1

Equilibrium States and Their Stability

In this chapter, as some of most important concepts to analyze dynamical systems, the definitions of *equilibrium states* and their *stability* are introduced. Considerations on the representative mechanical systems as a spring-mass system, a pendulum, and a magnetically levitated system, will make these concepts accessible. The stability of the equilibrium states is investigated in this chapter without mathematical analysis for the governing equations, but by inspecting the equation forms from a physical point of view. Then, a stabilization control method for the unstable equilibrium state will be briefly mentioned for the magnetically levitated system.

1.1 Equilibrium States

Let us start with the introduction of a *static equilibrium state* or an *equilibrium state* that is one of most important concepts to gain insights on the nonlinear dynamics of mechanical systems.

Definition 1.1 A point $x = x_{st}$ is said to be an equilibrium state, if it has the property of remaining at position x_{st} independently of the time at which the state of the system starts at x_{st}.

The equilibrium state $x = x_{st}$ has zero velocity and zero acceleration. Let us examine the equilibrium states of three fundamental mechanical systems.

1.1.1 Spring-Mass System

Consider a spring-mass system subject to gravity force. Figure 1.1a shows the position of the mass m at a certain instant while it is moving.

First, let us derive the equation of motion using Newton's second law. We introduce a static coordinate system x' whose origin is located at the lower end of the spring without mass as Figure 1.1b.

Figure 1.1c,d shows the free body diagrams at the state where the spring is elongated of Δx. $F_s^{\#}$ is the force acting on the spring from the mass and Δx is the resulting elongation

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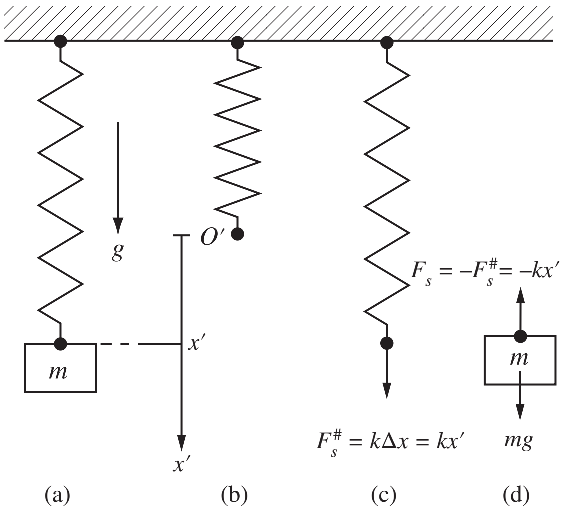


Figure 1.1 Spring-mass system and free body diagram. (a) Position of the vibrating mass m at a certain instant. (b) Coordinate x' whose origin O' is set to the lower end of the non-elongated spring; (c) and (d) Free body diagrams of spring and mass, respectively, where the elongation of the spring under the application of force $F_s^{\#}$ is denoted as Δx.

(see Figure 1.2a). In the ideal linear spring, Δx is proportional to $F_s^{\#}$ through a positive proportional constant k, i.e. a positive slope in the force–displacement characteristic of Figure 1.2b; the positive k means *positive stiffness*. Then, the mass is subject to the spring force $F_s = -F_s^{\#} = -k\Delta x$ representing the reaction force to $F_s^{\#}$ as shown in Figure 1.1d; the relationship between Δx and F_s is shown in Figure 1.2c, and the *negative* slope $(-k)$ indicates that the spring force direction is always opposite to that of Δx, i.e. the spring force acts on the mass as a *restoring* force. In this system, Δx is equal to x'.

Since the external force consists of the spring force F_s and the gravity force mg, the equation of motion is expressed as

$$m\frac{d^2x'}{dt^2} = -kx' + mg. \tag{1.1}$$

Thus, let us seek for the equilibrium state of the spring-mass system. The equilibrium state x_{st} that satisfies the following equation is obtained by assuming as conditions that velocity $\frac{dx'}{dt}$ and acceleration $\frac{d^2x'}{dt^2}$ are zero:

$$0 = -kx_{st} + mg. \tag{1.2}$$

Thus, the equilibrium state is

$$x_{st} = \frac{mg}{k}. \tag{1.3}$$

At this equilibrium state, the gravity force is balanced by the spring force.

Next, we rewrite the equation of motion by introducing a new coordinate x whose origin is shifted to the equilibrium state (see Figure 1.3). By substituting

$$x'(t) = x_{st} + x(t) \tag{1.4}$$

into Eq. (1.1) and taking into account that the equilibrium state x_{st} is time-independent, i.e. $\frac{dx_{st}}{dt} = \frac{d^2x_{st}}{dt^2} = 0$, we obtain

$$m\frac{d^2x}{dt^2} = -k(x + x_{st}) + mg. \tag{1.5}$$

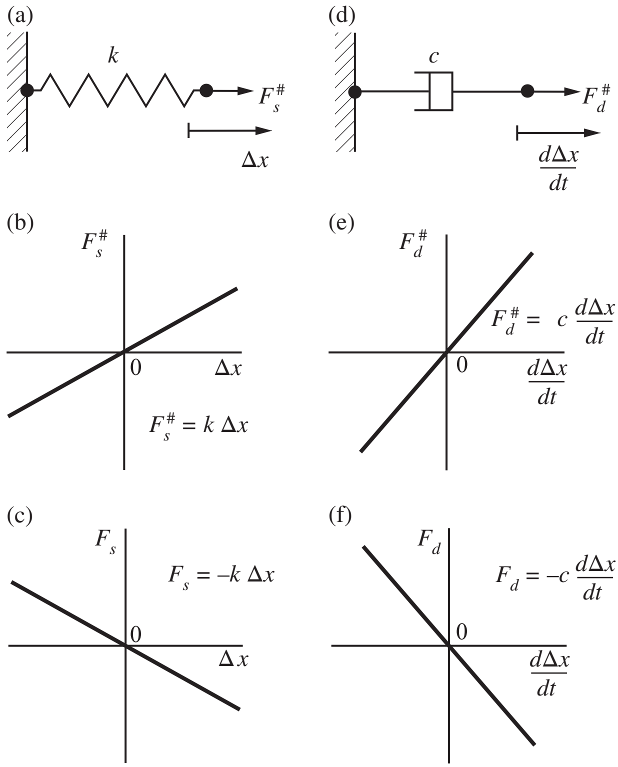


Figure 1.2 Characteristics of linear spring and linear damper. (a) Spring is elongated of Δx by applying force $F_s^\#$. (b) Force–displacement relationship $F_s^\# = k\Delta x$ showing that the ideal spring has a positive k. Such a stiffness effect is called *positive stiffness*. (c) Force $F_s(= -F_s^\# = -k\Delta x)$ produced by the spring with elongation Δx. When a mass is attached to the right end of the spring, the directions of the mass displacement and of the spring force acting on the mass are opposite. Thus, the spring force F_s acts on a mass as a *restoring* force. (d) By applying the force $F_d^\#$, the damper has velocity $\frac{d\Delta x}{dt}$. (e) Force–velocity relationship $F_d = c\frac{d\Delta x}{dt}$ showing that the ideal damper has a positive c. Such a damping effect is called a *positive damping*. (f) Force $F_d\left(= -F_d^\# = -c\frac{d\Delta x}{dt}\right)$ produced by the damper with velocity $\frac{d\Delta x}{dt}$. When a mass is attached to the right end of the damper, the directions of the mass velocity and the damping force acting on the mass are opposite.

By using the equilibrium equation (1.2), we can obtain another equation of motion with a different form as

$$m\frac{d^2x}{dt^2} = -kx. \tag{1.6}$$

By the way, unlike Eq. (1.1), the right-hand side does not include a constant term and Eq. (1.6) has the *trivial solution* ($x = 0$). In general, an equation having the trivial solution is called *homogeneous* equation. An equation not having trivial solution ($x' = 0$) as Eq. (1.1) is called *inhomogeneous* equation. The complete solution of the inhomogeneous equation consists of the homogeneous and particular solutions. Therefore, Eq. (1.6) describes the dynamics of the system in a simpler form from a mathematical point of view. We will solve Eq. (1.6) in Section 2.4.4.

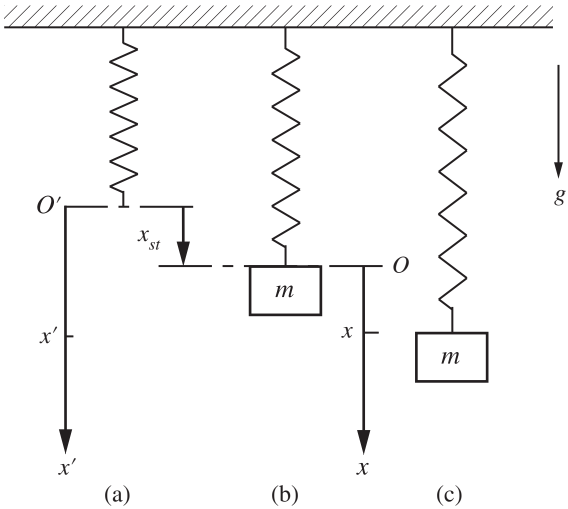


Figure 1.3 Coordinate transformation from x' to x ($x' = x + x_{st}$). (a) Origin O' of coordinate x' is set to the lower end of the non-elongated spring. (b) The origin O of coordinate x is located at the equilibrium state. (c) Position of mass at a certain instant while vibrating.

1.1.2 Magnetically Levitated System

As second example, we consider a simplified model of magnetically levitated vehicle (electro-magnetic suspension, EMS) shown in Figure 1.4. The vehicle is magnetically levitated from the guideway by using electromagnets attached to the vehicle. To investigate the dynamics of the levitated system in the vertical direction, we can focus on the motion of a magnet on the system, i.e. the part within the dotted circle in Figure 1.4a. The static coordinate x' is introduced and used to describe the position of the magnet. Its origin is located at the guideway as shown in Figure 1.4b. The electromagnetic force is assumed to be proportional to the square of the current i^2 and to be inversely proportional to the square of the gap between the guideway and the magnet x'^2 as (Dorf and Bishop 2011)

$$F_m = -k\frac{i^2}{x'^2}, \tag{1.7}$$

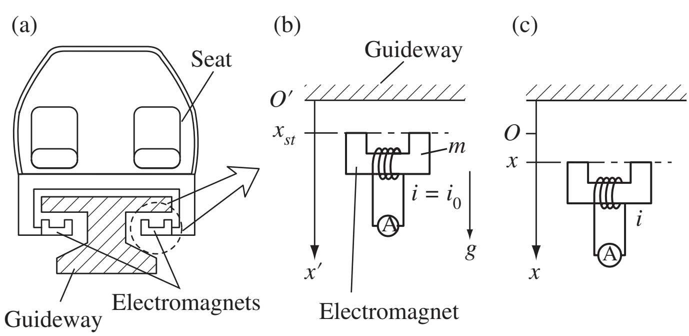


Figure 1.4 Magnetically levitated vehicle (MAGLEV). (a) Cross section of the magnetically levitated vehicle. (b) Static equilibrium state of the magnet expressed by coordinate x' in case of $i = i_0$. (c) Position of the magnet at a certain instant expressed by coordinate x whose origin is located at the static equilibrium state. Reprinted with permission Yabuno (2004).

where k is a positive constant which is determined by the materials of the magnet and the guideway, the shape of the magnet, and so on. Hence, the equation of motion is expressed as

$$m\frac{d^2x'}{dt^2} = -k\frac{i^2}{x'^2} + mg. \tag{1.8}$$

This equation is nonlinear since the first term of the magnetic force is not proportional to x'.

The equilibrium state x_{st} under the constant current of $i = i_0$ is determined by adopting a similar approach to the case of the spring-mass system. By assuming $d^2x'/dt^2 = 0$, Eq. (1.8) leads to the equilibrium equation

$$0 = -k\frac{i_0^2}{x_{st}^2} + mg. \tag{1.9}$$

Then, the equilibrium state $x_{st}(> 0)$ is expressed as

$$x_{st} = \sqrt{\frac{ki_0^2}{mg}}. \tag{1.10}$$

At the gap of $x' = x_{st}$, the gravity force is balanced by the magnetic force. Furthermore, we apply the coordinate transformation in the spring-mass system in Section 1.1.1, i.e.

$$x' = x_{st} + x, \tag{1.11}$$

where the origin of x is located at the equilibrium state O in Figure 1.4c. We obtain

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -k\left(\frac{i_0}{x_{st}+x}\right)^2 + mg \\ &= -k\frac{i_0^2}{x_{st}^2}\left\{\frac{1}{1+(x/x_{st})}\right\}^2 + mg. \end{aligned} \tag{1.12}$$

At this stage, unlike the spring-mass system, Eq. (1.8) cannot be transformed into the homogeneous equation due to the nonlinearity of the magnetic force.

By focusing on the motion in the neighborhood of the equilibrium state, i.e. in the state of $|x/x_{st}| \ll 1$, we approximate Eq. (1.7) to obtain the linear expression. Using Taylor expansion, we represent the magnetic force in terms of a power series with respect to the small value $|x/x_{st}|$ as (see Section 2.3)

$$F = -k\frac{i_0^2}{x_{st}^2}\left\{1 - 2\left(\frac{x}{x_{st}}\right) + 3\left(\frac{x}{x_{st}}\right)^2 - \cdots\right\}. \tag{1.13}$$

In the region of $|x/x_{st}| \ll 1$, this uniform expansion can be truncated by a finite number of terms depending on the desired accuracy. Neglecting higher order terms than the first order term with respect to x/x_{st}, we obtain the linearized magnetic force as

$$F = -k\frac{i_0^2}{x_{st}^2}\left\{1 - 2\left(\frac{x}{x_{st}}\right)\right\} = -k\frac{i_0^2}{x_{st}^2} + \frac{2ki_0^2}{x_{st}^3}x + O\left(\left(\frac{x}{x_{st}}\right)^2\right), \tag{1.14}$$

where O is the Landau symbol (Nayfeh 1981; Witelski and Bowen 2015); see Appendix D. Hence, the approximated equation of motion is written as

$$m\frac{d^2x}{dt^2} = -k\frac{i_0^2}{x_{st}^2} + \frac{2ki_0^2}{x_{st}^3}x + mg. \tag{1.15}$$

Recalling the equilibrium equation (1.9), we obtain the equation of motion in the form of linear homogeneous equation as

$$m\frac{d^2x}{dt^2} = k_m x, \tag{1.16}$$

where $k_m \overset{\text{def}}{=} \frac{2ki_0^2}{x_{st}^3} > 0$ is constant. It is the same form as Eq. (1.6), but the sign of the coefficient k_m of x in the right-hand side is different from that of x in Eq. (1.6). Figure 1.2c for the spring-mass system corresponds to Fig. 1.5c for the magnetically levitated system because of $F_m = k_m x$. The positive slope indicates that the magnetic force does not act on the levitated system as a restoring force but augments the deviation $|x|$ from the equilibrium state because such force acts in the same direction of the deviation from the equilibrium state. The corresponding force–displacement relationship is reported in Figure 1.5b, and it is characterized by *negative* slope ($F_m^{\#} = -k_m x$ in which x corresponds to Δx in Figure 1.5b). Such a feature is called *negative stiffness*, which produces a buckling phenomenon as mentioned in Section 5.1. We will solve Eq. (1.16) in Section 2.4.2.

1.1.3 Simple Pendulum

Let us move on to the third example of a simple pendulum as shown in Figure 1.6. The equation of motion is

$$ml\frac{d^2\theta'}{dt^2} = -mg\sin\theta'. \tag{1.17}$$

Letting $d^2\theta'/dt^2 = 0$, we obtain the equilibrium equation

$$0 = -mg\sin\theta_{st}. \tag{1.18}$$

The pendulum has two equilibrium states corresponding to the downward vertical position $\theta_{st} = 2n\pi \overset{\text{def}}{=} \theta_{st1}$ and the upright position $\theta_{st} = (2n+1)\pi \overset{\text{def}}{=} \theta_{st2}$ ($n = \dots, -1, 0, 1, 2, \dots$).

Similar to the preceding two examples, we transform the coordinate as follows:

$$\theta' = \theta_{st} + \theta. \tag{1.19}$$

Under the assumption of $|\theta| \ll 1$, we can linearize the equation of motion in the neighborhood of each equilibrium state separately. By substituting Eq. (1.19) into Eq. (1.17) and considering Eq. (1.18), the result is expanded with respect to the small term θ:

$$ml\frac{d^2\theta}{dt^2} = -mg\left.\frac{d\sin(\theta_{st}+\theta)}{d\theta}\right|_{\theta=0}\theta + O(\theta^2). \tag{1.20}$$

In the neighborhood of θ_{st1} and θ_{st2}, Eq. (1.17) is expressed respectively as:

$$ml\frac{d^2\theta}{dt^2} = -mg\theta \tag{1.21}$$

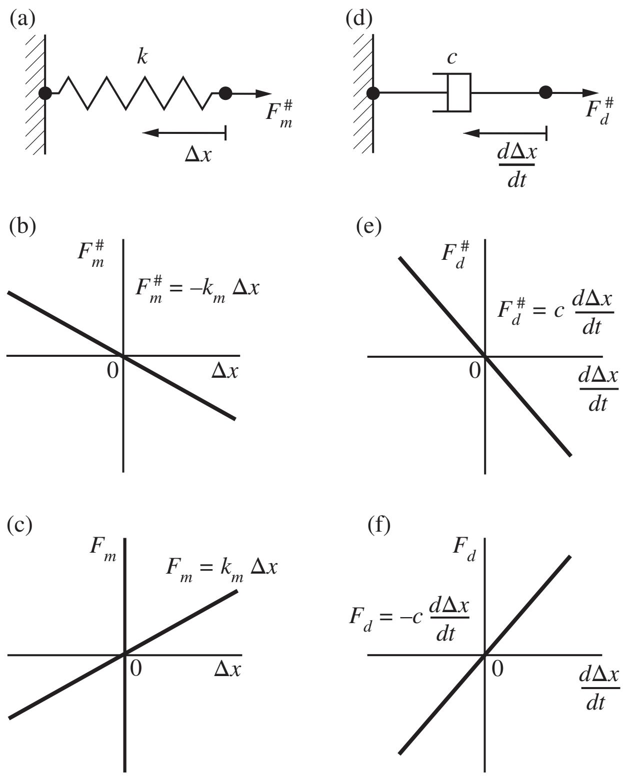
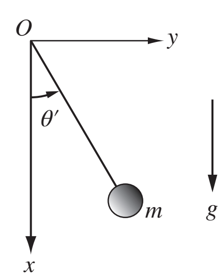


Figure 1.5 Negative stiffness and negative damping. (a) Virtual spring is shrunk with Δx by applying force $F_m^{\#}$. (b) Force–displacement relationship $F_m = -k_m \Delta x$, where $-k_m$ is negative. Such a stiffness effect is called *negative stiffness*. (c) Corresponding force $F_m (= -F_m^{\#} = k_m \Delta x)$ produced by the spring with elongation Δx. If a mass is attached to the right end of the virtual spring, the displacement directions of mass and spring force acting on the mass are the same. Therefore, the virtual spring force F_m does not act on the mass as a restoring force. (d) Virtual damper has velocity $\frac{d\Delta x}{dt}$ by applying the force $F_d^{\#}$. (e) Force–velocity relationship $F_d = -c\frac{d\Delta x}{dt}$, where $-c$ is negative. Such a damping effect is called *negative damping*. (f) Corresponding force $F_d \left(= -F_d^{\#} = c\frac{d\Delta x}{dt}\right)$ produced by the damper with $\frac{d\Delta x}{dt}$. If a mass is attached to the right end of the damper, the directions of the velocity of the mass and the damping force acting on the mass are the same.

Figure 1.6 Pendulum.



and

$$ml\frac{d^2\theta}{dt^2} = mg\theta. \tag{1.22}$$

The right-hand side of these equations can be regarded as an external force in the systems as previously done in Sections 1.1.1 and 1.1.2.

In the neighborhood of the downward vertical position $\theta = \theta_{st1}$, as for the spring-mass system, the coefficient on the right-hand term in Eq. (1.21) is negative (see Figure 1.2c in which Δx corresponds to θ). The equivalent external force is a restoring force and is characterized by a positive stiffness (see Figure 1.2b). The external force causes a vibration around the equilibrium state $\theta = \theta_{st1}$.

On the other hand, in the neighborhood of the upright position $\theta = \theta_{st2}$, due to the positive sign of the coefficient on the right-hand side (see Figure 1.5c in which Δx corresponds to θ), the equivalent external force is not a restoring force as already observed for the magnetically levitated system and is characterized by a negative stiffness (see Figure 1.5b). The external force monotonically increases the deviation $|\theta|$ from the equilibrium state $\theta = \theta_{st2}$.

We dealt with the four kinds of equilibrium states. The dynamics in the neighborhood of every equilibrium state are governed by the second-order ordinary differential equations. The main difference among the cases is only in the sign of the coefficient placed on their right-hand side. In the case of a negative coefficient, the equivalent external force is a restoring force, whereas it is not in the case of a positive coefficient. Hence, the behaviors of the spring-mass system and the simple pendulum in the neighborhood of the downward vertical position are qualitatively the same as it is for the behaviors of the magnetically levitated system and the simple pendulum in the neighborhood of the upright position. The above feature is one of the many cases in which the mathematical form is directly related to the physical characteristics. It will be noticed throughout this book that there are such various analogies between the dynamics in mechanical systems and the mathematical forms of their governing equations.

However, we intuitively know the qualitative difference between the equilibrium states of the downward vertical position and the upright position in the simple pendulum. The difference cannot be distinguished only through the concept of equilibrium state. It will be physically characterized by introducing the concept of *stability* in Section 1.3.

1.2 Work and Potential Energy

We consider a particle of mass m moving from a position P_1 to another position P_2 under the action of an external force $\boldsymbol{F}$ as shown in Figure 1.7. Let the position vector of m be $\boldsymbol{r}$. The elementary work done by $\boldsymbol{F}$ in an infinitesimal segment $d\boldsymbol{r}$ of the path is

$$\overline{dW} = \boldsymbol{F} \cdot d\boldsymbol{r}, \tag{1.23}$$

where the dot stands for inner product. $\overline{dW}$ does not generally denote the perfect differential of W, but the infinitesimal work associated to the infinitesimal displacement $d\boldsymbol{r}$ (Lanczos 1986).

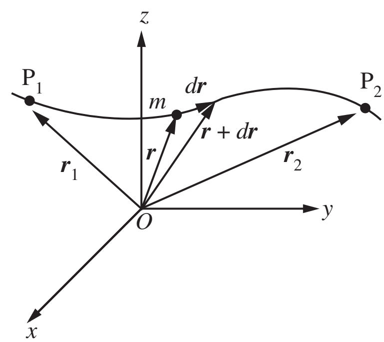


Figure 1.7 Path of particle m and an infinitesimal segment $d\boldsymbol{r}$ of the path.

Definition 1.2 If $\overline{dW}$ can be expressed in the form of the perfect differential of a scalar function V as

$$\overline{dW} = \boldsymbol{F} \cdot d\boldsymbol{r} = dV, \tag{1.24}$$

$\boldsymbol{F}$ is called a *conservative force* and $U \overset{\text{def}}{=} -V$ is called *potential energy*.

Then, $\boldsymbol{F}$ can be expressed as

$$\boldsymbol{F} = -\nabla U, \tag{1.25}$$

where ∇ is a differential operator called *nabla* and written in the terms of the Cartesian components, x, y, and z, in the form

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z. \tag{1.26}$$

Under the conservative force, the motion of the particle of mass m is governed by the following equation of motion:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -\nabla U. \tag{1.27}$$

At an equilibrium state, the potential energy satisfies

$$\nabla U = \boldsymbol{0}. \tag{1.28}$$

Problem 1.2 Derive Eq. (1.25) from $\boldsymbol{F} \cdot d\boldsymbol{r} = d(-U)$ in Cartesian coordinate.

Ans: Letting

$$d\boldsymbol{r} = dx\boldsymbol{e}_x + dy\boldsymbol{e}_y + dz\boldsymbol{e}_z, \tag{1.29}$$

we have the following relationship:

$$\begin{aligned} dV = d(-U) &= -\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) \\ &= -\left(\frac{\partial U}{\partial x}\boldsymbol{e}_x + \frac{\partial U}{\partial y}\boldsymbol{e}_y + \frac{\partial U}{\partial z}\boldsymbol{e}_z\right) \cdot (dx\boldsymbol{e}_x + dy\boldsymbol{e}_y + dz\boldsymbol{e}_z) \\ &= -\nabla\, U \cdot d\boldsymbol{r}. \end{aligned} \tag{1.30}$$

Comparing with Eq. (1.24) yields Eq. (1.25).

The work done from position $P_1(\boldsymbol{r} = \boldsymbol{r}_1)$ to position $P_2(\boldsymbol{r} = \boldsymbol{r}_2)$ by $\boldsymbol{F}$ is expressed using Eq. (1.23) as

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{1.31}$$

In particular, if $\boldsymbol{F}$ is a conservative force, the relationship $\boldsymbol{F} \cdot d\boldsymbol{r} = d(-U)$ holds, and Eq. (1.31) can be written as

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} dU = U(\boldsymbol{r}_1) - U(\boldsymbol{r}_2). \tag{1.32}$$

The work does not depend on the path between the points $P_1(\boldsymbol{r}_1)$ and $P_2(\boldsymbol{r}_2)$, but only on their positions. Then, the potential energy U is derived from $\boldsymbol{F}$ as

$$U(\boldsymbol{r}_2) - U(\boldsymbol{r}_1) = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{1.33}$$

The kinetic energy is defined using the velocity $\boldsymbol{v} = d\boldsymbol{r}/dt$ of the mass m as

$$K = \frac{1}{2}m\left(\frac{d\boldsymbol{r}}{dt} \cdot \frac{d\boldsymbol{r}}{dt}\right). \tag{1.34}$$

Here, considering Newton's second law for the case with constant mass

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}, \tag{1.35}$$

the first derivative of the kinetic energy with respect to time t is

$$\frac{dK}{dt} = m\frac{d\boldsymbol{r}}{dt} \cdot \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{r}}{dt} \cdot \boldsymbol{F}, \tag{1.36}$$

which is called *power*. We consider the case when $\boldsymbol{F}$ is conservative, i.e. $\boldsymbol{F} = -\nabla U$. Because of

$$\frac{d\boldsymbol{r}}{dt} \cdot \boldsymbol{F} = \left(\frac{dx}{dt}\boldsymbol{e}_x + \frac{dy}{dt}\boldsymbol{e}_y + \frac{dz}{dt}\boldsymbol{e}_z\right) \cdot \left(-\frac{\partial U}{\partial x}\boldsymbol{e}_x - \frac{\partial U}{\partial y}\boldsymbol{e}_y - \frac{\partial U}{\partial z}\boldsymbol{e}_z\right) = -\frac{dU}{dt}, \tag{1.37}$$

from Eq. (1.36), the *total energy* defined as the sum of kinetic energy and potential energy:

$$E = K + U \tag{1.38}$$

is constant, i.e.

$$\frac{dE}{dt} = \frac{d}{dt}(K + U) = 0. \tag{1.39}$$

As a result, if the external force is conservative, the total energy is conserved. Since the external forces in the three systems investigated in Section 1.1 can be expressed as Eq. (1.25), those systems are conservative. In the following sections, the potential energy is used for the determination of *stability*.

1.3 Stability of the Equilibrium State in Conservative Systems

We investigate the dynamics of conservative systems in the neighborhood of an equilibrium state. The variation of the total energy E can be expressed as

$$\Delta E = \Delta K + \Delta U = 0. \tag{1.40}$$

According to the definition of equilibrium state, $\boldsymbol{r}_{st}$ satisfying $\boldsymbol{F}(\boldsymbol{r}_{st}) = \boldsymbol{0}$, i.e. $\nabla U(\boldsymbol{r}_{st}) = \boldsymbol{0}$, is an *equilibrium state*. The potential energy, which is a function of the position vector $\boldsymbol{r}$, can be expanded in the neighborhood of the equilibrium state $\boldsymbol{r} = \boldsymbol{r}_{st}$

$$\begin{aligned}
U(\boldsymbol{r}) &= U(\boldsymbol{r}_{st}) + \left.\frac{\partial U}{\partial x}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta x + \left.\frac{\partial U}{\partial y}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta y + \left.\frac{\partial U}{\partial z}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta z \\
&\quad + \frac{1}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta x^2 + \frac{1}{2}\left.\frac{\partial^2 U}{\partial y^2}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta y^2 + \frac{1}{2}\left.\frac{\partial^2 U}{\partial z^2}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta z^2 \\
&\quad + \left.\frac{\partial^2 U}{\partial x \partial y}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta x \Delta y + \left.\frac{\partial^2 U}{\partial y\, \partial z}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta y \Delta z + \left.\frac{\partial^2 U}{\partial z \partial x}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \Delta z \Delta x + O(|\Delta \boldsymbol{r}|^3) \\
&= U(\boldsymbol{r}_{st}) + \left.\begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{bmatrix}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \\
&\quad + \frac{1}{2}[\Delta x \quad \Delta y \quad \Delta z] \left.\begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\, \partial y} & \frac{\partial^2 U}{\partial x\, \partial z} \\ \frac{\partial^2 U}{\partial y\, \partial x} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y\, \partial z} \\ \frac{\partial^2 U}{\partial z\, \partial x} & \frac{\partial^2 U}{\partial z\, \partial y} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + O(|\Delta \boldsymbol{r}|^3),
\end{aligned} \tag{1.41}$$

where $\Delta \boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_{st} = \Delta x \boldsymbol{e}_x + \Delta y \boldsymbol{e}_y + \Delta z \boldsymbol{e}_z$. By using the *Jacobian matrix* and *Hessian matrix* of the scalar valued function U at $\boldsymbol{r}_{st}$, respectively, as

$$DU(\boldsymbol{r}_{st}) \overset{\text{def}}{=} [\nabla U(\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{r}_{st}}]^T = \left.\begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{bmatrix}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}}, \tag{1.42}$$

$$HU(\boldsymbol{r}_{st}) \overset{\text{def}}{=} \left.\begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\, \partial y} & \frac{\partial^2 U}{\partial x\, \partial z} \\ \frac{\partial^2 U}{\partial y\, \partial x} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y\, \partial z} \\ \frac{\partial^2 U}{\partial z\, \partial x} & \frac{\partial^2 U}{\partial z\, \partial y} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix}\right|_{\boldsymbol{r}=\boldsymbol{r}_{st}}, \tag{1.43}$$

Eq. (1.41) is described as

$$U(\boldsymbol{r}) = U(\boldsymbol{r}_{st}) + DU(\boldsymbol{r}_{st})\Delta \boldsymbol{r} + \frac{1}{2}\Delta \boldsymbol{r}^T HU(\boldsymbol{r}_{st})\Delta \boldsymbol{r} + O(|\Delta \boldsymbol{r}|^3). \tag{1.44}$$

Since $DU(\boldsymbol{r}_{st}) = 0$, the variation of the potential energy $\Delta U = U(\boldsymbol{r}) - U(\boldsymbol{r}_{st})$ can be expressed as

$$\Delta U = \frac{1}{2}[\Delta x \quad \Delta y \quad \Delta z] HU(\boldsymbol{r}_{st}) \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{1.45}$$

In addition, from Eq. (1.40), we have

$$\Delta K = -\Delta U = -\frac{1}{2}[\Delta x \quad \Delta y \quad \Delta z] HU(\boldsymbol{r}_{st}) \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{1.46}$$

Therefore, if the Hessian matrix is positive-definite (e.g. Strang et al. 1993) at the equilibrium state, i.e. $\Delta \boldsymbol{r}^T HU(\boldsymbol{r}_{st}) \Delta \boldsymbol{r} > 0$ when $\Delta \boldsymbol{r} \neq 0$, the increase of U by the displacement, i.e. by the deviation from the equilibrium state $\Delta \boldsymbol{r}$, decreases the kinetic energy K. In this state, one says that the equilibrium state $\boldsymbol{r}_{st}$ is stable. In the positive-semidefinite case, i.e. $\Delta \boldsymbol{r}^T HU(\boldsymbol{r}_{st}) \Delta \boldsymbol{r} \geq 0$ when $\Delta \boldsymbol{r} \neq 0$, the terms in the order of $O(|\Delta \boldsymbol{r}^3|)$ or much higher order are required to determine the stability. If the Hessian matrix is not positive-definite or positive-semidefinite, one says that the equilibrium state $\boldsymbol{r}_{st}$ is unstable. We will return to the discussion of stability in Section 2.2.

1.4 Stability of Mechanical Systems

We recall the equilibrium states of the three systems investigated in Section 1.1 and discuss their stability using the method introduced in Section 1.3.

1.4.1 Stability of Spring-Mass System

Let us calculate the potential energy of the spring mass system using Eq. (1.6). Knowing the external force $F = -F_s^\# = -kx$, the potential energy is

$$U = -\int F \, dx = -\int (-kx) dx = \frac{1}{2} kx^2, \tag{1.47}$$

where the integral constant is set to zero. The Hessian matrix of U at the equilibrium state $x = 0$ is k and is positive definite. The potential energy curve is schematically depicted as the well in Figure 1.8a (see Problem 1.4.1) and has concave feature, which is due to the positive stiffness ($k > 0$), i.e. the restoring characteristic of the spring.

Problem 1.4.1 The potential energy of the spring-mass system under the gravity effect in Figure 1.1 is directly calculated by the definition as

$$U = -\int (-kx' + mg) dx' = \frac{1}{2} kx'^2 - mgx', \tag{1.48}$$

where the integral constant is set to zero. By the coordinate transformation from x' to x, derive Eq. (1.47).

Ans:

$$\begin{aligned} U &= \frac{1}{2} kx'^2 - mgx' \\ &= \frac{1}{2} k(x_{st} + x)^2 - mg(x_{st} + x) \\ &= \frac{1}{2} kx^2 - \frac{1}{2} mgx_{st}. \end{aligned} \tag{1.49}$$

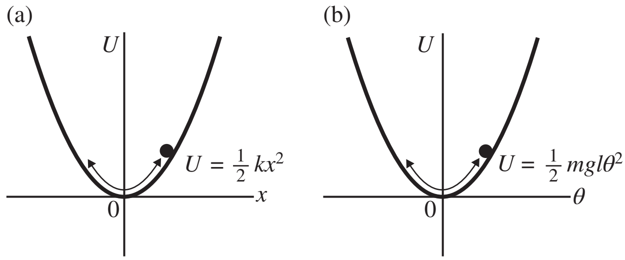


Figure 1.8 Potential energy curves. (a) Potential energy curve of the spring-mass system. (b) Potential energy curve of the simple pendulum in the neighborhood of the downward vertical equilibrium position.

Neglecting the constant second term yields Eq. (1.47).

By considering $\Delta x = x$, $\Delta y = 0$, and $\Delta z = 0$, Eq. (1.46) leads to

$$\Delta K = -\frac{1}{2}kx^2. \tag{1.50}$$

Because of positive k, as the deviation $|x(t)|$ from the equilibrium state $x = 0$ increases, the kinetic energy K decreases. The mass oscillates around the equilibrium state $x = 0$ as the motion of an imaginary particle on the potential energy curve in Figure 1.8a.

1.4.2 Stability of Magnetically Levitated System

Next, we examine Eq. (1.16) related to the motion of the magnetically levitated system. Also in this case, the potential energy U can be derived by the external force $F = -F_m^{\#} = k_m x$ as

$$U = -\int F\, dx = -\int k_m x\, dx = -\frac{1}{2}k_m x^2, \tag{1.51}$$

where the integral constant is set zero. The total energy

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}k_m x^2 \tag{1.52}$$

is conserved. Unlike the spring-mass system, because of the negative stiffness of magnetic suspension, i.e. $-k_m < 0$ corresponding to the negative slope in Figure 1.5b, the Hessian matrix of U at the equilibrium state $x = 0$ is $-k_m$ and negative definite.

As the deviation $|x(t)| \neq 0$ from the equilibrium state $x = 0$ increases, the kinetic energy increases. The absolute value of velocity increases monotonically as the motion of an imaginary particle on the potential energy curve in Figure 1.9a. Regardless of the magnitude of the initial deviation, the deviation $|\Delta x(t)|$ from the equilibrium state is not bounded. In other words, in the case when the position is changed downward from the equilibrium state due to a disturbance, the system continues going down and finally the levitated system falls to the guide rail. In the case when the position is changed upward from the equilibrium state due to a disturbance, the system continues going up and finally the electromagnet touches to the guide rail. Therefore, the system does not carry out the *stable levitation*. In practical systems, feedback control is equipped to stabilize the unstable equilibrium state as will be mentioned in Section 1.4.4.

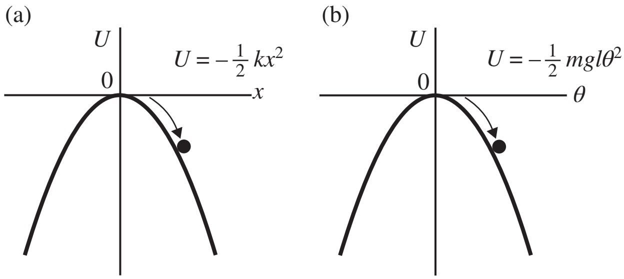


Figure 1.9 Potential energy curves. (a) Potential energy curve of the magnetically levitated system. (b) Potential energy curve of the simple pendulum in the neighborhood of the upright vertical equilibrium position.

1.4.3 Pendulum

The pendulum has two equilibrium states, i.e. the downward vertical and upright positions, $\theta_{st1} = 2n\pi$ and $\theta_{st2} = (2n+1)\pi$, respectively. The equations governing the motion near these equilibrium states are Eqs. (1.21) and (1.22), respectively. Considering the sign of the coefficient on their right-hand side, these equations are mathematically equivalent to Eqs. (1.6) and (1.16), respectively. The analogy with the governing equations of the spring-mass system and the magnetically levitated system leads to the determination of stability of the equilibrium states θ_{st1} and θ_{st2}. In fact, the potential energy curves are calculated in the neighborhood of $\theta_{st1} = 0$ and $\theta_{st2} = \pi$, respectively, as

$$U = -\int (-mg\theta)l\, d\theta = \frac{1}{2}mgl\theta^2, \tag{1.53}$$

$$U = -\int (mg\theta)l\, d\theta = -\frac{1}{2}mgl\theta^2, \tag{1.54}$$

where the integral constants are set to zero. The potential energy curves are shown as Figures 1.8b and 1.9b, respectively.

1.4.4 Stabilization Control of Magnetically Levitated System

As elucidated in Section 1.4.2, the equilibrium state of the magnetically levitated system is unstable. Let us propose a method to change the unstable equilibrium state into a stable one. For stabilization, the negative stiffness in the electromagnetic suspension needs to be turned into a positive one. To this end, we modify the current in the electromagnet depending on x as

$$i = i_0 + cx, \tag{1.55}$$

where c is a position feedback gain. Equation (1.12) is rewritten as

$$m\frac{d^2x}{dt^2} = -k\left(\frac{i_0 + cx}{x_{st} + x}\right)^2 + mg. \tag{1.56}$$

Under the condition of Eq. (1.55), the equilibrium state is $x' = x_{st} = \sqrt{\frac{ki_0^2}{mg}}$, i.e. $x = x' - x_{st} = 0$ is still the equilibrium state and Eq. (1.9) is still the equilibrium equation. Equation

(1.56) is linearized in the neighborhood of the equilibrium state $x = 0$ as

$$m\frac{d^2x}{dt^2} = -k_m\left(\frac{x_{st}c}{i_0} - 1\right)x + O(2), \tag{1.57}$$

where $k_m = 2k\frac{i_0^2}{x_{st}^3} > 0$, and $O(2)$ denotes higher order terms than the second order of $\frac{cx}{i_0}$ and $\frac{x}{x_{st}}$.

Problem 1.4.4 Derive Eq. (1.57).

Ans: Assuming $\left|\frac{x}{x_{st}}\right| \ll 1$ and demanding small cost of the feedback, i.e. $\left|\frac{cx}{i_0}\right| \ll 1$, Eq. (1.56) can be expanded into the power series with respect to $\frac{x}{x_{st}}$ and $\frac{cx}{i_0}$ as

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -k\left(\frac{i_0 + cx}{x_{st} + x}\right)^2 + mg \\
&= -k\frac{i_0^2}{x_{st}^2}\left(\frac{1 + (cx/i_0)}{1 + (x/x_{st})}\right)^2 + mg \\
&= -k\frac{i_0^2}{x_{st}^2}\left(1 + \frac{cx}{i_0}\right)^2\left(1 + \frac{x}{x_{st}}\right)^{-2} + mg \\
&= -k\frac{i_0^2}{x_{st}^2}\left(1 + 2\times\frac{cx}{i_0}\right)\left(1 - 2\times\frac{x}{x_{st}}\right) + mg + O(2) \\
&= -k_m\left(\frac{cx_{st}}{i_0} - 1\right)x + O(2).
\end{aligned} \tag{1.58}$$

The sign of the right-hand side determines whether the stiffness is negative or positive. If $k_m\left(\frac{cx_{st}}{i_0} - 1\right) > 0$, the stiffness is positive and the magnetic force expressed by the right-hand side in Eq. (1.57) acts as an ideal spring. Then, the equilibrium position $x = 0$ becomes stable. Therefore, by setting the feedback gain as $c > i_0/x_{st}$, we can stabilize the equilibrium state of the magnetically levitated vehicle.

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