1

Microwave Amplifier Fundamentals

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Microwave Amplifier and Active Circuit Design Using the Real Frequency Technique, First Edition.
Pierre Jarry and Jacques N. Beneat.

1.1 Introduction

In many high-speed applications, there is a need for microwave amplifier circuits. For example, satellite communications can be used when radio signals are blocked between two terrestrial transceiver stations as shown in Figure 1.1. The satellite then acts as a repeater, and the signal being repeated must be amplified before being sent back.

Important amplifier characteristics are center frequency and span of the pass band, gain, stability, input and output matching to the rest of the communication system, and noise figure [1].

At microwave frequencies, a common amplification component that has minimum noise is a field effect transistor (FET) as shown in Figure 1.2.

In this book, the FET will be typically modeled as a two-port network, where the input is on the gate and the output is on the drain. The source is mainly used for biasing of the transistor.

1.2 Scattering Parameters and Signal Flow Graphs

At high frequencies, voltages and currents are difficult to measure directly. However, scattering parameters determined from incident and reflected waves can be measured with resistive terminations. The scattering matrix of a two-port system provides relations between input and output reflected waves b_1 and b_2 and input and output incident waves a_1 and a_2 when the structure is

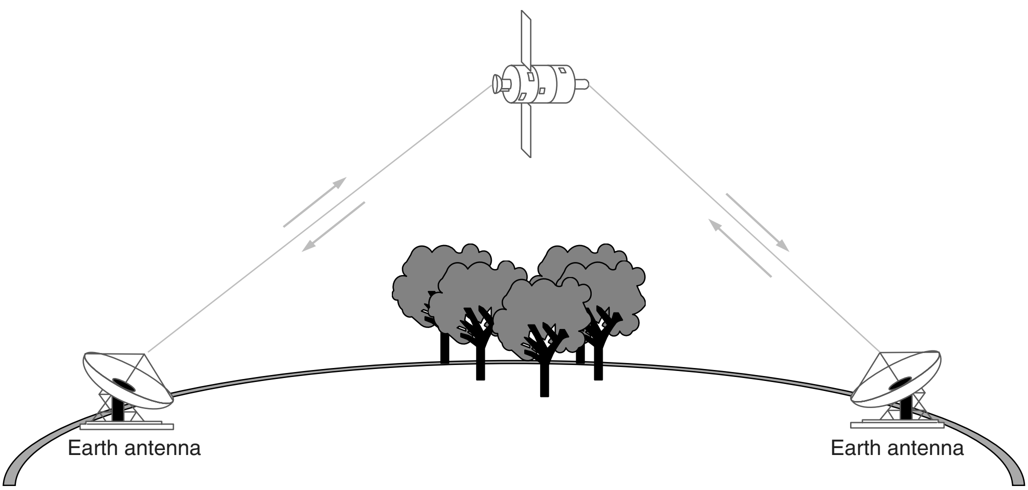

Figure 1.1 High-speed signals must be amplified in a satellite repeater.

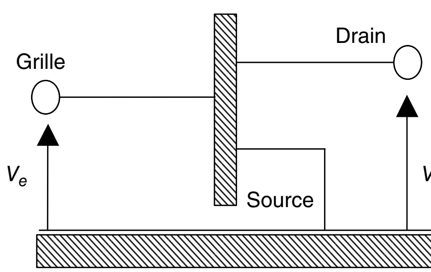

Figure 1.2 A FET modeled as a two-port network.

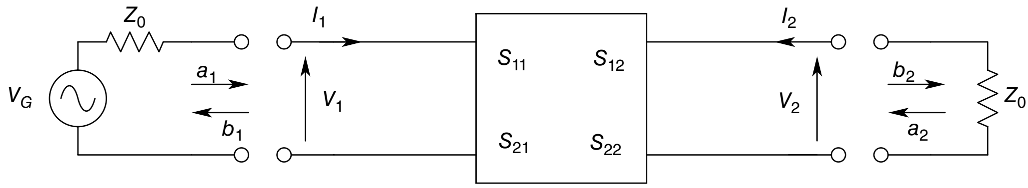

Figure 1.3 Scattering matrix of a two-port system terminated on characteristic impedance Z_0.

terminated on its characteristic impedance Z_0 as shown in Figure 1.3. Typically the reference source and load Z_0 used in commercial network analyzers is 50 Ω.

In the case of a two-port system, the equations relating incident and reflected waves and the scattering parameters are given by

$$\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 \\ b_2 &= S_{21}a_1 + S_{22}a_2 \end{aligned} \tag{1.1}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1.2}$$

The incident and reflected waves are related to the voltages and currents in Figure 1.3.

$$a_1 = \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}} \tag{1.3}$$

$$a_2 = \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} \tag{1.4}$$

$$b_1 = \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}} \tag{1.5}$$

$$b_2 = \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} \tag{1.6}$$

The parameter S_{11} is the input reflection coefficient and is the ratio of input reflected wave over input incident wave when the output incident wave is equal to zero. The output incident wave a_2 is equal to zero when the output of the system is connected to the characteristic impedance Z_0:

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \tag{1.7}$$

The parameter S_{21} is the forward transmission coefficient and is the ratio of the output reflected wave over the input incident wave when the output incident wave is equal to zero:

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \tag{1.8}$$

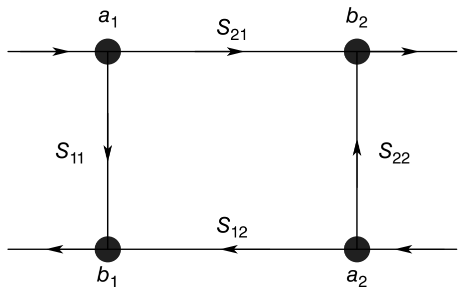

Figure 1.4 Signal flow graph representation of a two-port network.

The parameter S_{22} is the output reflection coefficient and is the ratio of the output reflected wave over the output incident wave when the input incident wave is equal to zero. The input incident wave a_1 is equal to zero when the input of the system is connected to the characteristic impedance Z_0:

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \tag{1.9}$$

The parameter S_{12} is the reverse transmission coefficient and is the ratio of the input reflected wave over the output incident wave when the input incident wave is equal to zero:

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \tag{1.10}$$

The two-port network and scattering parameters can be modeled using the signal flow graph representation of Figure 1.4.

A useful tool when defining system gains using signal flow graphs is the Mason gain formula [2]. It provides the gain T of a system between a source node and an output node:

$$T = \frac{\sum T_k \Delta_k}{\Delta} \tag{1.11}$$

with

$$\Delta = 1 - \sum L_i + \sum L_i L_j - \sum L_i L_j L_k \cdots$$

where

T_k is the gain of the kth forward path between the source node and the output node

$\sum L_i$ is the sum of all individual loop gains

$\sum L_i L_j$ is the sum of two loop gain products of any two nontouching loops

$\sum L_i L_j L_k$ is the sum of three loop gain products of any three nontouching loops

Δ_k is the part of Δ that does not touch the kth forward path

1.3 Reflection Coefficients

As shown in Figure 1.5, the input reflection coefficient when the output is connected to characteristic impedance Z_0 can be expressed in terms of the input impedance $Z_{IN} = V_1/I_1$ by replacing a_1 and b_1 by their expressions in terms of voltages and currents:

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{\dfrac{V_1 - Z_0 I_1}{2\sqrt{Z_0}}}{\dfrac{V_1 + Z_0 I_1}{2\sqrt{Z_0}}} = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0}$$

$$S_{11} = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} \tag{1.12}$$

The input impedance can be expressed in terms of the input reflection coefficient by

$$Z_{IN} = Z_0 \frac{1 + S_{11}}{1 - S_{11}} \tag{1.13}$$

Figure 1.6 defines additional reflection coefficients when the two-port is terminated on arbitrary loads Z_G and Z_L.

Reflection coefficient of the source:

$$\rho_G = \frac{a_1}{b_1} = \frac{Z_G - Z_0}{Z_G + Z_0} \tag{1.14}$$

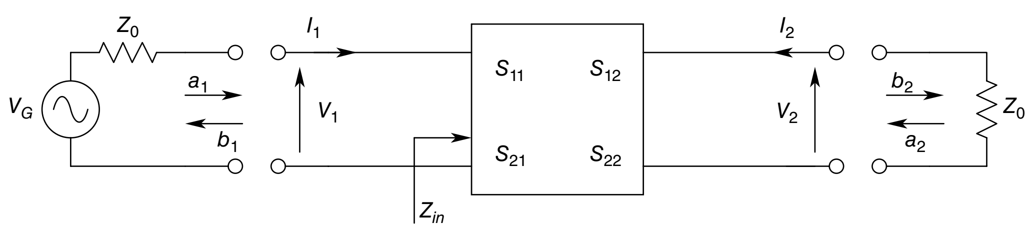

Figure 1.5 Input reflection coefficient and input impedance.

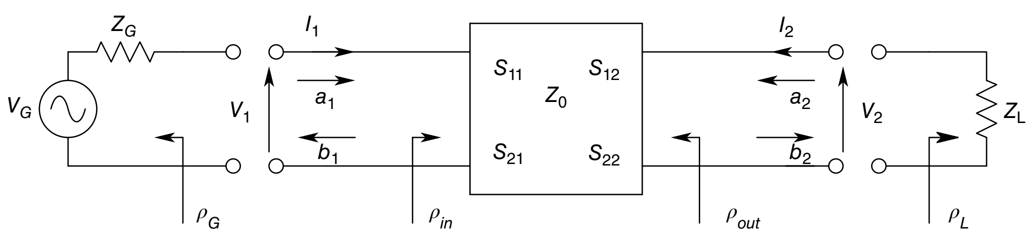

Figure 1.6 Reflection coefficients of a two-port when terminated on arbitrary loads.

Reflection coefficient of the load:

$$\rho_L = \frac{a_2}{b_2} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1.15}$$

Input reflection coefficient of the two-port when output loaded on ρ_L:

$$\rho_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L} \tag{1.16}$$

Output reflection coefficient of the two-port when input loaded on ρ_G:

$$\rho_{out} = \frac{b_2}{a_2} = S_{22} + \frac{S_{12}S_{21}\rho_G}{1 - S_{11}\rho_G} \tag{1.17}$$

For example, the expression of the input reflection coefficient when loaded on ρ_L is obtained by first using the general scattering parameter definition of the two-port:

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

Then, using the relation between incident and reflected waves $a_2 = \rho_L b_2$ gives

$$b_1 = S_{11}a_1 + S_{12}\rho_L b_2$$
$$b_2 = S_{21}a_1 + S_{22}\rho_L b_2$$

then from the second equation, $b_2(1 - S_{22}\rho_L) = S_{21}a_1$ and $b_2 = \frac{S_{21}}{(1 - S_{22}\rho_L)}a_1$, so that

$$b_1 = S_{11}a_1 + S_{12}\rho_L \frac{S_{21}}{1 - S_{22}\rho_L}a_1 = \left(S_{11} + \frac{S_{21}S_{12}\rho_L}{1 - S_{22}\rho_L}\right)a_1$$

and

$$\rho_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L}$$

The voltage standing wave ratio (VSWR) is given in terms of a reflection coefficient ρ by

$$\mathrm{VSWR} = \frac{1 + |\rho|}{1 - |\rho|} \tag{1.18}$$

The input VSWR for the two-port in Figure 1.6 is therefore

$$\mathrm{VSWR}_{IN} = \frac{1 + |\rho_{in}|}{1 - |\rho_{in}|} \tag{1.19}$$

The output VSWR for the two-port in Figure 1.6 is therefore

$$\text{VSWR}_{OUT} = \frac{1 + |\rho_{out}|}{1 - |\rho_{out}|} \tag{1.20}$$

1.4 Gain Expressions

Figure 1.7 shows the different reflection coefficients used to define various power gains.

The transducer power gain can be computed using the signal flow graph and the Mason gain formula as shown in Figure 1.8.

There is one forward path from node b_G to node b_2. The path gain of this path is $T_1 = 1 \times S_{21} = S_{21}$.

There are three individual loops: $\rho_G S_{11}$, $\rho_L S_{22}$, and $\rho_G S_{21} \rho_L S_{12}$.

This gives

$$\Delta = 1 - \sum L_i + \sum L_i L_j - \sum L_i L_j L_k \cdots = 1 - [S_{11}\rho_G + S_{22}\rho_L + S_{12}S_{21}\rho_G\rho_L] + [S_{11}\rho_G S_{22}\rho_L] - 0$$

and

$$T = \frac{\sum T_k \Delta_k}{\Delta} = \frac{T_1 \times (1-0)}{1 - S_{11}\rho_G - S_{22}\rho_L - S_{12}S_{21}\rho_G\rho_L + S_{11}\rho_G S_{22}\rho_L}$$

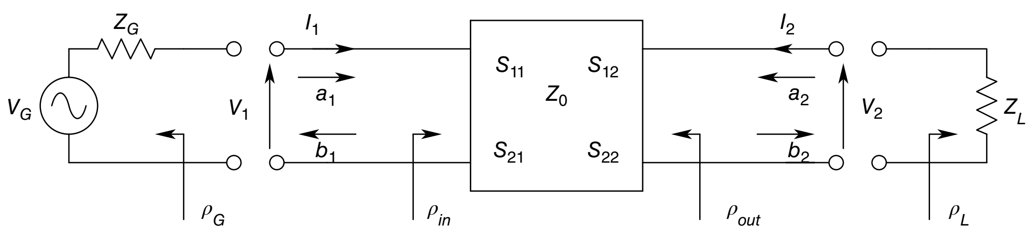

Figure 1.7 Gain definitions.

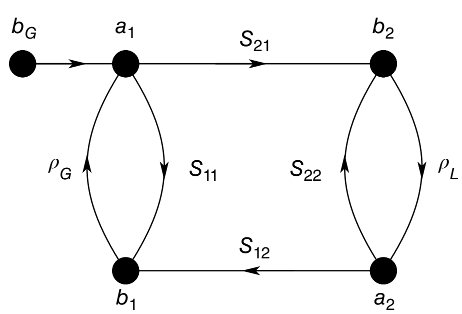

Figure 1.8 Signal flow graph representation for defining the gain.

$$T = \frac{S_{21}}{1 - S_{11}\rho_G - S_{22}\rho_L + S_{11}S_{22}\rho_G\rho_L - S_{12}S_{21}\rho_G\rho_L}$$

The transducer power gain is defined as

$$G_T = \frac{\text{power delivered to the load}}{\text{maximum available power from the source}}$$

so that

$$G_T = \frac{|b_2|^2\left(1 - |\rho_L|^2\right)}{\dfrac{|b_G|^2}{\left(1 - |\rho_G|^2\right)}} = |T|^2\left(1 - |\rho_G|^2\right)\left(1 - |\rho_L|^2\right)$$

and

$$G_T = \frac{\left(1 - |\rho_G|^2\right)\left(1 - |\rho_L|^2\right)|S_{21}|^2}{|1 - S_{11}\rho_G - S_{22}\rho_L + S_{11}S_{22}\rho_G\rho_L - S_{12}S_{21}\rho_G\rho_L|^2} \tag{1.21}$$

Note that G_T can also be written as

$$G_T = \frac{\left(1 - |\rho_G|^2\right)\left(1 - |\rho_L|^2\right)|S_{21}|^2}{|(1 - S_{11}\rho_G)(1 - S_{22}\rho_L) - S_{12}S_{21}\rho_G\rho_L|^2} \tag{1.22}$$

or as

$$G_T = \frac{\left(1 - |\rho_G|^2\right)\left(1 - |\rho_L|^2\right)|S_{21}|^2}{|1 - \rho_G\rho_{in}|^2|1 - S_{22}\rho_L|^2} \tag{1.23}$$

or as

$$G_T = \frac{\left(1 - |\rho_G|^2\right)\left(1 - |\rho_L|^2\right)|S_{21}|^2}{|1 - S_{11}\rho_G|^2|1 - \rho_L\rho_{out}|^2} \tag{1.24}$$

Note that when $S_{12} = 0$, the transducer power gain reduces to the unilateral transducer power gain G_{TU} given by [3]

$$G_{TU} = G_G G_0 G_L$$

where

$$G_G = \frac{1 - |\rho_G|^2}{|1 - S_{11}\rho_G|^2}, \quad G_0 = |S_{21}|^2 \quad \text{and} \quad G_L = \frac{1 - |\rho_L|^2}{|1 - S_{22}\rho_L|^2}$$

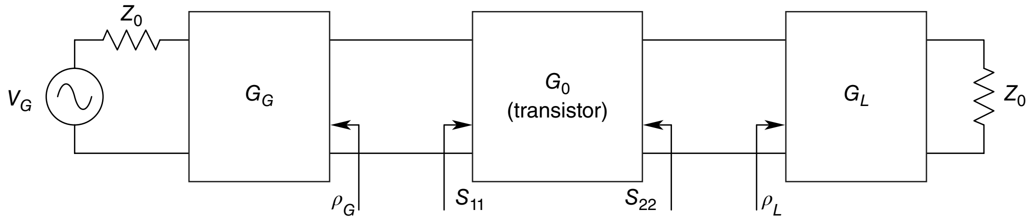

Figure 1.9 Representation in the case of a unilateral amplifier ($S_{12} = 0$).

In this case, G_G represents the losses in the source, G_0 is the intrinsic gain, and G_L represents the losses in the load and can be modeled as in Figure 1.9.

The maximum unilateral gain occurs when there is perfect matching of source and load impedances. For maximum unilateral gain, one would match the source to the input of the transistor by making $\rho_G = S_{11}^*$ and match the load to the output of the transistor by making $\rho_L = S_{22}^*$.

The maximum unilateral gain $G_{TU\ MAX}$ is then given by

$$G_{TU\ MAX} = \frac{1}{1-|S_{11}|^2}|S_{21}|^2\frac{1}{1-|S_{22}|^2} \tag{1.25}$$

The available power gain is defined as

$$G_A = \frac{\text{maximum power the amplifier can deliver to the load}}{\text{maximum available power from the source}}$$

and it is given by

$$G_A = \frac{\left(1-|\rho_G|^2\right)|S_{21}|^2}{|1-S_{11}\rho_G|^2\left(1-|\rho_{out}|^2\right)} \tag{1.26}$$

Note that when the input is perfectly matched then $Z_G = Z_0$ and $\rho_G = 0$ and

$$\rho_{out} = S_{22} + \frac{S_{12}S_{21}\rho_G}{1-S_{11}\rho_G} = S_{22}$$

so that the available power gain becomes

$$G_A = \frac{|S_{21}|^2}{1-|S_{22}|^2} \tag{1.27}$$

1.5 Stability

The stability of the amplifier depends on the scattering parameters of the transistor but also on the matching networks and terminations [4].

For the two-port shown in Figure 1.10, ρ_{in} is the input reflection coefficient of the transistor when output loaded on ρ_L and ρ_{out} is the output reflection coefficient of the transistor when input

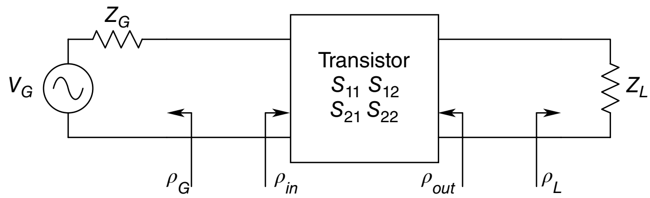

Figure 1.10 Stability of a two-port.

loaded on ρ_G. The system is said to be unconditionally stable if the amplitude of ρ_{in} and ρ_{out} are less than unity for all the real parts of load impedance Z_L and source impedance Z_G:

$$\forall Z_L(\text{with}\,\mathrm{Re}\{Z_L\}>0);\ |\rho_{in}|^2<1$$
$$\forall Z_G(\text{with}\,\mathrm{Re}\{Z_G\}>0);\ |\rho_{out}|^2<1$$

It can be shown that for unconditional stability one must satisfy three conditions:

$$\begin{cases} K>1 \\ B_1>0 \\ B_2>0 \end{cases} \tag{1.28}$$

where

$$K=\frac{1+|\Delta|^2-|S_{11}|^2-|S_{22}|^2}{2|S_{12}S_{21}|} \tag{1.29}$$

$$B_1=1-|S_{22}|^2-|S_{12}S_{21}| \tag{1.30}$$

$$B_2=1-|S_{11}|^2-|S_{12}S_{21}| \tag{1.31}$$

It is seen that these conditions only depend on the scattering parameters of the transistor. When these three conditions are met the amplifier can be connected to the loads without risk of becoming unstable and producing oscillations.

1.6 Noise

Figure 1.11 shows an active two-port between input impedance Z_G and load impedance Z_G.

The noise in the amplifier can be characterized by the noise figure F defined by

$$F=F_{\min}+\frac{R_n}{G_G}|Y_G-Y_{G\min}|^2 \tag{1.32}$$

where

$F_{\min}$ is the minimum noise figure obtained when $Y_G=Y_{G\min}$
R_n is the equivalent noise resistance of the active device

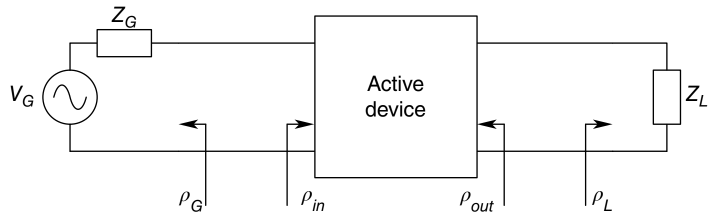

Figure 1.11 Active device and source and load impedances.

$Y_{G\,\min}$ is the source admittance that makes the noise figure minimum
Y_G is the source admittance such that $Y_G = G_G + jB_G$

The Section **1.7** consists of the rewritten noise figure formula in terms of reflection coefficients rather than in terms of admittances.

Referring to Figure 1.11, the reflection coefficient ρ_G from the source admittance is given by

$$\rho_G = \frac{Y_0 - Y_G}{Y_0 + Y_G} \tag{1.33}$$

where Y_0 is the characteristic admittance used.

This gives the source admittance in terms of the source reflection coefficient such that

$$Y_G = \frac{(1-\rho_G)}{(1+\rho_G)} Y_0 \tag{1.34}$$

In (1.32), taking $Y_G = Y_{G\min}$ makes the noise figure become minimum. This translates to a reflection coefficient $\rho_{G\,\min}$, where the noise figure is at a minimum such that

$$\rho_{G\min} = \frac{Y_0 - Y_{G\min}}{Y_0 + Y_{G\min}} \tag{1.35}$$

$$Y_{G\min} = \frac{(1-\rho_{G\min})}{(1+\rho_{G\min})} Y_0 \tag{1.36}$$

Then, replacing $Y_{G\,\min}$ and Y_G by their expressions in terms of ρ_G and $\rho_{G\,\min}$ gives us

$$\begin{aligned}|Y_G - Y_{G\min}|^2 &= {Y_0}^2 \left| \frac{(1-\rho_G)}{(1+\rho_G)} - \frac{(1-\rho_{G\min})}{(1+\rho_{G\min})} \right|^2 \\ &= {Y_0}^2 \left| \frac{1-\rho_G+\rho_{G\min}-\rho_G\rho_{G\min}-(1-\rho_{G\min}+\rho_G-\rho_G\rho_{G\min})}{(1+\rho_G)(1+\rho_{G\min})} \right|^2\end{aligned}$$

and

$$|Y_G - Y_{G\min}|^2 = {Y_0}^2 \left| \frac{2(\rho_{G\min}-\rho_G)}{(1+\rho_G)(1+\rho_{G\min})} \right|^2 = 4{Y_0}^2 \frac{|\rho_{G\min}-\rho_G|^2}{|1+\rho_G|^2|1+\rho_{G\min}|^2}$$

So that the noise figure can first be expressed as

$$F = F_{\min} + \frac{4R_n}{G_G} {Y_0}^2 \frac{|\rho_G - \rho_{G\min}|^2}{|1+\rho_G|^2 |1+\rho_{G\min}|^2}$$

Then, one expresses G_G as the real part of the source admittance in terms of reflection coefficients:

$$G_G = \mathrm{Re}\{Y_G\} = \frac{Y_G + {Y_G}^*}{2} = \frac{Y_0}{2}\left[\frac{(1-\rho_G)}{(1+\rho_G)} + \frac{(1-{\rho_G}^*)}{(1+{\rho_G}^*)}\right]$$
$$= \frac{Y_0}{2}\left[\frac{1-\rho_G+{\rho_G}^* - |\rho_G|^2 + 1 + \rho_G - {\rho_G}^* - |\rho_G|^2}{|1+\rho_G|^2}\right]$$

and

$$G_G = Y_0 \frac{1-|\rho_G|^2}{|1+\rho_G|^2}$$

so that

$$F = F_{\min} + 4R_n {Y_0}^2 \frac{1}{Y_0} \frac{|1+\rho_G|^2}{1-|\rho_G|^2} \frac{|\rho_G - \rho_{G\min}|^2}{|1+\rho_G|^2 |1+\rho_{G\min}|^2}$$

and the noise figure is given in terms of the source reflection parameter ρ_G and the optimum source reflection parameter $\rho_{G\min}$ by

$$F = F_{\min} + 4R_n Y_0 \frac{|\rho_G - \rho_{G\min}|^2}{\left(1-|\rho_G|^2\right)|1+\rho_{G\min}|^2} \tag{1.37}$$

Typically, the manufacturer provides the three parameters $\rho_{G\min}$, $F_{\min}$, and $r_n = R_n/R_0$, the normalized equivalent noise resistance. Note that the reflection coefficient $\rho_{G\min}$ is complex and is often given as magnitude and phase. Note that these parameters do change with frequency so they are provided in table form.

Next, we provide the noise figure corresponding to a cascade of active devices. In Figure 1.12, a first active device is characterized by a gain G_1 and noise figure F_1, and a second active device is characterized by a gain G_2 and noise figure F_2.

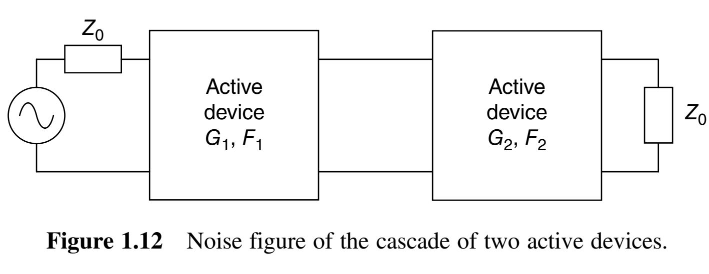

Figure 1.12 Noise figure of the cascade of two active devices.

It can be shown [5] that the noise figure F_2' corresponding to the cascade of the two systems is given by

$$F_2' = F_1 + \frac{1}{G_1}(F_2 - 1) \tag{1.38}$$

and the gain G_2' of the cascaded system is given by

$$G_2' = G_1 G_2 \tag{1.39}$$

The system is then placed in cascade with a third active device characterized by a gain G_3 and noise figure F_3 as shown in Figure 1.13.

Then, the noise figure F_3' corresponding to the cascade of the three systems is given by

$$F_3' = F_2' + \frac{1}{G_2'}(F_3 - 1)$$

and the gain G_3' of the cascaded system is given by

$$G_3' = G_2' G_3$$

We repeat this approach for the case of a cascade of k active devices as shown in Figure 1.14.

In this case, the noise figure F_k' corresponding to the cascade of the k systems is given by

$$F_k' = F_{k-1}' + \frac{1}{G_{k-1}'}(F_k - 1) \tag{1.40}$$

and the gain G_k' of the cascaded system is given by

$$G_k' = G_{k-1}' G_k \tag{1.41}$$

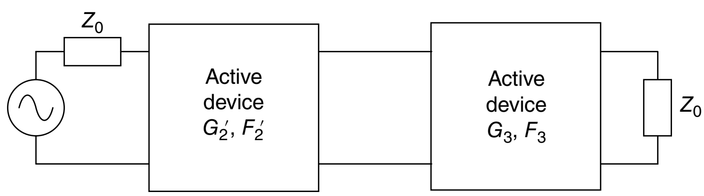

Figure 1.13 Noise figure of the cascade of three active devices.

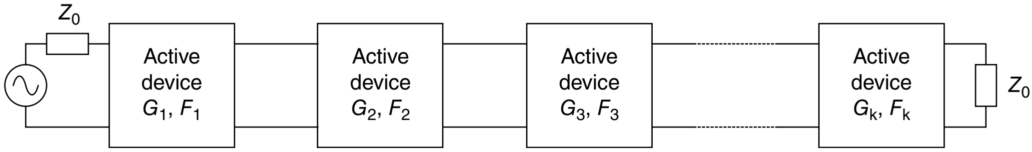

Figure 1.14 Noise figure of the cascade of k active devices.

where $k \geq 2$ with $F_1' = F_1$ and $G_1' = G_1$.

Note that this can also be written as

$$F_k' = F_1 + \frac{1}{G_1}(F_2 - 1) + \frac{1}{G_1 G_2}(F_3 - 1) + \cdots + \frac{1}{G_1 G_2 \cdots G_{k-1}}(F_k - 1) \tag{1.42}$$

1.7 *ABCD* Matrix

The *ABCD* matrix of a two-port is defined using the voltages and currents shown in Figure 1.15 [6].

The *ABCD* matrix shows the relation between input and output voltages and currents. It is given by

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \tag{1.43}$$

What follows are the *ABCD* matrices of several common elements and configurations found in microwave structures.

1.7.1 ABCD *Matrix of a Series Impedance*

Figure 1.16 shows the case of a two-port network made of impedance placed in series.

The equations of this system are

$$\begin{aligned} I_1 &= (-I_2) \\ V_1 &= V_2 + Z(-I_2) \end{aligned} \quad \text{or} \quad \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

Therefore, the *ABCD* matrix representing a two-port system made of impedance connected in series is given by

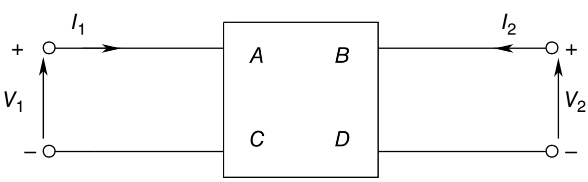

Figure 1.15 Notations for defining the *ABCD* matrix of a two-port system.

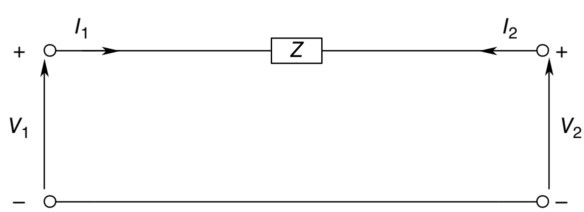

Figure 1.16 Impedance placed in series.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \tag{1.44}$$

1.7.2 ABCD *Matrix of a Parallel Admittance*

Figure 1.17 shows the case of a two-port network made of admittance placed in parallel.

The equations of this system are given by

$$\begin{aligned} V_1 &= V_2 \\ YV_2 &= I_1 + I_2 \end{aligned} \quad \text{or} \quad \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

Therefore, the *ABCD* matrix representing a two-port system made of admittance connected in parallel is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \tag{1.45}$$

1.7.3 Input Impedance of Impedance Loaded Two-Port

When a two-port is connected to load Z_L as shown in Figure 1.18, the output voltage V_2 and output current I_2 are such that $V_2 = Z_L(-I_2)$.

The equations of the system are given by

$$\begin{cases} V_1 = AV_2 + B(-I_2) = AZ_L(-I_2) + B(-I_2) \\ I_1 = CV_2 + D(-I_2) = CZ_L(-I_2) + D(-I_2) \end{cases}$$

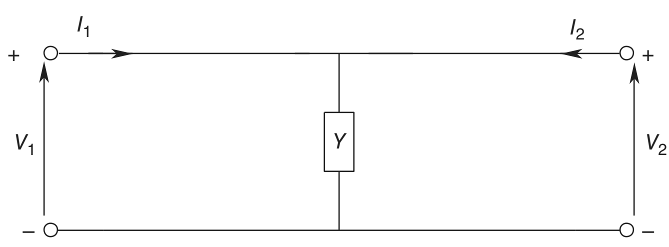

Figure 1.17 Admittance placed in parallel.

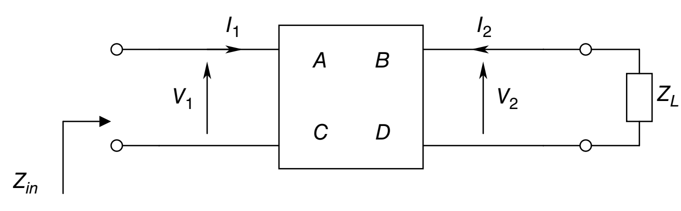

Figure 1.18 Two-port connected to load impedance Z_L.

from which it is straightforward to extract the input impedance of the two-port network:

$$Z_{IN} = \left.\frac{V_1}{I_1}\right|_{Z_L} = \frac{AZ_L + B}{CZ_L + D} \tag{1.46}$$

1.7.4 *Input Admittance of Admittance Loaded Two-Port*

When a two-port is connected to load admittance Y_L as shown in Figure 1.19, the output voltage V_2 and output current I_2 are such that $-I_2 = Y_L V_2$.

The equations of the system are given by

$$\begin{cases} V_1 = AV_2 + B(-I_2) = AV_2 + BY_L V_2 \\ I_1 = CV_2 + D(-I_2) = CV_2 + DY_L V_2 \end{cases}$$

from which it is straightforward to extract the input admittance of the two-port network:

$$Y_{IN} = \left.\frac{I_1}{V_1}\right|_{Y_L} = \frac{C + DY_L}{A + BY_L} \tag{1.47}$$

1.7.5 ABCD *Matrix of the Cascade of Two Systems*

One of the main advantages of the *ABCD* representation is that the *ABCD* matrix of the cascade of two systems as shown in Figure 1.20 is equal to the multiplication of the individual *ABCD* matrices.

The equations of this system are given by

$$\begin{cases} V_1 = A_1 V_2 + B_1(-I_2) \\ I_1 = C_1 V_2 + D_1(-I_2) \end{cases} \quad \text{and} \quad \begin{cases} V_1' = A_2 V_2' + B_2\left(-I_2'\right) \\ I_1' = C_2 V_2' + D_2\left(-I_2'\right) \end{cases}$$

In Figure 1.19, $V_2 = V_1'$ and $(-I_2) = I_1'$ so that the equations of the overall system are

$$\begin{cases} V_1 = A_1\left(A_2 V_2' + B_2\left(-I_2'\right)\right) + B_1\left(C_2 V_2' + D_2\left(-I_2'\right)\right) = (A_1 A_2 + B_1 C_2)V_2' + (A_1 B_2 + B_1 D_2)\left(-I_2'\right) \\ I_1 = C_1\left(A_2 V_2' + B_2\left(-I_2'\right)\right) + D_1\left(C_2 V_2' + D_2\left(-I_2'\right)\right) = (C_1 A_2 + D_1 C_2)V_2' + (C_1 B_2 + D_1 D_2)\left(-I_2'\right) \end{cases}$$

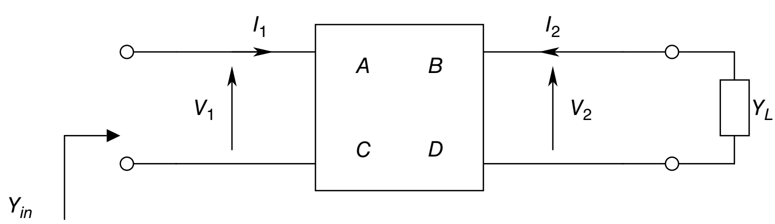

Figure 1.19 Two-port connected to load admittance Y_L.

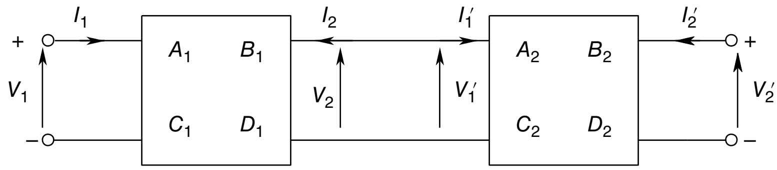

Figure 1.20 Cascade of two systems.

If we multiply the *ABCD* matrices representing the two systems, we get

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2+B_1C_2 & A_1B_2+B_1D_2 \\ C_1A_2+D_1C_2 & C_1B_2+D_1D_2 \end{pmatrix}$$

Since both techniques provide the same answers, the *ABCD* matrix of the cascade of two systems will be given by the multiplication of their *ABCD* matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2+B_1C_2 & A_1B_2+B_1D_2 \\ C_1A_2+D_1C_2 & C_1B_2+D_1D_2 \end{pmatrix} \tag{1.48}$$

1.7.6 ABCD *Matrix of the Parallel Connection of Two Systems*

In this case, the two-port networks are connected in parallel as shown in Figure 1.21.

The *ABCD* matrix representing two systems connected in parallel is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \dfrac{A_1B_2+B_1A_2}{B_1+B_2} & \dfrac{B_1B_2}{B_1+B_2} \\ C_1+C_2+\dfrac{(A_1-A_2)(D_2-D_1)}{B_1+B_2} & \dfrac{D_1B_2+B_1D_2}{B_1+B_2} \end{pmatrix} \tag{1.49}$$

1.7.7 ABCD *Matrix of the Series Connection of Two Systems*

In this case, the two-port networks are connected in series as shown in Figure 1.22.

The *ABCD* matrix representing two systems connected in series is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \dfrac{A_1C_2+C_1A_2}{C_1+C_2} & B_1+B_2+\dfrac{(A_1-A_2)(D_2-D_1)}{C_1+C_2} \\ \dfrac{C_1C_2}{C_1+C_2} & \dfrac{D_1C_2+C_1D_2}{C_1+C_2} \end{pmatrix} \tag{1.50}$$

1.7.8 ABCD *Matrix of Admittance Loaded Two-Port Connected in Parallel*

In this case, a two-port loaded on admittance Y_{L1} is connected in parallel to form the two-port of network as shown in Figure 1.23.

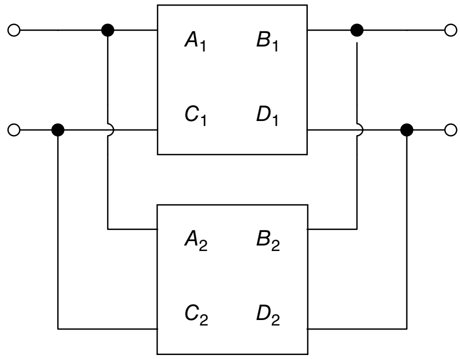

Figure 1.21 Two systems connected in parallel.

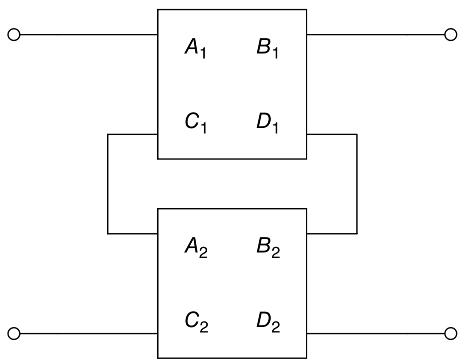

Figure 1.22 Two systems connected in series.

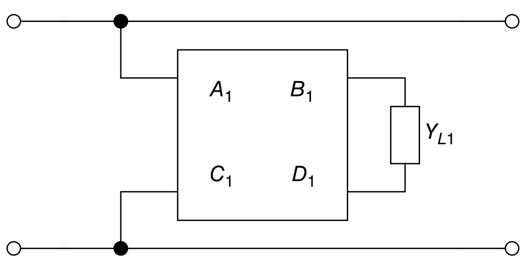

Figure 1.23 Two-port system made of admittance loaded two-port connected in parallel.

The *ABCD* matrix is that of an admittance in parallel and where the admittance is equal to the input admittance of the two-port loaded on Y_{L1}. It is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{C_1 + D_1 Y_{L1}}{A_1 + B_1 Y_{L1}} & 1 \end{pmatrix} \tag{1.51}$$

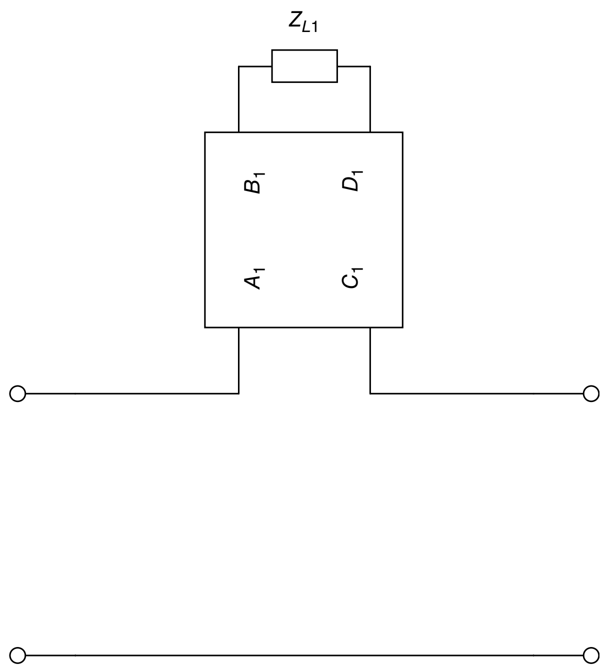

Figure 1.24 Two-port system made of impedance loaded two-port connected in series.

1.7.9 ABCD *Matrix of Impedance Loaded Two-Port Connected in Series*

In this case, a two-port loaded on impedance Z_{L1} is connected in series to form the two-port network as shown in Figure 1.24.

The *ABCD* matrix is that of an impedance in series and where the impedance is equal to the input impedance of the two-port loaded on Z_{L1}. It is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{A_1 Z_{L1} + B_1}{C_1 Z_{L1} + D_1} \\ 0 & 1 \end{pmatrix} \tag{1.52}$$

1.7.10 Conversion Between Scattering and ABCD *Matrices*

It is often needed to convert from scattering parameters to *ABCD* parameters and vice versa. Referring to Figure 1.3, the *ABCD* matrix of the two-port is given in terms of the scattering parameters by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \dfrac{(1+S_{11})(1-S_{22})+S_{21}S_{12}}{2S_{21}} & Z_0\dfrac{(1+S_{11})(1+S_{22})-S_{21}S_{12}}{2S_{21}} \\ Y_0\dfrac{(1-S_{11})(1-S_{22})-S_{21}S_{12}}{2S_{21}} & \dfrac{(1-S_{11})(1+S_{22})+S_{21}S_{12}}{2S_{21}} \end{pmatrix} \tag{1.53}$$

In return, the scattering matrix of the two-port can be expressed in terms of the *ABCD* parameters using the conversion formulas:

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{A+BY_0-CZ_0-D}{A+BY_0+CZ_0+D} & \dfrac{2(AD-BC)}{A+BY_0+CZ_0+D} \\ \dfrac{2}{A+BY_0+CZ_0+D} & \dfrac{-A+BY_0-CZ_0+D}{A+BY_0+CZ_0+D} \end{pmatrix} \tag{1.54}$$

1.8 Distributed Network Elements

For planar technologies, it is often more practical to obtain circuit layouts given in terms of distributed elements rather than given in terms of lumped elements. This section provides a background review of key aspects needed to better understand the subsequent chapters in this book.

1.8.1 Uniform Transmission Line

A uniform transmission line is a fundamental element when defining distributed networks. There are many ways to symbolize it, and in this book, we will generally represent it as shown in Figure 1.25.

The transmission line is defined by the characteristic impedance of the line Z_0, the propagation constant of the transmission line γ, and the physical length of the transmission line l.

The *ABCD* matrix of a uniform transmission line with physical length l is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cosh(\gamma l) & Z_0 \sinh(\gamma l) \\ \dfrac{\sinh(\gamma l)}{Z_0} & \cosh(\gamma l) \end{pmatrix} \tag{1.55}$$

Note that the propagation constant can be expressed in terms of the attenuation constant α and the phase constant β such that

$$\gamma = \alpha + j\beta \tag{1.56}$$

The electrical length is defined as the product of the phase constant and the physical length:

$$\theta = \beta l \tag{1.57}$$

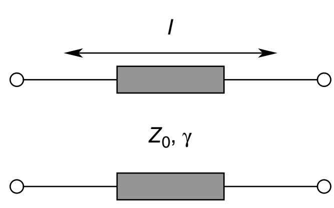

Figure 1.25 Uniform transmission line.

Since the phase rotates by 2π for one wavelength λ, the phase constant can be expressed as

$$\beta = \frac{2\pi}{\lambda} \tag{1.58}$$

The phase constant can also be expressed in terms of the operating radian frequency ω and the phase velocity v such that

$$\beta = \frac{\omega}{v} \tag{1.59}$$

For transmission lines that do not contain ferromagnetic material, for a TEM mode

$$v = \frac{c}{\sqrt{\varepsilon_r}} \tag{1.60}$$

where c is the speed of light and ε_r is the relative permittivity of the medium.

In the case of an ideal lossless transmission line, the attenuation constant α is equal to 0 and the *ABCD* matrix of the transmission line can be expressed in terms of the electrical length θ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\theta & jZ_0 \sin\theta \\ jY_0 \sin\theta & \cos\theta \end{pmatrix}$$

where $Y_0 = 1/Z_0$.

1.8.2 Unit Element

A unit element (UE) is a two-port made of a single section of a uniform lossless transmission line with a fixed length l and characteristic impedance Z_0 as shown in Figure 1.26.

The *ABCD* matrix of this two-port is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\theta & jZ_0 \sin\theta \\ jY_0 \sin\theta & \cos\theta \end{pmatrix} \tag{1.61}$$

where $\theta = \beta l$ is the electrical length of the transmission line.

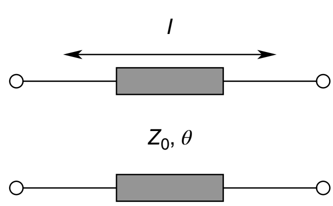

Figure 1.26 The unit element.

1.8.3 Input Impedance and Input Admittance

First, a UE is connected to load impedance Z_L as shown in Figure 1.27, and we are interested in the loaded input impedance of the UE.

The input impedance is given in terms of the load impedance Z_L and *ABCD* parameters of the two-port by

$$Z_{IN} = \left.\frac{V_1}{I_1}\right|_{Z_L} = \frac{AZ_L + B}{CZ_L + D}$$

with the *ABCD* parameters of the UE given by

$$\begin{pmatrix} \cos\theta & jZ_0 \sin\theta \\ jY_0 \sin\theta & \cos\theta \end{pmatrix}$$

so that

$$Z_{IN} = \frac{\cos\theta Z_L + jZ_0 \sin\theta}{jY_0 \sin\theta Z_L + \cos\theta} = \frac{\cos\theta(Z_L + jZ_0 \tan\theta)}{\cos\theta(1 + jY_0 \tan\theta Z_L)} = \frac{(Z_L + jZ_0 \tan\theta)}{Y_0(Z_0 + jZ_L \tan\theta)} = Z_0 \frac{Z_L + jZ_0 \tan\theta}{Z_0 + jZ_L \tan\theta}$$

and the input impedance of a UE loaded on impedance Z_L is given by

$$Z_{IN} = Z_0 \frac{Z_L + jZ_0 \tan\theta}{Z_0 + jZ_L \tan\theta} \tag{1.62}$$

Note that when $Z_L = Z_0$, then $Z_{IN} = Z_0$ regardless of the electrical length θ.

Second, a UE is connected to load admittance Y_L as shown in Figure 1.28, and we are interested in the loaded input admittance of the UE.

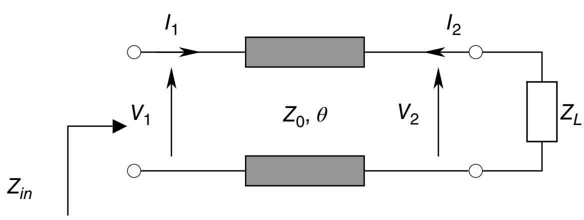

Figure 1.27 Input impedance of a UE loaded on impedance Z_L.

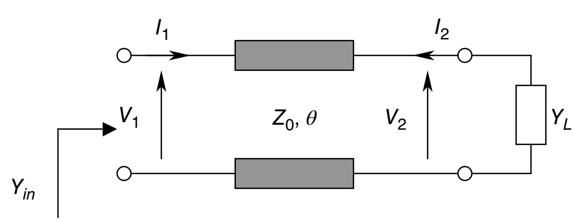

Figure 1.28 Input admittance of a UE loaded on impedance Y_L.

The input admittance is given in terms of the load admittance Y_L and *ABCD* parameters of the two-port by

$$Y_{IN} = \left.\frac{I_1}{V_1}\right|_{Y_L} = \frac{C + DY_L}{A + BY_L}$$

This gives

$$Y_{IN} = \frac{C + DY_L}{A + BY_L} = \frac{jY_0 \sin\theta + \cos\theta Y_L}{\cos\theta + jZ_0 \sin\theta Y_L} = \frac{\cos\theta(jY_0 \tan\theta + Y_L)}{\cos\theta(1 + jZ_0 \tan\theta Y_L)} = \frac{(jY_0 \tan\theta + Y_L)}{Z_0(Y_0 + j\tan\theta Y_L)} = Y_0\frac{jY_0 \tan\theta + Y_L}{Y_0 + j\tan\theta Y_L}$$

and the input admittance of a UE loaded on admittance Y_L is given by

$$Y_{IN} = Y_0\frac{Y_L + jY_0 \tan\theta}{Y_0 + j\tan\theta Y_L} \tag{1.63}$$

1.8.4 Short-Circuited Stub Placed in Series

In this case, we have a short-circuited stub that is placed in series to form the two-port of Figure 1.29.

In this case, the stub is loaded with impedance $Z_L = 0$ and the input impedance of the short-circuited stub is given by

$$Z_{IN} = Z_0\frac{0 + jZ_0 \tan\theta}{Z_0 + j0\tan\theta} = jZ_0 \tan\theta$$

and the *ABCD* matrix of the two-port system made of this short-circuited stub placed as a series impedance is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z_{IN} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & jZ_0 \tan\theta \\ 0 & 1 \end{pmatrix} \tag{1.64}$$

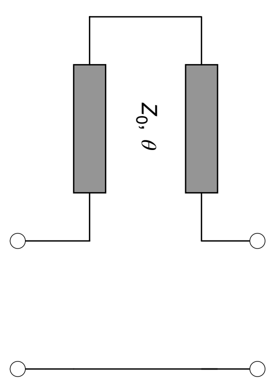

Figure 1.29 Short-circuited stub placed in series.

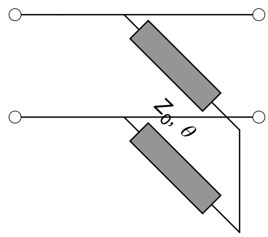

Figure 1.30 Short-circuited stub placed in parallel.

1.8.5 Short-Circuited Stub Placed in Parallel

In this case, we have a short-circuited stub that is placed in parallel to form the two-port of Figure 1.30.

In this case, the stub is loaded by admittance $Y_L = \infty$ and the input admittance of the short-circuited stub is given by

$$Y_{IN} = Y_0 \left.\frac{Y_L + jY_0 \tan\theta}{Y_0 + j\tan\theta Y_L}\right|_{Y_L \to \infty} = Y_0 \frac{Y_L}{j\tan\theta Y_L} = \frac{Y_0}{j\tan\theta}$$

and the *ABCD* matrix of the two-port system made of this short-circuited stub placed as a admittance in parallel is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y_{IN} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{Y_0}{j\tan\theta} & 1 \end{pmatrix} \tag{1.65}$$

1.8.6 Open-Circuited Stub Placed in Series

In this case, we have an open-circuited stub that is placed in series to form the two-port of Figure 1.31.

In this case, the stub is loaded by impedance $Z_L = \infty$ and the input impedance of the open-circuited stub is given by

$$Z_{IN} = Z_0 \left.\frac{Z_L + jZ_0 \tan\theta}{Z_0 + jZ_L \tan\theta}\right|_{Z_L = \infty} = Z_0 \frac{Z_L}{jZ_L \tan\theta} = \frac{Z_0}{j\tan\theta}$$

and the *ABCD* matrix of the two-port system made of this open-circuited stub placed as a series impedance is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z_{IN} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{Z_0}{j\tan\theta} \\ 0 & 1 \end{pmatrix} \tag{1.66}$$

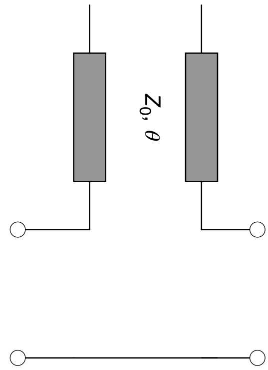

Figure 1.31 Open-circuited stub placed in series.

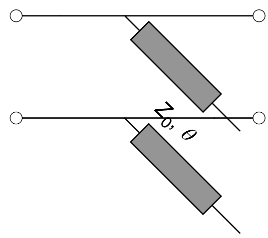

Figure 1.32 Open-circuited stub placed in parallel.

1.8.7 Open-Circuited Stub Placed in Parallel

In this case we have an open-circuited stub that is placed in parallel to form the two-port of Figure 1.32.

In this case the stub is loaded with admittance $Y_L = 0$ and the input admittance of the open-circuited stub is given by

$$Y_{IN} = Y_0 \frac{0 + jY_0 \tan\theta}{Y_0 + j\tan\theta 0} = jY_0 \tan\theta$$

and the *ABCD* matrix of the two-port system made of this open-circuited stub placed as an admittance in parallel is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y_{IN} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ jY_0 \tan\theta & 1 \end{pmatrix} \tag{1.67}$$

1.8.8 Richard's Transformation

It is not always easy to see the frequency dependency in the *ABCD* matrix of a UE when expressed in terms of electrical length θ. When the length of the transmission line is $\lambda/4$, there are special properties that can be exploited for distributed network design and synthesis.

The Richard's variable t is defined as the hyperbolic tangent of the delay τ_0 and the Laplace variable s as shown below [7]:

$$t = \tanh(\tau_0 s) \tag{1.68}$$

In (1.68), τ_0 is the delay of the UE and is proportional to the physical length l of the UE and the phase velocity v such that

$$\tau_0 = \frac{l}{v} \tag{1.69}$$

The UEs are assumed to be lossless so that $s \rightarrow j\omega$ and

$$t = \tanh(j\tau_0\omega) = j\tan(\tau_0\omega) = j\tan\left(\frac{\omega}{v}l\right) = j\tan(\beta l) = j\tan\theta = j\Omega \tag{1.70}$$

For example, in the case of quarter-wavelength UEs, we have

$$l = \frac{\lambda_0}{4} \tag{1.71}$$

where λ_0 is the wavelength given by the ratio of the velocity in the medium divided by a normalization frequency f_0 such that

$$\lambda_0 = \frac{v}{f_0} \tag{1.72}$$

so that

$$\tau_0 = \frac{\frac{1}{4}\lambda_0}{v} = \frac{\frac{1}{4}\frac{v}{f_0}}{v} = \frac{1}{4}\frac{1}{f_0} = \frac{1}{4}\frac{2\pi}{\omega_0}$$

$$\tau_0 = \frac{\pi}{2}\frac{1}{\omega_0}$$

where ω_0 is the normalization radian frequency.

So that in the case of quarter-wavelength UEs, we have

$$\Omega = \tan(\tau_0\omega) = \tan\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right) \tag{1.73}$$

1.8.8.1 *ABCD* Matrix of a UE in terms of Richard's Variable

The *ABCD* matrix of a general UE is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\theta & jZ_0\sin\theta \\ jY_0\sin\theta & \cos\theta \end{pmatrix}$$

and can be rewritten as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \cos\theta \begin{pmatrix} 1 & jZ_0 \tan\theta \\ jY_0 \tan\theta & 1 \end{pmatrix}$$

and since $t = j\tan\theta$, this matrix can be expressed in terms of the Richard's variable t to give the *ABCD* matrix of a UE to be

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\sqrt{1-t^2}} \begin{pmatrix} 1 & tZ_0 \\ tY_0 & 1 \end{pmatrix} \tag{1.74}$$

1.8.8.2 *ABCD* Matrices in Terms of Richard's Variable

The input impedance Z_{IN} of a quarter-wavelength UE loaded on impedance Z_L (see Fig. 1.27) can be expressed in terms of the Richard's variable by

$$Z_{IN} = Z_0 \frac{Z_L + jZ_0 \tan\theta}{Z_0 + jZ_L \tan\theta} = Z_0 \frac{Z_L + tZ_0}{Z_0 + tZ_L} \tag{1.75}$$

The input admittance Y_{IN} of a quarter-wavelength UE loaded on impedance Y_L (see Fig. 1.28) can be expressed in terms of the Richard's variable by

$$Y_{IN} = Y_0 \frac{Y_L + jY_0 \tan\theta}{Y_0 + j\tan\theta Y_L} = Y_0 \frac{Y_L + tY_0}{Y_0 + tY_L} \tag{1.76}$$

The *ABCD* matrix of a two-port system made of a short-circuited stub placed as series impedance (see Fig. 1.29) can be expressed in terms of the Richard's variable by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z_{IN} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & jZ_0 \tan\theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tZ_0 \\ 0 & 1 \end{pmatrix} \tag{1.77}$$

The *ABCD* matrix of a two-port system made of a short-circuited stub placed as parallel admittance (see Fig. 1.30) can be expressed in terms of the Richard's variable by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y_{IN} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{Y_0}{j\tan\theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{Y_0}{t} & 1 \end{pmatrix} \tag{1.78}$$

The *ABCD* matrix of a two-port system made of an open-circuited stub placed as series impedance (see Fig. 1.31) can be expressed in terms of the Richard's variable by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z_{IN} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{Z_0}{j\tan\theta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{Z_0}{t} \\ 0 & 1 \end{pmatrix} \tag{1.79}$$

The *ABCD* matrix of a two-port system made of an open-circuited stub placed as parallel admittance (see Fig. 1.32) can be expressed in terms of the Richard's variable by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y_{IN} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ jY_0 \tan\theta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ tY_0 & 1 \end{pmatrix} \tag{1.80}$$

1.8.9 Kuroda Identities

There are situations when a synthesis technique can lead to the direct cascade of stubs. This situation is often impossible to replicate in an actual circuit. There are also times where one might prefer one type of stub to the other. In these situations, one can transform a given set of stub configurations into another set of stub configurations. In this subsection, we recall the four classic Kuroda identities that can help solve these problems.

1.8.9.1 First Low-Pass Kuroda Identity

The first low-pass Kuroda identity consists of a UE with characteristic impedance Z_{01} placed left of a series short-circuited stub with characteristic impedance Z_{02}. This is equivalent to a UE with characteristic impedance Z_{04} placed right of a parallel open-circuited stub with characteristic impedance Z_{03} as shown in Figure 1.33 [7].

The two systems are equivalent if

$$\begin{cases} Z_{04} = Z_{01} + Z_{02} \\ Z_{03} = \dfrac{Z_{01}}{Z_{02}}(Z_{01} + Z_{02}) \end{cases} \tag{1.81}$$

The proof is obtained by equating the *ABCD* matrix of the system on the left

$$\begin{pmatrix} \cos\theta & jZ_{01}\sin\theta \\ jY_{01}\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & jZ_{02}\tan\theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & jZ_{01}\sin\theta + jZ_{02}\tan\theta\cos\theta \\ jY_{01}\sin\theta & \cos\theta + jY_{01}\sin\theta jZ_{02}\tan\theta \end{pmatrix}$$

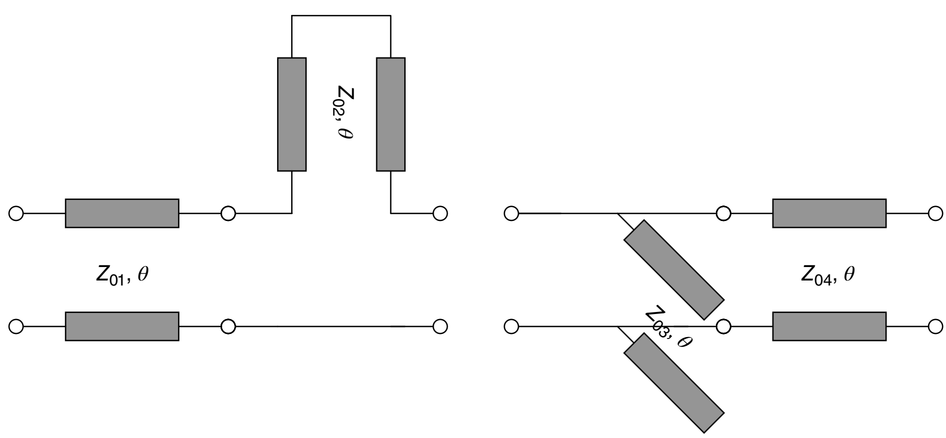

Figure 1.33 First low-pass Kuroda identity.

to the *ABCD* matrix of the system on the right:

$$\begin{pmatrix} 1 & 0 \\ jY_{03}\tan\theta & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & jZ_{04}\sin\theta \\ jY_{04}\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & jZ_{04}\sin\theta \\ jY_{04}\sin\theta + jY_{03}\tan\theta\cos\theta & \cos\theta + jY_{03}\tan\theta jZ_{04}\sin\theta \end{pmatrix}$$

Equating *B* parameters gives

$$jZ_{01}\sin\theta + jZ_{02}\tan\theta\cos\theta = jZ_{04}\sin\theta$$

and dividing both sides by $\sin\theta$ gives

$$Z_{01} + Z_{02} = Z_{04}$$

Equating *C* parameters gives

$$jY_{01}\sin\theta = jY_{04}\sin\theta + jY_{03}\tan\theta\cos\theta$$

and dividing both sides by $\sin\theta$ gives

$$Y_{01} = Y_{04} + Y_{03}$$

which can be expressed in terms of impedances by

$$Z_{01} = \frac{Z_{03}Z_{04}}{Z_{03} + Z_{04}}$$

which can also be expressed as

$$Z_{03} = \frac{Z_{01}Z_{04}}{Z_{04} - Z_{01}}$$

and replacing Z_{04} by $Z_{01} + Z_{02}$ gives

$$Z_{03} = \frac{Z_{01}(Z_{01} + Z_{02})}{Z_{01} + Z_{02} - Z_{01}} = \frac{Z_{01}}{Z_{02}}(Z_{01} + Z_{02})$$

1.8.9.2 Second Low-Pass Kuroda Identity

The second low-pass Kuroda identity consists of a UE with characteristic impedance Z_{02} placed right of a series short-circuited stub with characteristic impedance Z_{01}. This is equivalent to a UE with characteristic impedance Z_{03} placed left of a parallel open-circuited stub with characteristic impedance Z_{04} as shown in Figure 1.34.

The two systems are equivalent if

$$\begin{cases} Z_{03} = Z_{01} + Z_{02} \\ Z_{04} = \dfrac{Z_{02}}{Z_{01}}(Z_{01} + Z_{02}) \end{cases} \tag{1.82}$$

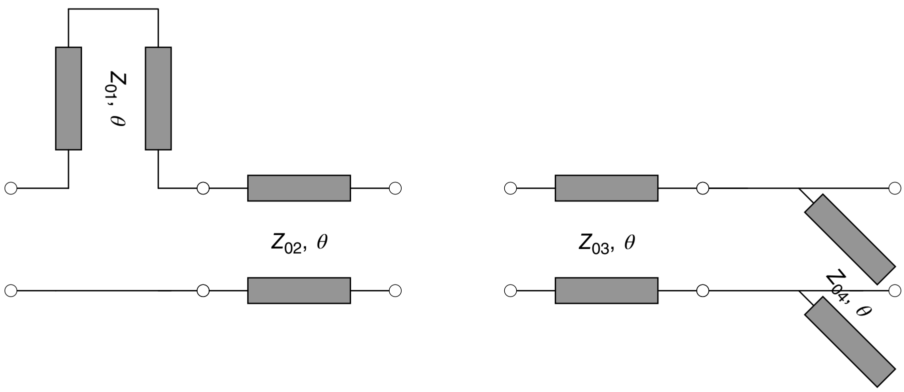

Figure 1.34 Second low-pass Kuroda identity.

The proof is obtained by equating the *ABCD* matrix of the system on the left

$$\begin{pmatrix} 1 & jZ_{01}\tan\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & jZ_{02}\sin\theta \\ jY_{02}\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + jY_{02}\sin\theta jZ_{01}\tan\theta & jZ_{02}\sin\theta + jZ_{01}\tan\theta\cos\theta \\ jY_{02}\sin\theta & \cos\theta \end{pmatrix}$$

to the *ABCD* matrix of the system on the right:

$$\begin{pmatrix} \cos\theta & jZ_{03}\sin\theta \\ jY_{03}\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ jY_{04}\tan\theta & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + jY_{04}\tan\theta jZ_{03}\sin\theta & jZ_{03}\sin\theta \\ jY_{03}\sin\theta + jY_{04}\tan\theta\cos\theta & \cos\theta \end{pmatrix}$$

Equating *B* parameters gives

$$jZ_{02}\sin\theta + jZ_{01}\tan\theta\cos\theta = jZ_{03}\sin\theta$$

and dividing both sides by $\sin\theta$ gives

$$Z_{02} + Z_{01} = Z_{03}$$

Equating *C* parameters gives

$$jY_{02}\sin\theta = jY_{03}\sin\theta + jY_{04}\tan\theta\cos\theta$$

and dividing both sides by $\sin\theta$ gives

$$Y_{02} = Y_{03} + Y_{04}$$

which can be expressed in terms of impedances by

$$Z_{02} = \frac{Z_{03}Z_{04}}{Z_{03} + Z_{04}}$$

which can also be expressed as

$$Z_{04} = \frac{Z_{02}Z_{03}}{Z_{03} - Z_{02}}$$

and replacing Z_{03} by $Z_{01} + Z_{02}$ gives

$$Z_{04} = \frac{Z_{02}(Z_{01} + Z_{02})}{Z_{01} + Z_{02} - Z_{02}} = \frac{Z_{02}}{Z_{01}}(Z_{01} + Z_{02})$$

1.8.9.3 First High-Pass Kuroda Identity

The first high-pass Kuroda identity consists of a UE with characteristic impedance Z_{02} placed right of a parallel short-circuited stub with characteristic impedance Z_{01}. This is equivalent to a UE with characteristic impedance Z_{03} placed left of a parallel short-circuited stub with characteristic impedance Z_{04} followed by an ideal transformer n as shown in Figure 1.35.

The two systems are equivalent if

$$\begin{cases} n = 1 + \dfrac{Z_{02}}{Z_{01}} \\ Z_{03} = \dfrac{Z_{01}Z_{02}}{Z_{01} + Z_{02}} \\ Z_{04} = \dfrac{{Z_{01}}^2}{Z_{01} + Z_{02}} \end{cases} \tag{1.83}$$

The proof is obtained by equating the *ABCD* matrix of the system on the left

$$\begin{pmatrix} 1 & 0 \\ \dfrac{Y_{01}}{j\tan\theta} & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & jZ_{02}\sin\theta \\ jY_{02}\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & jZ_{02}\sin\theta \\ jY_{02}\sin\theta - jY_{01}\dfrac{\cos^2\theta}{\sin\theta} & \cos\theta + Z_{02}Y_{01}\cos\theta \end{pmatrix}$$

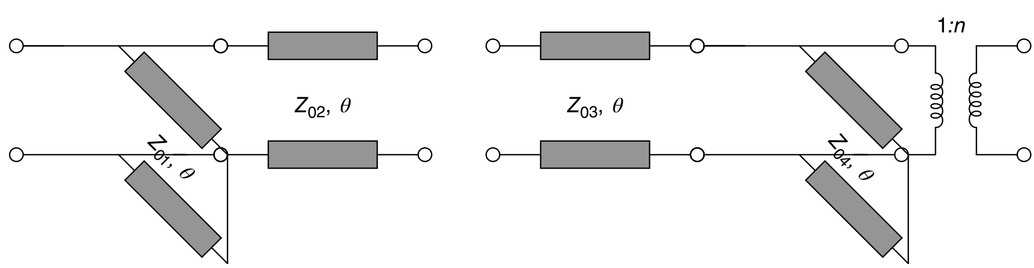

Figure 1.35 First high-pass Kuroda identity.

to the *ABCD* matrix of the system on the right:

$$\begin{pmatrix} \cos\theta & jZ_{03}\sin\theta \\ jY_{03}\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{Y_{04}}{j\tan\theta} & 1 \end{pmatrix}\begin{pmatrix} \dfrac{1}{n} & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} \dfrac{\cos\theta + Z_{03}Y_{04}\cos\theta}{n} & jnZ_{03}\sin\theta \\ \dfrac{jY_{03}\sin\theta - jY_{04}\dfrac{\cos^2\theta}{\sin\theta}}{n} & n\cos\theta \end{pmatrix}$$

Equating *D* parameters gives

$$\cos\theta + Z_{02}Y_{01}\cos\theta = n\cos\theta$$

and dividing both sides by $\cos\theta$ gives

$$1 + Z_{02}Y_{01} = n \quad \text{or} \quad n = 1 + \frac{Z_{02}}{Z_{01}}$$

Equating *B* parameters gives

$$jZ_{02}\sin\theta = jnZ_{03}\sin\theta$$

and dividing both sides by $\sin\theta$ gives

$$jZ_{02} = jnZ_{03}$$

then replacing n gives

$$Z_{03} = \frac{Z_{02}}{n} = \frac{Z_{02}}{1 + \dfrac{Z_{02}}{Z_{01}}} = \frac{Z_{01}Z_{02}}{Z_{01} + Z_{02}}$$

Equating *A* parameters gives

$$\cos\theta = \frac{\cos\theta + Z_{03}Y_{04}\cos\theta}{n}$$

and dividing both sides by $\cos\theta$ gives

$$1 = \frac{1 + Z_{03}Y_{04}}{n}$$

from which we find that

$$1 + \frac{Z_{03}}{Z_{04}} = n = 1 + \frac{Z_{02}}{Z_{01}} \quad \text{or} \quad Z_{04} = \frac{Z_{01}}{Z_{02}}Z_{03}$$

then replacing Z_{03} gives

$$Z_{04} = \frac{Z_{01}}{Z_{02}} \frac{Z_{01} Z_{02}}{Z_{01} + Z_{02}} = \frac{Z_{01}^{\ 2}}{Z_{01} + Z_{02}}$$

1.8.9.4 Second High-Pass Kuroda Identity

The second high-pass Kuroda identity consists of a UE with characteristic impedance Z_{02} placed right of a series open-circuited stub with characteristic impedance Z_{01}. This is equivalent to a UE with characteristic impedance Z_{03} placed left of a series open-circuited stub with characteristic impedance Z_{04} followed by an ideal transformer n as shown in Figure 1.36.

The two systems are equivalent if

$$\begin{cases} \dfrac{1}{n} = 1 + \dfrac{Z_{01}}{Z_{02}} \\ Z_{03} = Z_{01} + Z_{02} \\ Z_{04} = \dfrac{Z_{01}}{Z_{02}}(Z_{01} + Z_{02}) \end{cases} \tag{1.84}$$

The proof is obtained by equating the *ABCD* matrix of the system on the left

$$\begin{pmatrix} 1 & \dfrac{Z_{01}}{j\tan\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & jZ_{02}\sin\theta \\ jY_{02}\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + Z_{01}Y_{02}\cos\theta & jZ_{02}\sin\theta - jZ_{01}\dfrac{\cos^2\theta}{\sin\theta} \\ jY_{02}\sin\theta & \cos\theta \end{pmatrix}$$

to the *ABCD* matrix of the system on the right:

$$\begin{pmatrix} \cos\theta & jZ_{03}\sin\theta \\ jY_{03}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & \dfrac{Z_{04}}{j\tan\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{n} & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} \dfrac{\cos\theta}{n} & n\left(jZ_{03}\sin\theta - jZ_{04}\dfrac{\cos^2\theta}{\sin\theta}\right) \\ \dfrac{jY_{03}\sin\theta}{n} & n(\cos\theta + Z_{04}Y_{03}\cos\theta) \end{pmatrix}$$

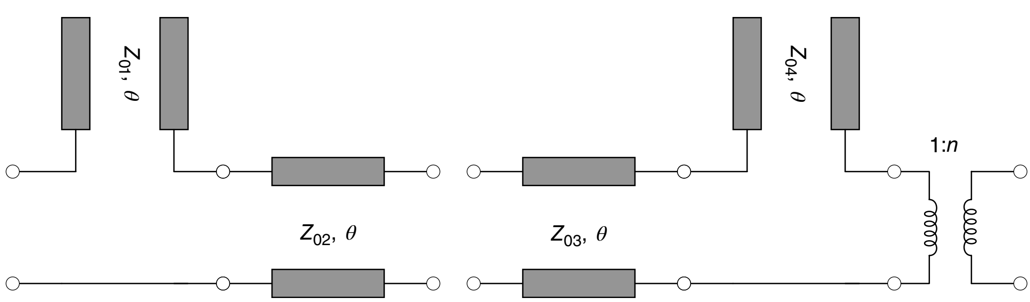

Figure 1.36 Second high-pass Kuroda identity.

Equating A parameters gives

$$\cos\theta + Z_{01}Y_{02}\cos\theta = \frac{\cos\theta}{n}$$

and dividing both sides by $\cos\theta$ gives

$$1 + Z_{01}Y_{02} = \frac{1}{n} \quad \text{or} \quad \frac{1}{n} = 1 + \frac{Z_{01}}{Z_{02}}$$

Equating C parameters gives

$$jY_{02}\sin\theta = \frac{jY_{03}\sin\theta}{n}$$

and dividing both sides by $\sin\theta$ gives

$$jY_{02} = \frac{jY_{03}}{n} \quad \text{or} \quad Z_{03} = \frac{1}{n}Z_{02}$$

and replacing n gives

$$Z_{03} = \left(1 + \frac{Z_{01}}{Z_{02}}\right)Z_{02} = Z_{01} + Z_{02}$$

Equating D parameters gives

$$\cos\theta = n(\cos\theta + Z_{04}Y_{03}\cos\theta)$$

and dividing both sides by $\cos\theta$ gives

$$1 = n(1 + Z_{04}Y_{03})$$

from which we find that

$$1 + \frac{Z_{04}}{Z_{03}} = \frac{1}{n} = 1 + \frac{Z_{01}}{Z_{02}} \quad \text{or} \quad Z_{04} = \frac{Z_{01}}{Z_{02}}Z_{03}$$

and replacing Z_{03} gives

$$Z_{04} = \frac{Z_{01}}{Z_{02}}Z_{03} = \frac{Z_{01}}{Z_{02}}(Z_{01} + Z_{02})$$

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