

CHAPTER 1

Essential Toolbox

In this book, we are concerned with the structure and properties of groups satisfying different rank conditions in various classes of generalized soluble groups. There are a variety of monographs in which the properties of different classes of generalized soluble groups are considered. Nevertheless, we have found it useful to collect together, in this first chapter, some concepts that will be useful in later chapters together with some standard results from the theory of generalized soluble groups. Some of these results are supplied with proof whilst others, which require specific techniques that may not be germane to the discussion, we just exhibit without proof. For such results, complete references are supplied for the book or article where the reader can find the proof.

1.1. Ascending and Descending Series in Groups

In this first section, we obtain some of the standard results concerning ascending and descending series in groups. We shall omit some of the proofs here since many of these results have been well documented in various other books.

Let H be a subgroup of a group G and suppose that γ is an ordinal number. An *ascending series* from H to G is a set of subgroups

$$(1.1) \quad H = V_0 \leq V_1 \leq \dots V_\alpha \leq V_{\alpha+1} \leq \dots V_\gamma = G$$

such that V_α is normal in $V_{\alpha+1}$ and $V_\lambda = \bigcup_{\beta < \lambda} V_\beta$ for all limit ordinals $\lambda < \gamma$. In this case, we call H an *ascendant subgroup* of G unless the series involved is of finite length in which case, of course, H is said to be *subnormal* in G . The subgroups V_α , for $\alpha \leq \gamma$, are called the *terms* of this series and the factor groups $V_{\alpha+1}/V_\alpha$ are called the *factors* of the series. An ascending series is called *normal* if each term of the series (1.1) is a normal subgroup of G .

Let $\mathfrak{X}$ be a class of groups (which can also be defined by means of a group theoretical property). A group G is called a *hyper- $\mathfrak{X}$ group* if it has an ascending normal series, which is infinite in general, starting

with 1, terminating in G itself, whose factors belong to $\mathfrak{X}$. We denote this class of groups by $\mathfrak{F}_n\mathfrak{X}$. Thus a group G is called *hyperabelian* if G has an ascending normal series whose factors are abelian. This class of groups is one of the many generalizations of the class of soluble groups.

We shall also meet the class of *hyperfinite groups*. Such groups have an ascending series of normal subgroups whose factors are all finite.

There are also generalizations of the notion of nilpotency. We let $\zeta(G)$ denote the centre of the group G . The ascending series

$$(1.2) \quad 1 = Z_0 \leq Z_1 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\gamma = G$$

is called *central* if the corresponding factors $Z_{\alpha+1}/Z_\alpha$ are central, that is $Z_{\alpha+1}/Z_\alpha \leq \zeta(G/Z_\alpha)$ for all $\alpha < \gamma$.

On the other hand, the *upper central series* of a group G is the ascending central series

$$1 = Z_0 \leq Z_1 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\gamma$$

of characteristic subgroups such that $Z_{\alpha+1}/Z_\alpha = \zeta(G/Z_\alpha)$, for all ordinals $\alpha < \gamma$, $Z_\lambda = \bigcup_{\beta < \lambda} Z_\beta$, for all limit ordinals λ and $\zeta(G/Z_\gamma) = 1$. For the upper central series, we shall often write $\zeta_\alpha(G)$ for the subgroup Z_α and we call $\zeta_\alpha(G)$ the α th *hypercentre* of G . The last term $\zeta_\gamma(G)$ of the upper central series is called the *upper hypercentre* of G and will be denoted by $\zeta_\infty(G)$. The ordinal γ is called the *central length* of G and is denoted by $\mathbf{zl}(G)$. A group G is called *hypercentral* if and only if it coincides with some term of its upper central series and so $\zeta_\infty(G) = G$ in this case. A hypercentral group having finite central length leads to the usual notion of nilpotence. We note that a group G is hypercentral if and only if it has an ascending central series and G is nilpotent if and only if it has a finite central series.

The following lemma shows one way to characterize hyperabelian and hypercentral groups. Its proof is quite straightforward and we leave it as an exercise.

1.1.1. LEMMA. *Let G be a group.*

- (i) *G is hyperabelian if and only if every nontrivial homomorphic image of G contains a nontrivial normal abelian subgroup.*
- (ii) *G is hypercentral if and only if every nontrivial homomorphic image of G has nontrivial centre.*

EXERCISE 1.1. *Prove Lemma 1.1.1.*

The following corollary is also easy to prove.

1.1.2. COROLLARY. *Let G be a hyperabelian (respectively hypercentral) group.*

- (i) If H is a subgroup of G , then H is hyperabelian (respectively hypercentral);
- (ii) If H is a normal subgroup of G , then G/H is hyperabelian (respectively hypercentral);
- (iii) If H, K are subgroups of G such that H is a normal subgroup of K , then K/H is hyperabelian (respectively hypercentral).

If H is a subgroup of a group G and if γ is an ordinal, then a *descending series* from G to H is a series of subgroups

$$G = V_0 \geq V_1 \geq V_2 \geq \dots V_\alpha \geq V_{\alpha+1} \geq \dots V_\gamma = H$$

such that $V_{\alpha+1}$ is normal in V_α and $V_\lambda = \bigcap_{\beta < \lambda} V_\beta$, for all limit ordinals λ . The subgroup H is then said to be a *descendant* subgroup of G ; when β is finite then, of course, H is called subnormal in G . Again, the subgroups V_α , for $\alpha \leq \gamma$, are called the *terms* of this series and the factor groups $V_\alpha/V_{\alpha+1}$ are called the *factors* of the series. A descending series is called normal if each term of the series is normal in G .

If $\mathfrak{X}$ is a class of groups then a group G is called a *hypo- $\mathfrak{X}$ group* if it has a descending normal series, which is infinite in general, terminating in 1, whose factors belong to $\mathfrak{X}$. We denote this class of groups by $\mathfrak{P}_n\mathfrak{X}$. Thus a group G is called *hypoabelian* if G has a descending normal series, whose factors are abelian.

We continue to set up notation and terminology by defining, for subsets X, Y of the group G , the subgroup

$$[X, Y] = \langle [x, y] = x^{-1}y^{-1}xy \mid x \in X, y \in Y \rangle.$$

If $Y = \{y\}$, then we simply write $[X, y]$ instead of $[X, \{y\}]$. For the subsets $X_1, \dots, X_n$ we inductively define

$$[X_1, X_2, \dots, X_n] = [[X_1, \dots, X_{n-1}], X_n]$$

and if $X = X_1 = X_2 = \dots = X_n$, then we write $[Y, X_1, X_2, \dots, X_n]$ as $[Y, {}_nX]$. We note that $H \leq \zeta_n(G)$ if and only if $[H, {}_nG] = 1$. With these definitions we now define the *lower central series* of a group G to be the descending central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \dots \gamma_\delta(G)$$

of characteristic subgroups defined by $\gamma_1(G) = G$, $\gamma_2(G) = [G, G]$ and recursively, $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$, for all ordinals α and $\gamma_\lambda(G) = \bigcap_{\beta < \lambda} \gamma_\beta(G)$, for all limit ordinals λ . If this series terminates in 1, then the group is called *hypocentral*.

As we will see shortly, when a group is defined by one or other infinite ascending series, this can have a significant effect on the structure and properties of the group, while the presence of an infinite descending

series is usually less influential. For example, by a theorem of Magnus (see, for example, [172, Chapter IX, §36]), every non-abelian free group is hypocentral of length ω , the first infinite ordinal. Furthermore, every free group has a descending chain of normal subgroups whose factors are finite p -groups, for each prime p , a result first obtained by K. Iwasawa [119].

1.1.3. LEMMA. *Let G be a group and let A be an abelian normal subgroup of G . Then*

- (i) $[A, x] = \{[a, x] | a \in A\}$;
- (ii) $[A, \langle x \rangle] = [A, x]$, for each $x \in G$;
- (iii) If $M \subseteq G$ then $[A, \langle M \rangle]$ is the product of the subgroups $[A, x]$, for $x \in M$;
- (iv) If $M \subseteq G$ and if $G = \langle C_G(A), M \rangle$ then $[A, G]$ is the product of the subgroups $[A, x]$ where $x \in M$.

PROOF. (i) Let $S = \{[a, x] | a \in A\}$, so that $[A, x] = \langle S \rangle$, by definition and it now suffices to prove that S is a subgroup. Using the well-known properties of commutators and the fact that A is an abelian normal subgroup if $a, b \in A$, we have

$$\begin{aligned} [ab, x] &= [a, x]^b [b, x] = [a, x][b, x], \text{ and} \\ [a^{-1}, x] &= (a[a, x]a^{-1})^{-1} = [a, x]^{-1}, \end{aligned}$$

which proves (i).

(ii) It is clear that $[A, x] \leq [A, \langle x \rangle]$. Furthermore, for each natural number $n \geq 2$, we have

$$[a, x^n] = [a, x^{n-1}x] = [a, x][a, x^{n-1}]^x = [a, x][a^x, x^{n-1}] \in [A, x],$$

since A is normal in G , using a natural induction argument. Also

$$[a, x^{-1}] = (x[a, x]x^{-1})^{-1} = [xax^{-1}, x]^{-1} \in [A, x],$$

since $[A, x]$ is a subgroup and it is now easy to see that (ii) holds.

(iii) It is clearly sufficient to prove that the assertion holds in the case when $M = \{x_1, x_2, \dots, x_n\}$ is a finite set. We note that a general element of $\langle M \rangle$ can be written in the form $u = x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}$, where $n_i \in \mathbb{Z}$, for $i = 1, \dots, k$. If $k = 1$ then $[a, u] \in [A, x_{i_1}]$, for all $a \in A$, which now forms the basis for an induction on k . Indeed we have

$$(1.3) \quad [a, u] = [a, x_{i_2}^{n_2} \dots x_{i_k}^{n_k}][a, x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}].$$

The first term on the right-hand side of (1.3) lies in the product $[A, x_1][A, x_2] \dots [A, x_n]$ by the induction hypothesis. The second term can be written as $[a, x_{i_1}^{n_1}][a, x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}]$. We have $[a, x_{i_1}^{n_1}] \in [A, x_{i_1}]$ by (ii) and, since A is normal in G , we also have $[a, x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}] \in$

$[A, x_{i_2}^{n_2} \dots x_{i_k}^{n_k}]$. Again the induction hypothesis can be applied and the result then follows.

(iv) Since A is normal in G , we have $C_G(A) \triangleleft G$ and hence $G = C_G(A)\langle M \rangle$. If $g \in G$ then $g = cx$, where $c \in C_G(A)$, $x \in \langle M \rangle$, and we have for each $a \in A$,

$$[a, g] = [a, cx] = [a, x][a, c]^x = [a, x].$$

Hence $[A, G] = [A, \langle M \rangle]$ and the result now follows by (iii). $\square$

It is an easy exercise to prove the following result due, we believe, to D. H. McLain.

1.1.4. COROLLARY. *Let G be a group and let M be a subset of G such that $G = G'\langle M \rangle$. Then $\gamma_n(G) = \langle \gamma_{n+1}(G), [x_1, \dots, x_n] | x_i \in M, \text{ for all } i \rangle$, for every positive integer n .*

EXERCISE 1.2. *Prove Corollary 1.1.4.*

1.1.5. COROLLARY. *Let G be a group. If G/G' is finitely generated, then $\gamma_n(G)/\gamma_{n+1}(G)$ is finitely generated for all natural numbers n .*

We shall need a number of properties concerning the terms of the upper and lower central series. We collect some of these well-known results in the following lemma.

1.1.6. PROPOSITION. *Let G be a group. Then*

- (i) $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$, for all $x, y, z \in G$;
- (ii) Let X, Y, Z be subgroups of G and let L be a normal subgroup of G . If L contains two of the subgroups $[X, Y, Z]$, $[Y, Z, X]$ and $[Z, X, Y]$, then L also contains the third of these subgroups;
- (iii) If m, n are natural numbers and if $n \geq m$, then $[\gamma_m(G), \zeta_n(G)] \leq \zeta_{n-m}(G)$;
- (iv) If m, n are natural numbers, then $\zeta_n(G/\zeta_m(G)) = \zeta_{m+n}(G)/\zeta_m(G)$.

We use this to prove the following fact.

1.1.7. LEMMA. *Let G be a group and let K be a normal subgroup of G . Then $[\gamma_n(G), K] \leq [K, {}_nG]$ for all natural numbers n . Furthermore, if H is a subgroup of G such that $G = HK$ then $\gamma_{n+1}(G) = \gamma_{n+1}(H)[K, {}_nG]$, for all natural numbers n .*

PROOF. We prove the first statement using induction on n . If $n = 1$, then $\gamma_1(G) = G$ and $[\gamma_1(G), K] = [G, K] = [K, G]$. Suppose that $n > 1$ and that we have proved that $[\gamma_{n-1}(G), K] \leq [K, {}_{n-1}G]$, for all normal subgroups K of G . Note that $[\gamma_n(G), K] = [[\gamma_{n-1}(G), G], K]$. We consider the subgroups $[[G, K], \gamma_{n-1}(G)] = [\gamma_{n-1}(G), [G, K]]$ and

$[[K, \gamma_{n-1}(G)], G] = [[\gamma_{n-1}(G), K], G]$. Let $L = [G, K] = [K, G]$ and note that L is normal in G . Hence, by the induction hypothesis,

$$[\gamma_{n-1}(G), L] \leq [L, {}_{n-1}G] = [[K, G], {}_{n-1}G] = [K, {}_nG].$$

Using the induction hypothesis we also have

$$[[\gamma_{n-1}(G), K], G] \leq [[K, {}_{n-1}G], G] = [K, {}_nG].$$

Hence $[K, {}_nG]$ contains both $[[G, K], \gamma_{n-1}(G)]$ and $[[K, \gamma_{n-1}(G)], G]$. Proposition 1.1.6(ii) now implies that $[K, {}_nG]$ contains $[\gamma_n(G), K]$, which completes the inductive step.

The second statement will also be proved using induction on n . When $n = 0$ we have $\gamma_1(G) = G = HK = \gamma_1(H)[K, {}_0G]$. For the inductive step, we let $n \geq 1$ and set $K_j = [K, {}_jG]$ so that $K_{j+1} = [K_j, G]$, for $j \in \mathbb{N}$. We note that $K_n \leq \gamma_{n+1}(G)$ and $\gamma_{n+1}(H) \leq \gamma_{n+1}(G)$ so $\gamma_{n+1}(H)K_n \leq \gamma_{n+1}(G)$.

Next we note that $[K, {}_nG]$ is normal in G . Let $x \in \gamma_{n+1}(H), y \in K$. Then $[x, y] \in [\gamma_n(G), K] \leq [K, {}_nG]$, by the first part of the proof. It follows that $x^y = x[x, y] \in \gamma_{n+1}(H)[K, {}_nG]$. Also, if $z \in H$ then $x^z \in \gamma_{n+1}(H)$. Since $G = HK$, it follows that $\gamma_{n+1}(H)[K, {}_nG] \triangleleft G$. Finally, using the induction hypothesis, we note that

$$\gamma_{n+1}(G) = [\gamma_n(G), G] = [\gamma_n(H)[K, {}_{n-1}G], G] = [\gamma_n(H)[K, {}_{n-1}G], HK].$$

Now we have

$$\begin{aligned} [\gamma_n(H), H] &= \gamma_{n+1}(H) \leq \gamma_{n+1}(H)[K, {}_nG]; \\ [\gamma_n(H), K] &\leq [\gamma_n(G), K] \leq [K, {}_nG] \leq \gamma_{n+1}(H)[K, {}_nG] \text{ and} \\ [[K, {}_{n-1}G], HK] &= [[K, {}_{n-1}G], G] = [K, {}_nG] \leq \gamma_{n+1}(H)[K, {}_nG]. \end{aligned}$$

Let $a \in \gamma_n(H), b \in [K, {}_{n-1}G], y \in K, z \in H$. Then

$$[ab, zy] = [ab, y][ab, z]^y = [a, y]^b[b, y][a, z]^{by}[b, z]^y.$$

Since $\gamma_{n+1}(H)[K, {}_nG]$ is normal in G , these equalities and the preceding inclusions imply that $\gamma_{n+1}(G) \leq \gamma_{n+1}(H)[K, {}_nG]$. Thus $\gamma_{n+1}(G) = \gamma_{n+1}(H)[K, {}_nG]$, which completes the induction step and the proof. $\square$

The following very useful consequence is very easy to deduce.

1.1.8. COROLLARY. *Let G be a group and let H be a subgroup of G such that $G = H\zeta_n(G)$. Then $\gamma_{n+1}(G) = \gamma_{n+1}(H)$.*

1.2. Generalized Soluble Groups

In this section, we shall collect together some well-known facts concerning generalized soluble groups. We begin with the class of hypercentral groups. We let $N_G(H)$ denote the normalizer of the subgroup H of G .

1.2.1. LEMMA. *Let G be a hypercentral group.*

- (i) *If H is a proper subgroup of G then $H \neq N_G(H)$;*
- (ii) *Every subgroup of G is ascendant.*

In particular, every subgroup of a nilpotent group is subnormal.

EXERCISE 1.3. *Prove Lemma 1.2.1.*

The next two properties of hypercentral groups are very important. Using Zorn's Lemma it is easy to see that every group has maximal abelian normal subgroups.

1.2.2. LEMMA. *Let G be a hypercentral group and suppose that A is a nontrivial normal subgroup of G . Then $A \cap \zeta(G) \neq 1$. If A is a maximal abelian normal subgroup of G , then $C_G(A) = A$.*

PROOF. Let

$$1 = Z_0 \leq Z_1 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\gamma = G$$

be the upper central series of G .

Since $A \neq 1$ there is a least ordinal α such that $A \cap Z_\alpha \neq 1$. If α is a limit ordinal, then using the definition of α , we have

$$A \cap Z_\alpha = A \cap \left(\bigcup_{\beta < \alpha} Z_\beta \right) = \bigcup_{\beta < \alpha} (A \cap Z_\beta) = 1,$$

which is a contradiction. Hence $\alpha - 1$ exists, so $A \cap Z_{\alpha-1} = 1$ by definition of α . However, it then follows that

$$[A \cap Z_\alpha, G] \leq A \cap Z_{\alpha-1} = 1,$$

so that $A \cap Z_\alpha \leq \zeta(G)$. Hence $A \cap \zeta(G) \neq 1$, which proves the first part of the lemma.

Suppose now that A is a maximal normal abelian subgroup of G . It is clear that $A \leq C_G(A)$. We let $C = C_G(A)$ and suppose that $A \neq C$. Then C/A is a nontrivial normal subgroup of the hypercentral group G/A , so there exists $xA \in C/A \cap \zeta(G/A)$. Hence $\langle x, A \rangle$ is an abelian normal subgroup of G , so that $x \in A$, by choice of A , a contradiction which proves the lemma. $\square$

A different type of characterization of hypercentral groups is illustrated in the following useful result due to S. N. Chernikov [39].

1.2.3. LEMMA. *A group G is hypercentral if and only if for each element $a \in G$ and every countable subset $\{x_n | n \in \mathbb{N}\}$ of elements of G there exists an integer k such that*

$$[\dots [[a, x_1], x_2], \dots, x_k] = 1.$$

PROOF. Let G be a hypercentral group and let

$$1 = Z_0 \leq Z_1 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\gamma = G$$

be the upper central series of G . Put $a_n = [\dots [[a, x_1], x_2], \dots, x_n]$, for $n \in \mathbb{N}$, and suppose, for a contradiction, that $a_n \neq 1$ for all $n \in \mathbb{N}$. Since $a_n \neq 1$ for each n , it follows that there is a least ordinal α such that Z_α contains a_t for some t . Clearly α cannot be a limit ordinal so $\alpha - 1$ exists. Then, by definition of α , $a_t \notin Z_{\alpha-1}$ for all $t \in \mathbb{N}$. However, this implies the desired contradiction since $a_{t+1} = [a_t, x_{t+1}] \in Z_{\alpha-1}$, contrary to the choice of α .

To prove sufficiency of the condition, we first note that the condition is inherited by each factor group of G . Hence, by Lemma 1.1.1, it suffices to prove that $\zeta(G) \neq 1$. We suppose, for a contradiction, that this is false. Let $a = a_0 \in G$ and suppose that we have constructed elements $x_1, \dots, x_n \in G$ such that $[a_j, x_{j+1}] = a_{j+1} \neq 1$, for $0 \leq j \leq n - 1$. Then, by assumption, $a_n \notin \zeta(G)$ so there exists $x_{n+1} \in G$ such that $[a_n, x_{n+1}] = a_{n+1} \neq 1$. In this way we construct a countable subset $\{x_n | n \in \mathbb{N}\}$ such that $[\dots [[a, x_1], x_2], \dots, x_n] \neq 1$ for all $n \in \mathbb{N}$. This contradicts our hypothesis and it follows that $\zeta(G) \neq 1$. The proof is complete. $\square$

1.2.4. PROPOSITION (Chernikov [39]). *A group G is hypercentral if and only if each countable subgroup of G is hypercentral.*

PROOF. If G is hypercentral, then each subgroup of G is hypercentral and hence every countable subgroup is certainly hypercentral.

Conversely, suppose that every countable subgroup of G is hypercentral, but G is not hypercentral. Factoring by the hypercentre we may suppose that $\zeta(G) = 1$. By Lemma 1.2.3 there exists $a \in G$ and a countable subset $\{x_n | n \in \mathbb{N}\}$ such that $[\dots [[a, x_1], x_2], \dots, x_n] \neq 1$, for each $n \in \mathbb{N}$. The subgroup $\langle a, x_n | n \in \mathbb{N} \rangle$ is countable and hence is hypercentral. Lemma 1.2.3 shows that there exists $k \in \mathbb{N}$ such that $[\dots [[a, x_1], x_2], \dots, x_k] = 1$ and we obtain a contradiction. Consequently G is hypercentral. $\square$

We note that a class $\mathfrak{X}$ of groups is a *countably recognizable* class if the group G is an $\mathfrak{X}$ -group whenever every countable subset of elements of G is contained in some $\mathfrak{X}$ -subgroup of G . Thus Proposition 1.2.4 asserts that the class of hypercentral groups is a countably recognizable

class and there is a similar result, due to Baer [12], for the class of hyperabelian groups.

In a similar vein, if $\mathfrak{X}$ is a class of groups then a group G is called *locally $\mathfrak{X}$* if every finite subset of G is contained in some $\mathfrak{X}$ -subgroup of G . In particular, a group is called locally nilpotent, locally soluble or locally finite in the cases when $\mathfrak{X}$ denotes, respectively, the class $\mathfrak{N}$ of all nilpotent groups, the class $\mathfrak{S}$ of all soluble groups and the class $\mathfrak{F}$ of all finite groups. We denote the class of locally $\mathfrak{X}$ -groups by $\mathbf{L}\mathfrak{X}$.

Of course, every locally nilpotent group is locally soluble. It is a well-known result of A. I. Maltsev [187] that every hypercentral group is locally nilpotent, a result which we now prove.

1.2.5. PROPOSITION. *Every finitely generated hypercentral group is nilpotent.*

PROOF. Let G be a finitely generated hypercentral group. Let

$$1 = Z_0 \leq Z_1 \leq Z_2 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\eta = G$$

be the upper central series of G and suppose that $G = \langle M \rangle$, where $M = \{g_1, g_2, \dots, g_n\}$. Let $\lambda(j)$ denote the least ordinal such that $g_j \in Z_{\lambda(j)}$, for $1 \leq j \leq n$. It is easy to see that none of the $\lambda(j)$ are limit ordinals. Thus η is the maximal ordinal in the set $\{\lambda(1), \lambda(2), \dots, \lambda(n)\}$. Suppose, for a contradiction, that η is infinite. Then there is a limit ordinal $\beta \geq \omega$ such that $\eta = \beta + m$, for some positive integer m . Since G/Z_β is then nilpotent of class at most m we have $\gamma_{m+1}(G) \leq Z_\beta$. It follows that $[x_1, \dots, x_{m+1}] \in Z_\beta$ for all elements $x_1, x_2, \dots, x_{m+1} \in M$. Since M is finite there is non-limit ordinal $\nu < \beta$ such that Z_ν contains all commutators $[x_1, x_2, \dots, x_{m+1}]$, for all $x_i \in M$. Corollary 1.1.4 now shows that $\gamma_{m+1}(G) \leq Z_\nu$, and since $G/\gamma_{m+1}(G)$ is nilpotent of class at most m it follows that $G = Z_{\nu+m}$. However, $\nu + m < \beta$ and we obtain a contradiction. Hence η is finite so that G is nilpotent. $\square$

1.2.6. COROLLARY. *Every hypercentral group is locally nilpotent.*

However, the classes of hyperabelian and locally soluble groups are distinct as can be seen from [193]. In particular, a finitely generated hyperabelian group need not be soluble and a locally soluble group need not be hyperabelian.

We shall need some properties of torsion-free locally nilpotent groups.

1.2.7. LEMMA. *Let G be a torsion-free locally nilpotent group. If $x, y \in G$ are elements such that $x^n = y^n$, for some $n \in \mathbb{N}$ then $x = y$.*

EXERCISE 1.4. *Prove Lemma 1.2.7.*

From this we deduce another important result of Maltsev [187].

1.2.8. COROLLARY. *Let G be a torsion-free locally nilpotent group and let $g, x \in G$. If there exist natural numbers k, t such that $g^t x^k = x^k g^t$ then $gx = xg$.*

PROOF. Let $y = x^k$, so that $g^t y = y g^t$. It follows that $g^t = y^{-1} g^t y = (y^{-1} g y)^t$ and by Lemma 1.2.7 we deduce that $g = y^{-1} g y$. Thus $x^k g = g x^k$ and the same argument allows us to deduce that $xg = gx$, as required. $\square$

Let G be a group. We recall that a subgroup H of G is called *pure*, or *isolated*, in G if either $\langle g \rangle \cap H = \langle g \rangle$ or $\langle g \rangle \cap H = 1$, for each $g \in G$. This leads us to a further result of Maltsev [187].

1.2.9. COROLLARY. *Let G be a torsion-free locally nilpotent group.*

- (i) *If M is a subset of G then $C_G(M)$ is a pure subgroup of G ;*
- (ii) *If A is a normal subgroup of G then A is pure in G if and only if G/A is torsion-free;*
- (iii) *The intersection of every family of pure subgroups is pure;*
- (iv) *Every term of the upper central series of G is pure.*

In particular, if G is hypercentral, then the factors of the upper central series of G are torsion-free.

For our next results we need the following notation. If X, Y are subsets of a group G , then $\langle X^Y \rangle = \langle x^y \mid x \in X, y \in Y \rangle$ is the *normal closure* of X by Y .

The following result appears in Plotkin [216, Corollary to Theorem 1].

1.2.10. PROPOSITION. *Let G be a group and let π be a set of primes.*

- (i) *The product of the periodic normal π -subgroups of G is a periodic normal π -subgroup of G ;*
- (ii) *If H is a periodic ascendant π -subgroup of G , then H^G is a periodic normal π -subgroup of G .*

PROOF. (i) Let $\mathcal{M}$ be a family of normal periodic π -subgroups of G and let $M = \langle T \mid T \in \mathcal{M} \rangle$. Clearly $M \triangleleft G$. If $x \in M$ then there are subgroups $T_1, T_2, \dots, T_n \in \mathcal{M}$ such that $x \in T_1 T_2 \dots T_n$. We use induction on n to prove that $T_1 \dots T_n$ is a periodic π -subgroup. Suppose first that $n = 2$. Then $T_1 \cap T_2$ is a periodic π -subgroup and $T_1 T_2 / (T_1 \cap T_2) = T_1 / (T_1 \cap T_2) \times T_2 / (T_1 \cap T_2)$ is also a periodic π -group. It follows that $T_1 T_2$ is a periodic π -subgroup. Suppose now that $n > 2$ and we have already proved that $U = T_1 \dots T_{n-1}$ is a periodic π -subgroup. Then $T_1 \dots T_n = UT_n$, and by the case $n = 2$, the result now follows.

(ii) Let

$$H = V_0 \leq V_1 \leq V_2 \leq \dots V_\alpha \leq V_{\alpha+1} \leq \dots V_\gamma = G$$

be an ascending series from H to G . We prove, using transfinite induction, that H^{V_α} is a normal π -subgroup of V_α for each ordinal $\alpha \leq \gamma$. This is very easy to see when α is a limit ordinal, so we suppose that the result is true when $\alpha - 1$ exists and then prove it for α . Let $H_{\alpha-1} = H^{V_{\alpha-1}}$ and let $x \in V_\alpha$. Then $(H_{\alpha-1})^x \triangleleft V_{\alpha-1}$, since $V_{\alpha-1} \triangleleft V_\alpha$, and hence $H_\alpha = H^{V_\alpha}$ is a π -subgroup of V_α , using (i). Since normality is clear, we have that H^{V_α} is a periodic normal π -subgroup of V_α and the result follows. $\square$

If π is a set of primes and G is a group, then we let $\mathbf{O}_\pi(G)$ denote the largest normal π -subgroup of G . If $\pi = \Pi(G)$, the set of prime divisors of the orders of elements of G , then $\mathbf{O}_\pi(G)$ is the largest normal periodic subgroup of G , which we denote by $\mathbf{Tor}(G)$. This subgroup is called the *periodic part* of G . In some cases, when $\mathbf{Tor}(G)$ consists of all the elements of G of finite order, then it is called the *torsion subgroup* of G .

Locally nilpotent groups enjoy some of the properties of abelian groups as the following result shows.

1.2.11. PROPOSITION. *Let G be a locally nilpotent group and let π be a set of primes. Then*

- (i) *The set of elements of G whose order is a π -number corresponds to $\mathbf{O}_\pi(G)$;*
- (ii) *The set of elements of finite order in G is a characteristic subgroup of G and coincides with $\mathbf{Tor}(G)$. Also $G/\mathbf{Tor}(G)$ is torsion-free;*
- (ii)i *For each prime p , the set $\mathbf{Tor}_p(G)$, consisting of all elements of p -power order, is a characteristic subgroup of $\mathbf{Tor}(G)$ and*

$$\mathbf{Tor}(G) = \text{Dr}_{p \in \Pi(G)} \mathbf{Tor}_p(G).$$

PROOF. (i) Suppose first that G is nilpotent. Let P be a maximal π -subgroup of G and note that, by Lemma 1.2.1, P is subnormal in G . Then Proposition 1.2.10 implies that P^G is a π -group and hence $P = P^G$. Consequently, P is normal in G . Let x be a π -element. Then, as above, $\langle x \rangle^G$ is a π -subgroup of G and Proposition 1.2.10 shows that $\langle x \rangle^G P$ is a π -subgroup of G , containing P . By definition of P we see that $x \in P$. Hence P contains all the π -elements of G , and coincides with $\mathbf{O}_\pi(G)$.

Suppose now that G is locally nilpotent and let T denote the set of π -elements of G . If $y, z \in T$ then $\langle y, z \rangle$ is nilpotent and a π -subgroup,

by our work above. It follows that T is a π -subgroup, coinciding with $\mathbf{O}_\pi(G)$, which proves (i).

(ii) follows upon setting $\pi = \Pi(G)$ in (i) and observing that if $x \in G$ is such that $x \mathbf{Tor}(G)$ has finite order n then $x^n \in \mathbf{Tor}(G)$, so that $x \in \mathbf{Tor}(G)$.

(iii) follows upon setting $\pi = \{p\}$ in (i). Furthermore we note that if $x \in \mathbf{Tor}_p(G)$ and $y \in \mathbf{Tor}_q(G)$, for distinct primes p, q , then $[x, y] \in \mathbf{Tor}_p(G) \cap \mathbf{Tor}_q(G) = 1$ and the result now follows. $\square$

It is a well-known theorem due to K. A. Hirsch [116] and B. I. Plotkin [215] that the subgroup generated by the normal locally nilpotent subgroups of a group is also locally nilpotent. Thus every group G has a unique maximal normal locally nilpotent subgroup called the *locally nilpotent radical* or the *Hirsch–Plotkin radical* of G and denoted in this book by $\mathbf{Ln}(G)$. Of course $\mathbf{Ln}(G)$ is a characteristic subgroup of G .

We now recall some facts which we often use and which are derived from [216, Corollary to Theorem 1]. If H is an ascendant locally nilpotent subgroup of a group G , then it turns out that H^G is also locally nilpotent. It follows that if G contains a non-trivial ascendant locally nilpotent subgroup, then G contains a non-trivial normal locally nilpotent subgroup.

A group G is called *radical* if it is hyper (locally nilpotent) and our remarks above show that the class of radical groups is precisely the class of groups with an ascending series each factor of which is locally nilpotent. Every locally nilpotent group and every hyperabelian group is radical and it is often convenient to subsume the classes of hyperabelian groups and locally nilpotent groups into the single larger class of radical groups.

We define the *radical series*

$$1 = R_0 \leq R_1 \leq \dots R_\alpha \leq R_{\alpha+1} \leq \dots R_\gamma$$

of a group G by

$$\begin{aligned} R_1 &= \mathbf{Ln}(G) \\ R_{\alpha+1}/R_\alpha &= \mathbf{Ln}(G/R_\alpha) \text{ for all ordinals } \alpha < \gamma \\ R_\lambda &= \bigcup_{\beta < \lambda} R_\beta \text{ for limit ordinals } \lambda < \gamma \end{aligned}$$

$$\mathbf{Ln}(G/R_\gamma) = 1.$$

It may happen that $\mathbf{Ln}(G)$ is trivial. The group G is radical if and only if $G = R_\gamma$ for some term R_γ of this series. Since the Hirsch–Plotkin radical is always a characteristic subgroup, a group G is radical

if and only if G has an ascending series of *characteristic* subgroups with locally nilpotent factors. Here are some other useful properties of radical groups.

1.2.12. LEMMA. *Let G be a radical group.*

- (i) *If H is a subgroup of G , then H is a radical group;*
- (ii) *If H is a normal subgroup of G , then G/H is a radical group;*
- (iii) *If H, K are subgroups of G , such that H is normal in K , then K/H is a radical group;*
- (iv) $C_G(\mathbf{Ln}(G)) \leq \mathbf{Ln}(G)$.

PROOF. The assertions (i)–(iii) are left as an exercise. To prove (iv) we let $L = \mathbf{Ln}(G)$ and $C = C_G(L)$. Suppose, for a contradiction, that $C \not\leq L$. Then CL/L is a nontrivial normal subgroup of G/L . By (ii), CL/L is a radical group and hence $\mathbf{Ln}(CL/L) = K/L \neq 1$. Since $CL/L \triangleleft G/L$, it follows that $K/L \triangleleft G/L$ and hence K is normal in G . It follows easily that $K = L(K \cap C)$. Consider $K \cap C$. If H is a finitely generated subgroup of $K \cap C$, then $H/H \cap L \cong HL/L$ is nilpotent. Also $H \leq C = C_G(L)$, so $H \cap L \leq \zeta(H)$. It follows that H is nilpotent and hence $K \cap C$ is a locally nilpotent normal subgroup of G , from which we deduce that $K \cap C \leq L$. Hence $K \leq L$ and we obtain the contradiction desired. $\square$

We shall also need the following important and useful result.

1.2.13. PROPOSITION. *Let G be a finitely generated group and let C be a subgroup of G of finite index. Then C is also finitely generated.*

PROOF. Let M be a finite subset of G such that $G = \langle M \rangle$. We may assume that if $x \in M$, then $x^{-1} \in M$ also. Suppose that $k = |G : C|$ and let $\{g_1, \dots, g_k\}$ be a left transversal to C in G . Without loss of generality we may assume that $g_1 = 1$. If x is an arbitrary element of G , then $x(g_j C)$ is a left coset so there is an integer $x(j)$ such that $x(g_j C) = g_{x(j)} C$, for $1 \leq j \leq k$. Clearly the mapping $j \mapsto x(j)$ is a permutation of the set $\{1, \dots, k\}$. Furthermore, $xg_j = g_{x(j)}d(j, x)$ for some element $d(j, x) \in C$.

If c is an arbitrary element of C , then $c = x_m \dots x_1$ for certain $x_j \in M$, for $1 \leq j \leq m$. We have

$$\begin{aligned} c &= cg_1 = x_m \dots x_2 g_{x_1(1)} d(1, x_1) \\ &= g_{c(1)} d(x_{m-1} \dots x_1(1), x_m) \dots d(1, x_1). \end{aligned}$$

But $g_{c(1)}C = cg_1C = C$, so that $g_{c(1)} = 1$. It follows that $C = \langle d(j, x) | 1 \leq j \leq k, x \in M \rangle$ and hence C is finitely generated. $\square$

An immediate deduction we can make is as follows.

1.2.14. COROLLARY. *Let G be a group and let H be a normal locally finite subgroup of G such that G/H is also locally finite. Then G is locally finite.*

Our next result can be regarded as analogous to Proposition 1.2.10 and is also due to Plotkin [216, Corollary to Theorem 1].

1.2.15. PROPOSITION. *Let G be a group.*

- (i) *The product of the normal locally finite subgroups of G is a normal locally finite subgroup of G ;*
- (ii) *If H is an ascendant locally finite subgroup of G , then H^G is a normal locally finite subgroup of G ;*
- (iii) *The product of the normal radical subgroups of G is a normal radical subgroup of G ;*
- (iv) *If H is an ascendant radical subgroup of G , then H^G is a normal radical subgroup of G .*

PROOF. (i) As in the proof of Proposition 1.2.10 it suffices to prove this assertion for the product of two normal locally finite subgroups T_1, T_2 . The intersection $T_1 \cap T_2$ is locally finite and

$$T_1 T_2 / (T_1 \cap T_2) = T_1 / (T_1 \cap T_2) \times T_2 / (T_1 \cap T_2)$$

is clearly also locally finite. Then Corollary 1.2.14 can be applied to deduce the result that $T_1 T_2$ is locally finite.

The proof of (ii) is similar to the proof of Proposition 1.2.10(ii).

To complete the proof of (iii) and (iv) we note that a radical normal subgroup R of a group G has a characteristic series whose factors are locally nilpotent, since the Hirsch–Plotkin radical is always a characteristic subgroup. This observation now makes it easy to prove the result. $\square$

It is also easy to see that a product of normal soluble subgroups is locally soluble. However an example due to P. Hall (see [225, Theorem 8.19.1], for example) shows that a product of two normal locally soluble groups need not be locally soluble in general and the same example shows that the product of two normal hyperabelian groups need not be hyperabelian.

Proposition 1.2.15 shows that if G contains a non-trivial ascendant locally finite subgroup, then it has a unique largest characteristic locally finite subgroup, the *locally finite radical*, which we denote by $\mathbf{Lf}(G)$. This is of considerable importance for the class of generalized radical groups, which we define next.

A group G is called *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. It should be noted

that in the past some authors have used a more restrictive definition where the factors are locally nilpotent or *finite*.

It follows easily from the definition that a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the former case, the Hirsch–Plotkin radical of G is nontrivial. In the latter case, G contains a nontrivial normal locally finite subgroup, by Proposition 1.2.15, so the locally finite radical is nontrivial. Thus every generalized radical group has an ascending series of *normal*, indeed characteristic, subgroups with locally nilpotent or locally finite factors. Consequently, every generalized radical group is hyper (locally nilpotent or locally finite).

1.2.16. LEMMA. *Let G be a group.*

- (i) *G is generalized radical if and only if every non-trivial homomorphic image of G contains a non-trivial ascendant subgroup which is either locally nilpotent or locally finite;*
- (ii) *If G is a generalized radical group and H is a subgroup of G , then H is generalized radical;*
- (iii) *If G is a generalized radical group and H is a normal subgroup of G , then G/H is generalized radical;*
- (iv) *If G is a generalized radical group and H, K are subgroups of G such that H is normal in K , then K/H is generalized radical.*

EXERCISE 1.5. *Prove Lemma 1.2.16.*

The next result shows that periodic subgroups of generalized radical groups are locally finite. We shall be interested in the class of *locally generalized radical groups*. A group G is a locally generalized radical group if every finitely generated subgroup is a generalized radical group. The class of locally generalized radical groups will play a very prominent role in this book.

1.2.17. LEMMA. *Let G be a finitely generated periodic group. If G is generalized radical, then G is finite. Consequently, every periodic locally generalized radical group is locally finite.*

PROOF. Let

$$1 = H_0 \leq H_1 \leq \dots H_\alpha \leq H_{\alpha+1} \leq \dots H_\gamma = G$$

be an ascending series in G whose factors are locally nilpotent or locally finite. We proceed by transfinite induction on γ . If $\gamma = 1$, then either G is locally nilpotent or locally finite. Since a finitely generated periodic nilpotent group is finite, in either case G is finite and the result follows.

Consequently, let $\gamma > 1$ and suppose inductively that if H is a finitely generated subgroup of H_β , where $\beta < \gamma$, then H is finite. If γ

is a limit ordinal then, since G is finitely generated, $G \leq H_\beta$ for some $\beta < \gamma$, and hence G is finite, by the induction hypothesis. If $\gamma - 1$ exists we let $L = H_{\gamma-1}$. Then G/L is a periodic finitely generated group that is locally nilpotent or locally finite. In either case G/L is finite, as above, and so L is also finitely generated by Proposition 1.2.13. By the induction hypothesis L is finite and therefore G is finite, as required. $\square$

Next we discuss the chief factors of a group. Suppose that G is a group and U, V are normal subgroups of G . If $U \leq V$, then V/U is called a *chief factor* of G if V/U contains no proper nontrivial normal subgroups of G/U . If $U = 1$, then we call V a *minimal normal subgroup* of G . It is a consequence of Zorn's Lemma that every group has chief factors although not every group possesses minimal normal subgroups. The chief factors of locally nilpotent and locally soluble groups are of particular interest and we now discuss these. These results are due to A. I. Maltsev, but we give here the method of D. H. McLain [192].

1.2.18. PROPOSITION. *Let G be a locally soluble group. Then the chief factors of G are abelian. Furthermore the chief factors are either elementary abelian p -groups for some prime p or are torsion-free divisible groups.*

PROOF. First we show that if M is a minimal normal subgroup of G , then M is abelian. If M is not abelian choose elements $x, y \in M$ such that $z = [x, y] \neq 1$. Then $\langle z^G \rangle = M$, since M is minimal normal in G and hence $\langle x, y \rangle \leq \langle z^K \rangle$ for some finite subset K of G . Let $H = \langle z, K \rangle$. Since H is finitely generated, and therefore soluble, it follows that $Z = \langle z^H \rangle$ is also soluble. On the other hand, $z = [x, y] \in Z'$, so $Z = Z'$, which is a contradiction. Hence M is abelian.

Since M is abelian it has a torsion subgroup $\mathbf{Tor}(M)$, which is characteristic in M , so either $\mathbf{Tor}(M) = 1$ or $\mathbf{Tor}(M) = M$. In the former case, it follows that M is torsion-free. In this case, for each natural number n , the subgroup M^n is normal in G and hence $M^n = M$, so that M is divisible. In the case when M is periodic it is the direct product of its p -components, each of which is characteristic in M (and hence normal in G). It follows that M is a p -group for some prime p . Moreover, $S = \{x \in M \mid x^p = 1\}$ is a characteristic subgroup of M , and hence normal in G so $M = S$ and M is elementary abelian. This completes the proof. $\square$

The immediate corollary is that simple locally soluble groups are not complicated.

1.2.19. COROLLARY. *Each simple locally soluble group has prime order.*

There is a similar result concerning the chief factors of a locally nilpotent group.

1.2.20. PROPOSITION. *The chief factors of a locally nilpotent group are central of prime order.*

PROOF. It is sufficient to prove that a minimal normal subgroup, M , of the locally nilpotent group G is central of prime order. If $\zeta(G)$ does not contain M , then there exists $x \in M$ and $y \in G$ such that $[x, y] = z \neq 1$. Since $z \in M$ and M is a minimal normal subgroup of G we see that $M = \langle z^G \rangle$. Hence there exists a finite subset S of G such that $x \in \langle z^S \rangle$. Let $H = \langle x, y, S \rangle$ and $K = \langle x^H \rangle \leq H$. Then $z = [x, y] \in [K, H]$, so $\langle z^S \rangle \leq [K, H]$. Hence $x \in [K, H]$ and it follows that $\langle x^H \rangle = K = [K, H]$. Thus $[K_r, H] = K$ for each $r \in \mathbb{N}$. However, H is a finitely generated nilpotent group so, for some natural number c , we have $[K_c, H] = 1$. Consequently, $K = 1$, so $x = 1$ and $z = 1$, a contradiction which implies that M is central. Since every subgroup of the centre of a group G is normal in G , it follows that M has prime order. $\square$

For the next result we need some more notation and terminology. If $\mathfrak{X}$ is a class of groups then the $\mathfrak{X}$ -residual of a group G is defined to be

$$G^{\mathfrak{X}} = \bigcap \{N \triangleleft G \mid G/N \in \mathfrak{X}\}.$$

If the set $\mathbf{Res}_{\mathfrak{X}}(G) = \{N \triangleleft G \mid G/N \in \mathfrak{X}\}$ has a least element L , then $L = G^{\mathfrak{X}}$ and $G/G^{\mathfrak{X}} \in \mathfrak{X}$ but in general $G/G^{\mathfrak{X}}$ need not be an $\mathfrak{X}$ -group.

When $\mathfrak{X} = \mathfrak{A}$, the class of abelian groups, then the $\mathfrak{A}$ -residual of a group G is precisely the derived subgroup G' and $G/G^{\mathfrak{A}} \in \mathfrak{A}$, in this case.

More generally, if $\mathfrak{X} = \mathfrak{N}_c$, the class of nilpotent groups of nilpotency class at most c , then the $\mathfrak{N}_c$ -residual is precisely $\gamma_{c+1}(G)$ and again $G/G^{\mathfrak{N}_c} \in \mathfrak{N}_c$.

However, when $\mathfrak{X} = \mathfrak{N}$, the class of all nilpotent groups then, in general, $G/G^{\mathfrak{N}} \notin \mathfrak{N}$. Indeed this factor group need not even be locally nilpotent. For example, as we remarked earlier, if G is a free group then, by the theorem of Magnus mentioned earlier, $\gamma_{\omega}(G) = 1$. It follows that $G^{\mathfrak{N}} = 1$ and that $G/G^{\mathfrak{N}}$ is a free group.

For the class of groups $\mathfrak{X}$, a group G is called a *residually $\mathfrak{X}$ -group* if, for each non-trivial element $g \in G$, there is a normal subgroup H_g such that $g \notin H_g$ and $G/H_g \in \mathfrak{X}$.

EXERCISE 1.6. *Let G be a group and let $\mathfrak{X}$ be a class of groups. Prove that G is a residually $\mathfrak{X}$ -group if and only if $\cap \mathbf{Res}_{\mathfrak{X}}(G) = 1$.*

We shall denote the class of residually $\mathfrak{X}$ -groups by $\mathbf{r}\mathfrak{X}$. If $\mathfrak{X} = \mathfrak{F}$, the class of all finite groups, then we obtain the familiar class $\mathbf{r}\mathfrak{F}$ of *residually finite groups*. If p is a prime, we let $\mathfrak{F}_p$ denote the class of all finite p -groups and in this way we obtain the class $\mathbf{r}\mathfrak{F}_p$ of residually $\mathfrak{F}_p$ -groups. It is well known that every free group is residually $\mathfrak{F}_p$, for each prime p , a theorem originally due to Iwasawa [119].

It is easy to see that $G^{\mathfrak{X}}$ is always a characteristic subgroup of G and that $G/G^{\mathfrak{X}}$ is always a residually $\mathfrak{X}$ -group. For example when $\mathfrak{X} = \mathfrak{F}$, the class of all finite groups, then $G^{\mathfrak{X}} = G^{\mathfrak{F}}$ is the finite residual of G and $G/G^{\mathfrak{F}}$ is residually finite.

For the group G , we let $\mathbf{Frat}(G)$ denote the Frattini subgroup of G , the intersection of all maximal subgroups of G , with the understanding that if G has no maximal subgroups then $\mathbf{Frat}(G) = G$. It is well known, since a maximal subgroup of a finite p -group G is a normal subgroup, that $\mathbf{Frat}(G) = G'G^p$ in this case. This observation is very useful in our next result.

1.2.21. PROPOSITION (Kurdachenko [139]). *Let p be a prime and let G be a locally finite and residually $\mathfrak{F}_p$ -group. If $G/G'G^p$ is finite, then G is a finite p -group.*

PROOF. First we note that if K is a finite subgroup of G then, for each $g \in K$, there is a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g is a finite p -group. If $N = \cap \{N_g | g \in K\}$, then G/N is also a finite p -group, by Remak's Theorem and $K \cap N = 1$, so that K too is a finite p -group. Consequently, G is a p -group.

Let $L = G'G^p$ so that G/L is a finite elementary abelian p -group. Then $G = KL$ for some finite subgroup K . Let

$$\mathcal{M} = \{H | H \triangleleft G \text{ such that } G/H \text{ is a finite } p\text{-group}\}.$$

If $H \in \mathcal{M}$, then $LH/H = \mathbf{Frat}(G/H)$ since

$$\mathbf{Frat}(G/H) = (G/H)'(G/H)^p = G'G^pH/H,$$

and hence $G/H = (KH/H)\mathbf{Frat}(G/H)$. It is well known that the Frattini subgroup of a group is the set of non-generators (see [229, 5.2.12]), so we deduce that $G/H = KH/H$. Since $KH/H \cong K/(K \cap H)$ we see that $|G/H| \leq |K|$ and this is true for every subgroup H of $\mathcal{M}$. Consequently, the family $\mathcal{M}$ must be finite and the embedding of G into $\text{Dr}_{H \in \mathcal{M}} G/H$ shows that G is also finite. $\square$

Finally in this section, we shall also need some information concerning *stability groups*. Suppose that

$$(1.4) \quad 1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_j \triangleleft G_{j+1} \triangleleft \dots \triangleleft G_n = G$$

is a finite chain of subgroups of the group G . An automorphism α of G is said to *stabilize* this chain if $\alpha(xG_j) = xG_j$ for every $x \in G_{j+1}$ and all j such that $0 \leq j \leq n-1$. Thus α acts trivially on each of the factors of the series.

The set of all such automorphisms stabilizing the chain in Equation (1.4) is a subgroup of $\mathbf{Aut}(G)$ called the *stability group* of the chain.

1.2.22. THEOREM. *Let G be a group having a finite series*

$$1 = G_0 \leq G_1 \leq \dots \leq G_{n-1} \leq G_n = G$$

of normal subgroups. Then the stability group of this series is nilpotent of class at most $n-1$.

L. A. Kaloujnine first obtained this result in the paper [122]; it was further improved by P. Hall [106] where it is shown that the stability group of any chain of subgroups of length n (without any normality assumption) is nilpotent of class at most $\frac{1}{2}n(n-1)$.

1.3. Chernikov Groups and the Minimum Condition

In this section we discuss the class $\mathcal{C}$ of Chernikov groups. Such groups have been characterized in many ways and have been discussed in great detail in a number of books. For more detail than we provide here the reader should consult one or more of the books written by S. N. Chernikov [41], M. R. Dixon [49], M. I. Kargapolov and Yu. I. Merzlyakov [129], O. H. Kegel and B. A. F. Wehrfritz [134], A. G. Kurosh [172], or D. J. S. Robinson [223, 225, 229], which contain the highlights of the theory.

A *subgroup theoretical property* $\mathcal{P}$ is a property of certain subgroups of a group G so that always the identity subgroup of G has $\mathcal{P}$, and whenever a subgroup H of G has the property $\mathcal{P}$, then $\theta(H)$ also has $\mathcal{P}$ for each isomorphism θ of G with some other group. Typical examples of $\mathcal{P}$ are the properties of being normal, subnormal, abelian, subnormal abelian, central and so on.

Let $\mathcal{M}$ be an ordered set, with ordering $\leq$. Recall that $\mathcal{M}$ is said to satisfy the *minimum (or minimal) condition* if every non-empty subset of $\mathcal{M}$ has a minimal element.

We say that $\mathcal{M}$ has the *descending chain condition* if for every descending chain $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots$ of elements of $\mathcal{M}$, there is

some $k \in \mathbb{N}$ such that $a_k = a_{k+n}$ for all $n \in \mathbb{N}$. In this regard, we have the following well-known result.

1.3.1. PROPOSITION. *The ordered set $\mathcal{M}$ satisfies the minimal condition if and only if $\mathcal{M}$ satisfies the descending chain condition.*

EXERCISE 1.7. *Prove Proposition 1.3.1.*

Let $\mathcal{P}$ be a subgroup theoretical property. The group G is said to satisfy the *minimum (or minimal) condition on $\mathcal{P}$ -subgroups* ($\min\text{-}\mathcal{P}$ for short) if the set of $\mathcal{P}$ -subgroups of G , ordered by inclusion, satisfies the minimal condition.

For example, if $\mathcal{P}$ is the property of being a subgroup, the condition $\min\text{-}\mathcal{P}$ is called the minimum condition and often abbreviated to $\min$. If $\mathcal{P}$ represents the property of being an abelian subgroup we obtain the condition $\min\text{-ab}$, the minimum condition on abelian subgroups. The conditions $\min\text{-}p$ and $\min\text{-sn}$ denote the cases when $\mathcal{P}$ represents the property of being a p -subgroup (for the prime p) and, respectively, the condition of being a subnormal subgroup.

The minimal condition has played an important role both in ring theory and group theory for many years. However, the structure of groups with $\min$ is not as well understood as the corresponding structure of rings. Since groups with the minimal condition are highly relevant to the topic in which we are interested, we shall spend some time obtaining some of the positive results concerning the structure of groups with $\min$. Of course, all finite groups have $\min$. The following proposition lists some of the properties of groups with the minimal condition.

1.3.2. PROPOSITION. *Let G be a group.*

- (i) *If G satisfies $\min$, then every subgroup H of G satisfies $\min$;*
- (ii) *If G satisfies $\min$ and L is a normal subgroup of G , then G/L satisfies $\min$;*
- (iii) *If L is a normal subgroup of G such that L and G/L satisfy $\min$, then G satisfies $\min$;*
- (iv) *If G has a finite series of subgroups*

$$1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G,$$

where every factor H_j/H_{j-1} satisfies $\min$, for $1 \leq j \leq n$, then G satisfies $\min$;

- (v) *If $G = H_1 \times H_2 \times \dots \times H_n$, where each group H_j satisfies $\min$, for $1 \leq j \leq n$, then G satisfies $\min$;*
- (vi) *Let H be a subgroup of G and $H = \text{Dr}_{\lambda \in \Lambda} H_\lambda$. If G satisfies $\min$, then Λ is finite;*

- (vii) *G satisfies the minimum condition if and only if every countable subgroup of G satisfies the minimum condition.*

PROOF. The assertions (i) and (ii) are clear so we proceed to part (iii). Let

$$D_1 \geq D_2 \geq \cdots \geq D_n \geq \cdots$$

be a descending chain of subgroups of G . Since L and G/L satisfy min, there exists a natural number m such that $D_m \cap L = D_{m+n} \cap L$ and $D_m L = D_{m+n} L$ for all $n \in \mathbb{N}$. Therefore,

$$D_m = D_m \cap (D_{m+n} L) = D_{m+n} (D_m \cap L) = D_{m+n} (D_{m+n} \cap L) = D_{m+n},$$

for all $n \in \mathbb{N}$, and (iii) follows. Assertions (iv) and (v) are now immediate consequences of (iii). To prove (vi) we suppose that Λ is infinite and note that, in this case, Λ contains a countably infinite subset Γ which we denote by $\{\mu_n | n \in \mathbb{N}\}$. We let $\Delta = \Lambda \setminus \Gamma$, $\Lambda_n = \{\mu_k | k > n\} \cup \Delta$ and $G_n = \text{Dr}_{\lambda \in \Lambda_n} H_\lambda$, for each $n \in \mathbb{N}$. Then we obtain a strictly descending chain

$$G_1 > G_2 > \cdots > G_n > \cdots,$$

which gives a contradiction. It follows that Λ is finite.

Finally we will prove (vii) and remark that necessity is clear. To prove sufficiency, suppose that G has an infinite strictly descending chain of subgroups

$$G_1 > G_2 > \cdots > G_n > \cdots$$

For each $j \in \mathbb{N}$ choose $g_j \in G_j \setminus G_{j+1}$ and let $H = \langle g_j | j \in \mathbb{N} \rangle$. Then H is a countable subgroup of G so satisfies min, by hypothesis. However it is clear that

$$H \cap G_1 > H \cap G_2 > \cdots > H \cap G_n > \cdots$$

is a strictly descending chain of subgroups of H , which is a contradiction. Hence every descending chain of subgroups terminates so that G has the minimum condition, by Proposition 1.3.1. The result follows. $\square$

The most important examples of infinite groups with the minimal condition are the *Prüfer groups of type p^∞* . Initially such groups were called *quasicyclic p -groups*, but this latter terminology is now often used for groups whose proper subgroups are cyclic and among such groups there are some which are very different from Prüfer groups. We now describe the structure of Prüfer p -groups.

Let p be a prime. For each natural number $n \geq 1$ let $G_n = \langle a_n \rangle$ be a cyclic group of order p^n and, for each $n \in \mathbb{N}$, let $\theta_n : G_n \longrightarrow G_{n+1}$

be the monomorphism defined by $\theta_n(a_n) = a_{n+1}^p$. In this way, we can think of G_n as a subgroup of G_{n+1} and hence we can form the group

$$G = \bigcup_{n \in \mathbb{N}} G_n,$$

which is the union of a chain of cyclic p -groups of orders $p, p^2, \dots$.

The group G obtained above is called a *Prüfer p -group*, or a *Prüfer group of type p^∞* , and is denoted by C_{p^∞} .

There are numerous useful descriptions of this group. It can be realized in terms of generators and relations as

$$C_{p^\infty} = \langle a_n \mid a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbb{N} \rangle.$$

Prüfer p -groups also arise somewhat more concretely; they can be thought of as the multiplicative group of complex p th roots of unity, or as the set of elements of p -power order in the additive abelian group $\mathbb{Q}/\mathbb{Z}$. Thus $C_{p^\infty} \cong \mathbf{Tor}_p(\mathbb{Q}/\mathbb{Z})$. The Prüfer groups play a very important role in group theory. They are the main examples of groups in which every proper subgroup is finite; such groups are called *quasifinite*.

1.3.3. LEMMA. *Let p be a prime. Every proper subgroup of C_{p^∞} is finite. In particular, C_{p^∞} has the minimum condition on subgroups.*

EXERCISE 1.8. *Prove Lemma 1.3.3.*

Consequently, the group C_{p^∞} is a fundamental example of a group which satisfies the minimal condition on subgroups.

We recall that a group G is called *divisible* (many authors use this word only if the group is abelian, and reserve the term *radicable* in the general case) if, for each $g \in G$ and each $n \in \mathbb{Z}$, the equation $x^n = g$ always has a solution $x \in G$. Let $G^n = \langle g^n \mid g \in G \rangle$. If G is an abelian group then $G^n = \{g^n \mid g \in G\}$ and it is then clear that an abelian group G is divisible if and only if $G = G^n$ for each $n \in \mathbb{N}$.

Thus C_{p^∞} and $\mathbb{Q}$ are examples of divisible groups. In fact the structure of divisible abelian groups is well known: For the prime p , divisible abelian p -groups are direct products of Prüfer p -groups, whilst torsion-free divisible abelian groups are direct products of copies of $\mathbb{Q}$ (see [88, Theorem 23.1], for example). Every abelian group G has a maximal divisible subgroup $D = \mathbf{Div}(G)$, the *divisible (or radicable) part* of G . Certainly, $\mathbf{Div}(G)$ is a characteristic subgroup of G . Since divisible subgroups of abelian groups are well known to be direct summands it follows that each abelian group can be written in the form $G = D \times R$, for some subgroup R which contains no nontrivial divisible subgroups

(see [88, Theorem 21.3], for example). Such a subgroup R is said to be *reduced*.

The concept of divisibility can be generalized further. It is clear that no nontrivial finite group is divisible and since divisibility is inherited by factor groups a divisible group has no proper subgroups of finite index. Groups which contain no proper subgroups of finite index are called $\mathfrak{F}$ -*perfect*, so divisible groups are $\mathfrak{F}$ -perfect. Every abelian $\mathfrak{F}$ -perfect group is divisible. Indeed if $G \neq G^n$ for some positive integer n , then the order of the elements of G/G^n divides n and, in particular, these orders are bounded. By the first Prüfer Theorem (see [88, Theorem 17.2], for example), G/G^n is a direct product of finite cyclic subgroups and hence has a proper subgroup of finite index. This contradiction shows that $G = G^n$, for each $n \in \mathbb{N}$, so that G is divisible.

By considering a minimal subgroup of finite index in a group with the minimal condition it is possible to deduce the following result.

1.3.4. LEMMA. *Every group satisfying the minimal condition on subgroups has a normal $\mathfrak{F}$ -perfect subgroup of finite index.*

It follows from Proposition 1.3.2 that a finite direct product of Prüfer groups and finite cyclic groups has the minimum condition. This establishes the easy half of the following theorem due to Kurosh [170].

1.3.5. THEOREM. *Let G be an abelian group. Then G has the minimum condition if and only if G is a direct product of finitely many Prüfer p -groups and finite cyclic groups.*

PROOF. If G has min, then G must be periodic since the infinite cyclic group does not have min.

By Lemma 1.3.4, G contains an $\mathfrak{F}$ -perfect subgroup D of finite index. As we saw above D is divisible, so $G = D \times R$, for some reduced subgroup R . Clearly R is a finite abelian group and hence is a direct product of finitely many cyclic subgroups. Also, as we mentioned above, D is a direct product of Prüfer subgroups, and Proposition 1.3.2 shows that the set of direct factors in this direct product is finite. The result follows. $\square$

A group G is called a *Chernikov group* if G contains a normal subgroup D of finite index which is a direct product of finitely many Prüfer p -groups.

Such groups were named in honour of S. N. Chernikov, who made an extensive study of groups with the minimum condition. The subgroup D is easily seen to be the maximal divisible subgroup of G which we again call the *divisible (or radicable) part* of G , denoted by $\mathbf{Div}(G)$.

By Proposition 1.3.2 and Theorem 1.3.5 every Chernikov group satisfies the minimal condition. Using Theorem 1.3.5 and Proposition 1.3.2 we obtain the following properties of Chernikov groups. For an abelian p -group G and natural number n we let $\Omega_n(G) = \{x \in G \mid x^{p^n} = 1\}$.

1.3.6. COROLLARY. *Let G be a group.*

- (i) *If G is a Chernikov group, then every subgroup of G is also Chernikov;*
- (ii) *If G is a Chernikov group and L is a normal subgroup of G , then G/L is Chernikov;*
- (iii) *If L is a normal subgroup of G such that L and G/L are divisible Chernikov groups, then G is a divisible abelian Chernikov group;*
- (iv) *If L is a normal subgroup of G such that L and G/L are Chernikov groups, then G is a Chernikov group;*
- (v) *If G has a finite series of subgroups $1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$, in which every factor H_j/H_{j-1} is Chernikov, for $1 \leq j \leq n$, then G is also a Chernikov group.*

PROOF. The assertions (i) and (ii) are clear so we prove (iii). Let $L = \operatorname{Dr}_{p \in \Pi(L)} L_p$, where $L_p = \mathbf{Tor}_p(G)$, for $p \in \Pi(L)$. We also let $M_n = \operatorname{Dr}_{p \in \Pi(L)} \Omega_n(L_p)$, for $n \in \mathbb{N}$. Then M_n is a finite G -invariant subgroup of G , for each $n \in \mathbb{N}$. It follows that $G/C_G(M_n)$ is finite, for each $n \in \mathbb{N}$. Since L is abelian, $L \leq C_G(M_n)$. On the other hand, G/L is $\mathfrak{F}$ -perfect and hence contains no proper subgroups of finite index. It follows that $G = C_G(M_n)$, for all $n \in \mathbb{N}$ and since $L = \bigcup_{n \in \mathbb{N}} M_n$ we deduce that $L \leq \zeta(G)$. Let $G/L = \operatorname{Dr}_{1 \leq j \leq t} P_j/L$, where P_j/L is a Prüfer subgroup for $1 \leq j \leq t$. Since $L \leq \zeta(G)$ and P_j/L is locally cyclic, P_j is abelian. Clearly P_j is $\mathfrak{F}$ -perfect, and hence divisible. Using the argument above, we obtain the inclusion $P_j \leq \zeta(G)$, so $G = P_1 P_2 \dots P_t$ is abelian. Clearly G is $\mathfrak{F}$ -perfect and satisfies the minimal condition, by Proposition 1.3.2. Part (iii) now follows by Theorem 1.3.5.

(iv) Suppose first that L is finite. By Proposition 1.3.2, G satisfies min, and Lemma 1.3.4 shows that G contains a normal $\mathfrak{F}$ -perfect subgroup K of finite index. Since L is finite, $G/C_G(L)$ is finite so $K/C_K(L)$ is finite. As K is $\mathfrak{F}$ -perfect, we see that $K = C_K(L)$, so $K \cap L \leq \zeta(K)$. Being $\mathfrak{F}$ -perfect, KL/L coincides with the divisible part of the Chernikov group G/L . It follows that $K/(K \cap L) \cong KL/L$ is an abelian Chernikov group. As above $K/(K \cap L) = \bigcup_{n \in \mathbb{N}} K_n/(K \cap L)$, where the subgroups $K_n/(K \cap L)$ are finite for all $n \in \mathbb{N}$. It follows that $K = \bigcup_{n \in \mathbb{N}} K_n$, where the subgroups K_n are finite and normal for

all $n \in \mathbb{N}$. Using the arguments above we see that K is abelian and hence divisible. Since G/K is finite, Theorem 1.3.5 now shows that G is Chernikov.

Suppose now that L is infinite and let $D = \mathbf{Div}(L)$. Then D is a G -invariant subgroup and L/D is finite. Our proof above shows that G/D is a Chernikov group. Let $K/D = \mathbf{Div}(G/D)$. Then K/D is a divisible Chernikov subgroup. By (iii) K is a divisible Chernikov subgroup and the finiteness of G/K shows that G is Chernikov.

The assertion (v) is an immediate corollary of (iv). □

S. N. Chernikov formulated the problem of whether every group satisfying the minimum condition is Chernikov (albeit using different terminology) and one of his first results in this direction was to show that a locally soluble group satisfies the minimal condition if and only if it is Chernikov (see [41, Theorem 1.1], for example). One of the highlights of the theory of locally finite groups is the theorem of V. P. Shunkov and, independently, O. H. Kegel and B. A. F. Wehrfritz who showed that every locally finite group satisfying min is Chernikov (see [134, Theorem 5.8]). We shall address some of these results in more detail in Chapter 3.

However, a negative answer to Chernikov's problem was obtained by A. Yu. Ol'shanskii (see [209, Chapters 28, 35, 38]) who constructed exotic examples of periodic quasifinite groups. These examples, termed Tarski monsters, so named after A. Tarski who first hypothesized their existence, can be constructed for any sufficiently large prime p . They are infinite 2-generator simple p -groups whose proper subgroups all have order p . Clearly such groups have min and are not even locally finite, let alone Chernikov. It is clear that Chernikov groups are countable. However there exist uncountable groups satisfying the minimum condition, by a result of V. N. Obraztsov [207]. We shall see how the construction of such groups proceeds in Chapter 6.

1.4. Linear Groups

In this book much of our work is concerned with generalized soluble groups; such groups have many abelian sections. If A, B are subgroups of a group G such that $A \triangleleft B$ and B/A is abelian then we can think of $K = N_G(B)/C_G(B/A)$ as a subgroup of the group of automorphisms of the abelian group B/A . Here, two fundamental cases arise, when $B_0 = B/A$ is torsion-free and also when $B_0 = B/A$ is periodic. In either case B_0 is a $\mathbb{Z}$ -module. In the first case, the $\mathbb{Z}$ -*injective hull* (or *divisible hull*) [88, Chapter IV, Section 24], a divisible abelian torsion-free group D , containing B_0 , such that D/B_0 is periodic, plays an important role.

1.4.1. LEMMA. *Let A be a torsion-free abelian group and let G be a subgroup of $\mathbf{Aut}(A)$. If D is the divisible hull of A then G is isomorphic to a subgroup of $\mathbf{Aut}(D)$.*

EXERCISE 1.9. *Prove Lemma 1.4.1.*

We can think of D as a vector space over $\mathbb{Q}$ so that it is then possible to embed $K = N_G(B)/C_G(B/A)$ as a subgroup of the group $GL(\mathbb{Q}, D)$ of all non-singular linear transformations of D .

If B_0 is periodic, then our study often reduces to the case when B_0 is a p -group for some prime p . In this case, a very important role is played by the structure of the *lower layer*,

$$\Omega_1(B_0) = \{b \in B_0 | b^p = 1\}.$$

We may think of $\Omega_1(B_0)$ as a vector space over the prime field $\mathbb{F}_p$. Furthermore, suppose that $1 \neq z \in \zeta(K)$ and let $J = \mathbb{F}_p\langle x \rangle$ be the group ring of the infinite cyclic group $\langle x \rangle$ over the field $\mathbb{F}_p$. We make $\Omega_1(B_0)$ into a J -module by defining $b \cdot x = b^z$ for every $b \in \Omega_1(B_0)$. If this module is J -torsion-free, then we can construct $E = \Omega_1(B_0) \otimes_J F$, where F is the field of fractions of the integral domain J . Then E is a vector space over F and in this case $K = N_G(B)/C_G(B/A)$ can be considered as a subgroup of $GL(F, E)$.

Therefore, it very often happens that a generalized soluble group induces a group of automorphisms on its abelian factor groups that is isomorphic to a subgroup of $GL(F, V)$, for some field F and some vector space V . In many instances (and these are the ones usually of concern to us) $\dim_F(V)$ is finite. In this case $GL(F, V)$ is isomorphic to the group of all non-singular $n \times n$ matrices with coefficients in F , which we denote by $GL_n(F)$. The theory of matrix groups is well established; its scope is broad and not limited to mathematical disciplines alone. In our work, we have to use a variety of important results from this theory. For our considerations the most convenient reference is the excellent book of B. A. F. Wehrfritz [258].

We now list some of the results we shall need without proof, accompanied by the appropriate reference, so the reader has easy access to the proof. A group G is called *linear* (more precisely, finite dimensional linear) if it is isomorphic to a subgroup of $GL_n(F)$ for some natural number n and some field F . The first result we need is a result concerning the periodic subgroups of $GL_n(\mathbb{Q})$. A proof of this result can be found in [258, Theorem 9.33]. We shall let $\mathbb{Z}_p$ denote the field of p -adic numbers, for the prime p .

1.4.2. PROPOSITION. *Let T be a periodic subgroup of $GL_r(\mathbb{Q})$ or $GL_r(\mathbb{Z}_p)$, for some prime p . Then T is finite and there is a function $\rho: \mathbb{N} \rightarrow \mathbb{N}$ such that $|T| \leq \rho(r)$.*

In the theory of finite groups an important role is played by minimal normal subgroups. In the study of torsion-free groups such minimal normal subgroups need not exist. It is often useful to look at normal abelian subgroups of minimal 0-rank (the meaning of which will be explained later) however. These are the so-called *rationaly irreducible* groups. Much of the work we do concerning rank conditions is dependent upon such groups.

We mention next the following far-reaching theorem of J. Tits [252], which is known as the Tits alternative.

1.4.3. THEOREM. *Let G be a subgroup of $GL_n(F)$, for some field F . If G contains no non-abelian free group then G has a soluble normal subgroup D such that G/D is locally finite. Moreover if F has characteristic 0, then G/D is finite.*

1.4.4. COROLLARY. *Let G be a finitely generated subgroup of $GL_n(F)$ for some field F . If G contains no non-abelian free subgroups then G is soluble-by-finite.*

Since a non-abelian free group is certainly not locally generalized radical we deduce the following.

1.4.5. COROLLARY. *Let G be a locally generalized radical subgroup of $GL_n(F)$, for the field F . Then there is a soluble normal subgroup D of G such that G/D is locally finite. Moreover, if F has characteristic 0, then G/D is finite.,*

Let V be a vector space over a field F and let G be a subgroup of $GL_n(F)$. Suppose that C, D are G -invariant subspaces of V such that $D \leq C$. Then C/D is called a *G -chief factor* of V if the only G -invariant subspaces B such that $D \leq B \leq C$ are C and D . Since V is an FG -module, this means that C/D is a simple FG -module. The subgroup G is called *irreducible* if V is a simple FG -module.

If V is a finite dimensional vector space over F then the finiteness of $\dim_F(V)$ implies that V has a finite series

$$0 = B_0 \leq B_1 \leq \cdots \leq B_n = V$$

of G -invariant subspaces whose factors are G -chief factors. It then follows that the factor groups $G/C_G(B_j/B_{j-1})$ are irreducible, for $1 \leq j \leq n$. Let

$$Z = C_G(B_1) \cap C_G(B_2/B_1) \cap \cdots \cap C_G(B_n/B_{n-1}).$$

We can choose a basis for V in such a way that Z will be isomorphic to a subgroup of $UT_n(F)$, the group of unitriangular matrices with coefficients in F . Furthermore, by Remak's Theorem, G/Z is isomorphic to a subgroup of

$$G/C_G(B_1) \times G/C_G(B_2/B_1) \times \cdots \times G/C_G(B_n/B_{n-1})$$

which of course is a finite direct product of irreducible linear groups.

We require some information concerning the structure of the group $UT_n(F)$. For each integer k such that $1 \leq k \leq n$, we define subgroups $UT_n^k(F)$ as follows. If $k = 1$ we let $UT_n^1(F) = UT_n(F)$ and, if $1 < k < n$, we let $UT_n^k(F)$ be the subgroup of $UT_n(F)$ defined by

$$UT_n^k(F) = \{A = (\alpha_{i,j}) \in UT_n(F) \mid \alpha_{i,r} = 0 \text{ for } i+1 \leq r \leq i+k-1\}.$$

We set $UT_n^n(F) = E$, the identity matrix. Thus $UT_n^k(F)$ is precisely the subgroup of $UT_n(F)$ consisting of matrices in which the first $k-1$ superdiagonals are all 0. It is easy to show that if $A \in UT_n^k(F)$ and $B \in UT_n^l(F)$ then $[A, B] \in UT_n^{k+l}(F)$, so the identity $[UT_n^k(F), UT_n^l(F)] = UT_n^{k+l}$ holds (see [129, 3.2], for example). It then follows, again quite easily, that the series

$$UT_n(F) = UT_n^1(F) \geq UT_n^2(F) \geq \cdots \geq UT_n^{n-1}(F) \geq UT_n^n(F) = E$$

is simultaneously the upper and the lower central series of $UT_n(F)$. Thus the group $UT_n(F)$ is nilpotent of class at most $n-1$. Furthermore, if F_+ denotes the additive group of F then

$$UT_n^m(F)/UT_n^{m+1}(F) \cong \underbrace{F_+ \oplus \cdots \oplus F_+}_{n-m}$$

(see, [129, 4.2], for example).

Further, if $T_n(F)$ denotes the group of all upper triangular matrices over F with non-zero determinant, then it is clear that $T_n(F)$ is soluble of derived length at most n .

The structure of irreducible soluble linear groups has been described by A. I. Maltsev [188]. The reader can consult Wehrfritz [258, Theorem 3.6 and Lemma 3.5] for a proof. This structure is given in the next two theorems.

1.4.6. THEOREM. *Let G be a soluble subgroup of $GL_n(F)$ for the field F . If G is irreducible then there exists an integer valued function $\mu(n)$ such that G has an abelian normal subgroup of index dividing $\mu(n)$.*

It follows from the proof of Theorem 1.4.6 that $\mu(n) \leq n!(n^2(n^2)!)^n$. We shall call $\mu(n)$ the *Maltsev function*.

1.4.7. THEOREM. *Let G be a soluble subgroup of $GL_n(F)$ for some field F . Then there exists a finite field extension E of F such that G has a normal subgroup H of index dividing $\mu(n)$ which is conjugate in $GL_n(E)$ to a subgroup of $T_n(E)$. In particular, G is nilpotent-by-abelian-by-finite.*

If G is a soluble group then we let $\mathbf{dl}(G)$ denote the derived length of G . One crude estimate for the derived length of a finite soluble group is easy to establish.

If $n \in \mathbb{N}$ and $n = p_1^{k_1} \dots p_m^{k_m}$ is the primary decomposition of n , we let $e(n) = k_1 + \dots + k_m$ and clearly

$$\begin{aligned} e(n) &= \log_{p_1}(p_1^{k_1}) + \dots + \log_{p_m}(p_m^{k_m}) \\ &\leq \log_2(p_1^{k_1}) + \dots + \log_2(p_m^{k_m}) = \log_2 n. \end{aligned}$$

From this it follows that if G is a finite soluble group then $\mathbf{dl}(G) \leq e(|G|) \leq \log_2 |G|$.

A particular consequence of Theorem 1.4.7 is a theorem of Zassenhaus [278].

1.4.8. THEOREM. *Let G be a locally soluble linear group of degree n over a field F . Then G is soluble. Furthermore there is a function $\zeta : \mathbb{N} \rightarrow \mathbb{N}$ such that $\mathbf{dl}(G) \leq \zeta(n)$.*

PROOF. We may assume that F is algebraically closed. If H is a finitely generated subgroup of G then H is soluble and Theorem 1.4.7 implies that H contains a normal subgroup $L(H)$ of index dividing $\mu(n)$, and such that $L(H)$ is triangularizable. We noted earlier that $\mathbf{dl}(T_n(F)) \leq n$, so $L(H)$ therefore has derived length at most n . Thus each finitely generated subgroup of G has derived length at most $\zeta(n) = e(\mu(n)) + n$ and the result follows. $\square$

The function $\zeta(n)$ is called the *Zassenhaus function*. B. Huppert [117, Satz 9] proved that $\zeta(n) \leq 2n$ and furthermore if $n = 2$, then there is a soluble linear group of derived length 4 (see [117, p.#495]). The most precise bound for the function $\zeta(n)$ has been obtained by M. F. Newman [206]. His paper includes a table of values of $\zeta(n)$ for all $n \leq 74$ and he proved that for $r \geq 66$ the relation

$$5 \log_9(r-1) + D \leq \zeta(r) \leq 5 \log_9(r-2) + D + (3/2)$$

is valid, where $D = (17/2) - 15(\log 2)/(2 \log 3)$. The lower bound is actually attained whenever $r = 24 \cdot 9^k + 1$ and the upper bound is attained whenever $r = 8 \cdot 9^k + 2$, for $k \in 0 \cup \mathbb{N}$.

The following corollary to Zassenhaus's theorem will also be very useful.

1.4.9. COROLLARY. *Let G be a locally radical linear group of degree n over a field F . Then G is soluble of derived length at most $\zeta(n)$.*

PROOF. Let K be an arbitrary finitely generated subgroup of G and let

$$1 = R_0 \leq R_1 \leq \dots R_\alpha \leq R_{\alpha+1} \leq \dots R_\gamma = K$$

be the radical series of K . We use transfinite induction on the length of the series to show, using Theorem 1.4.8, that K is soluble of derived length at most $\zeta(n)$.

If $\gamma = 1$ then K is nilpotent, hence soluble and we may apply Theorem 1.4.8 to deduce the result. Suppose that $\gamma > 1$ and that we have already proved that R_α is soluble of derived length at most $\zeta(n)$ for all $\alpha < \gamma$. If γ is a limit ordinal then for each $\alpha < \gamma$ the group R_α is soluble of derived length at most $\zeta(n)$ and hence K must be soluble of this derived length.

If $\gamma - 1$ exists then, inductively, $R_{\gamma-1}$ is soluble. Thus K is soluble-by-nilpotent, since it is finitely generated, so K is soluble and Theorem 1.4.8 implies that K is of derived length at most $\zeta(n)$. Consequently the group G is locally soluble and Theorem 1.4.8 implies the result. This completes the proof. $\square$

We next describe one situation in which irreducible linear groups arise. Accordingly, let A be a torsion-free abelian group and let G be a group of automorphisms of A . We say that A is *rationally irreducible with respect to G* or that A is *G -rationally irreducible* if, for every non-trivial G -invariant subgroup B of A , the factor group A/B is periodic. We shall also say that G is *rationally irreducible on A* .

The reason for the terminology is as follows. Suppose that A is a torsion-free abelian group. As we have seen above, the action of G on A can be extended to an action of G on the vector space $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

1.4.10. PROPOSITION. *Let G be a group of automorphisms of the torsion-free abelian group A . Then G is rationally irreducible on A if and only if G is irreducible as a group of linear transformations of the vector space $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$.*

EXERCISE 1.10. *Prove Proposition 1.4.10.*

The following generalization of Theorem 1.4.6 will also be needed.

1.4.11. THEOREM. *Let G be a soluble-by-finite subgroup of $GL_n(F)$, for the field F . If G is irreducible, then G is abelian-by-finite.*

PROOF. Let V be a vector space of F -dimension n on which G acts irreducibly and let H be a soluble normal subgroup of finite index in G . Since $\dim_F(V)$ is finite, there is a non-zero H -invariant subspace, W , of minimal dimension over F . Then W is a simple FH -submodule. Theorem 1.4.6 shows that $H/C_H(W)$ is abelian-by-finite. If $W = V$, then $C_H(W) = C_H(V) = 1$, so that H and hence G is abelian-by-finite. Therefore we may suppose that $W \neq V$.

If $g \in G, a \in W, h \in H$ then we have $(ag)h = a(ghg^{-1})g \in Wg$, since $H \triangleleft G$, which shows that Wg is also H -invariant. Let C be a non-zero, H -invariant subspace of Wg . Then Cg^{-1} is a non-zero H -invariant subspace of W and the minimal choice of W implies that $Cg^{-1} = W$. It follows that $C = Wg$, so that Wg is a simple FH -submodule. Hence either $Wg = W$ or $Wg \cap W = 0$. Since $W \neq V$ it follows that there must be some element $g_1 \in G$ such that $Wg_1 \cap W = 0$. If $Wg \leq Wg_1 \oplus W$, for all $g \in G$, then $V = Wg_1 \oplus W$. Otherwise there is an element $g_2 \in G$ such that $Wg_2 \cap (Wg_1 \oplus W) = 0$ and we can then form $W \oplus Wg_1 \oplus Wg_2$. Proceeding in this fashion and using the fact that V has finite F -dimension, we deduce that there are elements $g_1, g_2, \dots, g_s$ of G such that $V = W \oplus Wg_1 \oplus \dots \oplus Wg_s$. Theorem 1.4.6 implies, as above, that $H/C_H(Wg_j)$ is abelian-by-finite for $1 \leq j \leq s$. Since $C_H(W) \cap C_H(Wg_1) \cap \dots \cap C_H(Wg_s) = 0$, Remak's Theorem gives the embedding

$$H \longrightarrow H/C_H(W) \times H/C_H(Wg_1) \times \dots \times H/C_H(Wg_s),$$

which in turn implies that H , and hence G , is abelian-by-finite. $\square$

Using [31, Theorem 2] and Corollary 1.4.5 we obtain the following consequence.

1.4.12. COROLLARY. *Let F be a finite field extension of $\mathbb{Q}$ and let G be a locally generalized radical subgroup of $GL_n(F)$. If G is irreducible, then G contains a countable free abelian normal subgroup of finite index.*

1.5. Some Relationships Between the Factors of the Upper and Lower Central Series

It is well known that the k th term, $\zeta_k(G)$, of the upper central series of a group G coincides with G if and only if the $(k+1)$ th term, $\gamma_{k+1}(G)$, of the lower central series of G is trivial. For infinite central series there is no such general result. For example, we noted above that, by Magnus's theorem [172, Chapter IX, §36], the lower central series of a non-abelian free group G has the property that $\gamma_\omega(G) = 1$, but of course $\zeta(G) = 1$. In a later chapter, we exhibit a hypercentral Chernikov p -group whose lower central series is finite and does not

terminate in the identity. A very natural problem arises for groups with a finite central series as follows:

Let G be a group such that $\mathbf{zl}(G) = n$ is finite. For which classes of groups $\mathfrak{X}$ does the relation $G/\zeta_n(G) \in \mathfrak{X}$ imply that $\gamma_{n+1}(G) \in \mathfrak{X}$ or, more generally, when does the (locally) nilpotent residual belong to $\mathfrak{X}$?

An important special case arises here when $n = 1$. In this case our question asks for which classes $\mathfrak{X}$ is it the case that the relation $G/\zeta(G) \in \mathfrak{X}$ implies that $G' = \gamma_2(G) \in \mathfrak{X}$?

The case when $\mathfrak{X} = \mathfrak{F}$, the class of finite groups, is one that naturally springs to mind first. In this case, we obtain groups G whose central factor group $G/\zeta(G)$ is finite. Such groups are naturally called *centre-by-finite* groups and are closely connected with an important, interesting class of groups known as FC-groups.

A group G is called an *FC-group* if the conjugacy class $x^G = \{x^g | g \in G\}$ is finite for each element $x \in G$. Thus x has finitely many conjugates in G and it is well known that this implies that $G/C_G(x^G)$ is finite. In particular, a group G is an FC-group if and only if the index $|G : C_G(x)|$ is finite, for all $x \in G$.

1.5.1. LEMMA. *Let G be an FC-group. Then every finitely generated subgroup of G is centre-by-finite.*

EXERCISE 1.11. *Prove Lemma 1.5.1.*

An important role is played in the study of FC-groups by the following result, often called Dietzmann's Lemma, due to A. P. Dietzmann [47].

1.5.2. PROPOSITION. *Let G be a group and let M be a finite subset of G . Suppose that every element of M has finite order and that $x^G \subseteq M$, for every element $x \in M$. Then $\langle M \rangle = \langle M \rangle^G$ is finite.*

PROOF. Let $|M| = k$ and let $d = \text{lcm}\{|x| | x \in M\}$. Every element $y \in \langle M \rangle$ can be written in the form $y = x_1 x_2 \dots x_m$, where $x_j \in M$ for $1 \leq j \leq m$. Suppose that $m > k(d-1)$. Then at least one of the elements of M occurs at least d times in this product and we denote this particular element by u . Let s be the first index for which $x_s = u$. We have

$$y = x_1 x_2 \dots x_m = u(u^{-1}x_1u)(u^{-1}x_2u) \dots (u^{-1}x_{s-1}u)(u^{-1}u)x_{s+1} \dots x_m.$$

Let $w_j = u^{-1}x_ju$, so $w_j \in M$ for $1 \leq j \leq s-1$, by hypothesis, and $y = uw_1 \dots w_{s-1}x_{s+1} \dots x_m$. Let t be the next index for which $x_t = u$.

By the same argument we have

$$\begin{aligned}
 y &= uw_1 \dots w_{s-1} x_{s+1} \dots x_m \\
 &= uu(u^{-1}w_1u) \dots (u^{-1}w_{s-1}u)(u^{-1}x_{s+1}u) \\
 &\dots (u^{-1}x_{t-1}u)(u^{-1}u)x_{t+1} \dots x_m \\
 &= u^2v_1 \dots v_{s-1}v_{s+1} \dots v_{t-1}x_{t+1} \dots x_m,
 \end{aligned}$$

suitably renaming the elements.

Continuing in this way we can write $y = u^d z_1 \dots z_{m-d} = z_1 \dots z_{m-d}$, for certain elements $z_1, \dots, z_{m-d}$ of M . Repeating the argument as needed, we will eventually obtain the expression $y = y_1 \dots y_n$, where $y_1, \dots, y_n \in M$ and $n \leq k(d-1)$. Hence every element of $\langle M \rangle$ can be written as a product of at most $k(d-1)$ elements so $\langle M \rangle$ is finite. The hypotheses clearly imply that $\langle M \rangle = \langle M \rangle^G$. $\square$

The following result is immediate.

1.5.3. COROLLARY. *Let G be an FC-group. Then $\mathbf{Tor}(G)$ coincides with the set of elements of finite order. Furthermore $G/\mathbf{Tor}(G)$ is torsion-free.*

We present several results concerning torsion-free FC-groups. The first of the results we obtain, due to G. A. Miller and H. Moreno [197], seems totally unrelated.

1.5.4. PROPOSITION. *Let G be a finite group whose proper subgroups are abelian. Then G is soluble.*

PROOF. Suppose the contrary and among all finite insoluble groups whose proper subgroups are abelian choose one, G , of least order. If G contains a proper normal subgroup H then the minimal choice of $|G|$ implies that G/H is soluble—here we note that the proper subgroups of G/H are abelian. Since H is abelian, G is soluble and we obtain a contradiction. Hence G is simple.

Let M be a maximal subgroup of G and suppose that there is a proper subgroup K such that $K \not\leq M$. Since M is maximal it follows that $G = \langle M, K \rangle$. Clearly, $[M \cap K, M] = [M \cap K, K] = 1$ and we deduce that $M \cap K \leq \zeta(G)$. Thus $M \cap K = 1$, since G is simple.

Let $p \in \Pi(M)$ and let M_p be a Sylow p -subgroup of M . Let P be a Sylow p -subgroup of G containing M_p . If $P \not\leq M$ then $M \cap P = 1$, as above. However, $M_p \leq M \cap P$, a contradiction. Hence M_p is a Sylow p -subgroup of G and it follows that $m = |M|$ and $t = |G : M|$ are relatively prime. Since M is a maximal, non-normal subgroup of G we have $M = N_G(M)$. It follows that M has exactly t conjugates.

We note that M^g is a maximal subgroup of G , for each $g \in G$ and, as above, $M \cap M^g = 1$, whenever $g \notin M$. Each of the subgroups M^g contains exactly $m - 1$ nontrivial elements and hence M and all its conjugates contain exactly $(m - 1)t$ nontrivial elements. This leaves $t - 1$ nontrivial elements to account for.

Let q be a prime such that $q \notin \Pi(M)$ and let Q be a Sylow q -subgroup of G . Let V be a maximal subgroup of G containing Q . As above $M \cap V = 1$. Suppose that $\Pi(M) \cap \Pi(V) \neq \emptyset$ and let $r \in \Pi(M) \cap \Pi(V)$. Let V_r (respectively M_r) be a Sylow r -subgroup of V (respectively of M). As above, V_r and M_r are Sylow r -subgroups of G and hence there exists $y \in G$ such that $V_r^y = M_r$. It follows that $M_r \leq M \cap V^y$. Since V^y is a maximal subgroup of G , our work above proves that $M = V^y$. On the other hand, $q \notin \Pi(M)$ and $q \in \Pi(V) = \Pi(V^y)$, which is a contradiction. Hence $\Pi(M) \cap \Pi(V) = \emptyset$. Thus V and M have relatively prime orders, so $v = |V|$ divides $|G : M| = t$ and hence $t = vd$ for some natural number d . As above V has md conjugates each containing exactly $v - 1$ nontrivial elements, all distinct. None of these conjugates coincides with any of the conjugates of M so this gives $(v - 1)md$ elements not previously counted. However,

$$(v - 1)md = (vm - m)d \geq vd = t, \text{ since } v, m \geq 2,$$

which now yields a final contradiction. $\square$

1.5.5. LEMMA. *Let G be an infinite group whose nontrivial subgroups all have finite index. Then $\zeta(G) \neq 1$.*

EXERCISE 1.12. *Prove Lemma 1.5.5.*

We next obtain a result due to Fedorov [83].

1.5.6. THEOREM. *Let G be an infinite group whose nontrivial subgroups all have finite index. Then G is cyclic.*

PROOF. Clearly, G is torsion-free and, by Lemma 1.5.5, $\zeta(G) \neq 1$, so $G/\zeta(G)$ is finite. We prove the result by induction on $n = |G/\zeta(G)|$. If $n = 2$ then $G/\zeta(G)$ is cyclic and it is well known that G is then abelian, hence clearly cyclic. Suppose now that $n > 2$ and that we assume the obvious induction hypothesis. If H is a proper nontrivial subgroup containing $\zeta(G)$ then $\zeta(G) \leq \zeta(H)$ and hence $|H/\zeta(H)| < |G/\zeta(G)|$. By the induction hypothesis, H is infinite cyclic. Hence every proper subgroup of $G/\zeta(G)$ is cyclic and Proposition 1.5.4 implies that $G/\zeta(G)$ is soluble. Then $G/\zeta(G)$ contains a normal subgroup $K/\zeta(G)$ of prime index p . Again by the induction hypothesis, $K = \langle g \rangle$ is cyclic. If we suppose that $K \not\leq \zeta(G)$ then there is an element y

such that $g^y = g^{-1}$. On the other hand, $G/\zeta(G)$ is finite, so $\langle g^k \rangle = K \cap \zeta(G) \neq 1$, for some k . Then $y^{-1}g^ky = g^k$ and also $y^{-1}g^ky = g^{-k}$, which is a contradiction. Consequently, $K \leq \zeta(G)$ and since G/K is cyclic, G is abelian and hence cyclic, as required. $\square$

1.5.7. COROLLARY. *Let G be a torsion-free group. If G contains a cyclic normal subgroup of finite index, then G is cyclic.*

PROOF. Let C be a cyclic normal subgroup of G and suppose that $|G : C| = k$. If $1 \neq g \in G$, then $g^k \in C$ and, since G is torsion-free, $g^k \neq 1$. Since every nontrivial subgroup of a cyclic group has finite index we have $|C : \langle g^k \rangle|$ is finite. Hence $|G : \langle g^k \rangle|$ is finite, so that $|G : \langle g \rangle|$ is also finite and Theorem 1.5.6 applies to give the result. $\square$

The following result occurs in a paper of R. Baer [6].

1.5.8. LEMMA. *Let G be a torsion-free group. If every cyclic subgroup of G is normal, then G is abelian.*

EXERCISE 1.13. *Prove Lemma 1.5.8.*

1.5.9. COROLLARY. *Let G be a finitely generated torsion-free FC-group. Then G is abelian.*

PROOF. By Lemma 1.5.1, $G/\zeta(G)$ is finite so if $g \in G$, there is a natural number k such that $g^k \in \zeta(G)$. Hence $C = \langle g^k \rangle$ is a normal subgroup of G . The element gC is of finite order in G/C and Proposition 1.5.2 shows that $K/C = \langle gC \rangle^{G/C}$ is finite. The subgroup K is torsion-free and contains a cyclic subgroup of finite index, so by Corollary 1.5.7 K is cyclic. Since every subgroup of a cyclic group is characteristic, $\langle g \rangle$ is normal in G and Lemma 1.5.8 gives the result. $\square$

The following corollary can be immediately deduced.

1.5.10. COROLLARY. *Every torsion-free FC-group is abelian.*

EXERCISE 1.14. *Let G be a finitely generated abelian group. Prove that the torsion subgroup of G is finite and deduce that every periodic finitely generated abelian group is finite.*

1.5.11. COROLLARY. *Let G be a finitely generated centre-by-finite group. Then G' is finite.*

PROOF. Proposition 1.2.13 shows that $\zeta(G)$ is also finitely generated. By Corollary 1.5.3, the subgroup $T = \mathbf{Tor}(G)$ consists of the set of elements of finite order, so Exercise 1.14 implies that $R = \mathbf{Tor}(\zeta(G)) = T \cap \zeta(G)$ is finite. Since $T\zeta(G)/\zeta(G)$ is finite it follows that T is finite. The group G/T is torsion-free and hence is abelian, by Corollary 1.5.10. Therefore $G' \leq T$, so G' is finite. $\square$

1.5.12. THEOREM. *Let G be a centre-by-finite group. Then G' is finite.*

PROOF. There is a finitely generated subgroup K of G such that $G = K\zeta(G)$ and clearly $G' = K'$. By Corollary 1.5.11, K' is finite. $\square$

This theorem plays an important role in infinite groups where it is used in the proofs of many important, interesting results. There are already a number of alternative proofs of this theorem based on different approaches. We have presented here a proof that is self-contained. Furthermore, Theorem 1.5.6 is seen to be equivalent to Theorem 1.5.12. The history of Theorem 1.5.12 is quite interesting. In the form given here it first appeared in an article of B. H. Neumann [202]. However, at the end of this paper Neumann writes that R. Baer had informed him that the result is a consequence of a more general result, proved by Baer in [7]. In fact in [7, Theorem 3] it was proved that if the normal subgroup H of the group G has finite index then $(G' \cap H)/[H, G]$ is also finite. In a later article, Baer [8] gives the statement of Theorem 1.5.12 in its usual form and gave a new proof. In his famous lectures on nilpotent groups, P. Hall [109, Theorem 8.7] obtained a further generalization of Theorem 1.5.12. In these lectures Hall called this theorem “Schur’s Theorem” but gave no specific references. Many algebraists then started calling Theorem 1.5.12 “Schur’s Theorem” often citing the paper [234] as the source. However, Schur’s paper is not concerned with infinite group theory. In this classic old paper, [234], Schur introduced (only for finite groups!) the idea of what is now called the Schur multiplier (or multiplier) and studied its properties. In modern terminology, the Schur multiplier $M(G)$ of a group G is exactly the second cohomology group $H^2(G, U(\mathbb{C}))$. The following property of the Schur multiplier holds and appears in [262, Lemma 4.1].

1.5.13. THEOREM. *Let G be a group and let C be a subgroup of $\zeta(G)$. If G/C is finite, then $G' \cap C$ is an epimorphic image of $M(G/C)$.*

From this theorem, we can deduce the original result of Schur [234].

1.5.14. COROLLARY. *Let G be a finite group and let C be a subgroup of $\zeta(G)$. Then $G' \cap C$ is an epimorphic image of $M(G/C)$.*

The following important property of the Schur multiplier also occurs in [262, Corollaries 4.3 and 4.4].

1.5.15. THEOREM. *Let G be a finite group and let p be a prime.*

- (i) *If G is a p -group then $M(G)$ is a p -group;*

- (ii) If $p \in \Pi(G)$ and P is a Sylow p -subgroup of G , then the Sylow p -subgroup of $M(G)$ is isomorphic to a subgroup of $M(P)$. In particular, $\Pi(M(G)) \subseteq \Pi(G)$.

An interesting consequence is the following result.

1.5.16. COROLLARY. *Let G be a group and suppose that C is a subgroup of $\zeta(G)$ such that G/C is finite. Then $\Pi(G') \subseteq \Pi(G/C)$. In particular, if G/C is a p -group for some prime p , then G' is a p -subgroup of G .*

PROOF. Let $\Pi(G/C) = \pi$. Then G/C is a π -group and, by Theorem 1.5.15, $M(G/C)$ is also a π -group. Theorem 1.5.13 implies that $G' \cap C$ is also a π -group and, since $G'C/C \cong G'/(G' \cap C)$, it follows that G' is a π -group, as required. $\square$

1.5.17. COROLLARY. *Let G be a group and let C be a subgroup of $\zeta(G)$. If G/C is locally finite, then G' is also locally finite and $\Pi(G') \subseteq \Pi(G/C)$.*

PROOF. Let F be an arbitrary finitely generated subgroup of G' . Then there exists a finitely generated subgroup K such that $F \leq K'$. Since G/C is locally finite, KC/C is finite, so that $K/(K \cap C)$ is finite. Clearly $K \cap C \leq \zeta(K)$, so Theorem 1.5.12 shows that K' is finite. Hence F is finite, so that G' is locally finite. Moreover, using Corollary 1.5.16 we have

$$\Pi(F) \subseteq \Pi(K') \subseteq \Pi(K/K \cap C) \subseteq \Pi(G/C)$$

and the second part also follows. $\square$

Theorem 1.5.12 suggests a natural question as to whether there is a relationship between $t = |G/\zeta(G)|$ and $|G'|$. This question was posed in the paper [202] of B. H. Neumann, who obtained the first estimates for $|G'|$. The best estimate has been obtained by J. Wiegold [261], as follows.

1.5.18. THEOREM. *Let G be a group such that $G/\zeta(G)$ is finite of order t . Then*

- (i) $|G'| \leq w(t)$, where $w(t) = t^m$ and $m = (1/2)(\log_2 t - 1)$;
- (ii) If p is a prime and $t = p^n$, then G' is a p -group of order at most $p^{\frac{n(n-1)}{2}}$;
- (iii) For each prime p and each integer $n > 1$ there is a group G such that $|G/\zeta(G)| = p^n$ and $|G'| = p^{\frac{n(n-1)}{2}}$.

When $G/\zeta(G)$ has more than one prime divisor, the full picture is still unclear here. Theorem 1.5.12 admits the following natural generalization, known as Baer's Theorem.

1.5.19. THEOREM. *Let G be a group and suppose that there is a natural number k such that $G/\zeta_k(G)$ is finite. Then $\gamma_{k+1}(G)$ is finite.*

Paradoxically, R. Baer did not explicitly state and prove this result. In his article [8], mentioned above, Baer notes that it can be obtained from Zusatz zum Endlichkeitssatz of that paper. The title “Baer’s Theorem” again appears for the first time in the lectures of P. Hall. Hall gave a generalization of Theorem 1.5.12 which has as a corollary Theorem 1.5.19, but again Hall gave no references. Later D. J. S. Robinson [229] used the results of Baer’s paper to give a direct proof of Theorem 1.5.19. We give a proof of this result, but also indicate the connection between the orders of $G/\zeta_k(G)$ and $\gamma_{k+1}(G)$.

The next lemma has several variations which we shall obtain in Chapter 7.

1.5.20. LEMMA. *Let G be a group and let A be a normal abelian subgroup of G . Suppose that G satisfies the following conditions:*

- (i) $G/C_G(A) = \langle x_1 C_G(A), \dots, x_m C_G(A) \rangle$, for certain elements $x_1, \dots, x_m \in G$;
- (ii) $A/(\zeta(G) \cap A)$ is finite of order t .

Then $[A, G]$ is finite and $|[A, G]| \leq t^m$.

EXERCISE 1.15. *Prove Lemma 1.5.20.*

We now prove the quantitative version of Theorem 1.5.19 which appears in [168].

1.5.21. THEOREM. *Let G be a group and suppose that there is a natural number k such that $G/\zeta_k(G)$ is finite of order t . Then there is a function β_1 of k, t only such that $\gamma_{k+1}(G)$ is finite of order at most $\beta_1(t, k)$.*

PROOF. Let

$$1 = Z_0 \leq Z_1 \leq \dots \leq Z_{k-1} \leq Z_k = Z$$

be the upper central series of G . We use induction on k to obtain the result. If $k = 1$ then G/Z_1 has order at most t and an application of Theorem 1.5.18 proves that $\gamma_2(G) = G'$ has order at most $w(t)$.

Assume now that $k > 1$. Then G/Z_1 has a shorter upper central series than G , so we may suppose inductively that we have found a function β_1 such that $|\gamma_k(G/Z_1)| \leq \beta_1(t, k-1)$. We note that $K/Z_1 = \gamma_k(G/Z_1) = \gamma_k(G)Z_1/Z_1$ and, letting $L = \gamma_k(G)$, we observe that $K = LZ_1$, so $L \leq K$ and $K' = L'$. Furthermore, $|L/(L \cap Z_1)| = |K/Z_1| \leq \beta_1(t, k-1)$. By Proposition 1.1.6, $[L, Z_k] = 1$, so that $G/C_G(L)$ is finite of order at most t . The subgroup K is centre-by-finite and an

application of Theorem 1.5.18 shows that K' is finite of order at most $w(\beta_1(t, k-1))$. Of course L/K' is abelian. We have

$$(L/K')/(L/K' \cap Z_1 K'/K') \cong L/(L \cap Z_1 K'),$$

which shows that $(L/K')/(L/K' \cap Z_1 K'/K')$ is an epimorphic image of $L/(L \cap Z_1)$ and hence $|(L/K')/(L/K' \cap Z_1 K'/K')| \leq \beta_1(t, k-1)$. We apply Lemma 1.5.20 to G/K' to deduce that

$$V/K' = [L/K', G/K'] = [\gamma_k(G)/K', G/K'] = \gamma_{k+1}(G)K'/K'$$

is finite of order at most $t^{\beta_1(t, k-1)}$. Thus,

$$|\gamma_{k+1}(G)| \leq |K'| \cdot |V/K'| \leq w(\beta_1(t, k-1)) \cdot t^{\beta_1(t, k-1)},$$

so we define $\beta_1(t, k) = w(\beta_1(t, k-1)) \cdot t^{\beta_1(t, k-1)}$ and this completes the proof. $\square$

The proof gives us the bound, defined recursively as follows:

$$\begin{aligned} \beta_1(t, 1) &= w(t), \\ \beta_1(t, k) &= w(\beta_1(t, k-1)) \cdot t^{\beta_1(t, k-1)}. \end{aligned}$$

Theorem 1.5.19 shows that if the upper hypercentre of a group G has finite index and $\mathbf{z}\mathbf{l}(G)$ is finite, then G contains a finite normal subgroup K such that G/K is nilpotent. The question then arises: What can be said about a group G when the upper hypercentre has finite index but $\mathbf{z}\mathbf{l}(G)$ is arbitrary? This question was discussed in [77] and we end this section with a short, simple proof of their main theorem, which appeared in [157]. We remind the reader that $\zeta_\infty(G)$ denotes the upper hypercentre of G .

1.5.22. THEOREM. *Let G be a group and suppose that $G/\zeta_\infty(G)$ is finite. Then G contains a finite normal subgroup L such that G/L is hypercentral.*

PROOF. We may suppose that G is not hypercentral. Let $Z = \zeta_\infty(G)$ and choose a finitely generated subgroup K with the property that $G = ZK$. Since $K/K \cap Z$ is finite, Proposition 1.2.13 implies that $K \cap Z$ is also finitely generated. It follows from Proposition 1.2.5 that $K \cap Z$ is nilpotent, and the proof of Proposition 1.2.5 shows that $t = \mathbf{z}\mathbf{l}(K)$ is finite. Since G/Z is not nilpotent, neither is K and we let $C = \zeta_\infty(K) = \zeta_t(K)$. If $C \cap Z \neq C$ then CZ/Z is nontrivial which implies that the upper hypercentre of G/Z is nontrivial, a contradiction. Hence $C \cap Z = C$ and Theorem 1.5.19 shows that $\gamma_{t+1}(K)$ is finite. Hence the nilpotent residual, $L = K^{\mathfrak{N}}$, of K is finite.

It remains to show that L is normal in G and that G/L is hypercentral. To this end, we let $\mathcal{L}$ denote the family of finitely generated

subgroups of G each member of which contains K . Let $V \in \mathcal{L}$. Since $G = ZV$ we note that, as above, $V \cap Z = \zeta_\infty(V) = \zeta_n(V)$, for some n . Since $V \leq KZ$ and $K \leq V$ we have $V = K(V \cap Z) = K\zeta_n(V)$. Corollary 1.1.8 shows that $\gamma_{n+1}(V) = \gamma_{n+1}(K)$ and hence $\gamma_{n+1}(K)$ is normal in V . Since L is a characteristic subgroup of $\gamma_{n+1}(K)$ it follows that $L \triangleleft V$. Since G is the union of all the subgroups $V \in \mathcal{L}$ we deduce that L is a normal subgroup of G . Moreover,

$$\begin{aligned} G/ZL &\cong (G/L)/(ZL/L) = (K/L)(ZL/L)/(ZL/L) \\ &\cong (K/L)/((K/L) \cap (ZL/L)), \end{aligned}$$

which is nilpotent since K/L is nilpotent. However, the hypercentre of G/L contains ZL/L , so G/L is also hypercentral. $\square$

1.6. Some Direct Decompositions in Abelian Normal Subgroups

The topic in this section arises naturally as follows. Let G be a group and let B, C be normal subgroups of G such that $B \leq C$. The factor group C/B is called *G-central* if $C_G(C/B) = G$ and *G-eccentric* if $C_G(C/B) \neq G$. Now suppose that A is a finite normal abelian subgroup of G . Then A has a finite *G-chief series*,

$$1 = A_0 \leq A_1 \leq \dots \leq A_n = A.$$

In this series, the *G-central* and *G-eccentric* factors may be arranged arbitrarily and the following question, roughly posed, arises: Under what conditions can all the *G-central* factors be gathered together and when can all the *G-eccentric* factors be gathered together? This question, in a more general form (for factors associated with formations of groups) was raised and studied in the works of L. A. Shemetkov [239] and R. Baer [16]. We will not go into the details and intricacies of this subject, but here limit ourselves only to essential results which we shall require for future use. A more detailed discussion is available in the book [153].

If G is a group and A is a normal subgroup of G then we define the *upper G-central series* of A ,

$$1 = A_0 \leq A_1 \leq \dots A_\alpha \leq A_{\alpha+1} \leq \dots A_\gamma,$$

where

$$\begin{aligned} A_1 &= \zeta_G(A) = \{a \in A \mid [a, g] = 1, \text{ for all } g \in G\}, \\ A_{\alpha+1}/A_\alpha &= \zeta_G(A/A_\alpha), \text{ for all ordinals } \alpha < \gamma \text{ and} \\ A_\lambda &= \bigcup_{\beta < \lambda} A_\beta, \text{ for all limit ordinals } \lambda < \gamma. \end{aligned}$$

Furthermore, $\zeta_G(A/A_\gamma) = 1$. We note that every subgroup in this series is G -invariant. Of course, $A_1 = C_A(G) = \zeta(G) \cap A$ and it is easy to see that $A_\alpha = A \cap \zeta_\alpha(G)$, for all ordinals α .

The last term A_γ of this series is called the *upper G -hypercentre* of A and is denoted by $\zeta_G^\infty(A)$. If $A = A_\gamma$, then A is called *G -hypercentral*; when γ is finite, A is called *G -nilpotent*. The normal subgroup A is called *G -hypercenric* if it has an ascending series

$$1 = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots \leq C_\gamma,$$

of G -invariant subgroups such that each factor $C_{\alpha+1}/C_\alpha$ is G -eccentric and G -chief, for every $\alpha < \gamma$.

A normal abelian subgroup A of G is said to have the *$Z(G)$ -decomposition* if

$$A = \zeta_G^\infty(A) \oplus \eta_G^\infty(A),$$

where $\eta_G^\infty(A)$ is the maximal G -hypercenric G -invariant subgroup of A . This concept was introduced by D. I. Zaitsev [272] (note that here we do not use the language of modules used by Zaitsev).

We note that in this case $\eta_G^\infty(A)$ contains every G -hypercenric, G -invariant subgroup of A and hence is unique. To see this, let B be a G -hypercenric, G -invariant subgroup of A and let $E = \eta_G^\infty(A)$. If $(BE)/E$ is nontrivial it contains a nontrivial minimal G -invariant subgroup U/E . Since $(BE)/E \cong B/(B \cap E)$ it follows that U/E is G -isomorphic to some G -chief factor of B and hence $G/C_G(U/E) \neq G$. On the other hand, $(BE)/E \leq A/E \cong \zeta_G^\infty(A)$ so that $G/C_G(U/E) = G$, which gives us a contradiction. Hence $B \leq E$ and our claim concerning $\eta_G^\infty(A)$ follows.

We now consider some results that give conditions for the existence of a $Z(G)$ -decomposition in abelian normal subgroups. These results have been obtained in the article [150] and will be very useful.

1.6.1. LEMMA. *Let G be a group and let A be an abelian normal subgroup of G . Suppose that A contains a G -invariant subgroup C such that $C \leq \zeta_G(A)$ and A/C is a G -eccentric chief factor of A . If $G/C_G(A)$ is hypercentral, then A contains a minimal G -invariant subgroup D such that $A = C \times D$.*

EXERCISE 1.16. *Prove Lemma 1.6.1.*

By using a suitable inductive argument we deduce the next result.

1.6.2. COROLLARY. *Let G be a group and let A be an abelian normal subgroup of G . Suppose that A contains a G -invariant subgroup C such that C is G -nilpotent and A/C is a G -eccentric chief factor of A . If $G/C_G(A)$ is hypercentral, then A contains a minimal G -invariant subgroup D such that $A = C \times D$.*

1.6.3. LEMMA. *Let G be a group and let A be an abelian normal subgroup of G . Suppose that A contains a minimal G -invariant eccentric subgroup C such that $A/C \leq \zeta_G(G/C)$. If $G/C_G(A)$ is hypercentral, then A contains a G -invariant subgroup D such that $A = C \times D$.*

PROOF. Let $Z = C_G(A)$ and suppose that $Z \neq zZ \in \zeta(G/Z)$. The mapping $\xi_z : A \rightarrow A$ defined by $\xi_z(a) = [a, z]$ is an endomorphism of A . Since $\xi_z(a)^g = \xi_z(a^g)$, for all $g \in G$, it follows that $\ker(\xi_z) = C_A(z)$ and $\mathbf{Im}(\xi_z) = [A, z]$ are G -invariant subgroups of A . Since $A/C \leq \zeta_G(G/C)$ we have $[A, z] \leq C$ and since C is a minimal G -invariant subgroup of A it follows that $[A, z] = C$ or $[A, z] = 1$. The latter case cannot arise since it implies that $z \in C_G(A) = Z$, contrary to the choice of z . Hence $\mathbf{Im}(\xi_z) = C$.

Assume, for a contradiction, that $[C, z] = 1$, so that $C \leq \ker(\xi_z)$. Since $G \neq C_G(C)$ there exist $c \in C, g \in G$ such that $c^g \neq c$ and since $C = \mathbf{Im}(\xi_z)$ there exists $b \in A$ such that $c = \xi_z(b)$. Since $[b, g] \in C$ it follows that $[b, g] \in \ker(\xi_z)$ and we have

$$c^g = \xi_z(b^g) = \xi_z(b[b, g]) = \xi_z(b)\xi_z([b, g]) = \xi_z(b) = c,$$

which is the contradiction sought. Hence $[C, z] \neq 1$. However, $[C, z]$ is a G -invariant subgroup of C , since $zZ \in \zeta(G/Z)$, and hence $[C, z] = C$, from which it follows that $[A, z] = [C, z]$.

Let $a \in A \setminus C$ and choose $c \in C$ such that $[a, z] = [c, z]$. Then $[ac^{-1}, z] = 1$, so $ac^{-1} \in C_A(z)$. Hence $A = C \cdot C_A(z)$, the product of two G -invariant subgroups of A . It remains to show that $C \cap C_A(z) = 1$, at which point we set $D = C_A(z)$. However, $C \cap C_A(z)$ is also a G -invariant subgroup of A and the minimal choice of C implies that either $C \cap C_A(z) = 1$ or $C \leq C_A(z)$. As we observed above, the latter possibility cannot happen. Hence $A = C \times C_A(z) = C \times D$. The result follows. $\square$

A straightforward argument gives us the following result.

1.6.4. COROLLARY. *Let G be a group and let A be an abelian normal subgroup of G . Suppose that A contains a G -invariant subgroup C*

such that $A/C \leq \zeta_G(G/C)$ and suppose that C has a finite series of G -invariant subgroups whose factors are G -chief and G -eccentric. If $G/C_G(A)$ is hypercentral, then A contains a G -invariant subgroup D such that $A = C \times D$.

Let A be a normal subgroup of a group G having a finite G -chief series. By the Jordan–Hölder Theorem, every pair of such series have the same length and isomorphic factors. In particular, the length of such a G -chief series is an invariant of A , which we denote by $\mathbf{cl}_G(A)$.

We next give some sufficient conditions for an abelian normal subgroup of a group G to have a $Z(G)$ -decomposition.

1.6.5. COROLLARY. *Let G be a group and let A be an abelian normal subgroup of G with a finite G -chief series. If $G/C_G(A)$ is hypercentral, then A has a $Z(G)$ -decomposition.*

PROOF. We use induction on $\mathbf{cl}_G(A)$. If $\mathbf{cl}_G(A) = 1$ then A is a minimal normal subgroup of G and the result follows, so suppose that $\mathbf{cl}_G(A) > 1$. Since A has a finite G -chief series, it contains a G -invariant subgroup B such that A/B is a G -chief factor. Clearly $\mathbf{cl}_G(A) = \mathbf{cl}_G(B) + \mathbf{cl}_G(A/B)$ and $\mathbf{cl}_G(B) < \mathbf{cl}_G(A)$. By the induction hypothesis, B has a $Z(G)$ -decomposition, say $B = E \times C$, where C is the upper G -hypercentre of B and E is a G -hypereccentric G -invariant subgroup.

Suppose first that A/B is G -eccentric and consider A/E . Since $B/E = CE/E \cong_G C$ we may apply Corollary 1.6.2 to A/E and deduce that it contains a minimal G -invariant subgroup D/E such that $A/E = B/E \times D/E$. Since $A/B \cong_G D/E$ it follows that D/E is G -eccentric and hence D is G -hypereccentric. Then $A = BD = ECD = CD$. Also $C \cap D \leq E$ so

$$C \cap D = (C \cap D) \cap E = C \cap E = 1.$$

Hence $A = C \times D$ and A has a $Z(G)$ -decomposition.

Next, suppose that A/B is G -central and consider A/C . Since $B/C = EC/C \cong_G E$, we may apply Corollary 1.6.4 to A/C and deduce that there is a G -invariant, G -central subgroup Z/C such that $A/C = B/C \times Z/C$. Then $A = BZ = ECZ = EZ$. Also $E \cap Z \leq B \cap Z \leq C$ so

$$E \cap Z = (E \cap Z) \cap C = E \cap C = 1.$$

Since C is G -hypercentral and Z/C is G -central, Z is also G -hypercentral, so $A = E \times Z$ has a $Z(G)$ -decomposition in this case as well. $\square$

1.6.6. PROPOSITION. *Let G be a group and let A be an abelian normal subgroup of G . Suppose that A contains a G -invariant subgroup*

C such that C is G-nilpotent and A/C has a finite G-chief series. If $G/C_G(A)$ is hypercentral, then A has a $Z(G)$ -decomposition. Furthermore $\mathbf{cl}_G(\eta_G^\infty(A)) \leq \mathbf{cl}_G(A/C)$. In particular, if A/C is finite then $\eta_G^\infty(A)$ is finite and $|\eta_G^\infty(A)| \leq |A/C|$.

PROOF. Let B be the upper G -hypercentre of A so that $C \leq B$. Since $\mathbf{cl}_G(A/C)$ is finite, $\mathbf{zl}_G(B)$ and $\mathbf{cl}_G(A/B)$ are both finite. If $B = 1$, then the result follows from Corollary 1.6.5. Hence we may assume that $B \neq 1$ and use induction on $\mathbf{cl}_G(A/B)$. If $\mathbf{cl}_G(A/B) = 0$, then $A = B$, so A is G -nilpotent and the result follows. Suppose then that $A \neq B$. Since $\mathbf{cl}_G(A/B)$ is finite it follows that A/B has a minimal G -invariant subgroup D/B . Since B is the upper G -hypercentre, D/B is G -eccentric and it follows from Corollary 1.6.2 that D contains a minimal G -invariant subgroup E such that E is G -eccentric and $D = BE$.

Consider A/E and let K/E be the upper G -hypercentre of A/E . Clearly $D/E = BE/E \leq K/E$ and it follows that $\mathbf{cl}_G((A/E)/(K/E)) < \mathbf{cl}_G(A/B)$. By the induction hypothesis A/E has a $Z(G)$ -composition, say $A/E = K/E \times L/E$, where $L/E = \eta_G^\infty(A/E)$. It is easy to see, using Corollary 1.6.4 repeatedly, that K contains a G -invariant subgroup Y such that $K = Y \times E$. Clearly, $Y \cap L = 1$ and we have $A = KL = (YE)L = Y \times L$, as required. $\square$

We now show how to apply the results of this section to the problems of Section 1.5 and obtain further information concerning them. We recall that in that section we made use of a function which we denoted by $w(t)$.

1.6.7. PROPOSITION. *Let G be a finitely generated group and suppose that for some natural number k , $G/\zeta_k(G)$ is finite of order t . Then the nilpotent residual $G^\mathfrak{m}$ is finite of order at most $tw(t)$. Furthermore, $G/G^\mathfrak{m}$ is nilpotent.*

PROOF. Let

$$1 = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = Z$$

be the upper central series of G , so that each of the subgroups Z_j is G -invariant and each factor Z_j/Z_{j-1} is G -central, for $1 \leq j \leq n$. By Theorem 1.2.22 $G/C_G(Z)$ is nilpotent of class at most $k-1$. Let $C = C_G(Z)$. Then $Z \leq C_G(C)$ and hence $G/C_G(C)$ is finite of order at most t . Of course $C \cap Z \leq \zeta(C)$ so that $C/(C \cap Z) \cong CZ/Z$ is finite of order at most t . Theorem 1.5.18 implies that C' has order at most $w(t)$. Of course, C' is G -invariant and C/C' is abelian. Furthermore,

$(G/C')/C_{G/C'}(C/C')$ is a finite nilpotent group. We have

$$(C \cap Z)C'/C' \leq \zeta_{G/C'}^\infty(C/C') \text{ and} \\ (C/C')/((C \cap Z)C'/C') \cong C/(C \cap Z)C'$$

is a finite group of order at most t . By Proposition 1.6.6, it follows that C/C' contains a finite G -invariant subgroup V/C' of order at most t such that $(C/C')/(V/C')$ is G -nilpotent. Since G/C is nilpotent, G/V is also nilpotent, so $G^\mathfrak{N} \leq V$. Hence $|V| = |C'| \cdot |V/C'| \leq tw(t)$, as required.

Finally, since $G^\mathfrak{N} = \cap \{W \triangleleft G \mid G/W \in \mathfrak{N}\}$ and V is finite, there exist finitely many normal subgroups $V_1, \dots, V_k$ so that $G^\mathfrak{N} = \cap_{i=1}^k \{V_i \mid G/V_i \in \mathfrak{N}\}$. Then, by Remak's Theorem,

$$G/G^\mathfrak{N} \longrightarrow \operatorname{Dr}_{i=1}^k G/V_i$$

so $G/G^\mathfrak{N}$ is nilpotent. $\square$

The technique of using the $Z(G)$ -decomposition allows us, then, to obtain the following generalization of Theorem 1.5.22, which appears in [157].

1.6.8. THEOREM. *Let G be a group and suppose that $G/\zeta_\infty(G)$ is finite of order t . Then G contains a finite normal subgroup L of order at most $tw(t)$ such that G/L is hypercentral.*

PROOF. Let $Z = \zeta_\infty(G)$ and let $\mathcal{L}$ be the family of all finitely generated subgroups of G . If $U \in \mathcal{L}$ then $U \cap Z \leq \zeta_\infty(U)$ and, since $U/(Z \cap U)$ is finite, $U/\zeta_\infty(U)$ is also finite. Proposition 1.6.7 shows that the nilpotent residual, $U^\mathfrak{N}$, of U is finite of order at most $tw(t)$. In particular, the orders of the subgroups $U^\mathfrak{N}$ are bounded, for $U \in \mathcal{L}$. It follows that there is a finitely generated subgroup V such that the order of $V^\mathfrak{N}$ is largest. Let $g \in G$ be arbitrary and consider $H = \langle g, V \rangle$. Now $V/(V \cap H^\mathfrak{N}) \cong VH^\mathfrak{N}/H^\mathfrak{N}$ is nilpotent, by Proposition 1.6.7, and it follows that $V^\mathfrak{N} \leq H^\mathfrak{N}$. By the choice of V we have $V^\mathfrak{N} = H^\mathfrak{N}$ and hence $V^\mathfrak{N} \triangleleft H$. Consequently, $g \in N_G(V^\mathfrak{N})$ which implies that $V^\mathfrak{N} \triangleleft G$.

Now let $K/V^\mathfrak{N}$ be an arbitrary finitely generated subgroup of $G/V^\mathfrak{N}$. Then K is finitely generated and, as above, $V^\mathfrak{N} = K^\mathfrak{N}$, which shows that $K/V^\mathfrak{N}$ is nilpotent. Hence $G/V^\mathfrak{N}$ is locally nilpotent. However, $ZV^\mathfrak{N}/V^\mathfrak{N}$ has finite index in $G/V^\mathfrak{N}$ and hence $(G/V^\mathfrak{N})/(ZV^\mathfrak{N}/V^\mathfrak{N})$ is nilpotent. It follows that $G/V^\mathfrak{N}$ is hypercentral. $\square$