1 Introduction and Mathematical Preliminaries

1.1 Introduction

1.1.1 Preliminary Comments

The phrase "energy principles" or "energy methods" in the present study refers to methods that make use of the total potential energy (i.e., strain energy and potential energy due to applied loads) of a system to obtain values of an unknown displacement or force, at a specific point of the system. These include Castigliano's theorems, unit dummy load and unit dummy displacement methods, and Betti's and Maxwell's theorems. These methods are often limited to the (exact) determination of generalized displacements or forces at fixed points in the structure; in most cases, they cannot be used to determine the complete solution (i.e., displacements and/or forces) as a function of position in the structure. The phrase "variational methods," on the other hand, refers to methods that make use of the variational principles, such as the principles of virtual work and the principle of minimum total potential energy, to determine approximate solutions as continuous functions of position in a body. In the classical sense, a *variational principle* has to do with the minimization or finding stationary values of a functional with respect to a set of undetermined parameters introduced in the assumed solution. The functional represents the total energy of the system in solid and structural mechanics problems, and in other problems it is simply an integral representation of the governing equations. In all cases, the functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions.

1.1.2 The Role of Energy Methods and Variational Principles

Variational principles have always played an important role in mechanics. Variational formulations can be useful in three related ways. First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational state-

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ments. Second, there are problems that can be formulated by other means, such as by vector mechanics (e.g., Newton's laws), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute to the equations of equilibrium of an elastic body, as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means for studying the existence and uniqueness of solutions to problems. In many cases they can also be used to establish upper and/or lower bounds on approximate solutions.

1.1.3 A Brief Review of Historical Developments

In modern times, the term "variational formulation" applies to a wide spectrum of concepts having to do with weak, generalized, or direct variational formulations of boundary- and initial-value problems. Still, many of the essential features of variational methods remain the same as they were over 200 years ago when the first notions of variational calculus began to be formulated.[1]

Although Archimedes (287–212 B.C.) is generally credited as the first to use work arguments in his study of levers, the most primitive ideas of variational theory (the minimum hypothesis) are present in the writings of the Greek philosopher Aristotle (384–322 B.C.), to be revived again by the Italian mathematician/engineer Galileo (1564–1642), and finally formulated into a principle of least time by the French mathematician Fermat (1601–1665). The phrase *virtual velocities* was used by Jean Bernoulli in 1717 in his letter to Varignon (1654–1722). The development of early variational calculus, by which we mean the classical problems associated with minimizing certain functionals, had to await the works of Newton (1642–1727) and Leibniz (1646–1716). The earliest applications of such variational ideas included the classical *isoperimetric problem* of finding among closed curves of given length the one that encloses the greatest area, and Newton's problem of determining the solid of revolution of "minimum resistance." In 1696, Jean Bernoulli proposed the problem of the *brachistochrone*: among all curves connecting two points, find the curve traversed in the shortest time by a particle under the influence of gravity. It stood as a challenge to the mathematicians of their day to solve the problem using the rudimentary tools of analysis then available to them or whatever new ones they were capable of developing. Solutions to this problem were presented by some of the greatest mathematicians of the time: Leibniz, Jean Bernoulli's older brother Jacques Bernoulli, L'Hopital, and Newton.

[1] Many of the developments came from European scientists, whose works appeared in their native language and were not accessible to the whole scientific community.

The first step toward developing a general method for solving variational problems was given by the Swiss genius Leonhard Euler (1707–1783) in 1732 when he presented a "general solution of the isoperimetric problem," although Maupertuis is credited to have put forward a law of minimal property of potential energy for stable equilibrium in his *Mémoires de lÁcadémie des Sciences* in 1740. It was in Euler's 1732 work and subsequent publication of the principle of least action (in his book *Methodus inveniendi lineas curvas ...*) in 1744 that variational concepts found a welcome and permanent home in mechanics. He developed all ideas surrounding the principle of minimum potential energy in his work on the *elastica*, and he demonstrated the relationship between his variational equations and those governing the flexure and buckling of thin rods.

A great impetus to the development of variational mechanics began in the writings of Lagrange (1736–1813), first in his correspondence with Euler. Euler worked intensely in developing Lagrange's method but delayed publishing his results until Lagrange's works were published in 1760 and 1761. Lagrange used D'Alembert's principle to convert dynamics to statics and then used the principle of virtual displacements to derive his famous equations governing the laws of dynamics in terms of kinetic and potential energy. Euler's work, together with Lagrange's *Mécanique analytique* of 1788, laid down the basis for the variational theory of dynamical systems. Further generalizations appeared in the fundamental work of Hamilton in 1834. Collectively, all these works have had a monumental impact on virtually every branch of mechanics.

A more solid mathematical basis for variational theory began to be developed in the eighteenth and early nineteenth century. Necessary conditions for the existence of "minimizing curves" of certain functionals were studied during this period, and we find among contributors of that era the familiar names of Legendre, Jacobi, and Weierstrass. Legendre gave criteria for distinguishing between maxima and minima in 1786, and Jacobi gave sufficient conditions for existence of extrema in 1837. A more rigorous theory of existence of extrema was put together by Weierstrass, who established in 1865 the conditions on extrema for variational problems.

During the last half of the nineteenth century, the use of variational ideas was widespread among leaders in theoretical mechanics. We mention the works of Kirchhoff on plate theory; Lamé, Green, and Kelvin on elasticity; and the works of Betti, Maxwell, Castigliano, Menabrea, and Engesser on discrete structural systems. Lamé was the first in 1852 to prove a work equation, named after his colleague Claperon, for deformable bodies. Lamé's equation was used by Maxwell [1][2] to the solution of redundant frame-works using the unit dummy load technique. In 1875 Castigliano published an extremum version of this technique but attributed the idea to Menabrea. A generalization of Castigliano's work is due to Engesser [2].

[2]The references are listed at the end of the book.

Among the prominent contributors to the subject near the end of the nineteenth century and in the early years of the twentieth century, particularly in the area of variational methods of approximation and their applications, were Rayleigh [3], Ritz [4], and Galerkin [5]. Modern variational principles began in the works of Hellinger [6], Hu [7], and Reissner [8–10] on mixed variational principles for elasticity problems. A short historical account of early variational methods in mechanics can be found in the book of Lanczos [11] and Truesdell and Toupin [12]; additional information can be found in Dugas [13] and Timoshenko [14], and historical development of energetical principles in elastomechanics can be found in the paper by Oravas and McLean [15, 16]. Reference to much of the relevant contemporary literature can be found in the books by Washizu [17] and Oden and Reddy [18]. Additional historical papers and textbooks on variational principles and methods can be found in [19-60].

1.1.4 Preview

The objective of the present book is to introduce energy methods and variational principles of solid and structural mechanics and to illustrate their use in the derivation and solution of the equations of applied mechanics, including plane elasticity, beams, frames, and plates. Of course, variational formulations and methods presented in this book are also applicable to problems outside solid mechanics. To keep the scope of the book within reasonable limits, mostly linear problems of solid and structural mechanics are considered.

In the remaining part of the chapter, we review the algebra and calculus of vectors and tensors. In Chapter 2, a brief review of the equations of solid mechanics is presented, and the concepts of work and energy and elements from calculus of variations are discussed in Chapter 3. Principles of virtual work and their special cases are presented in Chapter 4. The chapter also includes energy theorems of structural mechanics, namely, Castigliano's theorems I and II, dummy displacement and dummy force methods, and Betti's and Maxwell's reciprocity theorems of elasticity. Chapter 5 is dedicated to Hamilton's principle for dynamical systems of solid mechanics. In Chapter 6 we introduce the Ritz, Galerkin, and weighted-residual methods. Chapter 7 contains the applications of variational methods to the formulation of plate theories and their solution by variational methods. For the sake of completeness and comparison, analytical solutions of bending, vibration, and buckling of circular and rectangular plates are also presented. An introduction to the finite element method and its application to displacement finite element models of beams and plates are discussed in Chapter 8. Chapter 9 is devoted to the discussion of mixed variational principles and mixed finite element models of beams and plates. Finally, theories and analytical as well as finite element solutions of functionally graded beams and plates are presented in Chapter 10.

1.2 Vectors

1.2.1 Introduction

Our approach in this book is evolutionary, that is, we wish to begin with concepts that are simple and intuitive and then generalize these concepts to a broader and more abstract body of analysis. This is a natural inductive approach, more or less in accord with the development of the subject of variational methods.

In analyzing physical phenomena, we set up, with the help of physical principles, relations between various quantities that represent the phenomena. As a means of expressing a natural law, a coordinate system in a chosen frame of reference can be introduced, and the various physical quantities involved can be expressed in terms of measurements made in that system. The mathematical form of the law thus depends upon the chosen coordinate system and may appear different in another type of coordinate system. The laws of nature, however, should be independent of the artificial choice of a coordinate system, and we may seek to represent the law in a manner independent of a particular coordinate system. A way of doing this is provided by vector and tensor analysis. When vector notation is used, a particular coordinate system need not be introduced. Consequently, the use of vector notation in formulating natural laws leaves them *invariant* to coordinate transformations. A study of physical phenomena by means of vector equations often leads to a deeper understanding of the problem in addition to bringing simplicity and versatility into the analysis.

The term *vector* is used often to imply a *physical* vector that has "magnitude and direction" and obeys certain rules of vector addition and scalar multiplication. In the sequel we consider more general, abstract objects than physical vectors, which are also called vectors. It transpires that the physical vector is a special case of what is known as a "vector from a linear vector space." Then the notion of vectors in modern mathematical analysis is an abstraction of the elementary notion of a physical vector. While the definition of a vector in abstract analysis does not require the vector to have a magnitude, in nearly all cases of practical interest, the vector is endowed with a magnitude, in which case the vector is said to belong to a normed vector space.

Like physical vectors, which have direction and magnitude and satisfy the parallelogram law of addition, *tensors* are more general objects that are endowed with a magnitude and multiple direction(s) and satisfy rules of tensor addition and scalar multiplication. In fact, vectors are often termed the first-order tensors. As will be shown shortly, the stress (i.e., force per unit area) requires a magnitude and two directions – one normal to the plane on which the stress is measured and the other is the direction of the force – to specify it uniquely. For additional details, References [61–88] listed at the end of the book may be consulted.

1.2.2 Definition of a Vector

In the analysis of physical phenomena, we are concerned with quantities that may be classified according to the information needed to specify them completely. Consider the following two groups:

Scalars	**Nonscalars**
Mass	Force
Temperature	Moment
Density	Stress
Volume	Acceleration
Time	Displacement

After units have been selected, the scalars are given by a single number. Nonscalars need not only a magnitude specified but also additional information, such as direction. Nonscalars that obey certain rules (such as the parallelogram law of addition) are called *vectors*. Not all nonscalar quantities are vectors. The specification of a stress requires not only a force, which is a vector, but also an area upon which the force acts. A stress is a second-order tensor, as will be shown shortly.

In written or typed material, it is customary to place an arrow or a bar over the letter denoting the vector, such as $\vec{A}$. Sometimes the typesetter's mark of a tilde under the letter is used. In printed material the vector letter is denoted by a boldface letter, $\mathbf{A}$, such as used in this book. The magnitude of the vector $\mathbf{A}$ is denoted by $|\mathbf{A}|$ or just A. The computation of the magnitude of a vector will be defined in the sequel, after the concept of scalar product of vectors is discussed.[3]

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if their magnitudes are equal, $|\mathbf{A}| = |\mathbf{B}|$, and if their directions and sense are equal. Consequently a vector is not changed if it is moved parallel to itself. This means that the position of a vector in space may be chosen arbitrarily. In certain applications, however, the actual point of location of a vector may be important (for instance, a moment or a force acting on a body). A vector associated with a given point is known as a localized or bound vector.

Let $\mathbf{A}$ and $\mathbf{B}$ be any two vectors. Then we can add them as shown in Fig. 1.2.1(a). The combination of the two diagrams in Fig. 1.2.1(a) gives the parallelogram shown in Fig. 1.2.1(b). Thus we say the vectors add according to the *parallelogram law* of addition so that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{1.2.1}$$

We thus see that vector addition is *commutative*.

[3] Mathematically, the length of a vector can be computed only when its components with respect to a basis are known.

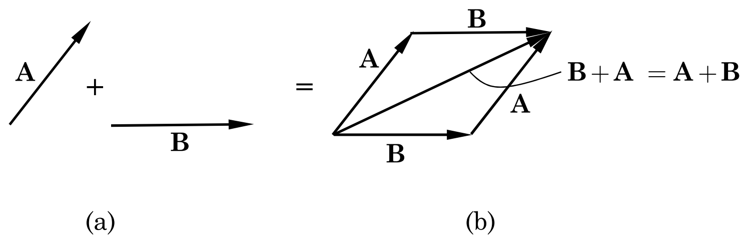


Fig. 1.2.1 (a) Addition of vectors. (b) Parallelogram law of addition.

Subtraction of vectors is carried out along the same lines. To form the difference $\mathbf{A} - \mathbf{B}$, we write

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{1.2.2}$$

and subtraction reduces to the operation of addition. The negative vector $-\mathbf{B}$ has the same magnitude as $\mathbf{B}$ but has the opposite *sense*.

With the rules of addition in place, we can define a (geometric) vector. *A vector is a quantity that possesses both magnitude and direction and obeys the parallelogram law of addition.* Obeying the law is important because there are quantities having both magnitude and direction that do not obey this law. A finite rotation of a rigid body is not a vector although infinitesimal rotations are. The definition given above is a *geometrical* definition. That vectors can be represented graphically is an *incidental* rather than a fundamental feature of the vector concept.

A vector of unit length is called a *unit vector*. The unit vector may be defined as follows:

$$\hat{\mathbf{e}}_A = \frac{\mathbf{A}}{A}. \tag{1.2.3}$$

We may now write

$$\mathbf{A} = A\hat{\mathbf{e}}_A. \tag{1.2.4}$$

Thus *any vector may be represented as a product of its magnitude and a unit vector*. A unit vector is used to designate direction. It does not have any physical dimensions. We denote a unit vector by a "hat" (caret) above the boldface letter.

A vector of zero magnitude is called a *zero vector* or a *null vector*. All null vectors are considered equal to each other without consideration as to direction:

$$\mathbf{A} + \mathbf{0} = \mathbf{A} \quad \text{and} \quad 0\mathbf{A} = \mathbf{0}. \tag{1.2.5}$$

The laws that govern addition, subtraction, and scalar multiplication of vectors are identical with those governing the operations of scalar algebra.

1.2.3 Scalar and Vector Products

Besides addition, subtraction, and multiplication by a scalar, we must consider the multiplication of two vectors. There are several ways the product of two vectors can be defined. We consider first the so-called scalar product. Let us recall the concept of work. When a force $\mathbf{F}$ acts on a mass point and moves through an infinitesimal displacement vector $\mathbf{ds}$, the work done by the force vector is defined by the *projection* of the force in the direction of the displacement times the magnitude of the displacement (see Fig. 1.2.2). Such an operation may be defined for any two vectors. Since the result of the product is a scalar, it is called the *scalar product.* We denote this product as follows:

$$\mathbf{F} \cdot \mathbf{ds} = F ds \cos\theta, \qquad 0 \leq \theta \leq \pi. \tag{1.2.6}$$

The scalar product is also known as the *dot product* or *inner product.*

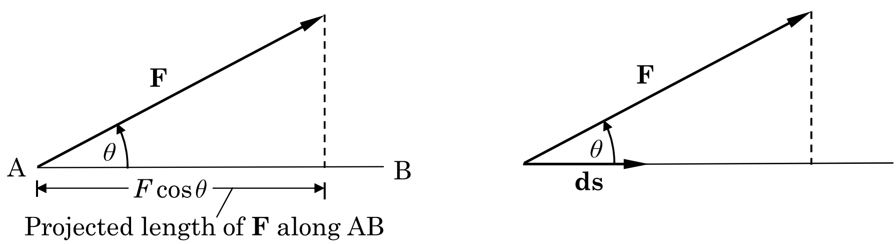


Fig. 1.2.2 Representation of work.

To understand the vector product, consider the concept of the *moment* due to a force. Let us describe the moment about a point O of a force $\mathbf{F}$ acting at a point P, as shown in Fig. 1.2.3(a). By definition, the magnitude of the moment is given by

$$M = F\,l, \quad F = |\mathbf{F}| = \sqrt{\mathbf{F} \cdot \mathbf{F}}, \tag{1.2.7}$$

where l is the lever arm for the force about the point O. If $\mathbf{r}$ denotes the vector $\mathbf{OP}$ and θ the angle between $\mathbf{r}$ and $\mathbf{F}$ as shown, such that $0 \leq \theta \leq \pi$, we have

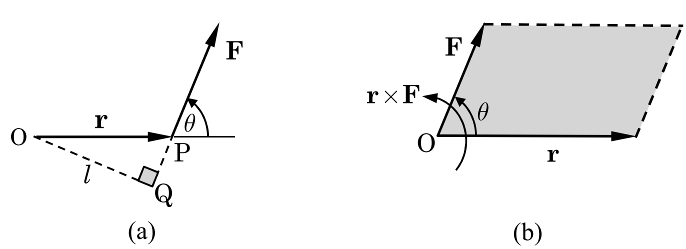


Fig. 1.2.3 (a) Representation of a moment. (b) Direction of rotation.

$l = r\sin\theta$, and thus

$$M = Fr\sin\theta. \tag{1.2.8}$$

A direction can now be assigned to the moment. Drawing the vectors **F** and **r** from the common origin O, we note that the rotation due to **F** tends to bring **r** into **F** [see Fig. 1.2.3(b)]. We now set up an axis of rotation perpendicular to the plane formed by **F** and **r**. Along this axis of rotation we set up a preferred direction as that in which a right-handed screw would advance when turned in the direction of rotation due to the moment [see Fig. 1.2.4(a)]. Along this axis of rotation, we draw a unit vector $\hat{\mathbf{e}}_M$ and agree that it represents the direction of the moment **M**. Thus we have

$$\mathbf{M} = Fr\,\sin\theta\,\hat{\mathbf{e}}_M = \mathbf{r}\times\mathbf{F}. \tag{1.2.9}$$

According to this expression, **M** may be looked upon as resulting from a special operation between the two vectors **F** and **r**. It is thus the basis for defining a product between any two vectors. Since the result of such a product is a vector, it may be called the *vector product.*

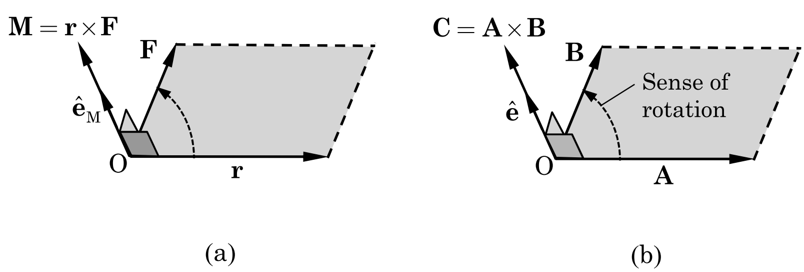


Fig. 1.2.4 (a) Axis of rotation. (b) Representation of the vector.

The vector product of two vectors **A** and **B** is a vector **C** whose magnitude is equal to the product of the magnitude of **A** and **B** times the sine of the angle measured from **A** to **B** such that $0 \le \theta \le \pi$, and whose direction is specified by the condition that **C** be perpendicular to the plane of the vectors **A** and **B** and points to the direction where a right-handed screw advances when turned so as to bring **A** into **B**.

The vector product is usually denoted by

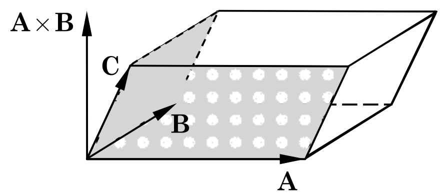
$$\mathbf{C} = \mathbf{A}\times\mathbf{B} = AB\,\sin(\mathbf{A},\mathbf{B})\,\hat{\mathbf{e}}, \tag{1.2.10}$$

where sin(**A**,**B**) denotes the sine of the angle between vectors **A** and **B**. This product is called the *cross product, skew product*, and also *outer product*, as well as the vector product [see Fig. 1.2.4(b)].

Now consider the various products of three vectors:

$$\mathbf{A}(\mathbf{B}\cdot\mathbf{C}),\quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}),\quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}). \tag{1.2.11}$$

The product $\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$ is merely a multiplication of the vector $\mathbf{A}$ by the scalar $\mathbf{B}\cdot\mathbf{C}$. The product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is a scalar. It can be seen that the product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, except for the algebraic sign, is the volume of the parallelepiped formed by the vectors **A, B**, and **C**, as shown in Fig. 1.2.5.



Volume of the parallelepiped is
$\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\equiv(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=-\mathbf{A}\times\mathbf{C}\cdot\mathbf{B}$

Fig. 1.2.5 Scalar triple product as the volume of a parallelepiped.

We also note the following properties:

1. The dot and cross can be interchanged without changing the value:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\equiv[\mathbf{ABC}]. \tag{1.2.12}$$

2. A cyclical permutation of the order of the vectors leaves the result unchanged:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}=\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}\equiv[\mathbf{ABC}]. \tag{1.2.13}$$

3. If the cyclic order is changed, the sign changes:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=-\mathbf{A}\cdot\mathbf{C}\times\mathbf{B}=-\mathbf{C}\cdot\mathbf{B}\times\mathbf{A}=-\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}. \tag{1.2.14}$$

4. A necessary and sufficient condition for any three vectors, **A**, **B**, and **C** to be coplanar is that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0$. Note also that the scalar triple product is zero when any two vectors are the same.

The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is a vector normal to the plane formed by $\mathbf{A}$ and $(\mathbf{B}\times\mathbf{C})$. The vector $(\mathbf{B}\times\mathbf{C})$, however, is perpendicular to the plane formed by $\mathbf{B}$ and $\mathbf{C}$. This means that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ lies in the plane formed by $\mathbf{B}$ and $\mathbf{C}$

and is perpendicular to **A** (see Fig. 1.2.6). Thus $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can be expressed as a linear combination of **B** and **C**:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = m_1\mathbf{B} + n_1\mathbf{C}. \tag{1.2.15}$$

Likewise, we would find that

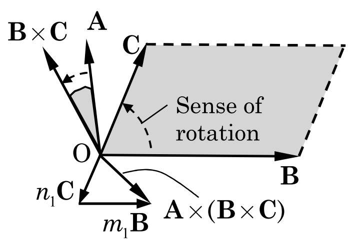
$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = m_2\mathbf{A} + n_2\mathbf{B}. \tag{1.2.16}$$

Thus the parentheses *cannot* be interchanged or removed. It can be shown that

$$m_1 = \mathbf{A} \cdot \mathbf{C}, \qquad n_1 = -\mathbf{A} \cdot \mathbf{B},$$

and hence that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{1.2.17}$$



$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = m_1\mathbf{B} + n_1\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

Fig. 1.2.6 The vector triple product.

Example 1.2.1

Find the equation of a plane perpendicular to a vector **A** and passing through the terminal point of vector **B** without the use of any coordinate system (see Fig. 1.2.7).

Solution: Let O be the origin and B the terminal point of vector **B**. Draw a directed line segment from O to Q, such that **OQ** is parallel to **A** and Q is in the plane. Then $\mathbf{OQ} = \alpha\,\mathbf{A}$, where α is a scalar. Let P be an arbitrary point on the line BQ. If the position vector of the point P is **r**, then

$$\mathbf{BP} = \mathbf{OP} - \mathbf{OB} = \mathbf{r} - \mathbf{B}.$$

Since **BP** is perpendicular to $\mathbf{OQ} = \alpha\mathbf{A}$, we must have

$$\mathbf{BP} \cdot \mathbf{OQ} = 0 \quad \text{or} \quad (\mathbf{r} - \mathbf{B}) \cdot \mathbf{A} = 0,$$

which is the equation of the plane in question.

The perpendicular distance from point O to the plane is the magnitude of **OQ**. However, we do not know its magnitude (or α is not known). The distance is also given by the projection

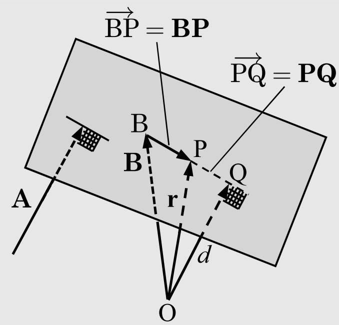


Fig. 1.2.7 Plane perpendicular to **A** and passing through the terminal point of **B**.

of vector **B** along **OQ**:

$$d = \mathbf{B} \cdot \frac{\mathbf{OQ}}{|\mathbf{OQ}|} = \mathbf{B} \cdot \hat{\mathbf{e}}_A,$$

where $\hat{\mathbf{e}}_A$ is the unit vector along **A**, $\hat{\mathbf{e}}_A = \mathbf{A}/A$.

Example 1.2.2

Let **A** and **B** be any two vectors in space. Then express the vector **A** in terms of components along (i.e., parallel) and perpendicular to **B**.

Solution: The component of **A** along **B** is given by $(\mathbf{A} \cdot \hat{\mathbf{e}}_B)$, where $\hat{\mathbf{e}}_B = \mathbf{B}/B$. The component of **A** perpendicular to **B** and in the plane of **A** and **B** is given by the vector triple product $\hat{\mathbf{e}}_B \times (\mathbf{A} \times \hat{\mathbf{e}}_B)$. Thus,

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}}_B)\hat{\mathbf{e}}_B + \hat{\mathbf{e}}_B \times (\mathbf{A} \times \hat{\mathbf{e}}_B). \tag{1}$$

Alternately, using Eq. (1.2.17) with $\mathbf{A} = \mathbf{C} = \hat{\mathbf{e}}_B$ and $\mathbf{B} = \mathbf{A}$, we obtain

$$\hat{\mathbf{e}}_B \times (\mathbf{A} \times \hat{\mathbf{e}}_B) = \mathbf{A} - (\hat{\mathbf{e}}_B \cdot \mathbf{A})\hat{\mathbf{e}}_B. \tag{2}$$

1.2.4 Components of a Vector

So far we have proceeded on a geometrical description of a vector as a directed line segment. We now embark on an analytical description of a vector and some of the operations associated with this description. Such a description yields a connection between vectors and ordinary numbers and relates operation on vectors with those on numbers. The analytical description is based on the notion of components of a vector.

In what follows, we shall consider a three-dimensional space, and the extensions to n dimensions will be evident (except for a few exceptions). A set of n vectors is said to be linearly dependent if a set of n numbers $\beta_1, \beta_2, \cdots, \beta_n$ can be found such that

$$\beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_n\mathbf{A}_n = \mathbf{0}, \tag{1.2.18}$$

where $\beta_1, \beta_2, \cdots, \beta_n$ cannot all be zero. If this expression cannot be satisfied, the vectors are said to be *linearly independent.*

In a three-dimensional space, a set of no more than three linearly independent vectors can be found. Let us choose any set and denote it as follows:

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3. \tag{1.2.19}$$

This set is called a *basis* (or a base system).

It is clear from the concept of linear dependence that we can represent any vector in three-dimensional space as a linear combination of the basis vectors (see Fig. 1.2.8):

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3. \tag{1.2.20}$$

The vectors $A_1\mathbf{e}_1$, $A_2\mathbf{e}_2$, and $A_3\mathbf{e}_3$ are called the *vector components* of $\mathbf{A}$, and A_1, A_2, and A_3 are called *scalar components* of $\mathbf{A}$ associated with the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Also, we use the notation $\mathbf{A} = (A_1, A_2, A_3)$ to denote a vector by its components.

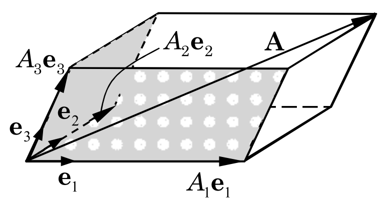


Fig. 1.2.8 Components of a vector.

1.2.5 Summation Convention

It is useful to abbreviate a summation of terms by understanding that a repeated index means summation over all values of that index. Thus the summation

$$\mathbf{A} = \sum_{i=1}^{3} A_i\mathbf{e}_i \tag{1.2.21}$$

can be shortened to

$$\mathbf{A} = A_i\mathbf{e}_i. \tag{1.2.22}$$

The repeated index is a *dummy index* and thus can be replaced by *any other symbol that has not already been used.* Thus we can also write

$$\mathbf{A} = A_i\mathbf{e}_i = A_m\mathbf{e}_m, \text{ and so on.}$$

When a basis is unit and orthogonal, that is, orthonormal, we have

$$[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = 1. \tag{1.2.23}$$

In many situations an *orthonormal basis* simplifies the calculations.

For an orthonormal basis, the vectors **A** and **B** can be written as

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3 = A_i\hat{\mathbf{e}}_i$$
$$\mathbf{B} = B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3 = B_i\hat{\mathbf{e}}_i,$$

where $\mathbf{e}_i \equiv \hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$ is the orthonormal basis and A_i and B_i are the corresponding *physical components* (i.e., the components have the same physical dimensions as the vector).

It is convenient at this time to introduce the Kronecker delta δ_{ij} and alternating symbol ε_{ijk} for representing the dot product and cross product of two orthonormal vectors in a right-handed basis system. We define the dot product $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ between the orthonormal basis vectors of a right-handed system as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \equiv \delta_{ij} = \begin{cases} 1, \text{ if } i = j, & \text{for any fixed value of } i, j \\ 0, \text{ if } i \neq j, & \text{for any fixed value of } i, j, \end{cases} \tag{1.2.24}$$

where δ_{ij} is called the *Kronecker delta symbol*. Similarly, we define the cross product $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$ for a right-handed system as

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \equiv \varepsilon_{ijk}\hat{\mathbf{e}}_k, \tag{1.2.25}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, \text{ if } i, j, k \text{ are in cyclic order} \\ \quad \text{and not repeated } (i \neq j \neq k), \\ -1, \text{ if } i, j, k \text{ are not in cyclic order} \\ \quad \text{and not repeated } (i \neq j \neq k), \\ 0, \text{ if any of } i, j, k \text{ are repeated.} \end{cases} \tag{1.2.26}$$

The symbol ε_{ijk} is called the *alternating symbol* or *permutation symbol*.

In an orthonormal basis, the scalar and vector products can be expressed in the index form using the Kronecker delta and alternating symbols as

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \cdot (B_j\hat{\mathbf{e}}_j) = A_iB_j\delta_{ij} = A_iB_i, \\ \mathbf{A} \times \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \times (B_j\hat{\mathbf{e}}_j) = A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k. \end{aligned} \tag{1.2.27}$$

Thus, the length of a vector in an orthonormal basis can be expressed as $A = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_iA_i}$. The Kronecker delta and the permutation symbol are related by the identity, known as the ε-δ *identity*:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{1.2.28}$$

The permutation symbol and the Kronecker delta prove to be very useful in proving vector identities. Since a vector form of any identity is invariant (i.e., valid in any coordinate system), it suffices to prove it in one coordinate system. In particular, an orthonormal system is very convenient because of the permutation symbol and the Kronecker delta. The following example illustrates some of the uses of δ_{ij} and ε_{ijk}.

Example 1.2.3

Express the vector operation $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$ in an alternate vector form.

Solution: We have

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (A_i B_j \varepsilon_{ijk} \hat{\mathbf{e}}_k) \cdot (C_m D_n \varepsilon_{mnp} \hat{\mathbf{e}}_p) \\
&= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnp} \delta_{kp} \\
&= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnk} \\
&= A_i B_j C_m D_n (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\
&= A_i B_j C_m D_n \delta_{im}\delta_{jn} - A_i B_j C_m D_n \delta_{in}\delta_{jm},
\end{aligned}$$

where we have used the ε–δ identity in Eq. (1.2.28). Since $C_m \delta_{im} = C_i$ (or $A_i \delta_{im} = A_m$, etc.), we have

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= A_i B_j C_i D_j - A_i B_j C_j D_i \\
&= A_i C_i B_j D_j - A_i D_i B_j C_j \\
&= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).
\end{aligned}$$

Although the above vector identity is established in an orthonormal coordinate system, it holds in a general coordinate system. That is, the vector identity is invariant.

We can establish the relationship between the components of two different orthonormal coordinate systems, say, unbarred and barred. Consider the unbarred coordinate basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and the barred coordinate basis $(\hat{\bar{\mathbf{e}}}_1, \hat{\bar{\mathbf{e}}}_2, \hat{\bar{\mathbf{e}}}_3)$. Then, we can express the same vector in the two coordinate systems as

$$\begin{aligned}
\mathbf{A} &= A_j \hat{\mathbf{e}}_j \quad \text{in unbarred basis,} \\
&= \bar{A}_j \hat{\bar{\mathbf{e}}}_j \quad \text{in barred basis.}
\end{aligned}$$

Now taking the dot product of the both sides with the vector $\hat{\bar{\mathbf{e}}}_i$ (from the left), we obtain the following relation between the components of a vector in two different coordinate systems:

$$\bar{A}_i = \beta_{ij}\, A_j, \quad \beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j. \tag{1.2.29}$$

Thus, the relationship between the components $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$ and (A_1, A_2, A_3) is called the *transformation rule* between the barred and unbarred components in the two orthogonal coordinate systems. The coefficients β_{ij} are the *direction cosines* of the barred coordinate system with respect to the unbarred coordinate system:

$$\beta_{ij} = \text{cosine of the angle between } \hat{\bar{\mathbf{e}}}_i \text{ and } \hat{\mathbf{e}}_j. \tag{1.2.30}$$

Note that the first subscript of β_{ij} comes from the barred coordinate system and the second subscript from the unbarred system. Obviously, β_{ij} is not symmetric (i.e., $\beta_{ij} \neq \beta_{ji}$). The direction cosines allow us to relate components of a vector (or a tensor) in the unbarred coordinate system to components of the same vector (or tensor) in the barred coordinate system. **Example 1.2.4** illustrates the computation of direction cosines.

Example 1.2.4

Let $\hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$ be a set of orthonormal base vectors, and define new right-handed coordinate base vectors by $(\hat{\bar{\mathbf{e}}}_1 . \hat{\bar{\mathbf{e}}}_2 = 0)$:

$$\hat{\bar{\mathbf{e}}}_1 = \frac{2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{3}, \qquad \hat{\bar{\mathbf{e}}}_2 = \frac{\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2}{\sqrt{2}}.$$

Determine the direction cosines of the transformation between the two coordinate systems.

Solution: First we compute the third base vector in the barred coordinate system by

$$\hat{\bar{\mathbf{e}}}_3 = \hat{\bar{\mathbf{e}}}_1 \times \hat{\bar{\mathbf{e}}}_2 = \frac{\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3}{3\sqrt{2}}.$$

An arbitrary vector $\mathbf{A}$ can be represented in either coordinate system:

$$\mathbf{A} = A_i \hat{\mathbf{e}}_i = \bar{A}_i \hat{\bar{\mathbf{e}}}_i.$$

The components of the vector in the two different coordinate systems are related by

$$\{\bar{A}\} = [\beta]\{A\}, \qquad \beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j.$$

For the case at hand, we have

$$\begin{aligned}
\beta_{11} &= \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_1 = \frac{2}{3}, & \beta_{12} &= \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_2 = \frac{2}{3}, & \beta_{13} &= \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_3 = \frac{1}{3},\\
\beta_{21} &= \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}}, & \beta_{22} &= \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_2 = -\frac{1}{\sqrt{2}}, & \beta_{23} &= \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_3 = 0,\\
\beta_{31} &= \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_1 = \frac{1}{3\sqrt{2}}, & \beta_{32} &= \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_2 = \frac{1}{3\sqrt{2}}, & \beta_{33} &= \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_3 = -\frac{4}{3\sqrt{2}},
\end{aligned}$$

or

$$[\beta] = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix}.$$

When the basis vectors are constant, that is, with fixed lengths (with the same units) and directions, the basis is called *Cartesian*. The general Cartesian system is oblique. When the basis vectors are unit and orthogonal (orthonormal), the basis system is called *rectangular Cartesian*, or simply *Cartesian*. In much of our study, we shall deal with Cartesian bases.

Let us denote an orthonormal Cartesian basis by

$$\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} \qquad \text{or} \qquad \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}.$$

The Cartesian coordinates are denoted by (x, y, z) or (x^1, x^2, x^3). The familiar rectangular Cartesian coordinate system is shown in Fig. 1.2.9. We shall always use right-handed coordinate systems.

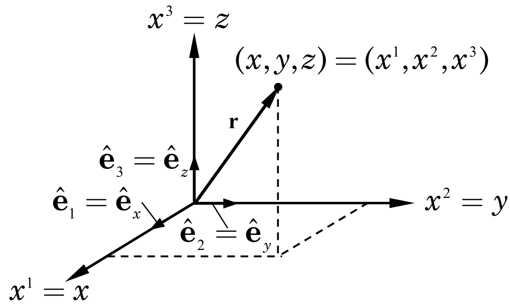


Fig. 1.2.9 Rectangular Cartesian coordinates.

A position vector to an arbitrary point (x, y, z) or (x^1, x^2, x^3), measured from the origin, is given by

$$\begin{aligned}\mathbf{r} &= x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z\\ &= x^1\hat{\mathbf{e}}_1 + x^2\hat{\mathbf{e}}_2 + x^3\hat{\mathbf{e}}_3,\end{aligned}$$

or, in summation notation, by

$$\mathbf{r} = x^j\hat{\mathbf{e}}_j. \tag{1.2.31}$$

The distance between two infinitesimally removed points is given by

$$\begin{aligned}d\mathbf{r}\cdot d\mathbf{r} = (ds)^2 &= dx^j dx^j\\ &= (dx)^2 + (dy)^2 + (dz)^2.\end{aligned} \tag{1.2.32}$$

1.2.6 Vector Calculus

The basic notions of vector and scalar calculus, especially with regard to physical applications, are closely related to the rate of change of a scalar field with distance. Let us denote a scalar field by $\phi = \phi(\mathbf{r})$. In general coordinates we can write $\phi = \phi(q^1, q^2, q^3)$. The coordinate system (q^1, q^2, q^3) is referred to as the *unitary system*.

We now define the unitary basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as follows:

$$\mathbf{e}_1 \equiv \frac{\partial\mathbf{r}}{\partial q^1}, \qquad \mathbf{e}_2 \equiv \frac{\partial\mathbf{r}}{\partial q^2}, \qquad \mathbf{e}_3 \equiv \frac{\partial\mathbf{r}}{\partial q^3}. \tag{1.2.33}$$

Hence, an arbitrary vector $\mathbf{A}$ is expressed as

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3, \tag{1.2.34}$$

and a differential distance is denoted by

$$d\mathbf{r} = dq^1\mathbf{e}_1 + dq^2\mathbf{e}_2 + dq^3\mathbf{e}_3 = dq^i\mathbf{e}_i. \tag{1.2.35}$$

Observe that the A's and dq's have superscripts, whereas the unitary basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has subscripts. The dq^i are referred to as the *contravariant components* of the differential vector $d\mathbf{r}$, and A^i are the contravariant components of vector $\mathbf{A}$. The unitary basis can be described in terms of the rectangular Cartesian basis $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$=$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ as follows:

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial q^1} = \frac{\partial x}{\partial q^1}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q^1}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q^1}\hat{\mathbf{e}}_z,$$
$$\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial q^2} = \frac{\partial x}{\partial q^2}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q^2}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q^2}\hat{\mathbf{e}}_z,$$
$$\mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial q^3} = \frac{\partial x}{\partial q^3}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q^3}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q^3}\hat{\mathbf{e}}_z.$$

In the summation convention, we have

$$\mathbf{e}_i \equiv \frac{\partial \mathbf{r}}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\hat{\mathbf{e}}_j, \qquad i = 1, 2, 3. \tag{1.2.36}$$

Associated with any arbitrary basis is another basis that can be derived from it. We can construct this basis in the following way: Taking the scalar product of the vector $\mathbf{A}$ in Eq. (1.2.34) with the cross product $\mathbf{e}_1 \times \mathbf{e}_2$, we obtain

$$\mathbf{A} \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = A^3 \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$$

since $\mathbf{e}_1 \times \mathbf{e}_2$ is perpendicular to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Solving for A^3 gives

$$A^3 = \mathbf{A} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \mathbf{A} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}.$$

In similar fashion, we can obtain expressions for A^1 and A^2. Thus, we have

$$A^1 = \mathbf{A} \cdot \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}, \quad A^2 = \mathbf{A} \cdot \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}, \quad A^3 = \mathbf{A} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}. \tag{1.2.37}$$

We thus observe that we can obtain the components A^1, A^2, and A^3 by taking the scalar product of the vector $\mathbf{A}$ with special vectors, which we denote as follows:

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}, \quad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}, \quad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]}. \tag{1.2.38}$$

The set of vectors $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ is called the *dual* or *reciprocal* basis. Notice from the basic definitions that we have the following relations:

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.2.39}$$

It is possible, since the dual basis is linearly independent (the reader should verify this), to express a vector $\mathbf{A}$ in terms of the dual basis:

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3. \tag{1.2.40}$$

Notice now that the components associated with the dual basis have subscripts, and A_i are the *covariant components* of $\mathbf{A}$.

By an analogous process as that above, we can show that the original basis can be expressed in terms of the dual basis in the following way:

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{[\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3]}, \qquad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{[\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3]}, \qquad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{[\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3]}. \tag{1.2.41}$$

Of course in the evaluation of the cross products, we shall always use the right-hand rule. It follows from the above expressions that

$$\begin{aligned} A^1 &= \mathbf{A}\cdot\mathbf{e}^1, \quad A^2 = \mathbf{A}\cdot\mathbf{e}^2, \quad A^3 = \mathbf{A}\cdot\mathbf{e}^3, \quad \text{or} \quad A^i = \mathbf{A}\cdot\mathbf{e}^i, \\ A_1 &= \mathbf{A}\cdot\mathbf{e}_1, \quad A_2 = \mathbf{A}\cdot\mathbf{e}_2, \quad A_3 = \mathbf{A}\cdot\mathbf{e}_3, \quad \text{or} \quad A_i = \mathbf{A}\cdot\mathbf{e}_i. \end{aligned} \tag{1.2.42}$$

Returning to the scalar field ϕ, the differential change is given by

$$d\phi = \frac{\partial\phi}{\partial q^1}dq^1 + \frac{\partial\phi}{\partial q^2}dq^2 + \frac{\partial\phi}{\partial q^3}dq^3. \tag{1.2.43}$$

The differentials dq^1, dq^2, and dq^3 are components of $d\mathbf{r}$ (see Eq. (1.2.35)). We would now like to write $d\phi$ in such a way that we elucidate *the direction* as well as the magnitude of $d\mathbf{r}$. Since $\mathbf{e}^1\cdot\mathbf{e}_1 = 1$, $\mathbf{e}^2\cdot\mathbf{e}_2 = 1$, and $\mathbf{e}^3\cdot\mathbf{e}_3 = 1$, we can write

$$\begin{aligned} d\phi &= \mathbf{e}^1\frac{\partial\phi}{\partial q^1}\cdot\mathbf{e}_1 dq^1 + \mathbf{e}^2\frac{\partial\phi}{\partial q^2}\cdot\mathbf{e}_2 dq^2 + \mathbf{e}^3\frac{\partial\phi}{\partial q^3}\cdot\mathbf{e}_3 dq^3 \\ &= (dq^1\mathbf{e}_1 + dq^2\mathbf{e}_2 + dq^3\mathbf{e}_3)\cdot\left(\mathbf{e}^1\frac{\partial\phi}{\partial q^1} + \mathbf{e}^2\frac{\partial\phi}{\partial q^2} + \mathbf{e}^3\frac{\partial\phi}{\partial q^3}\right) \\ &= d\mathbf{r}\cdot\left(\mathbf{e}^1\frac{\partial\phi}{\partial q^1} + \mathbf{e}^2\frac{\partial\phi}{\partial q^2} + \mathbf{e}^3\frac{\partial\phi}{\partial q^3}\right). \end{aligned} \tag{1.2.44}$$

Let us now denote the magnitude of $d\mathbf{r}$ by $ds \equiv |d\mathbf{r}|$. Then $\hat{\mathbf{e}} = d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$, and we have

$$\left(\frac{d\phi}{ds}\right)_{\hat{\mathbf{e}}} = \hat{\mathbf{e}}\cdot\left(\mathbf{e}^1\frac{\partial\phi}{\partial q^1} + \mathbf{e}^2\frac{\partial\phi}{\partial q^2} + \mathbf{e}^3\frac{\partial\phi}{\partial q^3}\right). \tag{1.2.45}$$

The derivative $(d\phi/ds)_{\hat{\mathbf{e}}}$ is called the *directional derivative* of ϕ. We see that it is the *rate of change* of ϕ with respect to distance and that it depends on the direction $\hat{\mathbf{e}}$ in which the distance is taken.

The vector that is scalar multiplied by $\hat{\mathbf{e}}$ can be obtained immediately whenever the scalar field is given. Because the magnitude of this vector is equal to the maximum value of the directional derivative, it is called the *gradient vector* and is denoted by grad ϕ:

$$\text{grad } \phi \equiv \mathbf{e}^1 \frac{\partial \phi}{\partial q^1} + \mathbf{e}^2 \frac{\partial \phi}{\partial q^2} + \mathbf{e}^3 \frac{\partial \phi}{\partial q^3}. \tag{1.2.46}$$

From this representation it can be seen that

$$\frac{\partial \phi}{\partial q^1}, \qquad \frac{\partial \phi}{\partial q^2}, \qquad \frac{\partial \phi}{\partial q^3}$$

are the *covariant components* of the gradient vector.

When the scalar function $\phi(\mathbf{r})$ is set equal to a constant, $\phi(\mathbf{r}) =$ constant, a family of surfaces is generated. A different surface is designated by different values of the constant, and each surface is called a *level surface* (see Fig. 1.2.10). If the direction in which the directional derivative is taken lies within a level surface, then $d\phi/ds$ is zero, since ϕ is a constant on a level surface. In this case the unit vector $\hat{\mathbf{e}}$ is tangent to a level surface. It follows, therefore, that if $d\phi/ds$ is zero, then grad ϕ must be perpendicular to $\hat{\mathbf{e}}$ and thus *perpendicular to a level surface*. Thus if any surface is given by $\phi(\mathbf{r}) =$ constant, the unit normal to the surface is determined by

$$\hat{\mathbf{n}} = \pm \frac{\text{grad } \phi}{|\text{grad } \phi|}. \tag{1.2.47}$$

The plus or minus sign appears because the direction of $\hat{\mathbf{n}}$ may point in either direction away from the surface. If the surface is closed, the usual convention is to take $\hat{\mathbf{n}}$ pointing outward.

It is convenient to write the gradient vector as

$$\text{grad } \phi \equiv \left(\mathbf{e}^1 \frac{\partial}{\partial q^1} + \mathbf{e}^2 \frac{\partial}{\partial q^2} + \mathbf{e}^3 \frac{\partial}{\partial q^3} \right) \phi \tag{1.2.48}$$

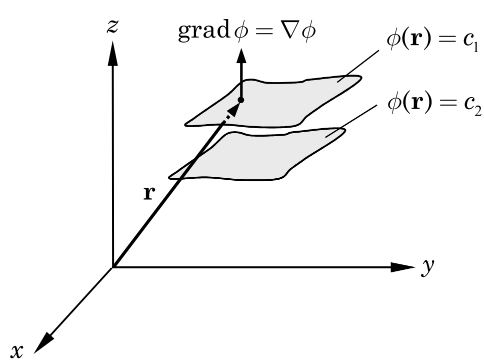


Fig. 1.2.10 Level surfaces and gradient to a surface.

and interpret grad ϕ as some operator operating on ϕ, that is, grad $\phi \equiv \boldsymbol{\nabla}\phi$. This operator is denoted by

$$\boldsymbol{\nabla} \equiv \mathbf{e}^1 \frac{\partial}{\partial q^1} + \mathbf{e}^2 \frac{\partial}{\partial q^2} + \mathbf{e}^3 \frac{\partial}{\partial q^3} \tag{1.2.49}$$

and is called the "del operator." The del operator is a *vector differential* operator, and the "components" $\partial/\partial q^1$, $\partial/\partial q^2$, and $\partial/\partial q^3$ appear as covariant components.

It is important to note that whereas the del operator has some of the properties of a vector, it does not have them all, because it is an operator. For instance, $\boldsymbol{\nabla} \cdot \mathbf{A}$ is a scalar (called the divergence of $\mathbf{A}$), whereas $\mathbf{A} \cdot \boldsymbol{\nabla}$ is a scalar *differential operator*. Thus the del operator does not commute in this sense.

In the rectangular Cartesian system, we have the simple form

$$\boldsymbol{\nabla} \equiv \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z},$$

or, in the summation convention, we have

$$\boldsymbol{\nabla} \equiv \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}. \tag{1.2.50}$$

The dot product of del operator with a vector is called the *divergence of a vector* and denoted by

$$\boldsymbol{\nabla} \cdot \mathbf{A} \equiv \text{div}\mathbf{A} = \frac{\partial A_i}{\partial x_i}. \tag{1.2.51}$$

If we take the divergence of the gradient vector, we have

$$\text{div}(\text{grad } \phi) \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi = \nabla^2\phi. \tag{1.2.52}$$

The notation $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ is called the *Laplacian operator*. In Cartesian systems this reduces to the simple form

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2\phi}{\partial x_i \partial x_i}. \tag{1.2.53}$$

The Laplacian of a scalar appears frequently in the partial differential equations governing physical phenomena.

The curl of a vector is defined as the del operator operating on a vector by means of the cross product:

$$\text{curl}\,\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \times \hat{\mathbf{e}}_k A_k = \frac{\partial A_k}{\partial x_j} (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) = \frac{\partial A_k}{\partial x_j} \varepsilon_{jki}\, \hat{\mathbf{e}}_i. \tag{1.2.54}$$

Thus the ith component of $(\boldsymbol{\nabla} \times \mathbf{A})$ is $\frac{\partial A_k}{\partial x_j} \varepsilon_{jki}$.

Example 1.2.5

Using the index-summation notation, prove the following vector identity:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2 \mathbf{v},$$

where $\mathbf{v}$ is a vector function of the coordinates, x_i.

Solution: Observe that

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \times \left(\hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \times v_k \hat{\mathbf{e}}_k \right) \\
&= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \times \left(\varepsilon_{jk\ell} \frac{\partial v_k}{\partial x_j} \hat{\mathbf{e}}_\ell \right) \\
&= \varepsilon_{i\ell m}\, \varepsilon_{jk\ell} \frac{\partial^2 v_k}{\partial x_i \partial x_j} \hat{\mathbf{e}}_m .
\end{aligned}$$

Using the ε-δ identity, we obtain

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) &\equiv (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial^2 v_k}{\partial x_i \partial x_j} \hat{\mathbf{e}}_m \\
&= \frac{\partial^2 v_i}{\partial x_i \partial x_j} \hat{\mathbf{e}}_j - \frac{\partial^2 v_k}{\partial x_i \partial x_i} \hat{\mathbf{e}}_k = \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_i} \right) - \frac{\partial^2}{\partial x_i \partial x_i} (v_k \hat{\mathbf{e}}_k) \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2 \mathbf{v}.
\end{aligned}$$

This result is sometimes used as the definition of the Laplacian of a vector, that is,

$$\nabla^2 \mathbf{v} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}).$$

A summary of vector operations in both general vector notation and in Cartesian component form is given in Table 1.2.1, and some useful vector operations for cylindrical and spherical coordinate systems (see Fig. 1.2.11) are presented in Table 1.2.2.

1.2.7 Gradient, Divergence, and Curl Theorems

Useful expressions for the integrals of the gradient, divergence, and curl of a vector can be established between volume integrals and surface integrals. Let Ω denote a region in space surrounded by the closed surface Γ. Let $d\Gamma$ be a differential element of surface and $\hat{\mathbf{n}}$ the unit outward normal, and let $d\Omega$ be a differential volume element. The following integral relations are proven to be useful in the coming chapters.

Gradient theorem:

$$\int_\Omega \text{grad}\, \phi \, d\Omega = \oint_\Gamma \hat{\mathbf{n}} \phi \, d\Gamma \quad \left[\int_\Omega \hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i} \, d\Omega = \oint_\Gamma \hat{\mathbf{e}}_i n_i \phi \, d\Gamma \right]. \tag{1.2.55}$$

Curl theorem:

$$\int_\Omega \text{curl } \mathbf{A}\, d\Omega = \oint_\Gamma \hat{\mathbf{n}} \times \mathbf{A}\, d\Gamma \quad \left[\int_\Omega \varepsilon_{ijk}\hat{\mathbf{e}}_k \frac{\partial A_j}{\partial x_i}\, d\Omega = \oint_\Gamma \varepsilon_{ijk}\hat{\mathbf{e}}_k n_i A_j\, d\Gamma\right]. \tag{1.2.56}$$

Divergence theorem:

$$\int_\Omega \text{div } \mathbf{A}\, d\Omega = \oint_\Gamma \hat{\mathbf{n}} \cdot \mathbf{A}\, d\Gamma \quad \left[\int_\Omega \frac{\partial A_i}{\partial x_i}\, d\Omega = \oint_\Gamma n_i A_i\, d\Gamma\right]. \tag{1.2.57}$$

Table 1.2.1 Vector expressions and their Cartesian component forms (**A**, **B**, and **C** are vector functions, U is a scalar function, **x** is the position vector, and $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ are the Cartesian unit vectors in a rectangular Cartesian coordinate system; see Fig. 1.2.9).

No.	Vector form and its equivalence	Component form
1.	$\mathbf{A}\cdot\mathbf{B}$	A_iB_i
2.	$\mathbf{A}\times\mathbf{B}$	$\varepsilon_{ijk}A_iB_j\hat{\mathbf{e}}_k$
3.	$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$	$\varepsilon_{ijk}A_iB_jC_k$
4.	$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$	$\varepsilon_{ijk}e_{klm}A_jB_lC_m\hat{\mathbf{e}}_i$
5.	$\boldsymbol{\nabla}\mathbf{A}$	$\frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$
6.	$\boldsymbol{\nabla}\cdot\mathbf{A}$	$\frac{\partial A_i}{\partial x_i}$
7.	$\boldsymbol{\nabla}\times\mathbf{A}$	$\varepsilon_{ijk}\frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_k$
8.	$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$	$\varepsilon_{ijk}\frac{\partial^2 A_j}{\partial x_i\partial x_k}$
9.	$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}U) = 0$	$\varepsilon_{ijk}\,\hat{\mathbf{e}}_k\,\frac{\partial^2 U}{\partial x_i\partial x_j}$
10.	$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$	$\varepsilon_{ijk}\frac{\partial}{\partial x_i}(A_jB_k)$
11.	$(\boldsymbol{\nabla}\times\mathbf{A})\times\mathbf{B} = \mathbf{B}\cdot[\boldsymbol{\nabla}\mathbf{A} - (\boldsymbol{\nabla}\mathbf{A})^{\mathrm{T}}]$	$\varepsilon_{ijk}\,\varepsilon_{klm}B_l\frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_m$
12.	$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{A}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}$	$\varepsilon_{nim}\,\varepsilon_{jkm}\,A_i\,\frac{\partial A_k}{\partial x_j}\,\hat{\mathbf{e}}_n$
13.	$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{A}) = \nabla^2\mathbf{A}$	$\frac{\partial^2 A_j}{\partial x_i\partial x_i}\hat{\mathbf{e}}_j$
14.	$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}$	$\varepsilon_{mil}\,\varepsilon_{jkl}\frac{\partial^2 A_k}{\partial x_i\partial x_j}\hat{\mathbf{e}}_m$

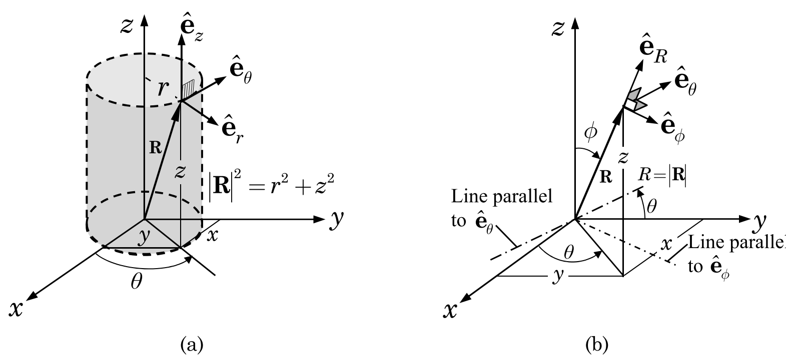


Fig. 1.2.11 (a) Cylindrical coordinate system. (b) Spherical coordinate system.

Table 1.2.2 Base vectors and operations with the del operator in cylindrical and spherical coordinate systems; see Fig. 1.2.11.

• *Cylindrical coordinate system* (r, θ, z)

$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\quad \mathbf{R} = r\,\hat{\mathbf{e}}_r + z\,\hat{\mathbf{e}}_z,\quad \mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$ (a vector)

$\hat{\mathbf{e}}_r = \cos\theta\ \hat{\mathbf{e}}_x + \sin\theta\ \hat{\mathbf{e}}_y,\ \ \hat{\mathbf{e}}_\theta = -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_y,\ \ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$

$\frac{\partial \hat{\mathbf{e}}_r}{\partial\theta} = -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\theta,\ \ \frac{\partial \hat{\mathbf{e}}_\theta}{\partial\theta} = -\cos\theta\ \hat{\mathbf{e}}_x - \sin\theta\ \hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_r$

All other derivatives of the base vectors are zero.

$\boldsymbol{\nabla} = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z},\qquad \nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial\theta^2} + r\frac{\partial^2}{\partial z^2}\right]$

$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\left[\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial\theta} + r\frac{\partial A_z}{\partial z}\right]$

$\boldsymbol{\nabla}\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{\mathbf{e}}_z$

$\boldsymbol{\nabla}\mathbf{A} = \frac{\partial A_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{\partial A_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial A_r}{\partial\theta} - A_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \frac{\partial A_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + \frac{\partial A_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r$
$\quad + \frac{1}{r}\left(A_r + \frac{\partial A_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial A_z}{\partial\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial A_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial A_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z$

• *Spherical coordinate system* (R, ϕ, θ)

$x = R\sin\phi\cos\theta,\ \ y = R\sin\phi\sin\theta,\ \ z = R\cos\phi,\ \ \mathbf{R} = R\,\hat{\mathbf{e}}_R,\ \ \mathbf{A} = A_R\hat{\mathbf{e}}_R + A_\phi\hat{\mathbf{e}}_\phi + A_\theta\hat{\mathbf{e}}_\theta$

$\hat{\mathbf{e}}_R = \sin\phi\,(\cos\theta\ \hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y) + \cos\phi\ \hat{\mathbf{e}}_z,\ \ \hat{\mathbf{e}}_\phi = \cos\phi\,(\cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y) - \sin\phi\ \hat{\mathbf{e}}_z$
$\hat{\mathbf{e}}_\theta = -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y$

$\hat{\mathbf{e}}_x = \cos\theta\,(\sin\phi\,\hat{\mathbf{e}}_R + \cos\phi\,\hat{\mathbf{e}}_\phi) - \sin\theta\ \hat{\mathbf{e}}_\theta,\ \ \hat{\mathbf{e}}_y = \sin\theta\,(\sin\phi\,\hat{\mathbf{e}}_R + \cos\phi\,\hat{\mathbf{e}}_\phi) + \cos\theta\ \hat{\mathbf{e}}_\theta$
$\hat{\mathbf{e}}_z = \cos\phi\ \hat{\mathbf{e}}_R - \sin\phi\ \hat{\mathbf{e}}_\phi$

$\frac{\partial \hat{\mathbf{e}}_R}{\partial\phi} = \hat{\mathbf{e}}_\phi,\ \ \frac{\partial \hat{\mathbf{e}}_R}{\partial\theta} = \sin\phi\ \hat{\mathbf{e}}_\theta,\ \ \frac{\partial \hat{\mathbf{e}}_\phi}{\partial\phi} = -\hat{\mathbf{e}}_R,\ \ \frac{\partial \hat{\mathbf{e}}_\phi}{\partial\theta} = \cos\phi\ \hat{\mathbf{e}}_\theta,\ \ \frac{\partial \hat{\mathbf{e}}_\theta}{\partial\theta} = -\sin\phi\ \hat{\mathbf{e}}_R - \cos\phi\ \hat{\mathbf{e}}_\phi$

All other derivatives of the base vectors are zero.

$\boldsymbol{\nabla} = \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \frac{\hat{\mathbf{e}}_\phi}{R}\frac{\partial}{\partial\phi} + \frac{\hat{\mathbf{e}}_\theta}{R\sin\phi}\frac{\partial}{\partial\theta},\ \nabla^2 = \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right]$

$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{2A_R}{R} + \frac{\partial A_R}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(A_\phi\sin\phi)}{\partial\phi} + \frac{1}{R\sin\phi}\frac{\partial A_\theta}{\partial\theta}$

$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{R\sin\phi}\left[\frac{\partial(\sin\phi A_\theta)}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right]\hat{\mathbf{e}}_R + \left[\frac{1}{R\sin\phi}\frac{\partial A_R}{\partial\theta} - \frac{1}{R}\frac{\partial(RA_\theta)}{\partial R}\right]\hat{\mathbf{e}}_\phi$
$\quad + \frac{1}{R}\left[\frac{\partial(RA_\phi)}{\partial R} - \frac{\partial A_R}{\partial\phi}\right]\hat{\mathbf{e}}_\theta$

$\boldsymbol{\nabla}\mathbf{A} = \frac{\partial A_R}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_R + \frac{\partial A_\phi}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_\phi + \frac{1}{R}\left(\frac{\partial A_R}{\partial\phi} - A_\phi\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_R + \frac{\partial A_\theta}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_\theta$
$\quad + \frac{1}{R\sin\phi}\left(\frac{\partial A_R}{\partial\theta} - A_\theta\ \sin\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_R + \frac{1}{R}\left(A_R + \frac{\partial A_\phi}{\partial\phi}\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial A_\theta}{\partial\phi}\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\theta$
$\quad + \frac{1}{R\sin\phi}\left(\frac{\partial A_\phi}{\partial\theta} - A_\theta\ \cos\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\phi + \frac{1}{R\sin\phi}\left(A_R\ \sin\phi + A_\phi\ \cos\phi + \frac{\partial A_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\theta$

Let $\mathbf{A}$ = grad ϕ in Eq. (1.2.57). Then the divergence theorem gives

$$\int_\Omega \text{div}(\text{grad}\ \phi)\,dv \equiv \int_\Omega \nabla^2\phi\,dv = \oint_\Gamma \hat{\mathbf{n}}\cdot\text{grad}\phi\,ds. \tag{1.2.58}$$

The quantity $\hat{\mathbf{n}}\cdot$grad ϕ is called the *normal derivative* of ϕ on the surface s and is denoted by (n is the coordinate along the unit normal vector $\hat{\mathbf{n}}$)

$$\frac{\partial\phi}{\partial n} \equiv \hat{\mathbf{n}}\cdot\text{grad}\ \phi = \hat{\mathbf{n}}\cdot\nabla\phi. \tag{1.2.59}$$

In a Cartesian system, this becomes

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y + \frac{\partial \phi}{\partial z} n_z,$$

where n_x, n_y and n_z are the direction cosines of the unit normal,

$$\hat{\mathbf{n}} = n_x \hat{\mathbf{e}}_x + n_y \hat{\mathbf{e}}_y + n_z \hat{\mathbf{e}}_z. \tag{1.2.60}$$

The next example illustrates the relation between the integral relations Eqs. (1.2.55) to (1.2.57) and the so-called integration by parts.

Example 1.2.6

Consider a rectangular region $R = \{(x, y) : 0 < x < a, 0 < y < b\}$ with boundary C, which is the union of line segments C_1, C_2, C_3, and C_4 (see Fig. 1.2.12). Evaluate the integral $\int_R \nabla^2 \phi dxdy$ over the rectangular region.

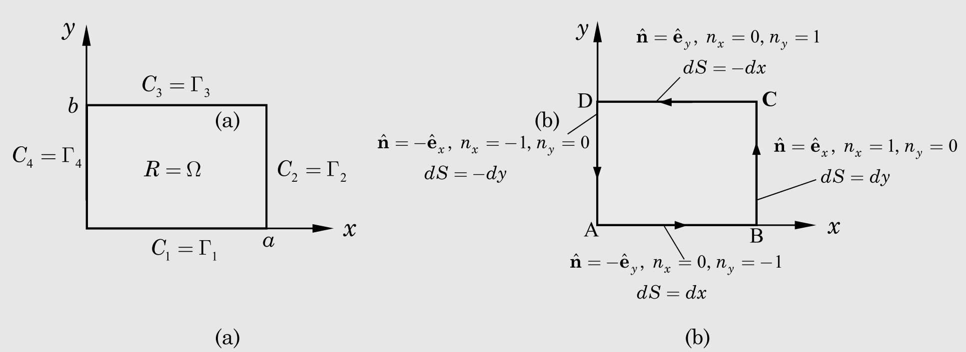


Fig. 1.2.12 Integration over rectangular regions.

Solution: From Eq. (1.2.58) we have

$$\int_R \nabla^2 \phi \, dxdy = \int_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \phi) \, dxdy = \oint_c \frac{\partial \phi}{\partial n} ds.$$

The line integral can be simplified for the region under consideration as follows (note that in two dimensions, the volume integral becomes an area integral):

$$\begin{aligned}
\oint_C \frac{\partial \phi}{\partial n} \, ds &= \int_{C_1} \frac{\partial \phi}{\partial n} \, ds + \int_{C_2} \frac{\partial \phi}{\partial n} \, ds + \int_{C_3} \frac{\partial \phi}{\partial n} \, ds + \int_{C_4} \frac{\partial \phi}{\partial n} \, ds \\
&= \int_0^a \left(-\frac{\partial \phi}{\partial y} \right) \Big|_{y=0} dx + \int_0^b \left(\frac{\partial \phi}{\partial x} \right) \Big|_{x=a} dy \\
&\quad + \int_a^0 \left(\frac{\partial \phi}{\partial y} \right) \Big|_{y=b} (-dx) + \int_b^0 \left(-\frac{\partial \phi}{\partial x} \right) \Big|_{x=0} (-dy) \\
&= \int_0^a \left[\left(\frac{\partial \phi}{\partial y} \right)_{y=b} - \left(\frac{\partial \phi}{\partial y} \right)_{y=0} \right] dx + \int_0^b \left[\left(\frac{\partial \phi}{\partial x} \right)_{x=a} - \left(\frac{\partial \phi}{\partial x} \right)_{x=0} \right] dy.
\end{aligned}$$

The same result can be obtained by means of integration by parts:

$$\begin{aligned}\int_R \nabla^2 \phi dxdy &= \int_0^b \int_0^a \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) dxdy \\ &= \int_0^b \int_0^a \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) dxdy + \int_0^a \int_0^b \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) dydx \\ &= \int_0^b \left(\frac{\partial \phi}{\partial x}\right)\Bigg|_{x=0}^{x=a} dy + \int_0^a \left(\frac{\partial \phi}{\partial y}\right)\Bigg|_{y=0}^{y=b} dx \\ &= \int_0^b \left[\left(\frac{\partial \phi}{\partial x}\right)_{x=a} - \left(\frac{\partial \phi}{\partial x}\right)_{x=0}\right] dy + \int_0^a \left[\left(\frac{\partial \phi}{\partial y}\right)_{y=b} - \left(\frac{\partial \phi}{\partial y}\right)_{y=0}\right] dx.\end{aligned}$$

Thus integration by parts is a special case of the gradient or the divergence theorem.

1.3 Tensors

1.3.1 Second-Order Tensors

To introduce the concept of a second-order tensor, also called a *dyad*, we consider the equilibrium of an element of a continuum acted upon by forces. The surface force acting on a small element of area in a continuous medium depends not only on the magnitude of the area but also upon the orientation of the area. It is customary to denote the direction of a plane area by means of a unit vector drawn normal to that plane [see Fig. 1.3.1(a)]. To fix the direction of the normal, we assign a *sense of travel* along the contour of the boundary of the plane area in question. The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the sense of travel along the boundary curve or contour [see Fig. 1.3.1(b)]. Let the unit normal vector be given by $\hat{\mathbf{n}}$. Then the area can be denoted by $\mathbf{s} = s\hat{\mathbf{n}}$.

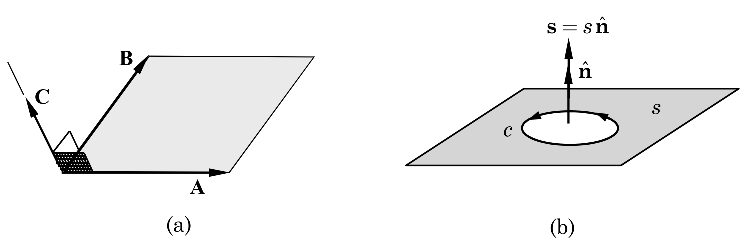


Fig. 1.3.1 (a) Plane area as a vector. (b) Unit normal vector and sense of travel.

If we denote by $\Delta\mathbf{F}(\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}\Delta s = \Delta\mathbf{s}$ located at the position $\mathbf{r}$ (see Fig. 1.3.2), the *stress vector* can be defined as follows:

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta s \to 0} \frac{\Delta\mathbf{F}(\hat{\mathbf{n}})}{\Delta s}. \tag{1.3.1}$$

We see that the stress vector is a point function of the unit normal $\hat{\mathbf{n}}$, which denotes the orientation of the surface Δs. The component of $\mathbf{t}$ that is in the direction of $\hat{\mathbf{n}}$ is called the *normal stress.* The component of $\mathbf{t}$ that is normal to $\hat{\mathbf{n}}$ is called a *shear* stress. Because of Newton's third law for action and reaction, we see that $\mathbf{t}(-\hat{\mathbf{n}}) = -\mathbf{t}(\hat{\mathbf{n}})$.

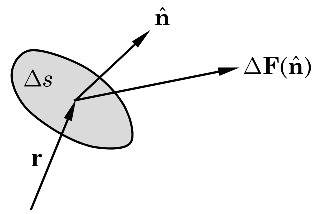


Fig. 1.3.2 Force on an area element.

At a fixed point $\mathbf{r}$ for each given unit vector $\hat{\mathbf{n}}$, there is a stress vector $\mathbf{t}(\hat{\mathbf{n}})$ acting on the plane normal to $\hat{\mathbf{n}}$. Note that $\mathbf{t}(\hat{\mathbf{n}})$ is, in general, not in the direction of $\hat{\mathbf{n}}$. It is fruitful to establish a relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$. To do this we now set up an infinitesimal tetrahedron in Cartesian coordinates, as shown in Fig. 1.3.3.

If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$, and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are Δs_1, Δs_2, Δs_3, and Δs, respectively, we have by Newton's second law for the mass inside the tetrahedron:

$$\mathbf{t}\Delta s - \mathbf{t}_1\Delta s_1 - \mathbf{t}_2\Delta s_2 - \mathbf{t}_3\Delta s_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}, \tag{1.3.2}$$

where Δv is the volume of the tetrahedron, ρ is the density, $\mathbf{f}$ is the body force per unit mass, and $\mathbf{a}$ is the acceleration. Since the total vector area of a closed surface is zero (see the gradient theorem; set $\phi = 1$ in Eq. (1.2.55)), we have

$$\Delta s\hat{\mathbf{n}} - \Delta s_1\hat{\mathbf{e}}_1 - \Delta s_2\hat{\mathbf{e}}_2 - \Delta s_3\hat{\mathbf{e}}_3 = \mathbf{0}.$$

It follows that

$$\Delta s_1 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\Delta s, \quad \Delta s_2 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\Delta s, \quad \Delta s_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\Delta s. \tag{1.3.3}$$

The volume of the element Δv can be expressed as

$$\Delta v = \frac{\Delta h}{3}\Delta s, \tag{1.3.4}$$

where Δh is the perpendicular distance from the origin to the slant face. The result in Eq. (1.3.4) can also be obtained from the divergence theorem in Eq. (1.2.57) by setting $\mathbf{A} = \mathbf{r}$.

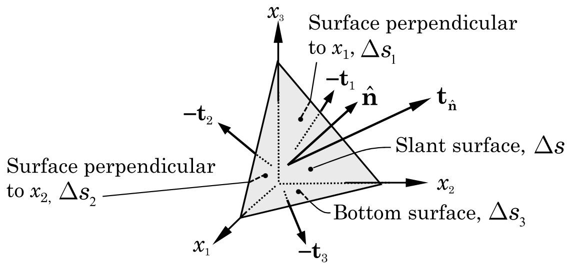


Fig. 1.3.3 Tetrahedral element in Cartesian coordinates.

Substitution of Eqs (1.3.3) and (1.3.4) in Eq. (1.3.2) and dividing throughout by Δs reduces it to

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho \frac{\Delta h}{3}(\mathbf{a} - \mathbf{f}).$$

In the limit when the tetrahedron shrinks to a point, $\Delta h \to 0$, we are left with

$$\begin{aligned} \mathbf{t} &= (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 \\ &= (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i. \end{aligned} \tag{1.3.5}$$

It is now convenient to display the above equation as

$$\mathbf{t} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3). \tag{1.3.6}$$

The terms in the parenthesis are to be treated as a dyad, called *stress dyad* or *stress tensor* $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3. \tag{1.3.7}$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$. Thus, we have

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \quad (t_i = n_j \sigma_{ji}) \tag{1.3.8}$$

and the dependence of $\mathbf{t}$ on $\hat{\mathbf{n}}$ has been explicitly displayed. Equation (1.3.8) is known as *Cauchy's formula.*

It is useful to resolve the stress vectors $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ into their orthogonal components. We have

$$\begin{aligned} \mathbf{t}_i &= \sigma_{i1}\hat{\mathbf{e}}_1 + \sigma_{i2}\hat{\mathbf{e}}_2 + \sigma_{i3}\hat{\mathbf{e}}_3 \\ &= \sigma_{ij}\hat{\mathbf{e}}_j \end{aligned} \tag{1.3.9}$$

for $i = 1, 2, 3$. Hence, the stress dyad can be expressed in summation notation as

$$\boldsymbol{\sigma} = \hat{\mathbf{e}}_i\mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{1.3.10}$$

The component σ_{ij} represents the stress (force per unit area) on an area perpendicular to the ith coordinate and in the jth coordinate direction (see Fig. 1.3.4). The stress vector $\mathbf{t}$ represents the vectorial stress on an area perpendicular to the direction $\hat{\mathbf{n}}$. Equation (1.3.8) is known as the *Cauchy stress formula* and $\boldsymbol{\sigma}$ is termed the *Cauchy stress tensor*.

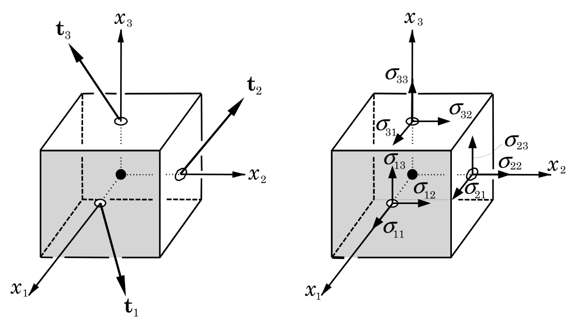


Fig. 1.3.4 Definition of stress components in Cartesian rectangular coordinates.

1.3.2 General Properties of a Dyadic

Because of its utilization in physical applications, a dyad is defined as two vectors standing side by side and acting as a unit. A linear combination of dyads is called a *dyadic*. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ and $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_n$ be arbitrary vectors. Then we can represent a dyadic as

$$\boldsymbol{\Phi} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \cdots + \mathbf{A}_n\mathbf{B}_n. \tag{1.3.11}$$

Here, we limit our discussion to Cartesian tensors. For a Cartesian tensor, the basis vectors are constants and thus do not take roles as variables in differentiation and integration.

One of the properties of a dyadic is defined by the dot product with a vector, say $\mathbf{V}$:

$$\begin{aligned} \boldsymbol{\Phi} \cdot \mathbf{V} &= \mathbf{A}_1(\mathbf{B}_1 \cdot \mathbf{V}) + \mathbf{A}_2(\mathbf{B}_2 \cdot \mathbf{V}) + \cdots + \mathbf{A}_n(\mathbf{B}_n \cdot \mathbf{V}), \\ \mathbf{V} \cdot \boldsymbol{\Phi} &= (\mathbf{V} \cdot \mathbf{A}_1)\mathbf{B}_1 + (\mathbf{V} \cdot \mathbf{A}_2)\mathbf{B}_2 + \cdots + (\mathbf{V} \cdot \mathbf{A}_n)\mathbf{B}_n. \end{aligned} \tag{1.3.12}$$

The dot operation with a vector produces another vector. In the first case, the dyadic acts as a *prefactor* and in the second case as a *postfactor*. The two operations in general produce different vectors.

The conjugate, or transpose, of a dyadic is defined as the result obtained by the interchange of the two vectors in each of the dyads:

$$\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{B}_1\mathbf{A}_1 + \mathbf{B}_2\mathbf{A}_2 + \cdots + \mathbf{B}_n\mathbf{A}_n. \tag{1.3.13}$$

It is clear that we have

$$\begin{aligned} \mathbf{V} \cdot \mathbf{\Phi} &= \mathbf{\Phi}^{\mathrm{T}} \cdot \mathbf{V}, \\ \mathbf{\Phi} \cdot \mathbf{V} &= \mathbf{V} \cdot \mathbf{\Phi}^{\mathrm{T}}. \end{aligned} \tag{1.3.14}$$

1.3.3 Nonion Form and Matrix Representation of a Dyad

We can display all of the components of a dyad $\mathbf{\Phi} = \phi_{ij}\, \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$ by letting the j index run to the right and the i index run downward:

$$\begin{aligned} \mathbf{\Phi} = \; & \phi_{11}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \phi_{12}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \phi_{13}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \\ & + \phi_{21}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \phi_{22}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \phi_{23}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 \\ & + \phi_{31}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \phi_{32}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + \phi_{33}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3. \end{aligned} \tag{1.3.15}$$

This form is called the *nonion* form. Equation (1.3.15) illustrates that a dyad in three-dimensional space, or what we shall call a second-order tensor, has nine independent components in general, each component associated with a certain dyadic pair. The components are thus said to be ordered. When the ordering is understood, such as suggested by the nonion form in Eq. (1.3.15), the explicit writing of the dyads can be suppressed and the dyadic written as an array:

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{\Phi} = \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}^{\mathrm{T}} [\Phi] \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}. \tag{1.3.16}$$

This representation is simpler than Eq. (1.3.15), but it is taken to mean the same.

This rectangular array $[\Phi]$ of scalars ϕ_{ij} is called a *matrix*, and the quantities ϕ_{ij} are called the *elements* of $[\Phi]$.[4] If a matrix has m rows and n columns, we say that is m by n ($m \times n$), the number of rows is always being listed first. The element in the ith row and jth column of a matrix $[A]$ is generally denoted by a_{ij}, and we will sometimes designate a matrix by $[A] = [a_{ij}]$. A square matrix is one that has the same number of rows as columns. An $n \times n$ matrix is said to be of *order n*. The elements of a square matrix for which the row number and the column number are the same (i.e., a_{ij} for $i = j$) are called *diagonal elements* or simply the *diagonal*. A square matrix is said to be a *diagonal matrix* if all of the off-diagonal elements are zero. An *identity matrix*, denoted by $[I]$ (i.e., matrix representation of the second-order identity tensor $\mathbf{I}$), is a diagonal matrix whose elements are all 1's.

[4]The word "matrix" was first used in 1850 by James Sylvester (1814–1897), an English algebraist. However, Arthur Caley (1821–1895), professor of mathematics at Cambridge, was the first one to explore properties of matrices. Significant contributions in the early years were made by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.

If the matrix has only one row or one column, we normally use only a single subscript to designate its elements. For example,

$$\{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \text{and} \quad \{Y\} = \{y_1\ y_2\ y_3\}$$

denote a column matrix and a row matrix, respectively. Row and column matrices can be used to represent the components of a vector.

The reader is expected to have a working knowledge of matrix theory, that is, addition of matrices, multiplication of a matrix by a scalar, and product of two matrices, determinant of a matrix, inverse of a matrix, and so on. Readers who wish to refresh their background on this topic may consult the textbooks [73, 74].

In the general scheme that is thus developed, vectors are called *first-order tensors* and dyads are called *second-order tensors*. Scalars are called *zeroth-order tensors*. The generalization to *third-order tensors* thus leads, or is derived from, *triads*, or three vectors standing side by side. It follows that higher-order tensors are developed from *polyads*.

Example 1.3.1

With reference to a rectangular Cartesian system (x_1, x_2, x_3), the components of the stress dyadic at a certain point of a continuous medium are given by

$$[\sigma] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{ psi.}$$

Determine the stress vector $\mathbf{t}$ at the point and normal to the plane, $p(x_1, x_2, x_3) = x_1 + 2x_2 + 2x_3 - 6 = 0$, and then compute the normal and tangential components of the stress vector at the point.

Solution: First we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal is given by (see Eq. (1.2.47))

$$\hat{\mathbf{n}} = \frac{\nabla p}{|\nabla p|}, \qquad p(x_1, x_2, x_3) = x_1 + 2x_2 + 2x_3 - 6,$$
$$\hat{\mathbf{n}} = \frac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3).$$

The components of the stress vector are displayed in an array

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ psi,}$$

or

$$\mathbf{t}(\hat{\mathbf{n}}) = \frac{1}{3}(1600\hat{\mathbf{e}}_1 + 400\hat{\mathbf{e}}_2 + 100\hat{\mathbf{e}}_3) \text{ psi.}$$

The normal component t_n of the stress vector $\mathbf{t}$ on the plane is given by

$$t_n = \mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = \frac{2600}{9} \text{ psi},$$

and the tangential component is given by (the Pythagorean theorem)

$$\begin{aligned} t_s = \sqrt{|\mathbf{t}|^2 - t_n^2} &= \frac{10^2}{9}\sqrt{(256+16+1)9 - 26 \times 26} \text{ psi} \\ &= 100\frac{\sqrt{1781}}{9} = 468.9 \text{ psi}. \end{aligned}$$

A second-order Cartesian tensor $\mathbf{\Phi}$ may be represented in unbarred and barred coordinate systems as

$$\begin{aligned} \mathbf{\Phi} &= \phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \\ &= \bar{\phi}_{kl}\hat{\bar{\mathbf{e}}}_k\hat{\bar{\mathbf{e}}}_l. \end{aligned} \tag{1.3.17}$$

The unit base vectors in the unbarred and barred systems are related by

$$\hat{\mathbf{e}}_i = \frac{\partial \bar{x}_j}{\partial x_i}\hat{\bar{\mathbf{e}}}_j \equiv \beta_{ji}\hat{\bar{\mathbf{e}}}_j, \quad \beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j, \tag{1.3.18}$$

where β_{ij} denote the direction cosines between unbarred and barred systems (see Eq. (1.2.29)). Thus the components of a second-order tensor transform according to

$$\bar{\phi}_{kl} = \phi_{ij}\beta_{ki}\beta_{lj} \quad \text{or} \quad [\bar{\phi}] = [\beta][\phi][\beta]^{\mathrm{T}}. \tag{1.3.19}$$

Equation (1.3.19) is used to define a second-order tensor, that is, $\mathbf{\Phi}$ is a second-order tensor if and only if its components ϕ_{ij} transform according to Eq. (1.3.19). In a right-handed orthogonal system, the determinant of the transformation matrix is unity, and we have

$$[\beta]^{-1} = [\beta]^{\mathrm{T}}. \tag{1.3.20}$$

The unit tensor is defined as

$$\mathbf{I} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i. \tag{1.3.21}$$

With the help of the Kronecker delta symbol, this can be written alternatively as

$$\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{1.3.22}$$

Clearly the unit tensor is symmetric.

The sum of the diagonal terms of a Cartesian tensor is called the *trace of the tensor*:

$$\text{trace}\, \mathbf{\Phi} = \phi_{ii}. \tag{1.3.23}$$

The trace of a tensor is *invariant*, called the first invariant, and it is denoted by I_1, that is, it is invariant with coordinate transformations ($\phi_{ii} = \bar{\phi}_{ii}$). The three invariants of a Cartesian tensor are given by

$$I_1 = \phi_{ii}, \quad I_2 = \frac{1}{2}\left(\phi_{ij}\phi_{ij} - \phi_{ii}\phi_{jj}\right), \quad I_3 = \det[\phi] = |\phi|. \tag{1.3.24}$$

The double-dot product between two dyadics is very useful in many problems. The double-dot product between a dyad ($\mathbf{AB}$) and another ($\mathbf{CD}$) is defined as the scalar:

$$(\mathbf{AB}) : (\mathbf{CD}) \equiv (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}). \tag{1.3.25}$$

The double-dot product, by this definition, is commutative. The double-dot product between two dyads is given by

$$\begin{aligned} \mathbf{\Phi} : \mathbf{\Psi} &= (\phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j) : (\psi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n) \\ &= \phi_{ij}\psi_{mn}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_n)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m) \\ &= \phi_{ij}\psi_{mn}\delta_{in}\delta_{jm} \\ &= \phi_{ij}\psi_{ji}. \end{aligned} \tag{1.3.26}$$

Note that the double-dot product of a Cartesian tensor $\mathbf{\Phi}$ with the unit tensor $\mathbf{I}$ produces its trace $I_1 = \phi_{ii}$.

We note that the gradient of a vector is a second-order tensor:

$$\begin{aligned} \nabla \mathbf{A} &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(A_j \hat{\mathbf{e}}_j) \\ &= \frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \end{aligned} \tag{1.3.27}$$

It can be expressed as the sum of

$$\nabla \mathbf{A} = \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j + \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{1.3.28}$$

Analogously to the divergence of a vector, the divergence of a (second-order) Cartesian tensor is defined as

$$\begin{aligned} \text{div}\mathbf{\Phi} &= \nabla \cdot \mathbf{\Phi} \\ &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \cdot (\phi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n) \\ &= \frac{\partial \phi_{mn}}{\partial x_i}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_m)\hat{\mathbf{e}}_n \\ &= \frac{\partial \phi_{in}}{\partial x_i}\hat{\mathbf{e}}_n. \end{aligned}$$

Thus the divergence of a second-order tensor is a vector.

The integral theorems of vectors presented in Section 1.2.7 are also valid for tensors (second-order and higher):

$$\begin{aligned}
\int_\Omega \text{grad}\mathbf{A}\, dv &= \oint_\Gamma \hat{\mathbf{n}}\mathbf{A}\, ds,\\
\int_\Omega \text{div}\boldsymbol{\Phi}\, dv &= \oint_\Gamma \hat{\mathbf{n}}\cdot\boldsymbol{\Phi}\, ds,\\
\int_\Omega \text{curl}\boldsymbol{\Phi}\, dv &= \oint_\Gamma \hat{\mathbf{n}}\times\boldsymbol{\Phi}\, ds.
\end{aligned} \tag{1.3.29}$$

It is important that the order of the operations be observed in the above expressions.

1.3.4 Eigenvectors Associated with Dyads

It is conceptually useful to regard a dyadic as an operator that changes a vector into another vector (by means of the dot product). In this regard it is of interest to inquire whether there are certain vectors that have only their lengths, and not their orientation, changed when operated upon by a given dyadic or tensor. If such vectors exist, they must satisfy the equation

$$\boldsymbol{\Phi}\cdot\mathbf{A} = \lambda\mathbf{A}. \tag{1.3.30}$$

The vectors $\mathbf{A}$ are called *characteristic vectors, or eigenvectors*, associated with $\boldsymbol{\Phi}$. The parameter λ is called an *eigenvalue*, and it characterizes the change in length (and possibly sense) of the eigenvector $\mathbf{A}$ after it has been operated upon by $\boldsymbol{\Phi}$. The eigenvalues of a stress tensor are known as the *principal stresses* and the eigenvectors are called the *principal planes*.

Since $\mathbf{A}$ can be expressed as $\mathbf{A} = \mathbf{I}\cdot\mathbf{A}$, Eq. (1.3.30) can also be written as

$$(\boldsymbol{\Phi} - \lambda\mathbf{I})\cdot\mathbf{A} = \mathbf{0}. \tag{1.3.31}$$

When written in matrix for Cartesian components, this equation becomes

$$\begin{bmatrix} \phi_{11}-\lambda & \phi_{12} & \phi_{13}\\ \phi_{21} & \phi_{22}-\lambda & \phi_{23}\\ \phi_{31} & \phi_{32} & \phi_{33}-\lambda \end{bmatrix} \begin{Bmatrix} A_1\\ A_2\\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0 \end{Bmatrix}. \tag{1.3.32}$$

Because this is a homogeneous set of equations for A_1, A_2, and A_3, a nontrivial solution will not exist unless the determinant of the matrix $[\boldsymbol{\Phi} - \lambda\mathbf{I}]$ vanishes. The vanishing of this determinant yields a cubic equation for λ, called the *characteristic equation*, the solution of which yields three values of λ, that is, three eigenvalues λ_1, λ_2, and λ_3. The character of these eigenvalues depends on the character of the dyadic $\boldsymbol{\Phi}$. At least one of the eigenvalues must be real. The other two may be real and distinct, real and repeated, or complex conjugates.

In the preponderance of practical problems, the dyadic $\mathbf{\Phi}$ is symmetric, that is, $\mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}}$ (e.g., Cauchy stress tensor). Of course, $\mathbf{\Phi}$ is always real in our considerations. For example, the moment-of-inertia dyadic is symmetric, and the stress tensor $\boldsymbol{\sigma}$ is usually but not always symmetric. We limit our discussion to symmetric dyadics.

The vanishing of the determinant assures that three eigenvectors are not unique to within a multiplicative constant; however, an infinite number of solutions exist having at least three different orientations. Since only orientation is important, it is thus useful to represent the three eigenvectors by three unit eigenvectors $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$, and $\hat{\mathbf{e}}_3^*$, denoting three different orientations, each associated with a particular eigenvalue.

Suppose now that λ_1 and λ_2 are two distinct eigenvalues and $\mathbf{A}_1$ and $\mathbf{A}_2$ are their corresponding eigenvectors:

$$\begin{aligned} \mathbf{\Phi} \cdot \mathbf{A}_1 &= \lambda_1 \mathbf{A}_1, \\ \mathbf{\Phi} \cdot \mathbf{A}_2 &= \lambda_2 \mathbf{A}_2. \end{aligned} \tag{1.3.33}$$

Scalar product of the first equation by $\mathbf{A}_2$ and the second by $\mathbf{A}_1$, and then subtraction, yields

$$\mathbf{A}_2 \cdot \mathbf{\Phi} \cdot \mathbf{A}_1 - \mathbf{A}_1 \cdot \mathbf{\Phi} \cdot \mathbf{A}_2 = (\lambda_1 - \lambda_2)\mathbf{A}_1 \cdot \mathbf{A}_2. \tag{1.3.34}$$

Since $\mathbf{\Phi}$ is symmetric, one can establish that the left-hand side of this equation vanishes. Thus

$$0 = (\lambda_1 - \lambda_2)\mathbf{A}_1 \cdot \mathbf{A}_2. \tag{1.3.35}$$

Now suppose that λ_1 and λ_2 are complex conjugates such that $\lambda_1 - \lambda_2 = 2i\lambda_{1i}$, where $i = \sqrt{-1}$ and λ_{1i} is the imaginary part of λ_1. Then $\mathbf{A}_1 \cdot \mathbf{A}_2$ is always positive since $\mathbf{A}_1$ and $\mathbf{A}_2$ are complex conjugate vectors associated with λ_1 and λ_2. It then follows from Eq. (1.3.35) that $\lambda_{1i} = 0$ and hence that the *three eigenvalues associated with a symmetric dyadic are all real.*

Now assume that λ_1 and λ_2 are real and distinct such that $\lambda_1 - \lambda_2$ is not zero. It then follows from Eq. (1.3.35) that $\mathbf{A}_1 \cdot \mathbf{A}_2 = 0$. Thus the *eigenvectors associated with distinct eigenvalues of a symmetric dyadic are orthogonal.* If the three eigenvalues are all distinct, then the three eigenvectors are mutually orthogonal.

If λ_1 and λ_2 are distinct, but λ_3 is repeated, say $\lambda_3 = \lambda_2$, then $\mathbf{A}_3$ must also be perpendicular to $\mathbf{A}_1$ as deducted by an argument similar to that for $\mathbf{A}_2$ stemming from Eq. (1.3.35). Neither $\mathbf{A}_2$ nor $\mathbf{A}_3$ is preferred, and they are both arbitrary, except insofar as they are both perpendicular to $\mathbf{A}_1$. It is useful, however, to select $\mathbf{A}_3$ such that it is perpendicular to both $\mathbf{A}_1$ and $\mathbf{A}_2$. We do this by choosing $\mathbf{A}_3 = \mathbf{A}_1 \times \mathbf{A}_2$ and thus establishing a mutually orthogonal set of eigenvectors. This sort of behavior arises when there is an axis of symmetry present in a problem.

In a Cartesian system the characteristic equation associated with a dyadic can be expressed in the form

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0, \tag{1.3.36}$$

where I_1, I_2, and I_3 are the invariants associated with the matrix of $\mathbf{\Phi}$. The invariants can also be expressed in terms of the eigenvalues:

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2 = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \quad I_3 = \lambda_1\lambda_2\lambda_3. \tag{1.3.37}$$

Finding the roots of the cubic Eq.(1.3.36) is not always easy. However, when the matrix under consideration is of the form

$$\begin{bmatrix} \phi_{11} & 0 & 0 \\ 0 & \phi_{22} & \phi_{23} \\ 0 & \phi_{32} & \phi_{33} \end{bmatrix},$$

one of the roots is $\lambda_1 = \phi_{11}$, and the remaining two roots can be found from the quadratic equation

$$(\phi_{22} - \lambda)(\phi_{33} - \lambda) - \phi_{23}\phi_{32} = 0.$$

That is,

$$\lambda_{2,3} = \frac{\phi_{22} + \phi_{33}}{2} \pm \tfrac{1}{2}\sqrt{(\phi_{22} + \phi_{33})^2 - 4(\phi_{22}\phi_{33} - \phi_{23}\phi_{32})}. \tag{1.3.38}$$

In cases where one of the roots is not obvious, an alternative procedure given below proves to be useful.

In the alternative method we seek the eigenvalues of the so-called *deviatoric tensor* associated with $\mathbf{\Phi}$:

$$\phi'_{ij} \equiv \phi_{ij} - \tfrac{1}{3}\phi_{kk}\delta_{ij}. \tag{1.3.39}$$

Note that

$$\phi'_{ii} = \phi_{ii} - \phi_{kk} = 0. \tag{1.3.40}$$

That is, the first invariant I'_1 of the deviatoric tensor is zero. As a result the characteristic equation associated with the deviatoric tensor is of the form,

$$(\lambda')^3 - I'_2\lambda' - I'_3 = 0, \tag{1.3.41}$$

where λ' is the eigenvalue of the deviatoric tensor. The eigenvalues associated with ϕ_{ij} itself can be computed from

$$\lambda = \lambda' + \tfrac{1}{3}\phi_{kk}. \tag{1.3.42}$$

The cubic equation in Eq. (1.3.41) is of a special form that allows a direct computation of its roots. Equation (1.3.41) can be solved explicitly by introducing the transformation

$$\lambda' = 2(\tfrac{1}{3}I'_2)^{1/2}\cos\alpha, \tag{1.3.43}$$

which transforms Eq. (1.3.41) into

$$2(\tfrac{1}{3}I'_2)^{3/2}[4\cos^3\alpha - 3\cos\alpha] = I'_3. \tag{1.3.44}$$

The expression in square brackets is equal to $\cos 3\alpha$. Hence

$$\cos 3\alpha = \frac{I'_3}{2}\left(\frac{3}{I'_2}\right)^{3/2}. \tag{1.3.45}$$

If α_1 is the angle satisfying $0 \leq 3\alpha_1 \leq \pi$ whose cosine is given by Eq. (1.3.45), then $3\alpha_1$, $3\alpha_1 + 2\pi$, and $3\alpha_1 - 2\pi$ all have the same cosine, and furnish three independent roots of Eq. (1.3.41):

$$\lambda'_i = 2\left(\tfrac{1}{3}I'_2\right)^{1/2}\cos\alpha_i, \quad i = 1, 2, 3, \tag{1.3.46}$$

where

$$\alpha_1 = \tfrac{1}{3}\left\{\cos^{-1}\left[\frac{I'_3}{2}\left(\frac{3}{I'_2}\right)^{3/2}\right]\right\}, \quad \alpha_2 = \alpha_1 + \tfrac{2}{3}\pi, \quad \alpha_3 = \alpha_1 - \tfrac{2}{3}\pi. \tag{1.3.47}$$

Finally we can compute λ_i from Eq. (1.3.42).

Example 1.3.2

Determine the eigenvalues and eigenvectors of the matrix:

$$[\phi] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Solution: The characteristic equation is obtained by setting det $(\phi_{ij} - \lambda\,\delta_{ij})$ to zero:

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 4-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(2-\lambda) - 1] - 1\cdot(2-\lambda) = 0,$$

or

$$(2-\lambda)[(4-\lambda)(2-\lambda) - 2] = 0.$$

Hence

$$\lambda_1 = 3 + \sqrt{3} = 4.7321, \quad \lambda_2 = 3 - \sqrt{3} = 1.2679, \quad \lambda_3 = 2.$$

Alternatively,

$$[\phi'] = \begin{bmatrix} 2-\frac{8}{3} & 1 & 0 \\ 1 & 4-\frac{8}{3} & 1 \\ 0 & 1 & 2-\frac{8}{3} \end{bmatrix}$$

$$I'_2 = \frac{1}{2}(\phi'_{ij}\phi'_{ij} - \phi'_{ii}\phi'_{jj}) = \frac{1}{2}\phi'_{ij}\phi'_{ij}$$

$$= \frac{1}{2}\left[\left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + 2 + 2\right] = \frac{10}{3}$$

$$I'_3 = \det(\phi'_{ij}) = \frac{52}{27}.$$

From Eq. (1.3.47),

$$\alpha_1 = \frac{1}{3}\left\{\cos^{-1}\left[\frac{52}{54}\left(\frac{9}{10}\right)^{3/2}\right]\right\} = 11.565^\circ$$

$$\alpha_2 = 131.565^\circ, \quad \alpha_3 = -108.435^\circ,$$

and from Eq. (1.3.46),

$$\lambda'_1 = 2.065384, \quad \lambda'_2 = -1.3987, \quad \lambda'_3 = -0.66667.$$

Finally, using Eq. (1.3.42), we obtain the eigenvalues

$$\lambda_1 = 4.7321, \quad \lambda_2 = 1.2679, \quad \lambda_3 = 2.00.$$

The eigenvector corresponding to $\lambda_3 = 2$, for example, is calculated as follows. From $(\phi_{ij} - \lambda_3\delta_{ij})A_j = 0$, we have

$$\begin{bmatrix} 2-2 & 1 & 0 \\ 1 & 4-2 & 1 \\ 0 & 1 & 2-2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

This gives

$$A_2 = 0, \quad A_1 = -A_3.$$

Using $A_1^2 + A_2^2 + A_3^2 = 1$ (called the normalization of the eigenvectors; the normalization of eigenvectors is not necessary as we are only interested in the planes represented by the vectors), we obtain

$$\hat{\mathbf{A}}_3 = \pm\frac{1}{\sqrt{2}}(1, 0, -1), \quad \text{for } \lambda_3 = 2.$$

Similarly, the eigenvectors corresponding to $\lambda_{1,2} = 3 \pm \sqrt{3}$ are calculated as

$$\hat{\mathbf{A}}_1 = \pm\frac{(3-\sqrt{3})}{12}\left(1, \left(1+\sqrt{3}\right), 1\right), \quad \text{for } \lambda_1 = 3 + \sqrt{3},$$

$$\hat{\mathbf{A}}_2 = \pm\frac{(3+\sqrt{3})}{12}\left(1, \left(1-\sqrt{3}\right), 1\right), \quad \text{for } \lambda_2 = 3 - \sqrt{3}.$$

When matrix $[\phi]$ represents the matrix associated with the stress tensor $[\sigma]$, the eigenvalues are called the *principal stresses* (i.e., maximum and minimum values of the stress at a point) and eigenvectors are called the *principal planes* (or directions).

1.4 Summary

In this chapter a brief review of vectors and tensors is presented. Operations with vectors and tensors, such as the scalar product (dot product) and vector product (cross product), and calculus of vectors and tensors are discussed. The index notation and summation convention are also introduced. The stress vector and Cauchy stress tensor are introduced and Cauchy's formula is derived. The determination of eigenvalues and eigenvectors of a second-order tensor is detailed, which provides a procedure for determining the principal values and principal planes of stress and strain tensors in solid and structural mechanics problems. The ideas presented in this chapter will be used in the coming chapters.

The main results of this chapter are summarized here using the rectangular Cartesian system.

Kronecker delta [Eq. (1.2.24)]:

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j, & \text{for any fixed value of } i, j \\ 0, \text{ if } i \neq j, & \text{for any fixed value of } i, j. \end{cases} \tag{1.4.1}$$

Permutation symbol [Eq. (1.2.26)]:

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order} \\ & \quad \text{and not repeated } (i \neq j \neq k), \\ -1, & \text{if } i, j, k \text{ are not in cyclic order} \\ & \quad \text{and not repeated } (i \neq j \neq k), \\ 0, & \text{if any of } i, j, k \text{ are repeated.} \end{cases} \tag{1.4.2}$$

ε-δ **identity** [Eq. (1.2.28)]:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{1.4.3}$$

Scalar and vector products of vectors [Eq. (1.2.27)]:

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i, \quad \mathbf{A} \times \mathbf{B} = A_i B_j \varepsilon_{ijk} \hat{\mathbf{e}}_k. \tag{1.4.4}$$

Transformation of the rectangular Cartesian components of vectors [Eq. (1.2.29)]:

$$\bar{A}_i = \beta_{ij} \, A_j, \quad \beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j. \tag{1.4.5}$$

The "nabla" operator in the rectangular Cartesian coordinate system [(Eq. (1.2.50)]:

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}. \tag{1.4.6}$$

The gradient, curl, and divergence operations in the rectangular Cartesian coordinate system [Eqs (1.2.46), (1.2.54), and (1.2.51)]:

$$\begin{aligned}\boldsymbol{\nabla}\phi &\equiv \text{grad}\phi = \hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i},\\ \boldsymbol{\nabla}\times\mathbf{A} &\equiv \text{curl}\mathbf{A} = \varepsilon_{ijk}\frac{\partial A_j}{\partial x_i}\,\hat{\mathbf{e}}_k,\\ \boldsymbol{\nabla}\cdot\mathbf{A} &\equiv \text{div}\mathbf{A} = \frac{\partial A_i}{\partial x_i}.\end{aligned} \tag{1.4.7}$$

The gradient, curl, and divergence theorems in the rectangular Cartesian coordinate system [Eqs (1.2.55), (1.2.56), and (1.2.57)]:

$$\int_\Omega \text{grad}\,\phi\, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\phi\, d\Gamma \quad \left[\int_\Omega \hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i}\, d\Omega = \oint_\Gamma \hat{\mathbf{e}}_i n_i \phi\, d\Gamma\right]. \tag{1.4.8}$$

$$\int_\Omega \text{curl}\,\mathbf{A}\, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\times\mathbf{A}\, d\Gamma \quad \left[\int_\Omega \varepsilon_{ijk}\hat{\mathbf{e}}_k \frac{\partial A_j}{\partial x_i}\, d\Omega = \oint_\Gamma \varepsilon_{ijk}\hat{\mathbf{e}}_k n_i A_j\, d\Gamma\right]. \tag{1.4.9}$$

$$\int_\Omega \text{div}\,\mathbf{A}\, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{A}\, d\Gamma \quad \left[\int_\Omega \frac{\partial A_i}{\partial x_i}\, d\Omega = \oint_\Gamma n_i A_i\, d\Gamma\right]. \tag{1.4.10}$$

Cauchy's formula and stress tensor [Eqs (1.3.8) and (1.3.10)]:

$$\mathbf{t} = \hat{\mathbf{n}}\cdot\boldsymbol{\sigma} \quad (t_i = n_j\sigma_{ji}); \quad \boldsymbol{\sigma} = \sigma_{ij}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{e}}_j. \tag{1.4.11}$$

Transformation of the rectangular Cartesian components of second-order tensors [Eq. (1.3.19)]:

$$[\bar{\phi}] = [\beta][\phi][\beta]^{\mathrm{T}}; \quad \bar{\phi}_{ij} = \beta_{im}\beta_{jn}\phi_{mn}. \tag{1.4.12}$$

Eigenvalues of a second-order tensor [Eqs (1.3.31) and (1.3.36)]:

$$|\mathbf{S} - \lambda\mathbf{I}| = 0 \;\Rightarrow\; \lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0, \tag{1.4.13}$$

where

$$I_1 = s_{kk}, \quad I_2 = -\frac{1}{2}(s_{ii}s_{jj} - s_{ij}s_{ji}), \quad I_3 = \det\mathbf{S} = |\mathbf{S}|. \tag{1.4.14}$$

are the three invariants of the tensor **S**.

Problems

1.1 Find the equation of a line (or a set of lines) passing through the terminal point of a vector **A** and in the direction of vector **B**.

1.2 Find the equation of a plane connecting the terminal points of vectors **A**, **B**, and **C**. Assume that all three vectors are referred to a common origin.

1.3 Prove with the help of vectors that the diagonals of a parallelogram bisect each other.

1.4 Prove the following vector identity without the use of a coordinate system:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

1.5 If $\hat{\mathbf{e}}$ is any unit vector and $\mathbf{A}$ an arbitrary vector, show that

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}} + \hat{\mathbf{e}} \times (\mathbf{A} \times \hat{\mathbf{e}}).$$

This identity shows that a vector can resolved into a component parallel to and one perpendicular to an arbitrary direction $\hat{\mathbf{e}}$.

1.6 Verify the following identities:

(a) $\delta_{ii} = 3.$
(b) $\delta_{ij}\delta_{ij} = \delta_{ii}.$
(c) $\delta_{ij}\delta_{jk} = \delta_{ik}.$
(d) $\varepsilon_{mjk}\varepsilon_{njk} = 2\delta_{mn}.$
(e) $\varepsilon_{ijk}\varepsilon_{ijk} = 6.$
(f) $A_i A_j \varepsilon_{ijk} = 0.$

1.7 Using the index notation, prove the identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))^2.$$

1.8 Prove the following vector identity in an orthonormal system using index-summation notation:

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{B} - [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D}]\mathbf{A}.$$

1.9 Determine whether the following set of vectors is linearly independent:

$$\mathbf{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2, \quad \mathbf{B} = \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_4, \quad \mathbf{C} = \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_4, \quad \mathbf{D} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_4.$$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in a four-dimensional space.

1.10 Determine whether the following set of vectors is linearly independent:

$$\mathbf{A} = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3, \quad \mathbf{B} = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3, \quad \mathbf{C} = -\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2.$$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in $\Re^3$.

1.11 Determine which of the following sets of vectors span $\Re^3$:

(a) $\mathbf{A} = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3, \quad \mathbf{B} = -4\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - 5\hat{\mathbf{e}}_3, \quad \mathbf{C} = 2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3.$
(b) $\mathbf{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2, \quad \mathbf{B} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3, \quad \mathbf{C} = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3.$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in $\Re^3$.

1.12 Consider two rectangular Cartesian coordinate systems that are rotated with respect to each other and have a common origin. Let one system be denoted as a barred system, so that a position vector can be written in each of the systems as

$$\begin{aligned} \mathbf{r} &= x_i \hat{\mathbf{e}}_i, \\ &= \bar{x}_j \hat{\bar{\mathbf{e}}}_j, \end{aligned}$$

where $\{\hat{\mathbf{e}}_j\}$ and $\{\hat{\bar{\mathbf{e}}}_j\}$ are the respective orthonormal Cartesian bases in the unbarred and barred systems. By requiring that the position vector $\mathbf{r}$ be invariant under a rotation of the coordinate systems, deduce that the transformation between the coordinates is given by

$$\begin{aligned}\bar{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3\\ \bar{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3\\ \bar{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3,\end{aligned}$$

or more compactly,

$$\bar{x}_i = a_{ij}x_j, \quad i,j = 1,2,3,$$

where the terms a_{ij} can be identified as the direction cosines

$$a_{ij} \equiv \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j = \cos\left(\hat{\bar{\mathbf{e}}}_i, \hat{\mathbf{e}}_j\right).$$

Deduce further that the basis vectors obey the same transformation

$$\hat{\bar{\mathbf{e}}}_i = a_{ij}\hat{\mathbf{e}}_j,$$

and that the following orthogonality conditions hold:

$$a_{ij}a_{kj} = \delta_{ik}.$$

1.13 Determine the transformation matrix relating the orthonormal basis vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and the orthonormal basis vectors $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$, when $\hat{\mathbf{e}}'_i$ are given by

(a) $\hat{\mathbf{e}}'_1$ along the vector $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}'_2$ is perpendicular to the plane $2x_1 + 3x_2 + x_3 - 5 = 0$.

(b) $\hat{\mathbf{e}}'_1$ along the line segment connecting point $(1, -1, 3)$ to $(2, -2, 4)$ and $\hat{\mathbf{e}}'_3 = (-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)/\sqrt{6}$.

(c) $\hat{\mathbf{e}}'_3 = \hat{\mathbf{e}}_3$, and the angle between x'_1-axis and x_1-axis is 30°.

1.14 The angles between the barred and unbarred coordinate lines are given as follows:

	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\bar{\mathbf{e}}}_1$	60°	30°	90°
$\hat{\bar{\mathbf{e}}}_2$	150°	60°	90°
$\hat{\bar{\mathbf{e}}}_3$	90°	90°	0°

Determine the direction cosines of the transformation.

1.15 The angles between the barred and unbarred coordinate lines are given as follows:

	x_1	x_2	x_3
$\bar{x}_1$	45°	90°	45°
$\bar{x}_2$	60°	45°	120°
$\bar{x}_3$	120°	45°	60°

Determine the transformation matrix.

1.16 In a rectangular Cartesian coordinate system, find the length and direction cosines of a vector $\mathbf{A}$ that extends from the point $(1, -1, 3)$ to the midpoint of the line segment from the origin to the point $(6, -6, 4)$.

1.17 The vectors $\mathbf{A}$ and $\mathbf{B}$ are defined as follows:

$$\mathbf{A} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{k}},$$
$$\mathbf{B} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}},$$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are an orthonormal basis.

(a) Find the orthogonal projection of $\mathbf{A}$ in the direction of $\mathbf{B}$.

(b) Find the angle between the positive directions of the vectors.

1.18 Prove the following identities (see Eq. (1.2.12) for the definition of $[\mathbf{ABC}]$):

(a) $\frac{d}{dt}[\mathbf{ABC}] = \left[\frac{d\mathbf{A}}{dt}\mathbf{BC}\right] + \left[\mathbf{A}\frac{d\mathbf{B}}{dt}\mathbf{C}\right] + \left[\mathbf{AB}\frac{d\mathbf{C}}{dt}\right]$.

(b) $\frac{d}{dt}\left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^2\mathbf{A}}{dt^2}\right] = \left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^3\mathbf{A}}{dt^3}\right]$.

1.19 Let $\mathbf{r}$ denote a position vector $\mathbf{r} = x_i\hat{\mathbf{e}}_i$ ($r^2 = x_i x_i$) and $\mathbf{A}$ an arbitrary constant vector. Show that (div $= \boldsymbol{\nabla}\cdot$; grad $= \boldsymbol{\nabla}$; curl $= \boldsymbol{\nabla}\times$):

(a) $\text{grad}(r) = \frac{\mathbf{r}}{r}$.

(b) $\text{grad}(r^n) = nr^{n-2}\mathbf{r}$.

(c) $\nabla^2(r^n) = n(n+1)r^{n-2}$.

(d) $\text{grad}\,(\mathbf{r}\cdot\mathbf{A}) = \mathbf{A}$.

(e) $\text{div}(\mathbf{r}\times\mathbf{A}) = 0$.

(f) $\text{curl}(\mathbf{r}\times\mathbf{A}) = -2\mathbf{A}$.

(g) $\text{div}(r\mathbf{A}) = \frac{1}{r}(\mathbf{r}\cdot\mathbf{A})$.

(h) $\text{curl}\,(r\mathbf{A}) = \frac{1}{r}(\mathbf{r}\times\mathbf{A})$.

1.20 Let $\mathbf{A}$ and $\mathbf{B}$ be continuous vector functions of the position vector $\mathbf{r}$ with continuous first derivatives, and let F and G be continuous scalar functions of $\mathbf{r}$ with continuous first and second derivatives. Show that (div $= \boldsymbol{\nabla}\cdot$; grad $= \boldsymbol{\nabla}$; curl $= \boldsymbol{\nabla}\times$):

(a) $\text{curl}\,(\text{grad}\,F) = 0$.

(b) $\text{div}\,(\text{curl}\,\mathbf{A}) = 0$.

(c) $\text{div}\,(\text{grad}\,F\times\,\text{grad}\,G) = 0$.

(d) $\text{grad}\,(FG) = F\,\text{grad}\,G + G\,\text{grad}\,F$.

(e) $\text{div}\,(F\mathbf{A}) = \mathbf{A}\cdot\,\text{grad}\,F + F\,\text{div}\,\mathbf{A}$.

(f) $\text{curl}\,(F\mathbf{A}) = F\,\text{curl}\,\mathbf{A}$ - $\mathbf{A}\times\,\text{grad}\,F$.

(g) $\text{grad}\,(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\text{grad}\mathbf{B} + \mathbf{B}\cdot\text{grad}\mathbf{A} + \mathbf{A}\times\text{curl}\mathbf{B} + \mathbf{B}\times\text{curl}\mathbf{A}$.

(h) $\text{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\text{curl}\mathbf{A} - \mathbf{A}\cdot\text{curl}\mathbf{B}$.

(i) $\text{curl}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{A}\text{div}\mathbf{B} - \mathbf{B}\text{div}\mathbf{A}$.

(j) $\nabla^2(FG) = F\nabla^2 G + 2\boldsymbol{\nabla}F\cdot\boldsymbol{\nabla}G + G\nabla^2 F$.

1.21 Find the gradient of a vector $\mathbf{A}$ in the (a) cylindrical and (b) spherical coordinate systems.

1.22 Show that the vector area of a closed surface is zero, that is,

$$\oint_\Gamma \hat{\mathbf{n}}\,ds = \mathbf{0}.$$

1.23 Show that the volume enclosed by a surface Γ is

$$\text{volume} = \frac{1}{6}\oint_\Gamma \text{grad}(r^2)\cdot\hat{\mathbf{n}}\,ds,$$

or

$$\text{volume} = \frac{1}{3}\oint_\Gamma \mathbf{r}\cdot\hat{\mathbf{n}}\,ds.$$

1.24 Let $\phi(\mathbf{r})$ be a scalar field. Show that

$$\int_\Omega \nabla^2 \phi \, dv = \oint_\Gamma \frac{\partial \phi}{\partial n} \, ds,$$

where $\partial\phi/\partial n \equiv \hat{\mathbf{n}} \cdot \text{grad}\phi$ is the derivative of ϕ in the outward direction normal to the boundary Γ of the domain Ω.

1.25 In the divergence theorems, set $\mathbf{A} = \phi \ \text{grad}\psi$ and $\mathbf{A} = \psi \ \text{grad}\phi$ successively and obtain the integral forms

$$\int_\Omega \left[\phi \nabla^2 \psi + \nabla\phi \cdot \nabla\psi\right] dr = \oint_\Gamma \phi \frac{\partial \psi}{\partial n} \, ds, \tag{1}$$

$$\int_\Omega \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi\right] dr = \oint_\Gamma \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right] ds, \tag{2}$$

$$\int_\Omega \left[\phi \nabla^4 \psi - \nabla^2\phi \nabla^2 \psi\right] dr = \oint_\Gamma \left[\phi \frac{\partial}{\partial n}(\nabla^2\psi) - \nabla^2\psi \frac{\partial \phi}{\partial n}\right] ds, \tag{3}$$

where Ω denotes a (2D or 3D) region with boundary Γ. The first two identities are sometimes called Green's first and second theorems.

1.26 Determine the rotation transformation matrix such that the new base vector $\hat{\bar{\mathbf{e}}}_1$ is along $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$, and $\hat{\bar{\mathbf{e}}}_2$ is along the normal to the plane $2x_1 + 3x_2 + x_3 = 5$. If $\mathbf{T}$ is the dyadic whose components in the unbarred system are given by $T_{11} = 1,\ T_{12} = 0,\ T_{13} = -1,\ T_{22} = 3,\ T_{23} = -2$, and $T_{33} = 0$, find the components in the barred coordinates.

1.27 Show that the characteristic equation for a second-order tensor σ_{ij} can be expressed as

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0,$$

where

$$I_1 = \sigma_{kk},$$
$$I_2 = -\frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}),$$
$$I_3 = \frac{1}{6}(2\sigma_{ij}\sigma_{jk}\sigma_{ki} - 3\sigma_{ij}\sigma_{ji}\sigma_{kk} + \sigma_{ii}\sigma_{jj}\sigma_{kk}) = \det\,(\sigma_{ij})$$

are the three invariants of the tensor.

1.28 Find the eigenvalues and eigenvectors of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} 2 & -\sqrt{3} & 0 \\ -\sqrt{3} & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

$$\text{(c)} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \qquad \text{(d)} \quad \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$\text{(e)} \quad \begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix}, \qquad \text{(f)} \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

1.29 Evaluate the three invariants of the matrices in **Problem 1.28** and check them against the invariants obtained by using the eigenvalues.

1.30 The components of a stress dyadic at a point, referred to the (x_1, x_2, x_3) system, are (in ksi = 1000 psi):

$$\text{(a)} \ \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \ \text{(b)} \ \begin{bmatrix} 9 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & 16 \end{bmatrix}, \ \text{(c)} \ \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}.$$

Find the following:

(a) The stress vector acting on a plane perpendicular to the vector $2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ passing through the point

(b) The magnitude of the stress vector and the angle between the stress vector and the normal to the plane

(c) The magnitudes of the normal and tangential components of the stress vector

1.31 If $\mathbf{A}$ is an arbitrary vector, $\mathbf{\Phi}$ is an arbitrary dyad, and $\mathbf{I}$ is the identity tensor, verify that

(a) $\mathbf{I} \cdot \mathbf{\Phi} = \mathbf{\Phi} \cdot \mathbf{I} = \mathbf{\Phi}$.

(b) $(\mathbf{I} \times \mathbf{A}) \cdot \mathbf{\Phi} = \mathbf{A} \times \mathbf{\Phi}$.

(c) $(\mathbf{A} \times \mathbf{I}) \cdot \mathbf{\Phi} = \mathbf{A} \times \mathbf{\Phi}$.

(d) $(\mathbf{\Phi} \times \mathbf{A})^{\mathrm{T}} = -\mathbf{A} \times \mathbf{\Phi}^{\mathrm{T}}$.

1.32 If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, and $[A]$ is any square matrix, we define the polynomial in $[A]$ by

$$p([A]) = a_0[I] + a_1[A] + a_2[A]^2 + \cdots + a_n[A]^n.$$

If

$$[A] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and $p(x) = 1 - 2x + x^2$, compute $p(A)$.

1.33 *Cayley–Hamilton theorem* Consider a square matrix $[S]$ of order n. Denote by $p(\lambda)$ the determinant of $[S] - \lambda[I]$ (i.e., $p(\lambda) \equiv |(S - \lambda I)|$), called the *characteristic polynomial.* Then the Cayley–Hamilton theorem states that $p([S]) = 0$ (i.e., every matrix satisfies its own characteristic equation). Here $p([S])$ is as defined in **Problem 1.32**. Use matrix computation to verify the Cayley–Hamilton theorem for each of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

1.34 Consider the matrix

$$[S] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Verify the Cayley–Hamilton theorem and use it to compute the inverse of $[S]$.

"If a man is in too big a hurry to give up an error, he is liable to give up some truth with it."

– *Wilbur Wright* (developed the world's first successful airplane)