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Electromagnetic Theory of Light

Introduction

Integrated photonics devices are based on optical waveguides with transversal dimensions of the order of microns, comparable to the wavelength of the optical radiation used in the integrated devices (visible and near infrared). This fact implies that the performance of the optical chips cannot be analysed in terms of ray optics, but instead the light must be treated as vectorial waves. Thus, to describe adequately the light propagation along the waveguide structures that define an integrated photonics device, the electromagnetic theory of light is required, which deals the light as optical waves in terms of their electric and magnetic fields. This treatment retains the vectorial character of the waves. Nevertheless, in some cases the vectorial nature of the electromagnetic waves can be simplified and a scalar treatment of the optical waves is enough for an accurate description of the light propagation through the optical waveguides.

Along this chapter the basics of the electromagnetic theory of light is described, which is the start point to derive the beam propagation equations to model the light propagation in optical waveguides and integrated photonic devices. First, the Maxwell's equations for light propagation in free space are presented in terms of the electric and magnetic field. Then, the electric displacement vector and the magnetic flux density vector are introduced to describe the optical propagation in material media. The constitutive relations allow then to establish a set of equations in terms of the electric and magnetic fields. Using these equations, the wave equations in inhomogeneous media are derived where the refractive index can be then defined. Then the wave equation for monochromatic waves in inhomogeneous media is obtained, where the temporal dependence of the fields is in the form of harmonic function. The especial cases of light propagation in absorbing media, anisotropic media and in second-order non-linear media are discussed, and wave equations for each case are derived. Finally, the wave equations in inhomogeneous isotropic and linear media in terms of the transverse field components are obtained, for both electric and magnetic fields. Also, wave equation for anisotropic media

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and second-order non-linear media are established in terms of the electric transverse field components. These equations will serve to derive in the subsequent chapters the beam propagation formalism.

1.1 Electromagnetic Waves

1.1.1 Maxwell's Equations

Light is, in terms of classical theory, the flow of electromagnetic (EM) radiation through free space or through a material medium in the form of oscillating electric and magnetic fields. Although electromagnetic radiation occurs over an extremely wide range from gamma rays to long radio waves, the term 'light' is restricted to the part of the electromagnetic spectrum that covers from the vacuum ultraviolet (UV) to the far infrared. This part of the spectrum is often also called optical range. The EM radiation propagates in the form of two mutually perpendicular and coupled vectorial waves: the electric field $\boldsymbol{\mathcal{E}}(\boldsymbol{r},t)$ and the magnetic field $\boldsymbol{\mathcal{H}}(\boldsymbol{r},t)$. These two vectorial magnitudes are dependent on the position ($\boldsymbol{r}$) and time (t). Therefore, to describe properly the light propagation in a medium, be it the vacuum or a material medium, it is necessary in general to know six scalar functions with their dependence of the position and the time. These functions are not independent but linked through Maxwell's equations.

Maxwell's equations form a set of four coupled equations involving the electric field vector and the magnetic field vector of the light and are based on experimental evidence, two of them being scalar equations and the other two vectorial equations. In their differential form, Maxwell's equations for light propagating in the free space are:

$$\nabla \cdot \boldsymbol{\mathcal{E}} = 0; \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{\mathcal{H}} = 0; \tag{1.1b}$$

$$\nabla \times \boldsymbol{\mathcal{E}} = -\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t}; \tag{1.1c}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \varepsilon_0 \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}, \tag{1.1d}$$

where the constants $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{m^{-3}\,kg^{-1}\,s^4\,A^2}$ and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{m\,kg\,s^{-2}\,A^{-2}}$ represent respectively the dielectric permittivity and the magnetic permeability of free space and the ∇ and $\nabla\times$ denote the divergence and curl operators, respectively.

The differential operator ∇ is defined as:

$$\nabla \equiv \left(\frac{\partial}{\partial x}\mathbf{u}_x + \frac{\partial}{\partial y}\mathbf{u}_y + \frac{\partial}{\partial z}\mathbf{u}_z \right), \tag{1.2}$$

where $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ represent the unitary vectors along the x-, y- and z-axis, respectively. This differential operator acting to a scalar field gives rise to a vector (gradient). In particular, if $\xi(x,y,z)$ represents a scalar field, we have:

$$\nabla\xi(x,y,z) \equiv \left(\frac{\partial\xi}{\partial x}\mathbf{u}_x + \frac{\partial\xi}{\partial y}\mathbf{u}_y + \frac{\partial\xi}{\partial z}\mathbf{u}_z\right). \tag{1.3}$$

On the other hand, if $\mathbf{A}(x,y,z) = A_x(x,y,z)\mathbf{u}_x + A_y(x,y,z)\mathbf{u}_y + A_z(x,y,z)\mathbf{u}_z$ is a vector field, the divergence operator $(\nabla\cdot)$ acts as follows:

$$\nabla\cdot\mathbf{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{1.4}$$

which is a scalar magnitude. Finally, the curl differential operator $(\nabla\times)$ acting on the vector field $\mathbf{A}$ gives another vector with the following components:

$$\nabla\times\mathbf{A} \equiv \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{u}_z. \tag{1.5}$$

For the description of the electromagnetic field in a material medium it is necessary to define two additional vectorial magnitudes: the electric displacement vector $\mathcal{D}(\boldsymbol{r},t)$ and the magnetic flux density vector $\mathcal{B}(\boldsymbol{r},t)$. Maxwell's equations in a material medium, involving these two magnitudes and the electric and magnetic fields, are expressed as:

$$\nabla\cdot\mathcal{D} = \rho; \tag{1.6a}$$

$$\nabla\cdot\mathcal{B} = 0; \tag{1.6b}$$

$$\nabla\times\mathcal{E} = -\frac{\partial\mathcal{B}}{\partial t}; \tag{1.6c}$$

$$\nabla\times\mathcal{H} = \mathcal{J} + \frac{\partial\mathcal{D}}{\partial t}, \tag{1.6d}$$

where $\rho(\boldsymbol{r},t)$ and $\mathcal{J}(\boldsymbol{r},t)$ denote the charge density and the current density vector, respectively. If the medium if free of charges, which is the most common situation in optics, Maxwell's equations simplify to the form:

$$\nabla\cdot\mathcal{D} = 0; \tag{1.7a}$$

$$\nabla\cdot\mathcal{B} = 0; \tag{1.7b}$$

$$\nabla\times\mathcal{E} = -\frac{\partial\mathcal{B}}{\partial t}; \tag{1.7c}$$

$$\nabla\times\mathcal{H} = \mathcal{J} + \frac{\partial\mathcal{D}}{\partial t}. \tag{1.7d}$$

Now, in order to solve these differential coupled equations it is necessary to establish additional relations between the vectors $\mathcal{D}$ and $\mathcal{E}$, $\mathcal{J}$ and $\mathcal{E}$ as well as the vectors $\mathcal{H}$ and $\mathcal{B}$. These relations are called constitutive relations and depend on the electric and magnetic properties of the considered medium. In the most simple case of linear and isotropic media, the constitutive relations are given by:

$$\mathcal{D} = \varepsilon \mathcal{E}; \tag{1.8a}$$

$$\mathcal{B} = \mu \mathcal{H}; \tag{1.8b}$$

$$\mathcal{J} = \sigma \mathcal{E}, \tag{1.8c}$$

where $\varepsilon = \varepsilon(\boldsymbol{r})$ is the dielectric permittivity, $\mu = \mu(\boldsymbol{r})$ is the magnetic permeability and $\sigma = \sigma(\boldsymbol{r})$ is the electrical conductivity of the medium. Here, their dependence on the position vector $\boldsymbol{r}$ has been explicitly indicated. If the medium is not linear, it is necessary to include additional terms involving power expansion of the electric and magnetic fields. Besides, in an isotropic medium (glasses for instance) these optical constants are scalar magnitudes and independent of the direction of the vectors $\mathcal{E}$ and $\mathcal{H}$, implying that the vectors $\mathcal{D}$ and $\mathcal{J}$ are parallel to the electric field $\mathcal{E}$ and the vector $\mathcal{B}$ is parallel to the magnetic field $\mathcal{H}$. By contrast, in an anisotropic medium (for instance, most of the dielectric crystals) the optical constants must be treated as tensorial magnitudes.

By using the constitutive relations for a linear and isotropic medium, Maxwell's equations can be written in terms of the electric field $\mathcal{E}$ and magnetic field $\mathcal{H}$ only:

$$\nabla \cdot (\varepsilon \mathcal{E}) = 0; \tag{1.9a}$$

$$\nabla \cdot \mathcal{H} = 0; \tag{1.9b}$$

$$\nabla \times \mathcal{E} = -\mu \frac{\partial \mathcal{H}}{\partial t}; \tag{1.9c}$$

$$\nabla \times \mathcal{H} = \sigma \mathcal{E} + \varepsilon \frac{\partial \mathcal{E}}{\partial t}. \tag{1.9d}$$

A perfect dielectric medium is defined as a material in which the conductivity is very low and can be neglected ($\sigma \approx 0$). In this category fall most of the materials used for integrated optical devices, such as glasses, ferroelectric crystals, polymers or even semiconductors, while metals do not belong to this category because of their high conductivity. In addition, in most of materials (non-magnetic materials) and in particular, dielectric media, the magnetic permeability is very close to that of free space and the approximation $\mu \approx \mu_0$ holds. Then, in dielectric and non-magnetic media, Maxwell's equations simplify in the form:

$$\nabla \cdot (\varepsilon \mathcal{E}) = 0; \tag{1.10a}$$

$$\nabla \cdot \mathcal{H} = 0; \tag{1.10b}$$

$$\nabla \times \mathcal{E} = -\mu_0 \frac{\partial \mathcal{H}}{\partial t}; \tag{1.10c}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \varepsilon \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}. \tag{1.10d}$$

In what follows, we will restrict ourselves to non-magnetic and low conductivity materials, where Maxwell's equations (1.10a)–(1.10d) apply.

1.1.2 Wave Equations in Inhomogeneous Media

Combining the four Maxwell's equations (1.10a)–(1.10d) it is possible to obtain an equation involving the electric field alone and another equation that involves only the magnetic field.

Taking the curl operation over the Eqs. (1.10c) and (1.10d) we obtain:

$$\nabla \times (\nabla \times \boldsymbol{\mathcal{E}}) = -\nabla \times \left(\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \right); \tag{1.11a}$$

$$\nabla \times (\nabla \times \boldsymbol{\mathcal{H}}) = \nabla \times \left(\varepsilon \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \right) = (\nabla \varepsilon) \times \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \varepsilon \left(\nabla \times \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \right). \tag{1.11b}$$

Having in mind the vectorial identity $\nabla \times \nabla \times \equiv \nabla(\nabla \cdot) - \nabla^2$, these equations transform to:

$$\nabla(\nabla \cdot \boldsymbol{\mathcal{E}}) - \nabla^2 \boldsymbol{\mathcal{E}} = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{\mathcal{H}}); \tag{1.12a}$$

$$\nabla(\nabla \cdot \boldsymbol{\mathcal{H}}) - \nabla^2 \boldsymbol{\mathcal{H}} = (\nabla \varepsilon) \times \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \frac{\partial}{\partial t}[\varepsilon \nabla \times \boldsymbol{\mathcal{E}}], \tag{1.12b}$$

where we have used the fact that the temporal and spatial derivatives commute and it is assumed that permittivity, ε, is time independent. On the other hand, expanding the first Maxwell's equation (1.10a):

$$\nabla \cdot (\varepsilon \boldsymbol{\mathcal{E}}) = (\nabla \varepsilon) \cdot \boldsymbol{\mathcal{E}} + \varepsilon \nabla \cdot \boldsymbol{\mathcal{E}} = 0 \quad \Rightarrow \quad \nabla \cdot \boldsymbol{\mathcal{E}} = -\boldsymbol{\mathcal{E}} \cdot \nabla \ln \varepsilon, \tag{1.13}$$

where we have used the relationship: $\nabla \ln \varepsilon \equiv \frac{\nabla \varepsilon}{\varepsilon}$.

Introducing Eqs. (1.13) and (1.10d) into Eq. (1.12a), we have:

$$\nabla^2 \boldsymbol{\mathcal{E}} + \nabla(\boldsymbol{\mathcal{E}} \cdot \nabla \ln \varepsilon) - \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} = 0, \tag{1.14}$$

and similarly using Eqs. (1.10b) and (1.10c) into Eq. (1.12b), we obtain:

$$\nabla^2 \boldsymbol{\mathcal{H}} + (\nabla \ln \varepsilon) \times (\nabla \times \boldsymbol{\mathcal{H}}) - \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2} = 0. \tag{1.15}$$

These last two differential equations are known as wave equations in inhomogeneous media, which are valid for linear, non-magnetic and isotropic material media. It is worth noting that, although we have obtained a wave equation for the electric field $\boldsymbol{\mathcal{E}}$ and another for the magnetic field $\boldsymbol{\mathcal{H}}$, the solution of both equations are not independent, because the electric and magnetic fields are related through the Maxwell's equations (1.10c) and (1.10d). The solutions of the wave equations are known as electromagnetic waves.

The electromagnetic waves transport energy and the flux of energy (measured in units of W/m^2) carried by the EM wave is given by the Poynting vector $\boldsymbol{\mathcal{S}}$, defined as:

$$\boldsymbol{\mathcal{S}} \equiv \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}. \tag{1.16}$$

On the other hand, the intensity (or irradiance) I of an EM wave, defined as the amount of energy passing through the unit area in the unit of time, is given by the time average of the Poynting vector modulus:

$$I = \langle |\boldsymbol{\mathcal{S}}| \rangle. \tag{1.17}$$

The reason for using an averaged value instead of an instant value for defining the intensity of an EM wave is because the electric and magnetic fields associated to the EM wave oscillate at very high frequency and the apparatus used to detect that intensity (light detectors) cannot follow the instant values of the Poynting vector modulus.

1.1.3 Wave Equations in Homogeneous Media: Refractive Index

An optically homogeneous medium is defined as a material in which its optical properties are independent on the position. Then, for homogeneous dielectric media the second terms in Eqs. (1.14) and (1.15) vanish:

$$\nabla \ln \varepsilon = \frac{\nabla \varepsilon}{\varepsilon} = 0, \tag{1.18}$$

and the wave equations simplify on the forms:

$$\nabla^2 \boldsymbol{\mathcal{E}} = \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2}; \tag{1.19a}$$

$$\nabla^2 \boldsymbol{\mathcal{H}} = \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2}. \tag{1.19b}$$

Each of these two vectorial wave equations can be split onto three scalar wave equations, expressed as:

$$\nabla^2 \xi = \mu_0 \varepsilon \frac{\partial^2 \xi}{\partial t^2}, \tag{1.20}$$

where the scalar variable $\xi(\boldsymbol{r},t)$ may represent each of the six Cartesian components of either the electric and magnetic fields. The solution of this scalar equation represents a wave that propagates with a speed v (phase velocity) given by:

$$v = \frac{1}{\sqrt{\varepsilon\mu_0}}. \tag{1.21}$$

Therefore, the complete solution of the vectorial wave equations (1.19a) and (1.19b) represents an electromagnetic wave, where each of the Cartesian components of the electric and magnetic fields propagate in the form of waves of equal speed v in the homogeneous medium.

For propagation in free space (vacuum) and using the values for ε_0 and μ_0 we obtain:

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \approx 3.00 \times 10^8\,\mathrm{m/s}, \tag{1.22}$$

which corresponds to the speed of light in free space measured experimentally. It is worth noting that here the speed of light is obtained by using only values of electric and magnetic constants.

Usually, it is convenient to express the propagation speed v of the electromagnetic waves in a medium as a function of the speed of light in free space, c, through the relation:

$$v \equiv \frac{c}{n}, \tag{1.23}$$

where n represents the refractive index of the dielectric medium. Taking into account the relations (1.21) and (1.22), the refractive index can be related to the dielectric permittivity of the medium and that of the free space by:

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r}, \tag{1.24}$$

where we have introduced the magnitude relative dielectric permittivity, ε_r, also called the dielectric constant, defined as the relation between the dielectric permittivity of the material medium and that of the free space. As we will see in the following chapters, the refractive index of a medium is the most important parameter for defining optical waveguide structures used in integrated photonic devices. As well as the refractive index of 1 corresponding to propagation through free space, the refractive index ranges from values close to 1.5 for glasses and some dielectric crystals (for instance, $n(SiO_2) = 1.55$) to values close to 4 for semiconductor materials (for instance, $n(\mathrm{Si}) = 3.75$).

1.2 Monochromatic Waves

The temporal dependence of the electric and magnetic fields within the wave equations admits solutions on the form of harmonic functions. The electromagnetic waves with such sinusoidal dependence on the time variable are called monochromatic waves, which are characterized by their angular frequency ω (in units of rad/s). In a general form, the electric and magnetic fields associated to a monochromatic wave can be expressed as:

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \boldsymbol{\mathcal{E}}_0(\boldsymbol{r})\cos[\omega t + \phi(\boldsymbol{r})]; \tag{1.25a}$$

$$\mathcal{H}(\boldsymbol{r},t) = \mathcal{H}_0(\boldsymbol{r})\cos[\omega t + \phi(\boldsymbol{r})], \tag{1.25b}$$

where the field amplitudes $\mathcal{E}_0(\boldsymbol{r})$ and $\mathcal{H}_0(\boldsymbol{r})$, and the initial phase $\varphi(\boldsymbol{r})$ have dependence on the position $\boldsymbol{r}$, and the time dependence of the fields is only in the cosine argument through ωt.

Usually, when dealing with monochromatic waves it is convenient to express the monochromatic fields using complex notation. Using this notation, the electric and magnetic fields are expressed as:

$$\mathcal{E}(\boldsymbol{r},t) = \mathrm{Re}\left[\boldsymbol{E}(\boldsymbol{r})e^{+i\omega t}\right]; \tag{1.26a}$$

$$\mathcal{H}(\boldsymbol{r},t) = \mathrm{Re}\left[\boldsymbol{H}(\boldsymbol{r})e^{+i\omega t}\right], \tag{1.26b}$$

where $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ denote the complex amplitudes of the electric and magnetic fields, respectively, i is the imaginary unity and Re stands for the real part. The angular frequency, ω, which characterizes the monochromatic wave, is related with the frequency, ν, and the period, T, by:

$$\omega = 2\pi\nu = 2\pi/T. \tag{1.27}$$

The electromagnetic spectrum covered by light (optical spectrum) ranges from frequencies of 3×10^5 Hz corresponding to the far infrared (IR), to 6×10^{15} Hz corresponding to vacuum UV, being the frequency of visible light in the range of 430–770 THz.

The average of the Poynting vector for monochromatic waves as a function of the complex fields amplitudes takes the form:

$$\langle \boldsymbol{\mathcal{S}} \rangle = \left\langle Re\left\{\boldsymbol{E}e^{+i\omega t}\right\} \times Re\left\{\boldsymbol{H}e^{+i\omega t}\right\}\right\rangle = Re\{\boldsymbol{S}\}, \tag{1.28}$$

where $\boldsymbol{S}$ is defined here as:

$$\boldsymbol{S} = 1/2\boldsymbol{E} \times \boldsymbol{H}^*, \tag{1.29}$$

which is called the complex Poynting vector. Using this definition, the intensity carried by a monochromatic EM wave can be expressed in a compact form as:

$$I = |Re\{\boldsymbol{S}\}|. \tag{1.30}$$

Maxwell's equations (1.10a)–(1.10d) using the complex fields amplitudes $\boldsymbol{E}$ and $\boldsymbol{H}$ simplify notably in the case of monochromatic waves, because the partial derivatives with respect to the time can be directly obtained by multiplying by the factor $i\omega$, resulting in:

$$\nabla \cdot (\varepsilon \boldsymbol{E}) = 0; \tag{1.31a}$$

$$\nabla \cdot \boldsymbol{H} = 0; \tag{1.31b}$$

$$\nabla \times \boldsymbol{E} = -i\mu_0\omega \boldsymbol{H}; \tag{1.31c}$$

$$\nabla \times \boldsymbol{H} = i\varepsilon\omega \boldsymbol{E}, \tag{1.31d}$$

where a dielectric and non-magnetic medium has been assumed in which $\sigma = 0$ and $\mu = \mu_0$. The corresponding wave equations for the electric and magnetic fields Eqs. (1.14) and (1.15) are given by:

$$\nabla^2 \boldsymbol{E} + \nabla(\boldsymbol{E} \cdot \nabla \ln \varepsilon) + \omega^2 \mu_0 \varepsilon \boldsymbol{E} = 0; \tag{1.32a}$$

$$\nabla^2 \boldsymbol{H} + (\nabla \ln \varepsilon) \times (\nabla \times \boldsymbol{H}) + \omega^2 \mu_0 \varepsilon \boldsymbol{H} = 0. \tag{1.32b}$$

Often, it is more convenient to rewrite these equations as a function of the refractive index $n(\boldsymbol{r})$ of the medium as follows:

$$\nabla^2 \boldsymbol{E} + \nabla\left(\frac{1}{n^2} \nabla n^2 \cdot \boldsymbol{E}\right) + n^2 k_0^2 \boldsymbol{E} = 0; \tag{1.33a}$$

$$\nabla^2 \boldsymbol{H} + \frac{1}{n^2}\left(\nabla n^2\right) \times (\nabla \times \boldsymbol{H}) + n^2 k_0^2 \boldsymbol{H} = 0, \tag{1.33b}$$

where we have defined the wavenumber k_0 as:

$$k_0 \equiv \omega / c \equiv 2\pi / \lambda, \tag{1.34}$$

λ being the wavelength of light in free space.

1.2.1 Homogeneous Media: Helmholtz's Equation

For light propagation in homogeneous media the wave equations are substantially simplified. If we substitute the solutions on the form of monochromatic waves ((1.26a) and (1.26b)) in the wave equations ((1.19a) and (1.19b)), we obtain a new wave equation, valid only for monochromatic waves, known as Helmholtz equation:

$$\nabla^2 \xi(\boldsymbol{r}) + k^2 \xi(\boldsymbol{r}) = 0, \tag{1.35}$$

where $\xi(\boldsymbol{r})$ now represents each of the six Cartesian components of the $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ vectors defined in Eqs. (1.26a) and (1.26b), and where we have defined the wavenumber in the medium k as:

$$k \equiv \omega (\varepsilon \mu_0)^{1/2} = n k_0, \tag{1.36}$$

n being the refractive index of the homogeneous medium.

1.2.2 Light Propagation in Absorbing Media

An absorbing medium is characterized by the fact that the energy of the EM radiation is dissipated in it. This would imply that the amplitude of a plane EM wave will decrease in exponential form as the wave propagates along the absorbing medium. The mathematical description of light propagation in absorbing media can be treated by considering that the dielectric permittivity is no longer a real number, but a complex quantity ε_c. In terms of the fields' descriptions, this implies that the electric displacement will not be generally in phase with the electric field. As the refractive index is defined as the function of the dielectric permittivity, in general it will be a complex number, now defined by:

$$n_c = \sqrt{\frac{\varepsilon_c}{\varepsilon_0}}, \tag{1.37}$$

where n_c is called the complex refractive index. It is useful to work separately with the real and imaginary part and for doing this the complex refractive index is put in the form:

$$n_c = n - i\kappa, \tag{1.38}$$

where n is now the real refractive index and κ is called the absorption index (both dimensionless quantities).

The most important aspect concerning light propagation in absorbing media is the intensity variation suffered by the electromagnetic radiation as it propagates. For the case of a monochromatic plane of the form:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{1.39}$$

and assuming that the propagation is along the z-axis without loss of generality, the intensity associated to that planar wave can be calculated by using Eq. (1.30), and takes the form:

$$I(z) = \frac{1}{2c\mu_0}|\boldsymbol{E}_0|^2 e^{-2\kappa k_0 z}. \tag{1.40}$$

If we define now I_0 as the intensity associated to the wave at the plane $z = 0$, it follows:

$$I_0 = \frac{1}{2c\mu_0}|\boldsymbol{E}_0|^2, \tag{1.41}$$

and the expression for $I(z)$ becomes more compact as:

$$I(z) = I_0 e^{-2\kappa k_0 z}. \tag{1.42}$$

This formula indicates that the intensity of a monochromatic plane wave propagating in an absorbing medium decreases exponentially as a function of the propagation distance.

In some applications it is convenient to deal with the absorption by using the absorption coefficient, α, defined as:

$$\alpha \equiv 2\kappa k_0 = 2\kappa\omega/c, \tag{1.43}$$

which has dimensions of m^{-1}. In this way, the attenuation of a light beam passing an absorbing medium is expressed in a compact manner by:

$$I(z) = I_0 e^{-\alpha z}. \tag{1.44}$$

When working with optical waveguides or optical fibres, light attenuation A is often referred to in decibels, whose relation with the absorption coefficient is:

$$A(\text{dB}) \equiv 10 \log_{10}(I_0/I) = 4.3\alpha d, \tag{1.45}$$

where I/I_0 represents the fraction of light intensity after a distance d.

1.2.3 Light Propagation in Anisotropic Media

In an isotropic medium, whose properties do not depend of the direction considered, we have seen that the electric displacement vector $\boldsymbol{D}$ and its associated electric field $\boldsymbol{E}$ are related through $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, where ε is the scalar electric permittivity. This relation implies that the vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ are always parallel, and the refractive index is given by Eq. (1.24). Nevertheless, in the most general case of light propagation in anisotropic media, whose optical properties depend on the polarization and propagation direction of the light, the displacement vector $\boldsymbol{D}$ and the electric field $\boldsymbol{E}$ are no longer necessary parallel, and they are related by:

$$\boldsymbol{D} = \boldsymbol{\varepsilon} \boldsymbol{E}, \tag{1.46}$$

where $\boldsymbol{\varepsilon}$ is the permittivity tensor, which can be expressed as a 3×3 matrix referring to three arbitrary orthogonal axes:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}. \tag{1.47}$$

The elements of the permittivity tensor fulfil the following relation:

$$\varepsilon_{ij} = \varepsilon_{ji}^{*}. \tag{1.48}$$

This implies that for materials that are neither absorbing nor amplifying, where the elements of the tensor are real, the permittivity tensor is symmetrical.

For some practical calculations concerning light propagation in anisotropic materials, it is often useful to work with the impermeability tensor $\boldsymbol{\eta}$, defined as:

$$\boldsymbol{\eta} \equiv \varepsilon_0 \boldsymbol{\varepsilon}^{-1}, \tag{1.49}$$

and then $\boldsymbol{E} = \boldsymbol{\varepsilon}^{-1} \boldsymbol{D} \equiv \frac{1}{\varepsilon_0} \boldsymbol{\eta} \boldsymbol{D}$. Taking advantage of the symmetry of the permittivity tensor for non-absorbing media, the impermeability matrix can be expressed as:

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 & \eta_6 & \eta_5 \\ \eta_6 & \eta_2 & \eta_4 \\ \eta_5 & \eta_4 & \eta_3 \end{pmatrix}, \tag{1.50}$$

where we have used the following contracted notation for the sub-indices:

$$xx \to 1 \qquad yy \to 2 \qquad zz \to 3\,;$$
$$yz, zy \to 4 \quad xz, zx \to 5 \quad xy, yx \to 6\,.$$

Using the impermeability, the optical properties of an anisotropic medium can be conveniently described through its index ellipsoid or optical indicatrix (Figure 1.1) defined by [1]:

$$\eta_1 x^2 + \eta_2 y^2 + \eta_3 z^2 + 2\eta_4 zy + 2\eta_5 xz + 2\eta_6 yx = 1. \tag{1.51}$$

Often, the components of the impermeability tensor are written as:

$$\eta_i \equiv \left(\frac{1}{n_i^2}\right), \tag{1.52}$$

but let us remember that the quantity n_i in expression (1.52) does not represent, in general, a true refractive index. Using this nomenclature, the index ellipsoid takes the form:

$$\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} + \frac{2zy}{n_4^2} + \frac{2xz}{n_5^2} + \frac{2yx}{n_6^2} = 1. \tag{1.53}$$

By making an appropriate choice of axes (x', y' and z'), called the principal axes of the material, the permittivity tensor (Eq. (1.47)) can be diagonalized and it becomes:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{x'} & 0 & 0 \\ 0 & \varepsilon_{y'} & 0 \\ 0 & 0 & \varepsilon_{z'} \end{pmatrix}, \tag{1.54}$$

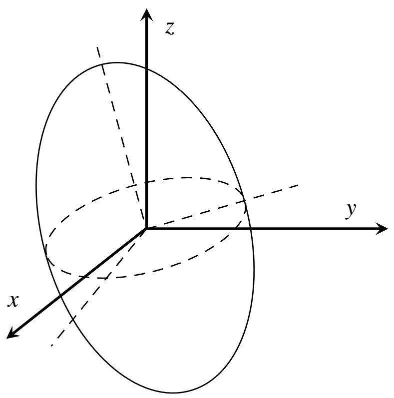

Figure 1.1 Index ellipsoid of an anisotropic medium. Its principal axes (dashed lines) do not coincide in general with the x-, y- and z-axes

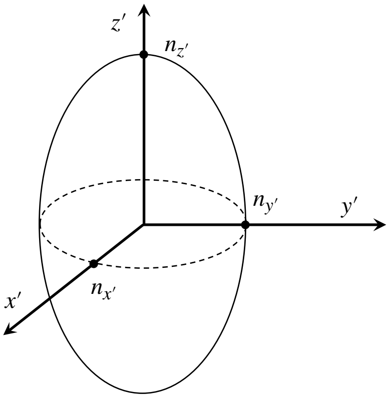

Figure 1.2 Index ellipsoid of an anisotropic medium, where the principal axes are coincident to the x'-, y'- and z'-axes

where all the elements are zero except to the diagonal. The $\varepsilon_{x'}$, $\varepsilon_{y'}$ and $\varepsilon_{z'}$ elements are called the principal permittivities along the x', y' and z' directions, respectively. In this reference system, the index ellipsoid (Figure 1.2) simplifies to the form:

$$\frac{x'^2}{n_{x'}^2} + \frac{y'^2}{n_{y'}^2} + \frac{z'^2}{n_{z'}^2} = 1, \tag{1.55}$$

where $n_{x'}$, $n_{y'}$, $n_{z'}$ now represent the true refractive indices that are related to the diagonal elements of the permittivity tensor by:

$$n_{x'} = \sqrt{\frac{\varepsilon_{x'}}{\varepsilon_0}}, n_{y'} = \sqrt{\frac{\varepsilon_{y'}}{\varepsilon_0}}, n_{z'} = \sqrt{\frac{\varepsilon_{z'}}{\varepsilon_0}}. \tag{1.56}$$

1.2.4 Light Propagation in Second-Order Non-Linear Media

For a non-linear medium, the constitutive relations given by Eq. (1.8) are no longer valid and it becomes necessary to include additional terms in serial powers of the fields. The non-linearity can be introduced by means of the polarization vector $\mathcal{P}$. In this way, Eq. (1.8a) may be rewritten as:

$$\mathcal{D} = \varepsilon_0 \mathcal{E} + \mathcal{P}. \tag{1.57}$$

For a non-linear medium, the polarization is split in a linear and a non-linear part through:

$$\mathcal{P} = \varepsilon_0 \chi_L \mathcal{E} + \mathcal{P}_{NL}, \tag{1.58}$$

where χ_L is the linear susceptibility and $\mathcal{P}_{NL}$ incorporates the non-linear polarization. In the case of second-order non-linear media, non-linear polarization is given by [2]:

$$(\mathcal{P}_{NL})_i = \sum_{j,k} 2d_{ijk}\mathcal{E}_j\mathcal{E}_k, \tag{1.59}$$

where d_{ijk} is the second-order non-linear tensor.

In this way, the displacement vector is expressed by:

$$\mathcal{D} = \varepsilon_0\mathcal{E} + \mathcal{P} = \varepsilon_0\mathcal{E} + \varepsilon_0\chi_L\mathcal{E} + \mathcal{P}_{NL} = \varepsilon\mathcal{E} + \mathcal{P}_{NL}, \tag{1.60}$$

where the linear permittivity is given by:

$$\varepsilon = \varepsilon_0(1+\chi_L). \tag{1.61}$$

Taking into account the non-linearity of the medium through relation (1.60), Maxwell's equations remain as:

$$\nabla\cdot(\varepsilon\mathcal{E} + \mathcal{P}_{NL}) = 0; \tag{1.62a}$$

$$\nabla\cdot\mathcal{H} = 0; \tag{1.62b}$$

$$\nabla\times\mathcal{E} = -\mu_0\frac{\partial\mathcal{H}}{\partial t}; \tag{1.62c}$$

$$\nabla\times\mathcal{H} = \frac{\partial}{\partial t}(\varepsilon\mathcal{E} + \mathcal{P}_{NL}). \tag{1.62d}$$

Combining the four Maxwell's equations it is possible to obtain an equation involving the electric field alone. Taking the curl operation over Eq. (1.62c) and having in mind the vectorial identity, $\nabla\times\nabla\times \equiv \nabla(\nabla\cdot) - \nabla^2$, that equation transforms to:

$$\nabla(\nabla\cdot\mathcal{E}) - \nabla^2\mathcal{E} = -\mu_0\frac{\partial}{\partial t}(\nabla\times\mathcal{H}), \tag{1.63}$$

where we have used the fact that the temporal and spatial derivatives commute. Inserting Eq. (1.62d) into Eq. (1.63) we have:

$$\nabla^2\mathcal{E} - \nabla(\nabla\cdot\mathcal{E}) - \mu_0\frac{\partial^2}{\partial t^2}(\varepsilon\mathcal{E} + \mathcal{P}_{NL}) = 0, \tag{1.64}$$

and assuming that the permittivity, ε, is time independent:

$$\nabla^2\mathcal{E} - \nabla(\nabla\cdot\mathcal{E}) - \mu_0\varepsilon\frac{\partial^2\mathcal{E}}{\partial t^2} - \mu_0\frac{\partial^2\mathcal{P}_{NL}}{\partial t^2} = 0. \tag{1.65}$$

This equation is known as the non-linear wave equation for light propagating in a non-linear medium.

Here, we will limit the analysis to monochromatic waves in a second-order non-linear medium, and we consider that only three frequencies ω_1, ω_2 and ω_3 are involved. The corresponding monochromatic fields are in the form:

$$\mathcal{E}^{\omega_1}(\boldsymbol{r},t) = \frac{1}{2}\left[\boldsymbol{E}_1(\boldsymbol{r})e^{i\omega_1 t} + \text{c.c.}\right]; \tag{1.66a}$$

$$\mathcal{E}^{\omega_2}(\boldsymbol{r},t) = \frac{1}{2}\left[\boldsymbol{E}_2(\boldsymbol{r})e^{i\omega_2 t} + \text{c.c.}\right]; \tag{1.66b}$$

$$\mathcal{E}^{\omega_3}(\boldsymbol{r},t) = \frac{1}{2}\left[\boldsymbol{E}_3(\boldsymbol{r})e^{i\omega_3 t} + \text{c.c.}\right], \tag{1.66c}$$

where $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ stand for the position dependent complex amplitudes of the fields at frequencies ω_1, ω_2 and ω_3, respectively, and c.c. denotes the complex conjugate. The electric field that appears in Eq. (1.65) is simply given by the sum of the fields at the three frequencies:

$$\mathcal{E}(\boldsymbol{r},t) = \mathcal{E}^{\omega_1}(\boldsymbol{r},t) + \mathcal{E}^{\omega_2}(\boldsymbol{r},t) + \mathcal{E}^{\omega_3}(\boldsymbol{r},t). \tag{1.67}$$

On the other hand, the i components of the non-linear polarization at frequencies ω_1, ω_2 and ω_3 are given by [3]:

$$[\mathcal{P}_{NL}^{\omega_1}(\boldsymbol{r},t)]_i = \sum_{j,k} d_{ijk}E_{3j}(\boldsymbol{r})E_{2k}^*(\boldsymbol{r})e^{i(\omega_3-\omega_2)t} + \text{c.c.}; \tag{1.68a}$$

$$[\mathcal{P}_{NL}^{\omega_2}(\boldsymbol{r},t)]_i = \sum_{j,k} d_{ijk}E_{3j}(\boldsymbol{r})E_{1k}^*(\boldsymbol{r})e^{i(\omega_3-\omega_1)t} + \text{c.c.}; \tag{1.68b}$$

$$[\mathcal{P}_{NL}^{\omega_3}(\boldsymbol{r},t)]_i = \sum_{j,k} d_{ijk}E_{2j}(\boldsymbol{r})E_{1k}(\boldsymbol{r})e^{i(\omega_2+\omega_1)t} + \text{c.c.}. \tag{1.68c}$$

Introducing the expression of the electric fields (Eqs. (1.66a)–(1.66c)) and the non-linear polarizations (Eqs. (1.68a)–(1.68c)) into Eq. (1.65), and after equating terms with the same frequency, the i-component of the wave equation corresponding to the frequency ω_1 yields:

$$\begin{aligned}&\frac{1}{2}\left\{\nabla^2 E_{1i}(\boldsymbol{r})e^{i\omega_1 t} + \text{c.c.}\right\} - \frac{1}{2}\left\{\frac{\partial}{\partial x_i}[\nabla\cdot\boldsymbol{E}_1(\boldsymbol{r})]e^{i\omega_1 t} + \text{c.c.}\right\}\\ &+ \frac{1}{2}\mu_0\varepsilon_1\omega_1^2\left\{E_{1i}(\boldsymbol{r})e^{i\omega_1 t} + \text{c.c.}\right\} + \mu_0\omega_1^2\left\{\sum_{j,k} d_{ijk}E_{3j}(\boldsymbol{r})E_{2k}^*(\boldsymbol{r})e^{i(\omega_3-\omega_2)t} + \text{c.c.}\right\} = 0,\end{aligned} \tag{1.69a}$$

where x_i indicates either x, y or z coordinates. Clearly, this equation couples the three Cartesian components of the field at frequency ω_1 through the second term and also depends on the components of the fields at frequencies ω_2 and ω_3 through the non-linear term. In addition to

Eq. (1.69a), two similar equations are obtained which involve oscillations at frequencies ω_2 and ω_3:

$$\frac{1}{2}\{\nabla^2 E_{2i}(\boldsymbol{r})e^{i\omega_2 t}+\text{c.c.}\}-\frac{1}{2}\left\{\frac{\partial}{\partial x_i}[\nabla\cdot\boldsymbol{E}_2(\boldsymbol{r})]e^{i\omega_2 t}+\text{c.c.}\right\}$$
$$+\frac{1}{2}\mu_0\varepsilon_2\omega_2^2\{E_{2i}(\boldsymbol{r})e^{i\omega_2 t}+\text{c.c.}\}+\mu_0\omega_2^2\left\{\sum_{j,k}d_{ijk}E_{3j}(\boldsymbol{r})E_{1k}^*(\boldsymbol{r})e^{i(\omega_3-\omega_1)t}+\text{c.c.}\right\}=0; \tag{1.69b}$$

$$\frac{1}{2}\{\nabla^2 E_{3i}(\boldsymbol{r})e^{i\omega_3 t}+\text{c.c.}\}-\frac{1}{2}\left\{\frac{\partial}{\partial x_i}[\nabla\cdot\boldsymbol{E}_3(\boldsymbol{r})]e^{i\omega_3 t}+\text{c.c.}\right\}$$
$$+\frac{1}{2}\mu_0\varepsilon_3\omega_3^2\{E_{3i}(\boldsymbol{r})e^{i\omega_3 t}+\text{c.c.}\}+\mu_0\omega_3^2\left\{\sum_{j,k}d_{ijk}E_{1j}(\boldsymbol{r})E_{2k}(\boldsymbol{r})e^{i(\omega_1+\omega_2)t}+\text{c.c.}\right\}=0. \tag{1.69c}$$

From Eqs. (1.69a)–(1.69c) we finally obtain:

$$\nabla^2 E_{1i}(\boldsymbol{r})-\frac{\partial}{\partial x_i}[\nabla\cdot\boldsymbol{E}_1(\boldsymbol{r})]+n_1^2k_{01}^2E_{1i}(\boldsymbol{r})+k_{01}^2\sum_{j,k}\chi_{ijk}E_{3j}(\boldsymbol{r})E_{2k}^*(\boldsymbol{r})=0; \tag{1.70a}$$

$$\nabla^2 E_{2i}(\boldsymbol{r})-\frac{\partial}{\partial x_i}[\nabla\cdot\boldsymbol{E}_2(\boldsymbol{r})]+n_2^2k_{02}^2E_{2i}(\boldsymbol{r})+k_{02}^2\sum_{j,k}\chi_{ijk}E_{3j}(\boldsymbol{r})E_{1k}^*(\boldsymbol{r})=0; \tag{1.70b}$$

$$\nabla^2 E_{3i}(\boldsymbol{r})-\frac{\partial}{\partial x_i}[\nabla\cdot\boldsymbol{E}_3(\boldsymbol{r})]+n_3^2k_{03}^2E_{3i}(\boldsymbol{r})+k_{03}^2\sum_{j,k}\chi_{ijk}E_{1j}(\boldsymbol{r})E_{2k}(\boldsymbol{r})=0, \tag{1.70c}$$

where we have used the definition of the non-linear susceptibility given by $\chi_{ijk}\equiv\frac{2d_{ijk}}{\varepsilon_0}$ and the wavenumber corresponding to the field at frequency ω_i ($i=1$–3) has been denoted by k_{0i}.

1.3 Wave Equation Formulation in Terms of the Transverse Field Components

For the description of light propagation in inhomogeneous media, such as optical waveguides, it is useful to rewrite the wave equations in terms of the transversal components of the electric or magnetic fields. If the refractive index profile of a given structure varies slowly along the direction of wave propagation, which is a common situation in integrated photonic devices and optical fibres, the transverse and longitudinal components of the electric and magnetic fields are decoupled [4]. Here we assume that the light is in the form of monochromatic waves and that the propagation is mainly along the *z*-direction. In the following we will develop the wave equations for the transverse components, in terms of the electric field and in terms of the magnetic field. These equations will be the starting point to obtain the paraxial wave equation for developing ‘beam propagation’ algorithms.

1.3.1 Electric Field Formulation

To obtain the wave equation in terms of the transversal electric field components, we start with the wave equation for monochromatic fields in terms of the electric field given by Eq. (1.32a):

$$\nabla^2 \boldsymbol{E} - \nabla(\nabla \cdot \boldsymbol{E}) + n^2 k_0^2 \boldsymbol{E} = 0, \tag{1.71}$$

where we have used the relationship $\boldsymbol{E} \cdot \nabla \ln \varepsilon = -\nabla \cdot \boldsymbol{E}$, derived from the first Maxwell equation. Next, we split the electric field in its transversal and longitudinal components in the following way:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_t(\boldsymbol{r}) + E_z(\boldsymbol{r})\mathbf{u}_z, \tag{1.72}$$

being:

$$\boldsymbol{E}_t(\boldsymbol{r}) = E_x(\boldsymbol{r})\mathbf{u}_x + E_y(\boldsymbol{r})\mathbf{u}_y. \tag{1.73}$$

Also, the differential operator ∇ is split in its transversal and longitudinal parts as:

$$\nabla = \nabla_t + \frac{\partial}{\partial \mathrm{z}}\mathbf{u}_z, \tag{1.74}$$

the transversal operator being defined by:

$$\nabla_t = \frac{\partial}{\partial x}\mathbf{u}_x + \frac{\partial}{\partial y}\mathbf{u}_y. \tag{1.75}$$

In this way, Eq. (1.71) expands as:

$$\nabla^2 \boldsymbol{E}_t - \nabla^2 E_z \mathbf{u}_z - \nabla\left(\nabla_t \cdot \boldsymbol{E}_t + \frac{\partial E_z}{\partial z}\right) + n^2 k_0^2 \boldsymbol{E} = 0. \tag{1.76}$$

Taking the transversal components of this equation, it yields:

$$\nabla^2 \boldsymbol{E}_t - \nabla_t\left(\nabla_t \cdot \boldsymbol{E}_t + \frac{\partial E_z}{\partial z}\right) + n^2 k_0^2 \boldsymbol{E}_t = 0. \tag{1.77}$$

On the other hand, the first Maxwell equation (1.9a) (valid for isotropic and non-linear media) can be expanded as:

$$\nabla \cdot \left(n^2 \boldsymbol{E}\right) = \nabla_t \cdot \left(n^2 \boldsymbol{E}_t\right) + \frac{\partial}{\partial z}\left(n^2 E_z\right) = 0, \tag{1.78}$$

or:

$$\nabla_t \cdot \left(n^2 \boldsymbol{E}_t\right) + n^2 \frac{\partial E_z}{\partial z} + E_z \frac{\partial n^2}{\partial z} = 0. \tag{1.79}$$

Now assuming that the refractive index change in the propagation direction is negligible, that is, there are not abrupt discontinuities in the z direction, it holds that:

$$\frac{\partial n^2}{\partial z} \approx 0. \tag{1.80}$$

Using this approximation, the partial derivative of the z component of the electric field respective to the z coordinate equation that appears in Eq. (1.79) can be expressed as:

$$\frac{\partial E_z}{\partial z} \approx -\frac{1}{n^2}\nabla_t \cdot \left(n^2 \boldsymbol{E}_t\right) = -\frac{\nabla_t n^2}{n^2} \cdot \boldsymbol{E}_t - \nabla_t \cdot \boldsymbol{E}_t, \tag{1.81}$$

which is exact for uniform (z invariant) structures. Substituting this partial derivative on Eq. (1.77) now, it becomes:

$$\nabla^2 \boldsymbol{E}_t + \nabla_t \left[\frac{\nabla_t n^2}{n^2} \cdot \boldsymbol{E}_t\right] + n^2 k_0^2 \boldsymbol{E}_t = 0. \tag{1.82}$$

Expressed explicitly in terms of the x and y components, it finally yields:

$$\nabla^2 E_x + \frac{\partial}{\partial x}\left[\frac{1}{n^2}\frac{\partial n^2}{\partial x} E_x\right] + \frac{\partial}{\partial x}\left[\frac{1}{n^2}\frac{\partial n^2}{\partial y} E_y\right] + n^2 k_0^2 E_x = 0; \tag{1.83a}$$

$$\nabla^2 E_y + \frac{\partial}{\partial y}\left[\frac{1}{n^2}\frac{\partial n^2}{\partial x} E_x\right] + \frac{\partial}{\partial y}\left[\frac{1}{n^2}\frac{\partial n^2}{\partial y} E_y\right] + n^2 k_0^2 E_y = 0. \tag{1.83b}$$

1.3.2 Magnetic Field Formulation

In an analogous way, we can proceed to obtain a wave equation in inhomogeneous media in terms of the transversal components of the magnetic field. Starting from Eq. (1.33b), and considering a structure in which the refractive index varies slowly along the direction of the wave propagation ($\partial n^2/\partial z \approx 0$), we obtain:

$$\nabla^2 \boldsymbol{H}_t + \frac{1}{n^2}\left(\nabla_t n^2\right) \times \left(\nabla_t \times \boldsymbol{H}_t\right) + n^2 k_0^2 \boldsymbol{H}_t = 0. \tag{1.84}$$

And by separating this vectorial equation in two scalar equations, we finally get:

$$\nabla^2 H_x - \frac{1}{n^2}\frac{\partial n^2}{\partial y}\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right) + n^2 k_0^2 H_x = 0; \tag{1.85a}$$

$$\nabla^2 H_y - \frac{1}{n^2}\frac{\partial n^2}{\partial x}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) + n^2 k_0^2 H_y = 0. \tag{1.85b}$$

It is worth mentioning that the Eqs. (1.82) and (1.84) are exact for z-invariant structures, for example, in straight optical waveguides. For structures where their refractive indices change along the propagation direction (z-coordinate), for example, in Y-junctions these equations are only approximated. Nevertheless, most of the optical waveguides used in integrated optical devices are designed with small angles in their geometry, typically in the order of 2–10°, where this approximation is very successfully fulfilled. Also, when sinusoidal index gratings are written in waveguides or fibres (where the refractive index along the propagation directions is periodically modulated), their index variation in a wavelength distance is very small and therefore the condition $\partial n^2/\partial z \approx 0$ is also fulfilled.

1.3.3 Wave Equation in Anisotropic Media

As in the case of isotropic media, if the transverse components of an electromagnetic field are known, then the longitudinal component can be obtained by using the first Maxwell equation $\nabla(\boldsymbol{\varepsilon E}) = 0$, but now considering that the permittivity is a tensor and not a scalar. This will allow us to describe the vectorial properties of the electromagnetic field in anisotropic media using the transverse components of the field [5]. For convenience, let us assume that the permittivity tensor (Eq. (1.47)) has the following not null components:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}. \tag{1.86}$$

To obtain a wave equation for the transversal components, we will follow a similar procedure to that described in the previous section. To start with, we take the transverse component of Eq. (1.71) now assuming the tensorial character of the medium permittivity, which is written as:

$$\nabla^2 \boldsymbol{E}_t - \nabla_t \left(\nabla_t \cdot \boldsymbol{E}_t + \frac{\partial E_z}{\partial z} \right) + (\boldsymbol{\varepsilon}_{tt}/\varepsilon_0) k_0^2 \boldsymbol{E}_t = 0, \tag{1.87}$$

where the sub-matrix $\boldsymbol{\varepsilon}_{tt}$, which includes the transverse components of the permittivity tensor, is defined as:

$$\boldsymbol{\varepsilon_{tt}} \equiv \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix}. \tag{1.88}$$

On the other hand, using Gauss' law:

$$\nabla \cdot (\boldsymbol{\varepsilon E}) = 0, \tag{1.89}$$

we obtain:

$$\nabla_t \cdot (\boldsymbol{\varepsilon}_{tt} \boldsymbol{E}_t) + \varepsilon_{zz} \frac{\partial E_z}{\partial z} + E_z \frac{\partial \varepsilon_{zz}}{\partial z} = 0. \tag{1.90}$$

If the permittivity component $\varepsilon_{zz}(x,y,z)$ varies slowly along the propagation direction z, which is valid for most photonic guided-wave devices, then the last term can be neglected and Eq. (1.90) simplifies to:

$$\frac{\partial E_z}{\partial z} \approx -\frac{1}{\varepsilon_{zz}} \nabla_t \cdot (\boldsymbol{\varepsilon}_{tt} \boldsymbol{E}_t). \tag{1.91}$$

This relation is exact for z-invariant structures, where $\dfrac{\partial \varepsilon_{zz}}{\partial z} = 0$.

By substituting Eq. (1.91) into Eq. (1.87) we finally derive the wave equation for the transverse electric field:

$$\nabla^2 \boldsymbol{E}_t + (\boldsymbol{\varepsilon}_{tt}/\varepsilon_0) k_0^2 \boldsymbol{E}_t = \nabla_t \left[\nabla_t \cdot \boldsymbol{E}_t - \frac{1}{\varepsilon_{zz}} \nabla_t \cdot (\boldsymbol{\varepsilon}_{tt} \boldsymbol{E}_t) \right]. \tag{1.92}$$

1.3.4 Second Order Non-Linear Media

In general, it is not possible to obtain a wave equation in terms of the transverse components for second-order non-linear media, as non-linear polarization can induce transverse and longitudinal components of the fields. Nevertheless, we can derive a general wave equation that will serve as the starting point to develop 'beam propagation' equations for non-linear media, where some extra approximation will be needed.

The first Maxwell equation (1.62a) considering a non-linear medium can be expanded as:

$$(\nabla \varepsilon) \cdot \boldsymbol{\mathcal{E}} + \varepsilon (\nabla \cdot \boldsymbol{\mathcal{E}}) + \nabla \cdot \boldsymbol{\mathcal{P}}_{NL} = (\nabla_t \varepsilon) \cdot \boldsymbol{\mathcal{E}}_t + \frac{\partial \varepsilon}{\partial z} \mathcal{E}_z + \varepsilon (\nabla \cdot \boldsymbol{\mathcal{E}}) + \nabla \cdot \boldsymbol{\mathcal{P}}_{NL} = 0. \tag{1.93}$$

Assuming that the structure is slowly changing with the propagation direction, $\dfrac{\partial \varepsilon}{\partial z} \approx 0$, and neglecting the last term in expression (1.93) $\nabla \cdot \boldsymbol{\mathcal{P}}_{NL} \approx 0$ (small non-linearity), we get:

$$\nabla \cdot \boldsymbol{\mathcal{E}} \approx -\frac{1}{\varepsilon} (\nabla_t \varepsilon) \cdot \boldsymbol{\mathcal{E}}_t. \tag{1.94}$$

Introducing this approximation in the wave equations ((1.70a)–(1.70c)) we obtain finally:

$$\nabla^2 \boldsymbol{E}_{1t} + \nabla_t \left[\frac{\nabla_t n_1^2}{n_1^2} \cdot \boldsymbol{E}_{1t} \right] + n_1^2 k_{10}^2 \boldsymbol{E}_{1t} + \left[k_{10}^2 \chi_{ijk} E_{3j} E_{2k}^* \right]_t = 0; \tag{1.95a}$$

$$\nabla^2 \boldsymbol{E}_{2t} + \nabla_t \left[\frac{\nabla_t n_2^2}{n_2^2} \cdot \boldsymbol{E}_{2t} \right] + n_2^2 k_{20}^2 \boldsymbol{E}_{2t} + \left[k_{20}^2 \chi_{ijk} E_{3j} E_{1k}^* \right]_t = 0; \tag{1.95b}$$

$$\nabla^2 \boldsymbol{E}_{3t} + \nabla_t \left[\frac{\nabla_t n_3^2}{n_3^2} \cdot \boldsymbol{E}_{3t} \right] + n_3^2 k_{30}^2 \boldsymbol{E}_{3t} + \left[k_{30}^2 \chi_{ijk} E_{1j} E_{2k} \right]_t = 0. \tag{1.95c}$$

Note that these equations involve the transversal components of the fields, but also may contain longitudinal components as well through the non-linear terms, depending on the non-null components of the non-linear tensor χ_{ijk}.

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