1

Forces

1.1 Terminology and Notation

A force exerted on a body tends to change the state of the body, that is, if the body is rigid the force tends to move the body, but when the body is elasto-plastic the force tends to deform the body.

A force can be defined as a vector quantity that is defined by magnitude and direction. The *direction* of a force is specified by its *orientation* (also known as the line of action) and *sense*. The *magnitude* of a force is a positive *scalar*. A *scalar* is a number expressed in specific units of measure.

Vectors (forces) are usually denoted by boldface letters. If the starting point O and the end point S of a vector (force) are given, the vector (force) could be denoted by $\mathbf{F}_{OS}$ or more simply $\mathbf{F}$. It is also usual to denote the magnitude of the vector (force) by F_{OS} or by $|\mathbf{F}_{OS}|$. Some other notations for vectorial quantities could be $\overrightarrow{F}_{OS}$, $\overrightarrow{OS}$, or $\overline{F}_{OS}$.

Graphically a force $\mathbf{F}_{OS}$ is represented by a straight arrow as shown in Figure 1.1. The point O is named the *application point* or the *origin* of the force $\mathbf{F}_{OS}$ and the line passing through O and S is named the action line of $\mathbf{F}_{OS}$.

There are some possible operations regarding vectors.

Equality of forces

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal to each other when they have the same magnitude and direction, that is

$$\mathbf{F}_1 = \mathbf{F}_2. \tag{1.1}$$

If the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal but are acting at different locations on the same body it will not cause identical motion.

Multiplication of a Force by a Scalar

The product between a force $\mathbf{F}$ and a scalar d written as $d\mathbf{F}$, is a force having the same orientation as $\mathbf{F}$, the same sense as if $\mathbf{F}$ if $d > 0$ and opposite sense if $d < 0$, and the magnitude $|d||\mathbf{F}|$.

Engineering Applications: Analytical and Numerical Calculation with MATLAB, First Edition.
Mihai Dupac and Dan B. Marghitu.

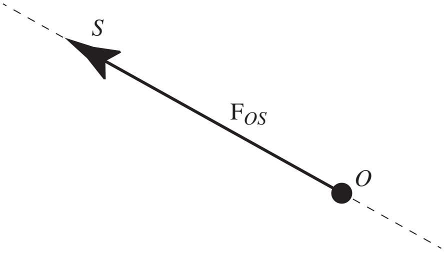

Figure 1.1 Vector representation.

Zero Force

A *zero force*, usually denoted by **0**, has a zero magnitude and an undefined direction.

Unit Vector (Force)

A *unit vector* **u** has its magnitude equal to unity, that is, $|\mathbf{u}| = 1$. Any force **F** can be written as a product of a unit vector **u** having the same orientation and sense as the force **F** and its magnitude $|\mathbf{F}|$ or equivalent

$$\mathbf{u} = \mathbf{F}\,\frac{1}{|\mathbf{F}|} = \frac{\mathbf{F}}{|\mathbf{F}|}\ . \tag{1.2}$$

Addition of Forces

The sum of a two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is a new force $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ named resultant. The sum of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is the force **F** represented graphically by the diagonal of the parallelogram shown in Figure 1.2 with its tail connecting the tail of the force $\mathbf{F}_1$ and head connecting the head of the force $\mathbf{F}_2$.

The sum $\mathbf{F}_1 + (-\mathbf{F}_2)$ is named the *difference* of the two forces as shown in Figure 1.3.

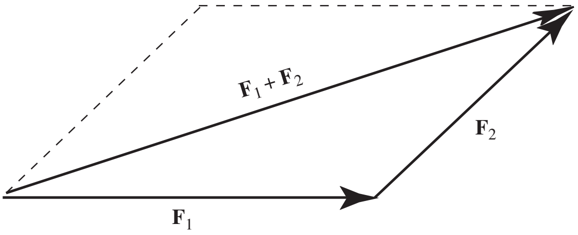

Figure 1.2 Parallelogram law of vector addition.

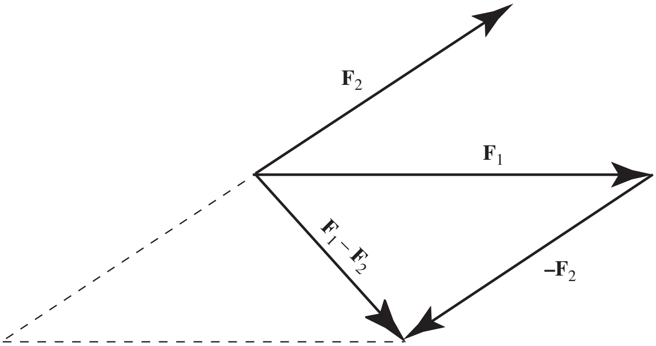

Figure 1.3 Parallelogram law of vector subtraction.

1.2 Resolution of Forces

If the unit vectors **ı**, **ȷ**, **k** have the same application point (origin) and are perpendicular to each other, as shown in Figure 1.4, they form a *Cartesian reference frame*.

Any force **F** can be expressed with respect to the unit vectors **ȷ**, **ı**, **k** by $\mathbf{F} = F_x\boldsymbol{\imath} + F_y\boldsymbol{\jmath} + F_z\mathbf{k}$ where F_x, F_y, and F_z are the **ı**, **ȷ**, **k** components of the force.

The magnitude of **F** can be written as

$$|\mathbf{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}.$$

Addition and subtraction of forces could be easily manipulated using the resolution of forces into components. Considering the forces $\mathbf{F} = F_x\ \boldsymbol{\imath} + F_y\ \boldsymbol{\jmath} + F_z\ \mathbf{k}$ and $\mathbf{P} = P_x\ \boldsymbol{\imath} + P_y\ \boldsymbol{\jmath} + P_z\ \mathbf{k}$, one can calculate

$$\mathbf{F} + \mathbf{P} = (F_x + P_x)\ \boldsymbol{\imath} + (F_y + P_y)\ \boldsymbol{\jmath} + (F_z + P_z)\ \mathbf{k}$$

and

$$\mathbf{F} - \mathbf{P} = (F_x - P_x)\ \boldsymbol{\imath} + (F_y - P_y)\ \boldsymbol{\jmath} + (F_z - P_z)\ \mathbf{k}.$$

1.3 Angle Between Two Forces

The angles between the forces **F** and **P**, and respectively **F** and **R** – in the range between 0° and 360° – are usually denoted by Greek letters such as α and θ, as shown in Figure 1.5.

The direction of a force $\mathbf{F} = F_x\boldsymbol{\imath} + F_y\boldsymbol{\jmath} + F_z\mathbf{k}$ in a Cartesian frame is given by the direction cosines (Figure 1.6) of the angles between by the force and the associated unit vectors **ı**, **ȷ**, **k**, written as $\cos\alpha = \frac{F_x}{|\mathbf{F}|};\ \cos\beta = \frac{F_y}{|\mathbf{F}|};\ \cos\gamma = \frac{F_z}{|\mathbf{F}|}$.

A unit force $\mathbf{F_u}$ (of magnitude 1) having the same direction as **F** can be written as

$$\mathbf{F_u} = \frac{F_x}{|\mathbf{F}|}\boldsymbol{\imath} + \frac{F_y}{|\mathbf{F}|}\boldsymbol{\jmath} + \frac{F_z}{|\mathbf{F}|}\ \mathbf{k} = \cos\alpha\boldsymbol{\imath} + \cos\beta\boldsymbol{\jmath} + \cos\gamma\mathbf{k}.$$

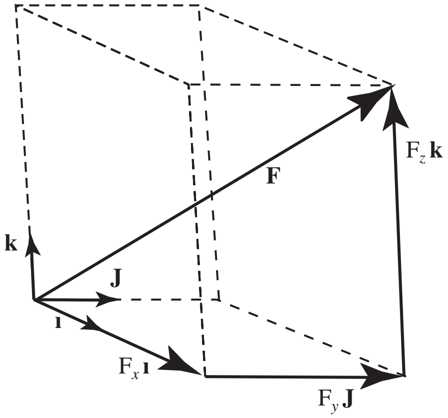

Figure 1.4 Resolution of a force.

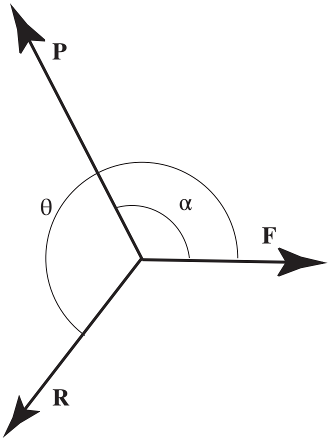

Figure 1.5 The angles α and θ between the forces $\mathbf{F}$ and $\mathbf{P}$, and respectively $\mathbf{F}$ and $\mathbf{R}$.

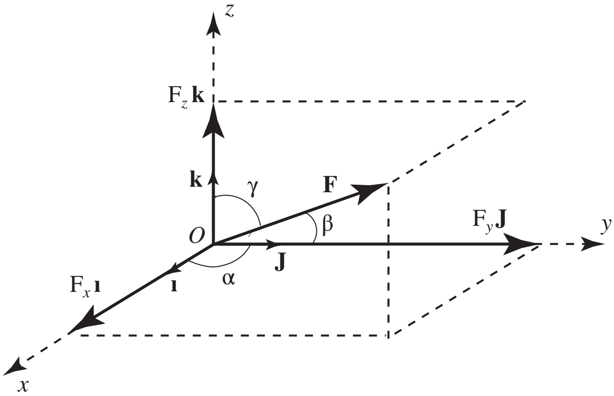

Figure 1.6 Direction cosines.

1.4 Force Vector

The position force (vector) $\mathbf{F}_{OS}$ shown in Figure 1.7 of a point $S(x_S,\ y_S,\ z_S)$ relative to a point $O(x_O,\ y_O,\ z_O) = O(0,\ 0,\ 0)$ can be written as

$$\mathbf{F}_{OS} = F_{x_S}\ \mathbf{\imath} + F_{y_S}\ \mathbf{\jmath} + F_{z_S}\ \mathbf{k}. \tag{1.3}$$

The position force (vector) $\mathbf{F}_{MS}$ shown in Figure 1.7 of the point $S(x_S,\ y_S,\ z_S)$ relative to a point $(x_M,\ y_M,\ z_M)$ is calculated with

$$\mathbf{F}_{MS} = (F_{x_S} - F_{x_M})\ \mathbf{\imath} + (F_{y_S} - F_{y_M})\ \mathbf{\jmath} + (F_{z_S} - F_{z_M})\ \mathbf{k}. \tag{1.4}$$

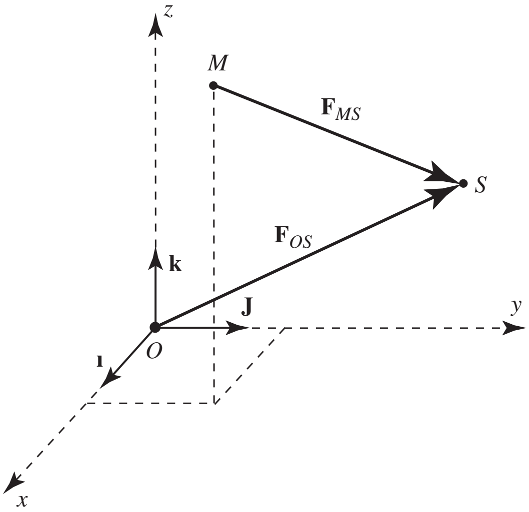

Figure 1.7 Position forces (vectors).

1.5 Scalar (Dot) Product of Two Forces

Definition. The dot product of two forces $\mathbf{F} = F_x\mathbf{\imath} + F_y\mathbf{\jmath} + F_z\mathbf{k}$ and $\mathbf{P} = P_x\mathbf{\imath} + P_y\mathbf{\jmath} + P_z\mathbf{k}$ is

$$\mathbf{F} \cdot \mathbf{P} = |\mathbf{F}|\ |\mathbf{P}| \cos(\theta) = F_xP_x + F_yP_y + F_zP_z \tag{1.5}$$

where θ is the angle between the forces **F** and **P**.

1.6 Cross Product of Two Forces

The cross product of two forces **F** and **P** is another force defined by (Figure 1.8)

$$\mathbf{F} \times \mathbf{P} = |\mathbf{F}|\ |\mathbf{P}| \sin(\mathbf{F}, \mathbf{P})\mathbf{u} \tag{1.6}$$

where **u** is a unit force normal to **F** and **P** having its direction given by the right-hand rule.

The magnitude of the cross product is given by

$$|\mathbf{F} \times \mathbf{P}| = |\mathbf{F}|\ |\mathbf{P}| \sin(\mathbf{F}, \mathbf{P}).$$

When $\mathbf{F} = F_x\ \mathbf{\imath} + F_y\ \mathbf{\jmath} + F_z\ \mathbf{k}$, and $\mathbf{P} = P_x\ \mathbf{\imath} + P_y\ \mathbf{\jmath} + P_z\ \mathbf{k}$, the cross product $\mathbf{F} \times \mathbf{P}$ can be calculated using

$$\mathbf{F} \times \mathbf{P} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ F_x & F_y & F_z \\ P_x & P_y & P_z \end{vmatrix} = (F_y\ P_z - F_z\ P_y)\mathbf{\imath} + (F_z\ P_x - F_x\ P_z)\mathbf{\jmath} + (F_x\ P_y - F_y\ P_x)\mathbf{k}. \tag{1.7}$$

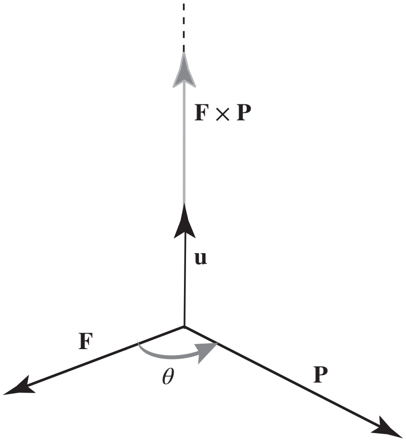

Figure 1.8 Cross product of two forces **F** and **P**.

1.7 Examples

Example 1.1
Figure 1.9 shows three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, and the angles of the forces with the horizontal $\theta_1 = \pi/6$, $\theta_2 = \pi/3$, and $\theta_3 = \pi$. The forces have the magnitudes $F_1 = 1$ kN, $F_2 = 3$ kN, and $F_3 = 2$ kN. Find the resultant of the planar forces and the angle of the resultant with the horizontal.

Solution
The input data are introduced in MATLAB with:

```
clear; clc; close all
F1 = 1; % kN
F2 = 3; % kN
F3 = 2; % kN
% angle of force F1_ with x-axis
theta1 = pi/6;
% angle of force F2_ with x-axis
```

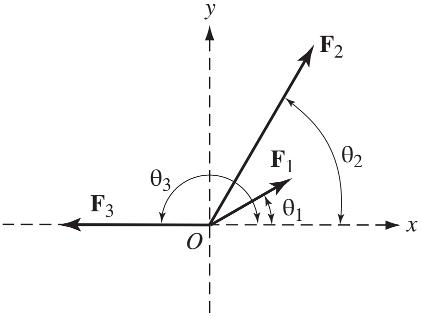

Figure 1.9 Graphical representation the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$.

```
theta2 = pi/3;
% angle of force F3_ with x-axis
theta3 = pi;
```

The components of the forces on x and y axes are

$$\mathbf{F}_i = F_i \ \cos\theta_i \ \mathbf{\imath} + F_i \ \sin\theta_i \ \mathbf{\jmath}, \ i = 1, \ 2, \ 3, \tag{1.8}$$

or in MATLAB:

```
% components of forces F1_, F2_, and F3_
F1x = F1*cos(theta1);
F1y = F1*sin(theta1);
F1_ = [F1x F1y];

F2x = F2*cos(theta2);
F2y = F2*sin(theta2);
F2_ = [F2x F2y];

F3x = F3*cos(theta3);
F3y = F3*sin(theta3);
F3_ = [F3x F3y];
```

The numerical values are:

```
F1_ = [ 0.866  0.500] (kN)
F2_ = [ 1.500  2.598] (kN)
F3_ = [-2.000  0.000] (kN)
```

The resultant is calculated with

$$\mathbf{R} = \sum \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3, \tag{1.9}$$

and the angle of the horizontal with the horizontal axis is

$$\phi = \tan^{-1} \frac{\mathbf{R} \cdot \mathbf{\jmath}}{\mathbf{R} \cdot \mathbf{\imath}}. \tag{1.10}$$

With MATLAB the resultant and the angle are calculated with:

```
R_ = F1_+F2_+F3_;
phi = atand(R_(2)/R_(1));
```

and the results are

```
% R_ = F1_+F2_+F3_ = [ 0.366  3.098] (kN)
% phi = atan(Ry, Rx) = 83.262 (deg)
```

The MATLAB representation of the forces is shown in Figure 1.10 and it is obtained with:

```
sa = 4;
hold on
axis([-sa sa -sa sa])
axis square
```

```
quiver(0,0,F1_(1),F1_(2),0,'Color','k','LineWidth',1.2)
quiver(0,0,F2_(1),F2_(2),0,'Color','k','LineWidth',1.2)
quiver(0,0,F3_(1),F3_(2),0,'Color','k','LineWidth',1.2)
quiver(0,0,R_(1),R_(2),0,'Color','r','LineWidth',2)

text(F1_(1),F1_(2),' F_1',...
    'fontsize',14,'fontweight','b')
text(F2_(1),F2_(2),' F_2',...
    'fontsize',14,'fontweight','b')
text(F3_(1),F3_(2),' F_3',...
    'fontsize',14,'fontweight','b')
text(R_(1),R_(2),' R',...
    'fontsize',14,'fontweight','b')
grid on
xlabel('x'), ylabel('y'),
```

Example 1.2
Figure 1.11 shows a system of spatial forces with the magnitudes $F_1 = 15$ N, $F_2 = 30$ N, $F_3 = 10$ N, and $F_4 = 15$ N. The parallelepiped has the dimensions $a = 2$ m, $b = 3$ m, and $c = 5$ m. Find:

(a) the resultant of the system of forces;
(b) the angle between the forces $\mathbf{F}_1$ and $\mathbf{F}_2$;
(c) the projection of the force $\mathbf{F}_1$ on the force $\mathbf{F}_4$; and
(d) calculate $\mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3)$.

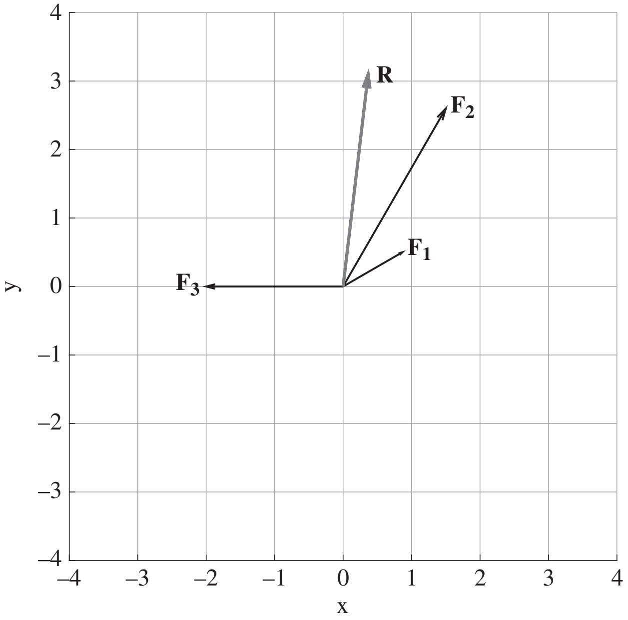

Figure 1.10 MATLAB representation of the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$.

Solution

1)

(a) The input data in MATLAB are:

```
clear; clc; close all
a = 2; % m
b = 3; % m
c = 5; % m

F1 = 15; % N
F2 = 30; % N
F3 = 10; % N
F4 = 15; % N
```

A Cartesian reference frame xyz is selected as shown in Figure 1.11. The position force (vector) of the application point of the force $\mathbf{F}_1$ is

$$\mathbf{r}_E = a\mathbf{\imath} + b\mathbf{\jmath} + c\mathbf{k}. \tag{1.11}$$

The position vector of the application point of the force $\mathbf{F}_2$ is

$$\mathbf{r}_D = b\mathbf{\jmath} + c\mathbf{k}. \tag{1.12}$$

The position (vector) of the application point of the force $\mathbf{F}_3$ is

$$\mathbf{r}_F = a\mathbf{\imath} + c\mathbf{k}. \tag{1.13}$$

The position vector of the application point of the force $\mathbf{F}_4$ is

$$\mathbf{r}_B = a\mathbf{\imath} + b\mathbf{\jmath}. \tag{1.14}$$

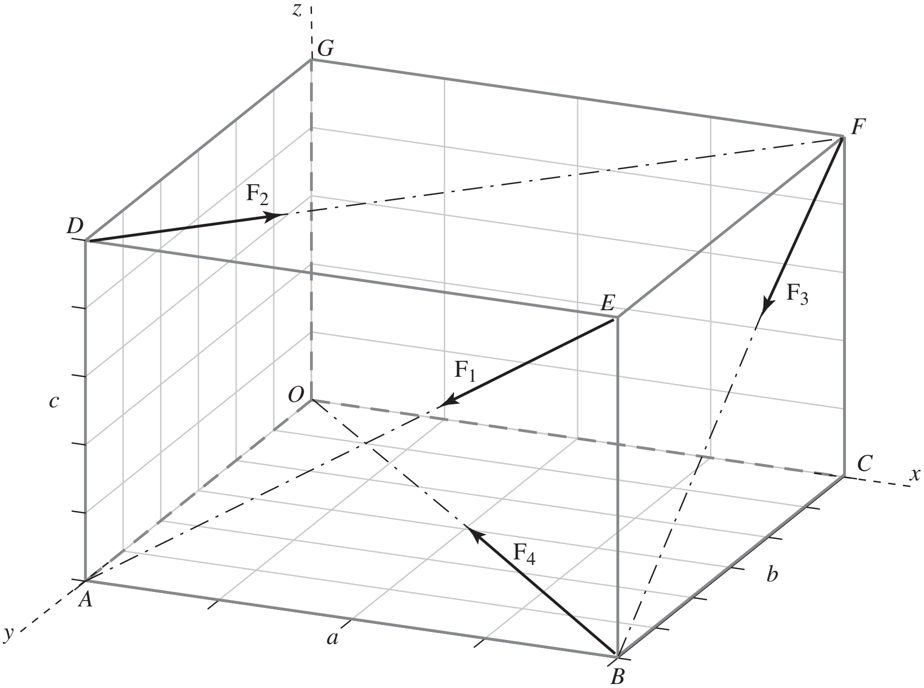

Figure 1.11 System of four spatial forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$.

The position vectors of the application points of the spatial forces are defined in MATLAB as:

```
% position force of F1_
rE_ = [a b c];
% position force of F2_
rD_ = [0 b c];
% position force of F3_
rF_ = [a 0 c];
% position force of F4_
rB_ = [a b 0];
```

Next the position vectors of points A, G, and C, shown in Figure 1.11, are defined in MATLAB as:

```
% position force of A
rA_ = [0 b 0];
% position force of G
rG_ = [0 0 c];
% position force of C
rC_ = [a 0 0];
```

The unit vector of the force $\mathbf{F}_1$ is

$$\mathbf{u}_1 = \frac{\mathbf{r}_A - \mathbf{r}_E}{|\mathbf{r}_A - \mathbf{r}_E|}, \tag{1.15}$$

and the force $\mathbf{F}_1$ is $\mathbf{F}_1 = F_1\ \mathbf{u}_1$. The unit vector of the force $\mathbf{F}_2$ is

$$\mathbf{u}_2 = \frac{\mathbf{r}_F - \mathbf{r}_D}{|\mathbf{r}_F - \mathbf{r}_D|}, \tag{1.16}$$

and the force $\mathbf{F}_2$ is $\mathbf{F}_2 = F_2\ \mathbf{u}_2$. The unit vector of the force $\mathbf{F}_3$ is

$$\mathbf{u}_3 = \frac{\mathbf{r}_B - \mathbf{r}_F}{|\mathbf{r}_B - \mathbf{r}_F|}, \tag{1.17}$$

and the force $\mathbf{F}_3$ is $\mathbf{F}_3 = F_3\ \mathbf{u}_3$. The unit vector of the force $\mathbf{F}_4$ is

$$\mathbf{u}_4 = \frac{-\mathbf{r}_B}{|\mathbf{r}_B|}, \tag{1.18}$$

and the force $\mathbf{F}_4$ is $\mathbf{F}_4 = F_4\ \mathbf{u}_4$. The unit vectors and the forces are calculated in MATLAB with

```
u1_ = (rA_-rE_)/norm(rA_-rE_);
F1_ = F1*u1_;

u2_ = (rF_-rD_)/norm(rF_-rD_);
F2_ = F2*u2_;

u3_ = (rB_-rF_)/norm(rB_-rF_);
F3_ = F3*u3_;

u4_ = (-rB_)/norm(-rB_);
F4_ = F4*u4_;
```

The numerical results are:

```
% unit forces
% u1 = [-0.371,0,-0.928]
% u2 = [ 0.555,-0.832,0]
% u3 = [0, 0.514,-0.857]
% u4 = [-0.555,-0.832,0]
%
% forces
% F1_ = [ -5.57,0, -13.9] (N)
% F2_ = [  16.6,   -25,0] (N)
% F3_ = [0,  5.14, -8.57] (N)
% F4_ = [ -8.32, -12.5,0] (N)
```

The plot of the unit vectors is shown in Figure 1.12, and it is obtained using the commands:

```
axis([0 a 0 b  0 c])
grid on, hold on
axis ij
xlabel('x'), ylabel('y'), zlabel('z')

text(0,0,0,'O','fontsize',12,'fontweight','b')

quiver3(rE_(1),rE_(2),rE_(3),u1_(1),u1_(2),u1_(3),0,...
    'Color','k','LineWidth',1.5)
text(rE_(1)+u1_(1),rE_(2)+u1_(2),rE_(3)+u1_(3),'u_1',...
    'fontsize',12,'fontweight','b')
```

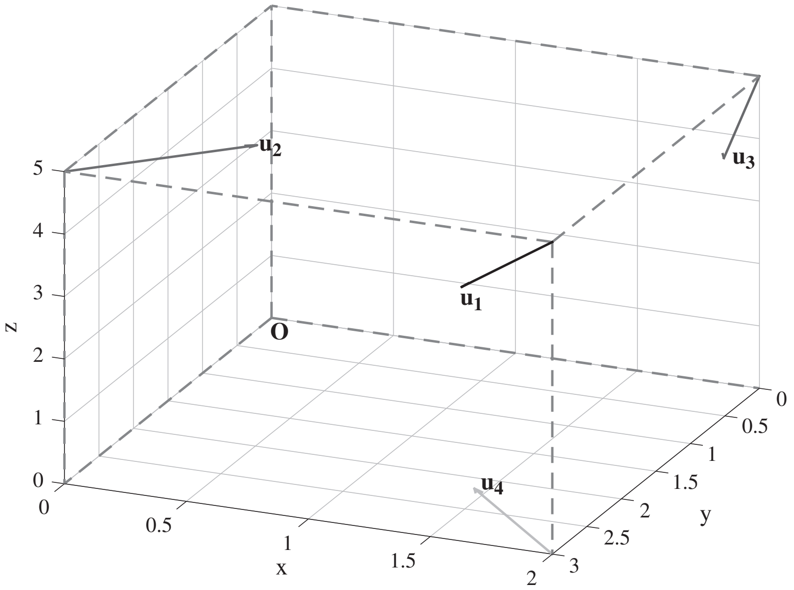

Figure 1.12 Plot of unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$

```
quiver3(rD_(1),rD_(2),rD_(3),u2_(1),u2_(2),u2_(3),0,...
    'Color','r','LineWidth',1.5)
text(rD_(1)+u2_(1),rD_(2)+u2_(2),rD_(3)+u2_(3),'u_2',...
    'fontsize',12,'fontweight','b')

quiver3(rF_(1),rF_(2),rF_(3),u3_(1),u3_(2),u3_(3),0,...
    'Color','b','LineWidth',1.5)
text(rF_(1)+u3_(1),rF_(2)+u3_(2),rF_(3)+u3_(3),'u_3',...
    'fontsize',12,'fontweight','b')

quiver3(rB_(1),rB_(2),rB_(3),u4_(1),u4_(2),u4_(3),0,...
    'Color','g','LineWidth',1.5)
text(rB_(1)+u4_(1),rB_(2)+u4_(2),rB_(3)+u4_(3),' u_4',...
    'fontsize',12,'fontweight','b')

line([rE_(1) rD_(1)],[rE_(2) rD_(2)],[rE_(3) rD_(3)],...
    'LineStyle','--','LineWidth',1.5)
line([rE_(1) rF_(1)],[rE_(2) rF_(2)],[rE_(3) rF_(3)],...
    'LineStyle','--','LineWidth',1.5)
line([rE_(1) rB_(1)],[rE_(2) rB_(2)],[rE_(3) rB_(3)],...
    'LineStyle','--','LineWidth',1.5)

line([rF_(1) rG_(1)],[rF_(2) rG_(2)],[rF_(3) rG_(3)],...
    'LineStyle','--','LineWidth',1.5)
line([rD_(1) rG_(1)],[rD_(2) rG_(2)],[rD_(3) rG_(3)],...
    'LineStyle','--','LineWidth',1.5)
line([rD_(1) rA_(1)],[rD_(2) rA_(2)],[rD_(3) rA_(3)],...
    'LineStyle','--','LineWidth',1.5)

line([0 rC_(1)],[0 rC_(2)],[0 rC_(3)],...
    'LineStyle','--','LineWidth',1.5)
line([0 rA_(1)],[0 rA_(2)],[0 rA_(3)],...
    'LineStyle','--','LineWidth',1.5)
line([0 rG_(1)],[0 rG_(2)],[0 rG_(3)],...
    'LineStyle','--','LineWidth',1.5)

view(23, 30)
```

The resultant of the spatial system, the magnitude of the resultant, and the direction cosines of the the resultant are calculated with:

```
R_ = F1_+F2_+F3_+F4_;
modR = norm(R_);
uR_ = R_/modR;
% R_=F1+F2+F3+F4=[ 2.750,-32.297,-22.502] (N)
% |R|=  39.5 (N)
% direction cosines=uR_=R_/|R|=[ 0.070,-0.819,-0.570]
```

2)
(b) The angle between the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is calculated in MATLAB with:

```
c12 = dot(F1_,F2_)/(F1*F2);
phi12= acosd(c12);

% F1.F2 = |F1||F2| cos(phi12)
% phi12 =    102 (deg)
```

3)
(c) The projection of the force $\mathbf{F}_1$ on the force $\mathbf{F}_4$ is obtained with

```
prF1F4 = dot(F1_, u4_);
% projection of F1 on F4 = F1_.u4_ =   3.09
```

4)
(d) The term $\mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3)$ is calculated in MATLAB:

```
T = dot(F1_,cross(F2_,F3_));
% F1_ . (F2_ x F3_) = -2384.809
```

Example 1.3
Show that $(\mathbf{p} + \mathbf{q}) \times (\mathbf{p} - \mathbf{q}) = 2\,\mathbf{q} \times \mathbf{p}$.

Solution
The two symbolical forces $\mathbf{p}$ and $\mathbf{q}$ are defined in MATLAB with:

```
syms px py pz qx qy qz

p_ = [px py pz];
q_ = [qx qy qz];
```

The left-hand side (LHS) is $(\mathbf{p} + \mathbf{q}) \times (\mathbf{p} - \mathbf{q})$ which is

```
LHS = cross(p_+q_, p_-q_);
```

The right-hand side (RHS) is $2\,\mathbf{q} \times \mathbf{p}$ which is

```
RHS = 2*cross(q_, p_);
```

The difference LHS−RHS is calculated with

```
simplify(LHS-RHS)
% [ 0, 0, 0]
```

and the equality is true. If $\mathbf{p}$ and $\mathbf{q}$ are the adjacent sides of a parallelogram then $\mathbf{p} + \mathbf{q} = \mathbf{d}_1$ and $\mathbf{p} - \mathbf{q} = \mathbf{d}_2$ are the two diagonals of a parallelogram.

The area of the parallelogram is the magnitude of the cross product $\mathbf{q} \times \mathbf{p}$. Notice that the area of a parallelogram is given by half of the magnitude of the cross product of its diagonals.