium positions of the inverted rigid-link pendulum found in Example 1.30.

□ Solution:

For equilibrium position 1, the linearized equation of motion found in Example 1.31 is

$$\ddot{\phi} + \frac{3g}{2L}\left(\frac{2kL}{mg} - 1\right)\phi = 0 \tag{a}$$

The equilibrium is stable only when

$$\frac{2kL}{mg} - 1 > 0 \qquad \text{or} \qquad 2kL > mg \tag{b}$$

For equilibrium positions 2 and 3, the linearized equation of motion found is

$$\ddot{\phi} + \frac{3g}{2L}\left[1 - \left(\frac{2kL}{mg}\right)^2\right]\phi = 0 \tag{c}$$

The equilibrium is stable only when

$$1 - \left(\frac{2kL}{mg}\right)^2 > 0 \qquad \text{or} \qquad 2kL < mg \tag{d}$$

Thus, when the spring is sufficiently strong such that $2kL > mg$, equilibrium position 1 is stable and equilibrium positions 2 and 3 do not exist. For a weaker spring, equilibrium position 1 become unstable and equilibrium positions 2 and 3 appear and are stable. According to eqn. (h) in Example 1.30, the weakest spring for equilibrium positions 2 and 3 to exist is $2kL > mg/\sqrt{2}$.

1.11.3.2 Steady Position versus Equilibrium Position

Some systems may undergo steady motions, especially steady rotations, as its regular operation mode while vibration problems arise. There are a few such systems described in the end-of-chapter problems, for example, a pendulum encased in a container and placed on a steady-rotating platform in Problem 1.14. When the rotation of the platform steadies, the pendulum stays in an inclined position if there is no disturbance. Such a position is called a *steady position*.

The process of finding the steady positions is exactly the same as finding the equilibrium position, except that we need to be observant that angular velocity of the rotating platform is constant.

In this book, steady positions are categorically referred to as equilibrium positions, in which equilibrium is a relative sense: relative to a rotating base.

1.12 Chapter Summary

The procedure for deriving the equation(s) of motion for mechanical systems has been summarized in Section 1.9. The step of crucial importance is defining a set of generalized coordinates and subsequently conducting tests to ensure that it is a complete and independent set. If this is not carefully tested, we might end up with a wrong number of degrees of freedom for the system. Then, everything that follows that conclusion would be wrong.

The following are a few other important observations we have made:

- For a holonomic system, the number of equations of motion equals to the number of generalized coordinates and, in turn, equals to the number of degrees of freedom.
- When generalized coordinates are defined from an equilibrium configuration, the constants in Lagrange's equation satisfy the equilibrium condition and cancel out.

- For small motions, the mathematics could be simplified if the "small motion" assumption is incorporated earlier. Specifically, in the Lagrangian, we need to keep up to second-order small terms. In generalized forces, we only need to keep up to the first-order terms.
- A system may have multiple equilibrium positions. These positions can be found from the equation(s) of motion by setting all time derivatives (velocities and accelerations) of the generalized coordinates to zero.
- Linearization of the equation(s) of motion is specific to an equilibrium position.
- When the linearized equation of motion is normalized to the following form:

$$\ddot{x} + \gamma\dot{x} + \lambda x = 0$$

  The equilibrium position is stable if and only if $\gamma \geq 0$ and $\lambda > 0$.

Problems

Problem 1.1: A planar pendulum is made of a single massless rigid link with two attached masses. The first mass m_1 is located at a distance a from the pivot point, and the second mass m_2 is at a distance b from the first mass, as shown in Fig. P1.1. Derive the equation(s) of motion for the system.

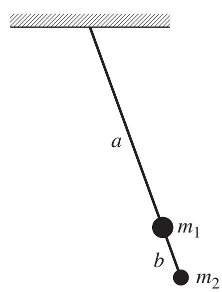

Figure P1.1 Two masses attached to a massless rigid-link pendulum

Problem 1.2: A particle of mass m can slide frictionlessly along a rigid fixed wire, whose shape is given by the equation $y = a + bx^2$, where both a and b are positive constants, as shown in Fig. P1.2. Derive the equation(s) of motion for the system.

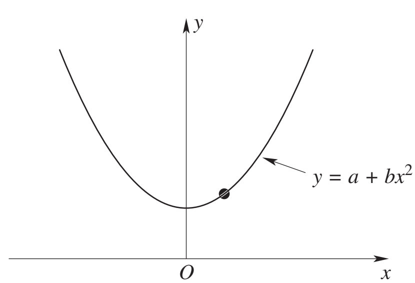

Figure P1.2 A particle slides along a wire of known shape

Problem 1.3: The rigid wire in Problem 1.2 is rotating about the y-axis at a constant angular velocity of Ω. Derive the equation(s) of motion for the system.

Problem 1.4: A particle of mass m can slide frictionlessly along a circular ring of radius R that is rotating about its vertical diameter at an angular velocity of Ω, as shown in Fig. P1.4. Note that a vertical stem holding the ring would prevent the particle from passing through it. Derive the equation(s) of motion for the system.

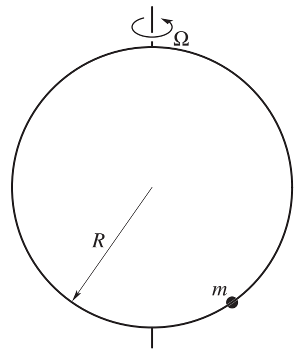

Figure P1.4 A particle slides along a circular ring that rotates about vertical axis

Problem 1.5: A particle of mass m slides frictionlessly inside a circular track of radius R, as shown in Fig. P1.5. Derive the equation(s) of motion for the system.

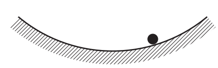

Figure P1.5 Particle slides frictionlessly inside a circular track

Problem 1.6: A pendulum of mass m and length l is attached to a mass M, which is confined to move within a vertical slot. The mass M is restrained by a spring k and a dashpot c, as shown in Fig. P1.6. Derive the equation(s) of motion for the system.

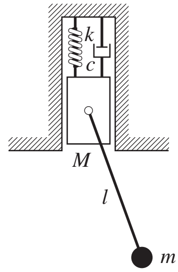

Figure P1.6 Pendulum attached to mass confined in a vertical slot

Problem 1.7: A *metronome*, also called a *rhythm timer* or a *music timer*, is modeled as comprising a massless hand (rigid link) pivoted at its root and a mass m near its tip. The hand is restrained by a torsional spring of spring constant k_t. The mass is located at a distance h from the pivot point, as shown in Fig. P1.7. When the hand is in its vertically upright position, the torsional spring is undeformed. Derive the equation(s) of motion for the system.

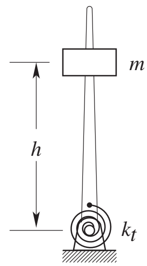

Figure P1.7 Metronome with mass attached to its hand

Problem 1.8: A planar pendulum of mass m and length l is restrained by a dashpot as shown in Fig. P1.8. The dashpot connected to a vertical wall is maintained in the horizontal orientation at all times. Derive the equation(s) of motion for the system.

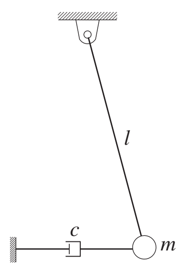

Figure P1.8 Pendulum constrained by a horizontal dashpot

Problem 1.9: A massless rigid link of length $2L$ is pivoted vertically at its center, with two masses, $2m$ and m, mounted at its two ends. The link can only move within the plane of paper and is further restrained by spring k and a dashpot c, which are attached at a distance a from the pivot, as shown in Fig. P1.9. When the link is in the vertical orientation, the spring and the dashpot are horizontal, the spring is undeformed, and of length l_0. Derive the equation(s) of motion for the system.

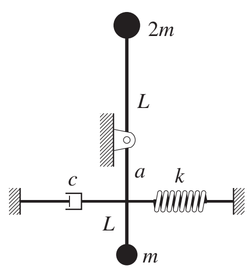

Figure P1.9 Masses mounted on rigid link and restrained by spring and dashpot

Problem 1.10: A mass m can move frictionlessly on the inclined surface of a cart, of mass M, which can move frictionlessly on the horizontal floor. Mass m is restrained by spring k and dashpot c, as shown in Fig. P1.10. The surface's incline angle is θ. Derive the equation(s) of motion for the system.

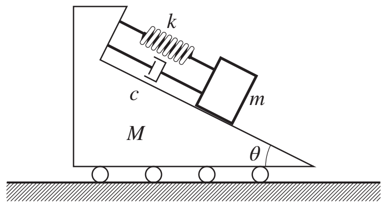

Figure P1.10 Mass on the inclined surface of a cart

Problem 1.11: A planar pendulum of mass m and length l is mounted on an L-shaped frame of mass M, which can move frictionlessly on the horizontal floor, as shown in Fig. P1.11. Assume that the pendulum never touches the frame, but its motion is not necessarily small. Derive the equation(s) of motion for the system.

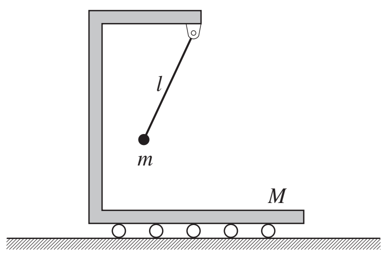

Figure P1.11 Planar pendulum mounted on a moving frame

Problem 1.12: The inextensible massless cable of a pendulum slides frictionlessly through a hole. Its upper end is held by a force $F(t)$, and the lower end attaches to a mass m, as shown in Fig. P1.12. Assume the pendulum is confined to move within the plane of paper, and the cable remains taut at all times. Derive the equation(s) of motion for the system.

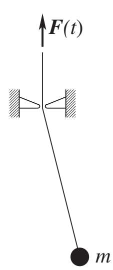

Figure P1.12 Planar pendulum with force exerted at the end of its cable

Problem 1.13: The inextensible massless cable of a planar pendulum of mass m passes through a pivoting hole, wraps around a massless pulley, and then connects to a spring k, which in turn is attached to a wall, as shown in Fig. P1.13. At the equilibrium configuration, the length of the cable below pivoting hole is l. The unstretched length of the spring is l_0. Derive the equation(s) of motion for the system.

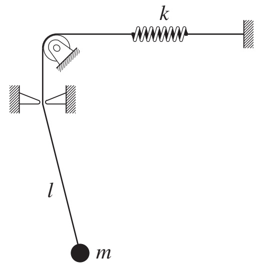

Figure P1.13 Planar pendulum with an inextensible cable and a spring

Problem 1.14: A planar pendulum of mass m and length l is mounted on an L-shaped frame that rests on a rotating platform, which rotates at a constant angular velocity of Ω. The pivot point is located at a distance e from the axis of rotation, as shown in Fig. P1.14. Assume that the pendulum never touches the frame, but its motion is not necessarily small. Derive the equation(s) of motion for the system.

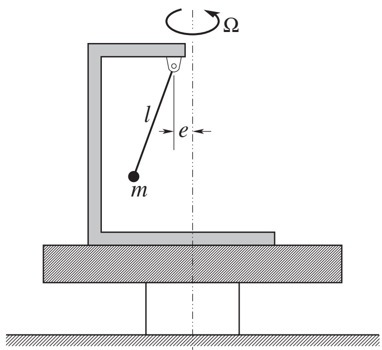

Figure P1.14 Planar pendulum mounted on a rotating frame

Problem 1.15: A mass m moves within a slot of length L located along a diameter of a horizontal rotating platform. It is further restrained by spring k and dashpot c, as shown in Fig. P1.15. The spring is unstretched when the mass is located at the center of the slot. The platform rotates at a constant angular velocity Ω. Derive the equation(s) of motion for the system.

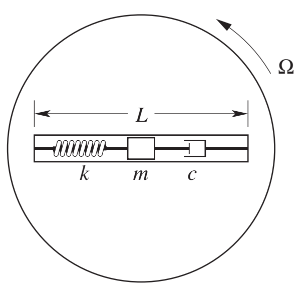

Figure P1.15 Mass–spring–dashpot system inside diametrical slot in a rotating platform

Problem 1.16: A mass m moves within a slot, of length L, in a horizontal rotating platform. It is further restrained by spring k and dashpot c, as shown in Fig. P1.16. The center line of the slot is at a distance a from the center of the platform, which rotates at a constant angular velocity Ω. The spring is unstretched when the mass is located at the center of the slot. Derive the equation(s) of motion for the system.

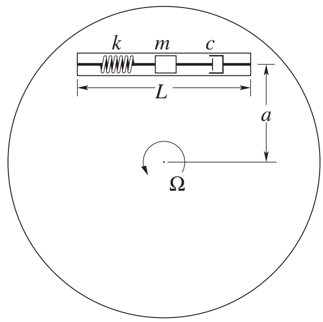

Figure P1.16 Mass–spring–dashpot system inside off-center slot in a rotating platform

Problem 1.17: Two masses m_1 and m_2 are attached to the ends of a rod that is rotating about the pivot at a constant angular velocity Ω. The collar of mass M, which houses the pivot, can slide frictionlessly along the vertical column but is restrained by two identical springs of spring constant k, as shown in Fig. P1.17. The masses are a distance r from the rotation center. Both springs have unstretched length of l_0, which is larger than the length of the entire column such that they are always compressed. Derive the equation(s) of motion for the system.

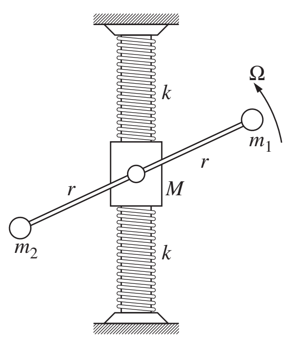

Figure P1.17 Masses attached to the ends of a rotating arm

Problem 1.18: A mass m is restrained by four identical springs, each of a spring constant k, between two perpendicular pairs of walls facing each other, as shown in Fig. P1.18. Assume that the distance between each pair of facing walls is $2L$ and the unstretched length of the springs is l_0 ($l_0 < L$). Consider only small motions of the mass within the horizontal plane of paper. Derive the equation(s) of motion for the system.

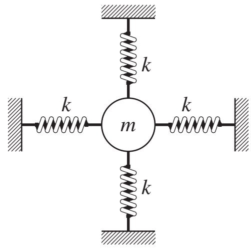

Figure P1.18 Mass restrained by four identical springs

Problem 1.19: A uniform rigid slender rod of mass m and length L is suspended from the ceiling by two identical inextensible cables of length l. It is free to swing within the plane of paper, as shown in Fig. P1.19. The distance between two suspension points is also L. Both cables remain taut at all times. Derive the equation(s) of motion for the system.

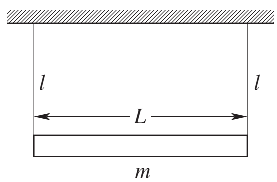

Figure P1.19 A rigid slender rod as a planar pendulum

Problem 1.20: A composite rigid-body pendulum is made of a uniform rigid slender rod of mass m and length L rigidly attached to a uniform circular disk, of mass M and radius r, as shown in Fig. P1.20. Assume the system is confined within the plane of paper. Derive the equation(s) of motion for the system.

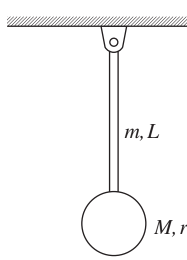

Figure P1.20 Composite rigid-body pendulum

Problem 1.21: A uniform rigid slender rod of mass m and length l has one of its ends attached to an inextensible massless string, also of length l, which, in turn, is attached to the ceiling, as shown in Fig. P1.21. Assume the system is confined to swing in the plane of paper. Derive the equation(s) of motion for the system.

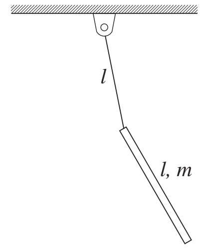

Figure P1.21 Another rigid slender rod as a planar pendulum

Problem 1.22: A planar double rigid-link pendulum comprises two identical uniform rigid links of mass m and length L joined in series and pivoted at one end, as shown in Fig. P1.22. They are confined within the plane of paper. Both the joint and the pivot are frictionless. A horizontal force $F(t)$ acts on the joint. Derive the equation(s) of motion for the system.

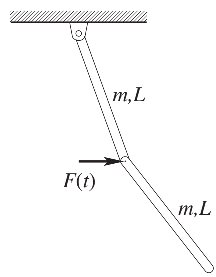

Figure P1.22 Planar double rigid-link pendulum

Problem 1.23: A U-shaped tube is filled with an inviscid and incompressible fluid of mass density ρ, as shown in Fig. P1.23. Its both ends are open to the atmosphere. At equilibrium, the fluid columns in both branches are of a height h, measured from its semicircular bottom of radius $R, h > R$. The tube has a uniform cross-sectional area A throughout. Derive the equation(s) of motion for the system.

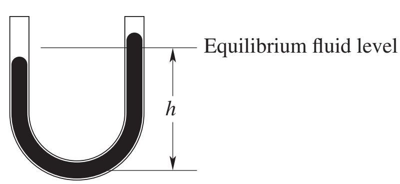

Figure P1.23 Fluid-filled U-shaped tube

Problem 1.24: A uniform rigid slender rod of mass M and length L is supported at its two ends by two springs k_1 and k_2, respectively. A mass m is attached to the center of the rod via spring k_3 and dashpot c, as shown in Fig. P1.24. Consider only small motions in the vertical direction. Derive the equation(s) of motion for the system.

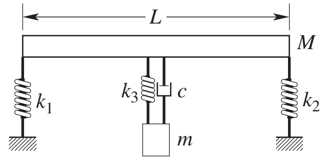

Figure P1.24 A mass attached to the center of a rod

Problem 1.25: A uniform rigid slender rod of mass m and length L is supported at its two ends by four springs and four dashpots, as shown in Fig. P1.25. Consider only small motions in the vertical direction. Derive the equation(s) of motion for the system.

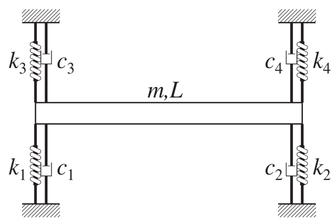

Figure P1.25 Rod supported by springs and dashpots at the ends

Problem 1.26: A uniform rigid slender rod of mass M is pivoted at its left end. Its right end rests on a spring k_1 and dashpot c_1. It is also attached to a mass m through a pulley and another spring k_2 and dashpot c_2, as shown in Fig. P1.26. The pulley can be treated as a uniform circular disk of mass m_0 and radius r. Assume small motions of the right end of the rod. Derive the equation(s) of motion for the system.

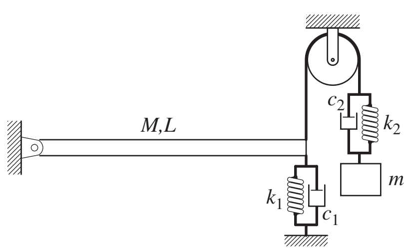

Figure P1.26 Mass attached to the end of pivoted rod via pulley

Problem 1.27: A uniform rigid slender rod of mass m_1 is pivoted at its left end. It is further supported by a spring k_1 and a dashpot c_1 at a distance a from the pivot point. A mass m_2 is attached to its right end, which is at a distance b from the supporting spring and dashpot, through another spring k_2 and another dashpot c_2, as shown in Fig. P1.27. Assume small motions. Derive the equation(s) of motion for the system.

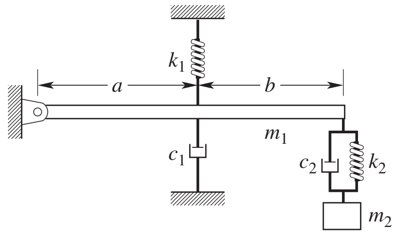

Figure P1.27 Mass attached to the end of pivoted rod

Problem 1.28: A circular rigid cylinder of mass m and radius r_1 can roll without slip on an inclined surface, of inclining angle θ. It is restrained by spring k whose end is wrapped around the core of the cylinder at radius r_2 and a dashpot is connected to its center, as shown in Fig. P1.28. The cylinder has a centroidal moment of inertia I. Derive the equation(s) of motion for the system.

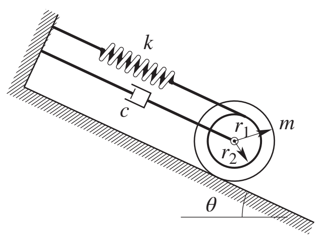

Figure P1.28 Cylinder restrained by spring and dashpot on an inclined surface

Problem 1.29: A uniform circular rigid cylinder of mass m and radius r is confined by spring k and dashpot c at its center on the inclined surface of a cart of mass M, as shown in Fig. P1.29. The cart can move frictionlessly on the horizontal floor, while the cylinder can roll without slip on the inclined surface. The surface's inclined angle is θ. Derive the equation(s) of motion for the system for the following two scenarios:

(a) A horizontal force $F(t) = F_0 \sin \Omega t$ acts on the cart, pointing to the right.
(b) The cart moves according to $x(t) = X_0 \sin \Omega t$, where x is the location of the left side of the cart measured from a reference configuration, which is positive to the right.

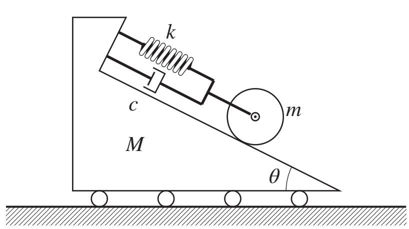

Figure P1.29 Mass restrained by spring and dashpot on the inclined surface of a cart

Problem 1.30: A uniform rigid slender rod of mass m and length L is pinned to the rim of a circular disk of mass M and radius R. The disk is constrained by spring k that connects the center of the disk to the wall, as shown in Fig. P1.30. When the spring is unstretched, the pin that connects the rod is located on the horizontal diameter. The disk rolls without slip on the horizontal floor, while the rod slides frictionlessly. Derive the equation(s) of motion for the system.

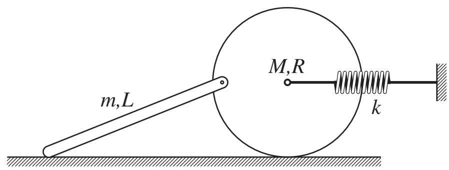

Figure P1.30 Slender rod pinned to the rim of a circular disk

Problem 1.31: A uniform semicircular rigid cylinder of mass m and radius r rolls without slip on the horizontal ground, as shown in Fig. P1.31. The centroid is located at a distance $a = \frac{4r}{3\pi}$ below the flat surface of the cylinder, and the centroidal moment of inertia is $I_C = m\left(\frac{r^2}{2} - a^2\right)$. Derive the equation(s) of motion for the system.

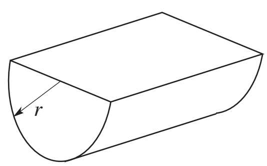

Figure P1.31 Semicircular cylinder rolls without slip on the horizontal ground

Problem 1.32: A uniform rigid slender rod of mass m and length L can slide frictionlessly inside a circular track of radius R, as shown in Fig. P1.32. The rod is confined to move within the plane of paper. Derive the equation(s) of motion for the system.

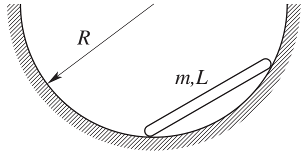

Figure P1.32 Slender rod slides inside a circular track

Problem 1.33: A uniform rigid slender rod of mass m and length L can roll without slip on top of a fixed circular cylindrical surface of radius R. At equilibrium, the rod is horizontal, as shown in Fig. P1.33. Derive the equation(s) of motion for the system.

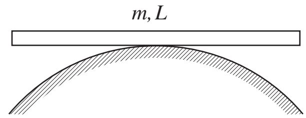

Figure P1.33 Rigid slender rod rolls without slip on a circular cylindrical surface

Problem 1.34: A uniform rigid circular disk of radius r and mass m_2 can roll without slip inside of a circular track of radius R. The disk is restrained by a rigid slender rod of mass m_1 to the center of the track, as shown in Fig. P1.34. All pin-connections are frictionless. Derive the equation(s) of motion for the system.

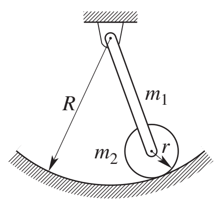

Figure P1.34 Restrained disk rolls without slip inside a circular track

Problem 1.35: Two identical circular disks of mass m and radius r are connected by a uniform rigid link of mass m_0 and length l. The disks roll without slip on a circular track of radius R, as shown in Fig. P1.35. Assume that the entire assembly moves in a vertical plane and remains below the center of the track; and all pin-connections are frictionless. Derive the equation(s) of motion for the system.

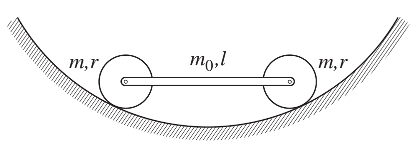

Figure P1.35 Circular disks connected by rod roll without slip on a circular track

Problem 1.36: Two identical circular disks of mass m and radius r are connected by a spring k and a dashpot c. The disks roll without slip on a circular track of radius R, as shown in Fig. P1.36. Assume that both disks move in a vertical plane and remain entirely below the center of the track at all times. The unstretched length of the spring is l_0. Derive the equation(s) of motion for the system.

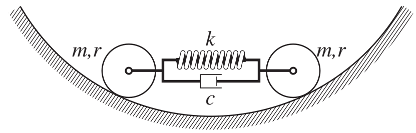

Figure P1.36 Disks connected by spring and dashpot roll without slip on a circular track

Problem 1.37: Two identical circular disks of mass m and radius r are pin-connected by a uniform rigid link of mass m_0 and length l. The assembly straddles on a fixed circular cylindrical surface of radius R, and is confined to move within the plane of paper, as shown in Fig. P1.37. The disks can roll without slip on the cylindrical surface. Derive the equation(s) of motion for the system.

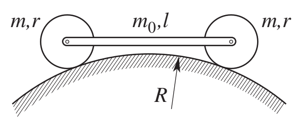

Figure P1.37 Disks connected by rod straddle on a circular cylindrical surface

Problem 1.38: Two identical circular disks of mass m and radius r are connected by a spring k. The assembly straddles on a fixed circular cylindrical surface of radius R and is confined to move within the plane of paper, as shown in Fig. P1.38. The disks can roll without slip on the cylindrical surface. The unstretched length of the spring is l_0. Derive the equation(s) of motion for the system.

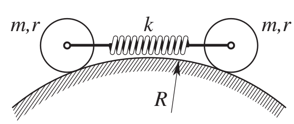

Figure P1.38 Disks connected by spring straddle on a circular cylindrical surface

Problem 1.39: Two uniform rigid slender beams of masses M and m_0, respectively, are pin-connected via two rigid links at their ends, as shown in Fig. P1.39. Both beams have a length of L; and both links have a length of l and mass m_1. The beam of mass M can slide frictionlessly on the horizontal surface. All pin-connections are frictionless. Derive the equation(s) of motion for the system.

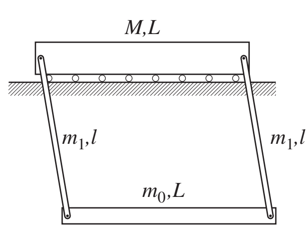

Figure P1.39 Two beams connected by two identical rigid links

Problem 1.40: A uniform rigid slender beam of mass M and length L is attached to two carts via rigid links at its ends. Two links are identical, and both are of mass m_1 and length l. The two carts are identical, and both are of mass m and can slide frictionlessly on the horizontal floor, as shown in Fig. P1.40. The carts are connected by spring k and dashpot c. The cart on the left is subjected to a prescribed motion $x_0(t)$ with respect to the ground. Assume small motions. Derive the equation(s) of motion for the system.

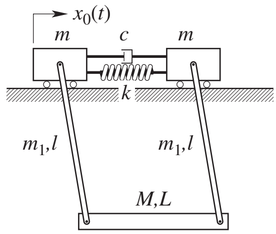

Figure P1.40 Uniform rigid slender beam connected to two carts via rigid links

Problem 1.41: Two identical circular rigid disks of mass M and radius R are pin-connected by a rigid link of mass m and length l. At equilibrium, the link is horizontal, and the pins are located at a distance a vertically below the centers, as shown in Fig. P1.41. A horizontal force $F(t)$ acts on the rim of a disk at the same level as the centers of the disks. Both pin-connections are frictionless; and both disks roll without slip on the horizontal ground. Derive the equation(s) of motion for the system.

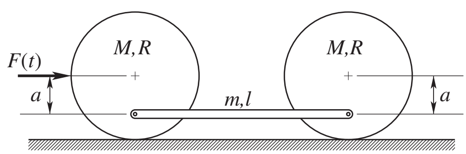

Figure P1.41 Two disks linked by rigid link roll without slip on the ground

Problem 1.42: A roller comprises a circular cylindrical core of mass m_2 and radius r_2, which is shrink-fitted into a circular annulus of outer radius r_1 and mass m_1. The protruded core rolls without slip on a circular track of radius R, as shown in Fig. P1.42. Derive the equation(s) of motion for the system.

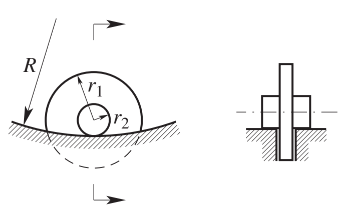

Figure P1.42 Roller's cylindrical core rolls without slip on a circular track

Problem 1.43: A rigid rocker having a concave circular cylindrical surface of radius R can roll without slip on a circular cylindrical support, of radius $r < R$, as shown in Fig. P1.43. The rocker has a mass m and a centroidal moment of inertia I. At equilibrium, the center of mass of the rocker is located vertically at a distance a above the contact point. Derive the equation(s) of motion for the system.

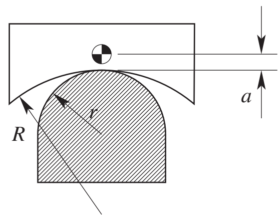

Figure P1.43 Rocker rolls without slip on a circular support

Problem 1.44: A uniform circular rigid cylinder of mass m_1 and radius r_1 rolls without slip on a horizontal surface. A pendulum comprises a circular disk of mass m_2 and radius r_2 and a rigid connection rod of mass m_3 and length L, rigidly joined together. The pendulum is attached to the cylinder as shown in Fig. P1.44. Derive the equation(s) of motion for the system for the following two scenarios:

(a) The connecting rod is welded to the cylinder m_1 rigidly.
(b) The connecting rod is pinned to the cylinder m_1 frictionlessly.

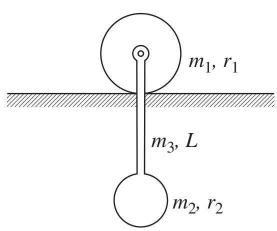

Figure P1.44 Pendulum connected to a circular cylinder that rolls on surface

Problem 1.45: A circular cylinder of radius r rolls without slip on the horizontal surface. The cylinder is made of two uniform semicircular cylinders welded together, as shown in Fig. P1.45. One has an areal mass density of ρ_1 and the other ρ_2 ($\rho_2 > \rho_1$). Derive the equation(s) of motion for the system. (*Hint: Refer to Problem* 1.31 *for the location of the centroid and the centroidal moment of inertia for a semicircular cylinder.*)

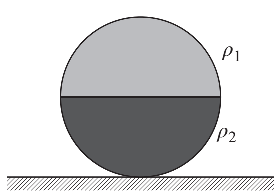

Figure P1.45 Cylinder made of two halves of uniform materials

Problem 1.46: A rotor blade of a helicopter is modeled as a uniform rigid slender rod of mass m and length L. It is pin-joined to a base of diameter a, which is rotating at a constant angular velocity Ω, as shown in Fig. P1.46. The rotor blade is free to "flap" about the pin-joint. Derive the equation(s) of motion for the system.

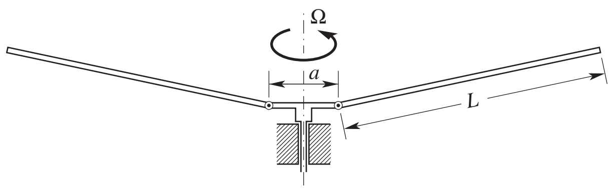

Figure P1.46 Rotor blades on a helicopter

Problem 1.47: A slender rod of mass m and length L is pin-joined to a rotating axle, as shown in Fig. P1.47. The pin-joint is planar, meaning that the rod can only rotate about the pin within a plane that is being rotated, at a given angular velocity Ω. Derive the equation(s) of motion for the system.

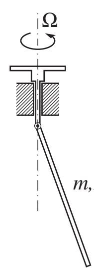

Figure P1.47 Rigid slender rod being rotated at an angular velocity Ω

Problem 1.48: Find all the equilibrium positions for the system described in Problem 1.3. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.49: Find all the equilibrium positions for the system described in Problem 1.4. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.50: Find all the equilibrium positions for the system described in Problem 1.7. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.51: Find all the equilibrium positions for the system described in Problem 1.9. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.52: Find all the equilibrium positions for the system described in Problem 1.14 when the offset $e = 0$. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.53: Find all the equilibrium positions for the system described in Problem 1.16. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.54: Find all the equilibrium positions for the system described in Problem 1.46 for the case $a = 0$. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.55: Find all the equilibrium positions for the system described in Problem 1.47. Linearize the equation(s) of motion for small motions about each equilibrium position.

Problem 1.56: If the linearized equation of motion for a single degree-of-freedom system about an equilibrium position can be written

$$\ddot{x} + \gamma \dot{x} + \lambda x = 0$$

show that the equilibrium position is stable if and only if $\gamma \geq 0$ and $\lambda > 0$.