Chapter 1

Moving towards Mathematical Mastery

In This Chapter

- Understanding the overlap with GCSE
- Doing advanced algebra
- Building on geometry
- Diving into calculus

It's a big step up from GCSE to A level – especially if you're coming in with a B or a marginal A. The pace is pretty frenetic, and there's a fair amount of A and A* material from GCSE that's assumed knowledge at A level. If you're not especially happy about algebraic fractions or sketching curves, for example, you're likely to have a bit of catching up to do.

Luckily, this book has a whole part devoted to catching up with the top end of GCSE, as well as the stuff you'll need to learn completely fresh. In this chapter, I note where the content overlaps with GCSE and introduce you to A level algebra, trigonometry and calculus.

Reviewing GCSE

The good news is that if you've got a solid understanding of everything in your GCSE, quite a lot of Core 1 and a fair amount of Core 2 will be old news to you. Possibly less good news is that if you've got gaps in your knowledge, you need to fill them in pretty sharpish.

The four key areas where there's an overlap between the two qualifications are algebra, graphs and powers (all in Core 1) and trigonometry (which comes up a lot in Core 2). There are other bits and pieces, too – your arithmetic needs to be pretty decent in Core 1, where you don't have a calculator, and parts of Core 2 are likely to test your knowledge of shapes other than triangles – but generally speaking, these are the big four. The first part of this book is all about making sure you're up to speed with them.

Setting up for study success

Forgive me if you think it's patronising to tell you how to study – after all, you must have done pretty well with exams to get this far. I go into studying because A level is a much tougher beast than GCSE – it's possible for a reasonably smart student to coast through GCSE and get a good grade without needing to work too hard; by contrast, it's unusual to see someone glide through A level. And presumably, if you were finding it straightforward, you wouldn't be buying books like this to help you through it.

In this kind of scenario, everything you can do to optimise your working environment, your note-taking and your revision translates to quicker understanding and more marks in the exam.

Also, if you're studying at sixth form, there's likely to be a bit more going on socially than at secondary school. The more quickly you can absorb your studies, the sooner you can get out to absorb the odd lemonade with your friends. That is what you're drinking, isn't it?

All about the algebra

You've probably been manipulating algebraic expressions for years by now, and some of it will be second nature. However, just as a checklist, here are some of the topics you need to have under your belt:

- Solving linear equations (such as $7x+9=3x-7$)
- Expanding quadratic brackets (for example, $\left(x+3\right)\left(2x-5\right)$)
- Factorising and solving quadratics (such as $x^2-5x-36=0$)
- Solving linear and nonlinear simultaneous equations
- Simplifying algebraic fractions

I recap all of these in Chapter 3.

All this algebra isn't just for the sake of jumbling letters around and feeling super-smug when your answer matches the one in the mark scheme (although that can be a nice motivator). Algebraic competence underpins just about everything in A level. Even in places where you'd normally expect to use only numbers (for example, Pythagoras's theorem), you may be asked to work with named constants (such as k) instead of given numbers (such as 3).

Grabbing graphs by the horns

Somewhat related to algebra are graphs. You rarely need to draw an *accurate* graph at A level; it's far more common to be asked to *sketch* a graph.

That's good news: sketching is much quicker and more generously marked than plotting. However, you can no longer rely on painstakingly working out the coordinates and joining them up with a nice curve. Instead, you need to know the shapes of several kinds of graphs you've come across at GCSE: the straight line, the quadratic and the cubic graphs as well as the reciprocal and squared-reciprocal graphs.

You'll frequently be asked to work out where a graph crosses either of the coordinate axes (which is really an algebra question), and you'll be expected to be on top of curve transformations. I cover all these in Chapters 4 and 10.

Taming triangles and other shapes

Triangles, obviously, are the best shape of all, which is why you spend so much time on SOH CAH TOA, the sine and cosine rules, and finding areas at GCSE.

Oh, and Pythagoras's theorem. If there were a usefulness scale, Pythagoras's theorem would be *way* off it. I can't think of a more important equation at A level, and you can read about it in Chapter 4.

Those skills are extremely useful in A level maths. Pretty much every Core 2 paper I've ever seen has used a triangle somewhere, and triangles frequently crop up in other modules. (If you're doing Mechanics, having strong trigonometry skills is a massive help.)

It's not just triangles you need to know about, though. Be sure you know the areas and perimeters of basic two-dimensional shapes like rectangles, trapeziums and circles as well as the surface areas and volumes of three-dimensional shapes such as cuboids and prisms. I recap these shapes in Chapter 15.

Attacking Advanced Algebra

As you'd expect, the algebra you're expected to do at A level gets a bit more involved than what you did at GCSE. It comes down to learning some new techniques and linking some new notation to ideas you may already have a decent grasp of.

You start with powers and surds, a GCSE topic that sometimes gets glossed over. You'll need to be fairly solid on these, as they come up over and over again in A level. In later modules, you have a calculator that will happily tell you that the square root of 98 can be written as $7\sqrt{2}$, but in Core 1, you need to be able to work that out on paper.

You also deal with sequences and series (extending the work you've done in the past), solidify the ideas of factorising polynomials, and do some work on functions – one of the most important ideas in maths.

Picking over powers and surds

Working out combinations of powers is one of the most critical skills for A level maths. I wouldn't say it's more important than topics like algebra, but a student's skill here is a strong indicator of how easy a student is going to find A level. If you're a bit rusty on the power laws, you're going to have to sort that out in fairly short order.

You also need to be pretty hot on your surds, especially in Core 1, when you're one calculator short of a pencil case. Throughout your course, you'll need to work square roots out in *simplified surd* form or, more generally, in *exact* form – examiners want to see things like $\pi\sqrt{2}$ rather than 4.443.

A step up from powers and surds are *logarithms*, which are handy functions for turning equations with unknown powers into equations with unknown multipliers. For example, without logarithms, $3^x = 100$ is hard to solve (you know the answer is 4-and-a-bit but not necessarily what the bit is), but with logarithms, getting the answer is a simple bit of algebra: $x\log(3) = \log(100)$, where log(3) and log(100) are just numbers you can get from your calculator.

Lastly, under the 'powers' heading, you need to work with one of the most interesting numbers in all of maths, *e*. It's a constant (exactly $1+\frac{1}{1}+\frac{1}{2\times 1}+\frac{1}{3\times 2\times 1}+\ldots$, or roughly $2.718281828459045\ldots$) with the lovely property that if you work out the tangent line of $y = e^x$ at any point, you find its gradient is equal to e^x.

Sorting out sequences and series

A *sequence* is simply a list of mathematical objects (often numbers, sometimes expressions). A *series* is what you get if you add them up.

You probably did some work on sequences in the past (finding the *n*th term, for instance, or deciding whether a term belonged to a sequence), and that will stand you in good stead. At A level, though, there's a lot more to it (who would have thought?).

As well as the arithmetic sequences you know and love, there are geometric sequences (where each term is a constant multiple of the one before). There are also explicitly and recursively defined functions, which *sound* like a horrible wild-card but in fact are quite nice because you're told precisely how they behave in the question.

And there are binomial expansions, which are a really neat way of expanding expressions like $(1+2x)^6$ without needing to multiply out huge numbers of brackets. It's a particularly useful technique when you get to Core 4 and have to expand monsters like $\frac{1}{\sqrt{4+3x}}$ and use the result to approximate $\frac{1}{\sqrt{4.03}}$.

Of course, the binomial expansion is one of many things you do much more often in exams than you ever will in the outside world. (We have machines for that.) However, the idea of approximating things using polynomial series is a powerful tool for doing serious maths if you take the subject beyond A level. Oh, and it's handy for doing mental arithmetic tricks that make you look like a god, too.

Finding factors

I keep coming back to a theme in this book: things in brackets are (usually) happy things. In most cases, if you can put something in brackets – a quadratic expression, or a cubic, or a fraction – you almost certainly should. If you have to solve for where an expression is 0, the factorising makes it very easy; if you need to sketch a curve, the bracketed form is much easier to work with than the expanded one.

All the work you did in learning to factorise quadratics over the last few years will serve you quite well with this – although, as you may expect, you take it a few steps further at A level.

In Core 2, you learn to identify factors of cubics (and higher-degree polynomials) using the factor theorem. You use polynomial division to take these factors out so you can factorise the remaining expression. You also use its cousin, the remainder theorem, to find out what's left over without having to work through the whole division.

Sometimes, though, you need to do the whole division or, depending on your preferred method, find a way to work around it. I like to turn the problem on its head by coming up with a template answer and seeing which numbers have to go in the template, but your mileage may vary.

All this factor fun shows up in Chapter 7, along with the Core 4 topic of partial fractions. Since you started working with fractions, you've been adding and subtracting them using common denominators. Partial fractions is the reverse process of taking a fraction that's been combined and splitting it up into the parts that once made it up. Why would you do such a thing? Two reasons: it makes things much easier to integrate, and it means you can apply the binomial expansion much more easily.

Functions

A mathematical *function* is, roughly speaking, a recipe for taking one or more values and spitting out another. They're a big deal, mathematically speaking: being able to talk about functions in the abstract, without explaining what the recipe is, means you can do interesting things with graphs and calculus without getting bogged down in the details. For example, if you compare the graph of $y = f\left(x - 2\right)$ with the graph of $y = f\left(x\right)$, you can say, 'The graph has moved two units to the right' without caring whether the function is quadratic, reciprocal, trigonometric or other – quite a handy trick!

In Chapter 9, you learn about the slightly esoteric notation you use for defining functions. You find out how to combine functions with each other, how to *invert* (undo) functions and how to solve equations involving functions.

You also get to play with *iteration*, which falls into the dull-but-usually-straightforward category. The idea is that you set up a recursive process, doing the same thing over and over again, until it converges on a specific value.

Getting to Grips with Geometry

Geometry – literally 'measuring the Earth' – has developed over time to mean the study of shapes. At A level, the most important shapes are triangles (clearly the best shape) and circles (which are really triangles in disguise), although you will need to deal with rectangles and trapeziums and all manner of three-dimensional shapes in good time.

The four main areas of A level geometry are

- **Coordinate geometry,** which is about dealing – as you might expect – with coordinates; that includes midpoints, distances and equations of curves
- **Circles,** including their equations and some theorems
- **Triangles**, including advanced trigonometry
- **Vectors**, including vector lines, angles between vectors, and triangles in three dimensions

Conquering coordinate geometry

Coordinate geometry is a big topic at A level. You need to be super-confident with your xs and ys. You've covered the basics at GCSE – the equation of a line, finding midpoints and distances between points using Pythagoras's theorem, and so on – but it all gets taken a bit further at A level.

There are curves to sketch and shapes whose areas need to be known. After you do some differentiation, there are tangents and normals to find the equations of.

Setting up circles and triangles

I'm always surprised when I ask for the equation of a circle and someone says, 'πr^2'. First of all, that's not an equation (he or she means $A = \pi r^2$, where A is the area and r is the radius), and second, that's the *area* of the circle, not the circle itself.

The equation of a circle, like the equation of a line, gives you a relationship between its coordinates. If the equation is $(x-3)^2 + (y-4)^2 = 25$, one of A level's favourite circles, you can tell whether a particular point is on the circle by putting its coordinates into the equation as x and y.

In Chapter 11, I show you how to work out the equation of a circle as well as work out the area of a sector, the length of an arc and other things related to a circle.

Circles are closely linked to triangles; apart from all the trigonometry you know from GCSE, you'll also need to be able to do it all in *radians*, a much better measurement of angles than the degree. Fortunately, much of it is just a case of switching your calculator mode and relabelling your graphs.

Taking trigonometry further

If you split the word *trigonometry* up into its parts, you get 'tri', meaning three; 'gon', meaning 'knee' or corner; 'o', which means nothing; and 'metry', meaning 'measuring.' Trigonometry is about measuring things with three corners.

However, that's not all you use it for. It also has applications in any situation where things are periodic – measuring tides, modelling daylight lengths and analysing sounds, just off the top of my head. For that reason, you need to be able to take trigonometry further. Some of the things you'll be doing include

- Finding all the possible solutions to simple trigonometric equations in a given interval
- Using trigonometric identities to turn trigonometric equations into something you can solve
- Exploring what happens to sine, cosine and their friends when you add angles together
- Adding sine and cosine waves together to get another sine or cosine wave
- Working with the minor trigonometric functions – the reciprocals of sine, cosine and tangent, which are the cosecant, secant and cotangent (respectively); usually, they're denoted $\text{cosec}(x)$, $\sec(x)$ and $\cot(x)$

You also need to be up to speed on proving that two trigonometric expressions are equivalent. That generally involves combining fractions, applying identities, factorising things cleverly and understanding the symmetries of the various functions. Because these problems bring so many areas together, they're one of the most demanding (but also most rewarding) bits of Core 3.

Vanquishing vectors

You've done some work with vectors at GCSE – although you may not have done as much as you'd like, because vectors are usually in the A* questions at the end of the paper. It's OK, though: your GCSE vectors work isn't essential to your A level studies.

Vectors at A level can seem intimidating, with new vocabulary, new ways of multiplying things together, new equations of lines. After you get past that and think about how vectors fit together, they're really powerful – and straightforward.

In Chapter 13, I show you the following:

- How to come up with the vector equation of a line
- How to find the distance between two points (the length of a vector)
- How to find the angle between two vectors
- How to tell whether two vector lines cross
- How to find a point on a line so that a vector involving it makes a right angle with the line

Additionally, in case you're studying for the OCR MEI board, I take you through the gory details of vector equations of a plane – usually a big part of a Section B question in Core 4.

Conquering Calculus

In terms of A level maths, calculus is pretty much what you've been building towards for your entire career. The discipline has plenty of rules to learn but is based very carefully on much of what you've done up until now in algebra, arithmetic and geometry.

You can think of calculus as the study of curves, very loosely speaking: How steep are they? How much area do they have underneath them?

Why is that important? In many processes in physics, chemistry, biology, economics, psychology and anywhere else you care to apply maths, either there's a relationship between how much of a thing there is and how quickly it's changing, or you're actually interested in how something is changing. For example, your speed is how quickly your position is changing, and your acceleration is how quickly your speed is changing.

To work that out, you *differentiate* an expression or an equation. *Integration* is the reverse process. And because Core maths isn't just about doing maths for the sake of it, you need to know how to apply calculus to situations, both in mathematical and real-life contexts.

Dashing off differentiation

Differentiation is the process of finding how a quantity is changing instantaneously – at a specific point in space or time. It's kind of a big deal. Two of the greatest scientists of the early eighteenth century, Isaac Newton and Gottfried Leibniz, squabbled for years over priority (Newton thought of it before Leibniz, but Leibniz published first, leading to Newton having a hissy fit and accusing Leibniz of pinching his ideas).

The history of calculus is interesting but not especially useful for understanding what to do. In the chapters on differentiation (14 and 15, to be precise), you learn the following:

- How to differentiate powers of x (or any other letter you put your mind to)
- How to differentiate the trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$
- How to differentiate exponential and logarithmic expressions
- What to do when expressions are multiplied together, divided by each other or applied to each other

Here, you also learn about the geometrical interpretation of differentiation – how to find tangents and normals, how to determine and classify turning points of a function and how to find whether a turning point is a maximum or a minimum.

Inspiring yourself to integrate

Integration is the reverse process of differentiation: taking a gradient and turning it into a curve. It's also used for finding the area underneath a curve. In fact, all the formulas for the area and volume of shapes can be worked out by integration, but I leave that as an exercise for the very interested reader.

In Chapters 16 and 17, I show you how to

- Integrate powers of x (apart from that pesky x^{-1})
- Remember to add a constant of integration
- Work with limits to evaluate definite integrals and areas
- Find the area between curves
- Deal with trigonometric functions such as sine and cosine
- Work with exponentials and (at last) that problematic $\frac{1}{x}$
- Manage functions of functions and expressions multiplied together

Applying the calculus

There are two main schools of thought when it comes to maths. The crazy pure mathematicians think maths should be done for its own sake, for the sheer beauty of it, because it's fun. The more sensible applied mathematicians do maths because it's useful.

And calculus, while it's beautiful and fun, is especially useful. It's useful geometrically: it allows you to draw tangents and normals, find turning points and classify them, and that sort of thing. But it's also useful in what exam boards think of as real life, too. Here are some examples:

- **Finding maxima and minima of functions:** This allows you to find the most efficient way to use material to package products, or at what price to sell something, or how to minimise the cost of a journey.
- **Finding rates of change:** Knowing how quickly your experiment is running or at what speed your car is going is often useful.
- **Finding areas and volumes:** You may need to know how big an irregularly shaped object is. For a slightly more abstract application, the area under a speed–time graph gives you the distance something has travelled.
- **Solving differential equations:** In many applications, the value of a function is linked to its rate of change. For instance, the amount of a radioactive substance left depends on how quickly it decays, and the position of a satellite depends on its acceleration. Solving the links between derivatives and values is a key skill, both in Core 4 and as you take maths further.