
1

SOME INTRODUCTORY PROBLEMS

The introductory problems in this chapter focus on developing students' abilities to reason and produce a reasonable model from a real-world situation. Photographs are used in several examples and the student may easily expand the problem assignments by providing their own photographs and related questions. The problems are not necessarily “beginners” problems; they are selected because they are “nonstandard,” and offer a glimpse into some of the different possibilities of modeling with GeoGebra. The problems presented in the next chapter are more standard and may for some beginners be a better choice.

Students should first try to solve some part of a problem “by head and hand,” using a calculator and paper and pencil and then expand to the other parts using GeoGebra. To learn mathematics in the present digital age, we need to recognize that old school content and methods have been overtaken by a digital technology that has moved us into a mathematical golden age. To be effective in teaching mathematics with technology-assisted instruction, we suggest that you study the TPACK framework and work toward an integrated approach where technology, mathematical content, and pedagogy are all integrated seamlessly in every lesson.

We begin with a couple of different introductory problems. As students enter their upper secondary mathematical studies, the core content suddenly needs to be mastered in greater depth than previously. Some examples include, but are not limited to, the following:

- Understanding numbers, arithmetic, and algebra.
- Properties of a range of whole numbers, different number bases, and the concepts of prime numbers and divisibility.

- Calculations in everyday life and with real numbers, written in different forms, including powers with real exponents, together with strategies for calculations when using digital tools.
- Generalization of the rules of arithmetic to handle algebraic expressions. The concept of linear inequality.
- Algebraic and graphical methods for solving linear equations and inequalities, and exponential equations.
- Concepts of sine, cosine, and tangent, and methods of calculating angles and lengths of right-angled triangles. The concept of a vector and its representations, such as direction, length, and points in a coordinate system.
- Addition and subtraction with vectors and scalar multiplication to produce a vector.
- Mathematical reasoning using basic logic, including implication and equivalence, and comparisons with how to argue in everyday contexts and in science subjects.
- Illustration of the concepts of definition, theorem, and proof, such as the Pythagorean Theorem and the sum of the angles of a triangle.
- Advanced percentage concepts: ppm and percentage points.
- Concepts of rate of change and indexes, as well as methods for calculating interest and mortgage payments for different types of loans.
- Concept of a function, its domain and range of a definition, and also properties of linear functions, exponential, and polynomial functions.
- Representations of functions in the form of words, functional expressions, tables, and graphs.
- Differences in the concepts of equation, inequality, algebraic expression, and function.
- Appreciation of how statistical methods and results are used in society and in science.
- Concepts of dependent and independent events, as well as methods for calculating probabilities in multi-stage random trials, using examples from games and from risk and safety assessments.
- Strategies for mathematical problem solving including the use of digital tools.
- Mathematical problems relevant to personal finances, social life, and applications in other subjects.
- Mathematical problems related to the cultural history of mathematics.

As the teacher you need to be clear about what your goals are and how your methods will reach those goals. Your students need to know where you are going with your teaching and how you aim to get there. You also need to structure your time but leave ample room for flexibility. The modeling tasks you plan to use should be simple to begin with, so that they may be solved in one or two lessons. Use detailed instructions for your students, preferably together with screencasts to explain technical details in GeoGebra and other digital tools that your students can watch

over and over again if necessary. You will also need detailed and clear summaries of your lessons to help the students who just want to keep on working without much reflection. It is beneficial to start with several short easy tasks in the beginning so that your students learn how to model, and when they have learned that, you can expand the mathematics in further modeling tasks.

1.1 TICKET PRICES

A travel agency is advertising a favorably priced sports holiday to the 2017 FIFA Confederation Cup in Russia. The transport to Russia from the United States will be by chartered flight with seating for 200 passengers. If a group intends to travel to the Confederation Cup fills the airplane, then the price for each person will be 400 USD. Understandably, the price for each sold ticket will rise for every empty seat on the airplane. The flight company owning the plane demands a minimum of 20,000 USD in rent for the plane taking this trip.

Make a model of the travel agency's total income possibilities for travel to this international football event. If the travel agency wants the same income for 199 tickets as they want for 200 passengers, what seat pricing would be reasonable then?

Instead of doing this calculation for every number of possible passengers and therefore presenting a table of possible prices for each customer, the travel agency chooses to simplify and add a standard fee for each empty seat. How will the model then describe the travel agency's total income for different numbers of passengers?

Also calculate the maximal price that a single passenger may need to pay. What is the least number of passengers that the travel agency must attract to avoid losing money on this trip?

Your model should clearly show the relation between the price of an air seat and the number of sold air seats as well as the relation between the total revenues and the number of sold seats.

Proposed Solution 1

With 200 air seats at 400 USD the total income is 80,000 USD. If the travel agency still wants the 80,000 USD even with just 199 sold air seats, the price needs to be $80,000 \text{ USD} / 199 \text{ air seats} = 402 \text{ USD}$. It therefore seems reasonable to ask for 2 USD more from every passenger for every empty air seat.

The number of sold air seats x , and the price in USD y , should exhibit a linear relationship $y = a \cdot x + b$. If the number of unsold seats increases, the price will increase, and the price will be reduced if more seats are sold, which in turn means that the slope is negative. If the number of sold air seats increase with 1, the price will decrease with 2 USD, so $a = -2$. You also know that when the number of sold seats = 200, so the price is 400 USD. This gives $400 = -2 \cdot 200 + b$, and hence $b = 1200$. The equation becomes $y = 1200 - 2x$. This is represented in Table 1.1.

TABLE 1.1 Linear Relationship between the Number of Sold Air Seats and the Price for a Seat

| Number of sold air seats | Price for one air seat (USD) |
|--------------------------|------------------------------|
| 200 | 400 |
| 199 | 402 |
| 198 | 404 |
| 197 | 406 |
| 196 | 408 |
| 195 | 410 |
| 190 | 420 |
| 180 | 440 |
| 170 | 460 |
| 160 | 480 |
| 150 | 500 |
| 100 | 600 |
| 50 | 700 |
| 0 | 800 |

The maximal price, with just one passenger in the plane, will be 798 USD.
If you label the total revenues with R , you can write

$$R = \text{number of sold seats} \cdot \text{price per seat}$$

The most convenient way to handle this is to put the values into a spreadsheet, for example, in Excel or GeoGebra. To show the spreadsheet in GeoGebra, you would press **Ctrl-Shift-S** or select **View > Spreadsheet** in the menu. When you have entered the values in the first column you may type

$$=1200 - 2 \mathbf{A2}$$

In cell B2 and press Enter. Observe that the space works as multiplication sign in GeoGebra. Then enter

$$=\mathbf{A2} \mathbf{B2}$$

in cell C2 and press **Enter**.

Now you may select cells **B2** and **C2** and then grab the small square in the lower right corner, the *fill handle*, dragging it downward, as indicated in Figure 1.1. This will copy the formulas downward. This is the standard procedure for copying formulas in spreadsheets.

It is very convenient to represent the relations graphically in GeoGebra. Select all the values in columns A and B (but not the titles) and right-click on the selection. Then select **Create...>List with points**, which will create both the data points and a list containing these points. You will not see them in the Graphic window yet, since the values are too large. Drag the background to pan the window. Hold down **Ctrl** and drag one axis at the time to zoom in or out until you see something similar to what you see in Figure 1.2.

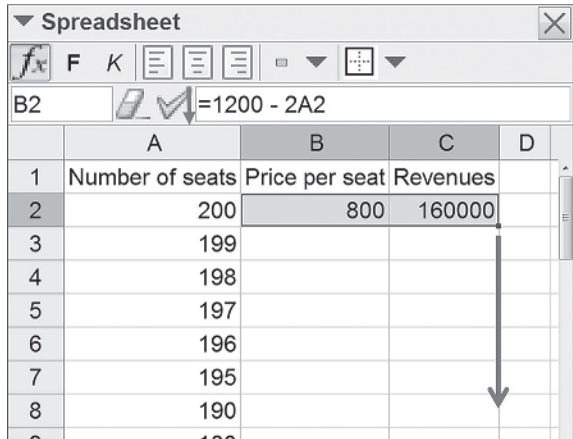


FIGURE 1.1 How to copy a formula using the fill handle.

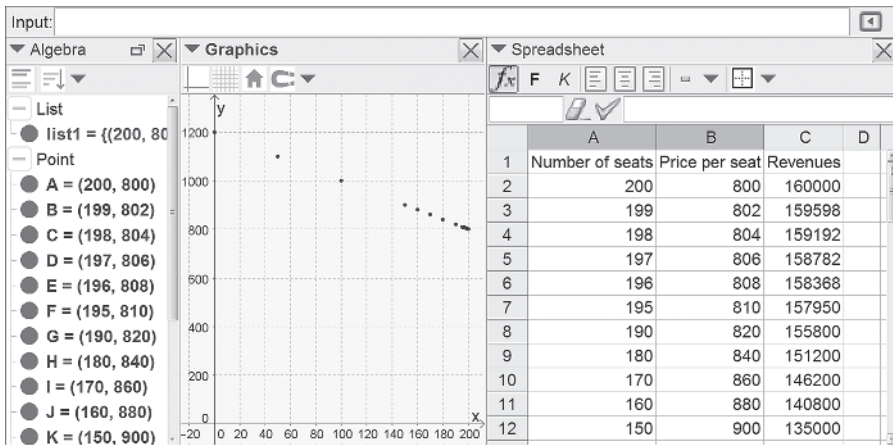


FIGURE 1.2 Table with values and corresponding points of a graphical representation.

In the Figure 1.2 the point's labels (name) are not shown. This is because of the setting **Settings > Name on object > No new object**. If you want to change this afterward, just right-click on the word **Point** in the Algebra window and select **Show label** to toggle the visibility of labels.

We would also like to create a diagram that shows how the revenues will vary with the number of air seats. Do this by choosing **Show > Graphic 2** from the menu. It might be shown as a free-floating window without axes or a grid, but if you want to, you can dock this window to the main window by clicking on the dock button in the upper right corner of the window shown in Figure 1.3. Note that the axes and the grid have their own buttons in the upper left corner in the Style Bar.

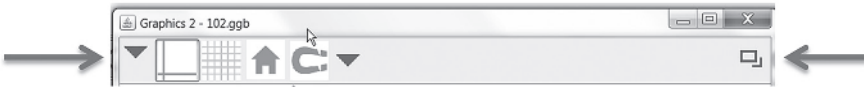


FIGURE 1.3 The dock button that allows you to dock a free window.

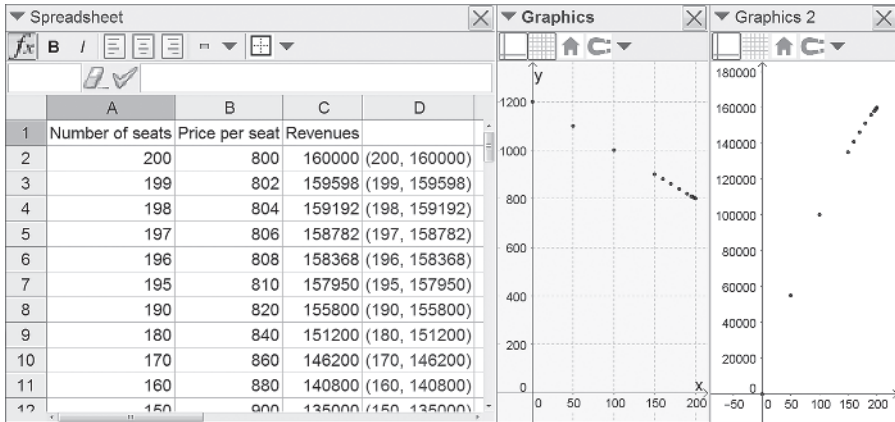


FIGURE 1.4 Graphic Window 2 showing another graph with a different scale.


When you have docked and arranged Graphic Window 2 the way you want it, you can create the points directly in the spreadsheet. Make sure that Graphic Area 2 is active and then type

$$= (A2, C2)$$

in cell **D2** and copy this formula downward by dragging the fill handle. Your result should be similar to Figure 1.4. This is a good way to create new points directly from values in the spreadsheet. The GeoGebra spreadsheet can handle all kinds of objects in its cells, while most other spreadsheets only handle real numbers, text, and formulas.

In order to find the number of passengers it takes to not lose money for the travel agency, you can do some investigations directly in the spreadsheet. Enter new values in columns A and B and look for the income that goes under USD. It takes at least 18 passengers to break even. They will then have to pay 1164 USD for their tickets.

Proposed Solution 2

There is, of course, nothing that can stop you from going directly to graphical solutions. When you have reached the relation $y = 1200 - 2x$, you can graph this directly in GeoGebra by entering $y = 1200 - 2x$ in the command line, and then pressing **Enter**. If you have already created the points, you could also create a line by using the Create New Line tool. Click once on the symbol  and then click on two of

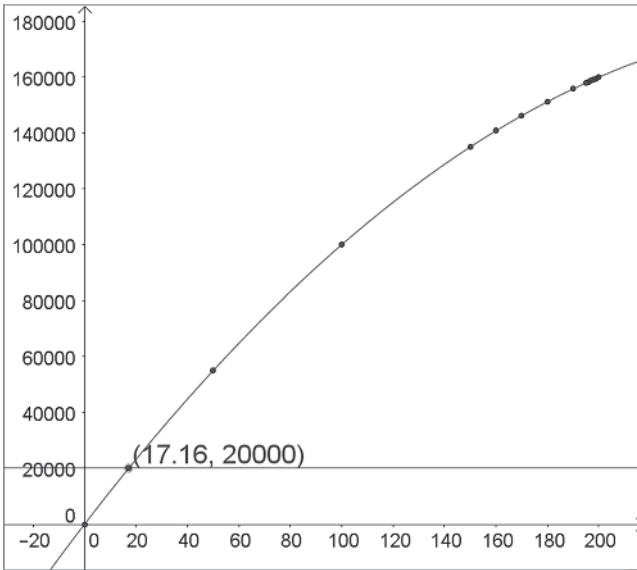



FIGURE 1.5 Income as it decreases dramatically after the first 50 empty seats.

the points. A line is created and you find the symbolic representation of it in the Algebra window.

By using substitution, you can write the relationship between income and sold air seats as $R(x)=x \cdot y=x(1200 - 2x)=1200x - 2x^2$. This model can also be represented graphically. In order to find the number of sold air seats that will give 20,000 USD as income you could draw a line $y=20,000$ and observe where the line intersects the graph. The intersection point is found by clicking on the Intersect tool  and then clicking on the graph and the line. You could also have written **Intersect [a, b]**, if a and b are names for the line and the graph. The coordinates for the intersection can be found in the Algebra window or if you right-click the point and select **Object Properties** and then on the **Basic** tab, **Show Label: Value**, as in Figure 1.5.

1.2 HOW LONG WILL THE PASTURE LAST IN A FIELD?

The pasture in a field like the one shown in Figure 1.6 has been known to last for 3 days for 6 cows and for 7 days for 3 cows.

Based on this experience, how long will the pasture likely last for a single cow?

Proposed Solution 1

Allow the students to think about a situation for a while, but in the end you will probably need to encourage the students to formulate the situation in a mathematical

language in some way. In all events where you translate a situation into a mathematical language, you are in some sense always using a mathematical model. A very simple model would be a straight line between the two points that are given by (6 cows, 3 days) and (3 cows, 7 days), shown in Figure 1.7. It gives you the possibility to follow the line to the point where 1 cow is and see that it points at ...



FIGURE 1.6 Cow in a dandelion field. Photo: Keith Weller/USDA (www.ars.usda.gov: Image Number K5176-3) [Public domain], via Wikimedia Commons. https://upload.wikimedia.org/wikipedia/commons/0/0c/Cow_female_black_white.jpg.

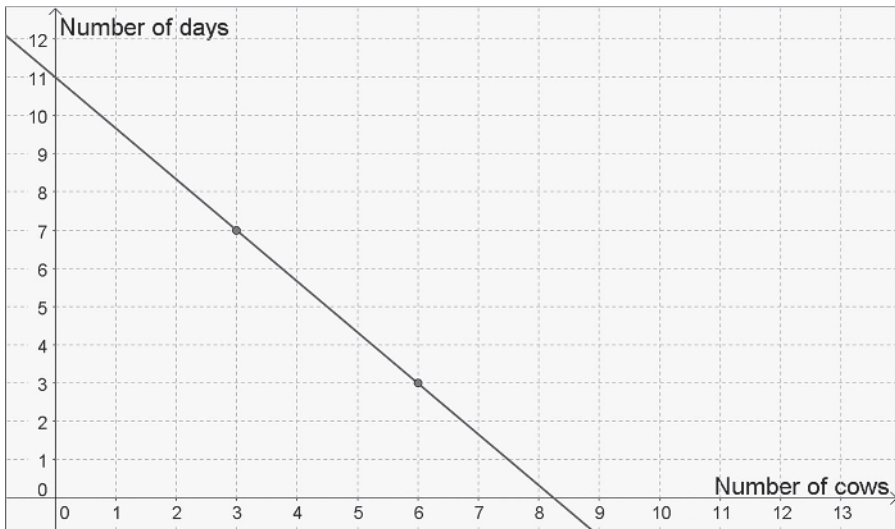


FIGURE 1.7 Linear model.

Proposed Solution 2

Another model could be found by reasoning around the concept of “cow days,” where one cow days is equal to a certain amount of food. The two solutions will then deliver two different answers. That helps when thinking about the validity of the model, errors, how the model could become better, and so on.

We find that the first bit of information in the problem indicates that you have enough food for 18 cow days. The next bit of information indicates that you have food for 21 cow days. So you can assume that there is food for about 20 cow days (the differences may have something to do with rounding errors). In Figure 1.8 you can see a mathematical model where

$$\text{Number of cows} \cdot \text{Number of days} = 20 \Leftrightarrow \text{Number of days} = 20 / \text{Number of cows}$$

We see that the prognosis in this case is remarkable higher. The food should be enough for approximately 20 days for a single cow, and possibly a lot longer if you factor in that grass is growing all the time.

Comments

This is a good example of a task that can be used even in lower secondary school to get students to think mathematically and see how the strength in using different representations might get a solution to reveal itself. The authors of this book, who teach mathematical modeling at an upper secondary school and at a university, usually start with a simple task, before they introduce more complex modeling activities. In this way students at all levels get a good introduction to the modeling process before they have to think through a complex, complicated, and mathematically more demanding situation.

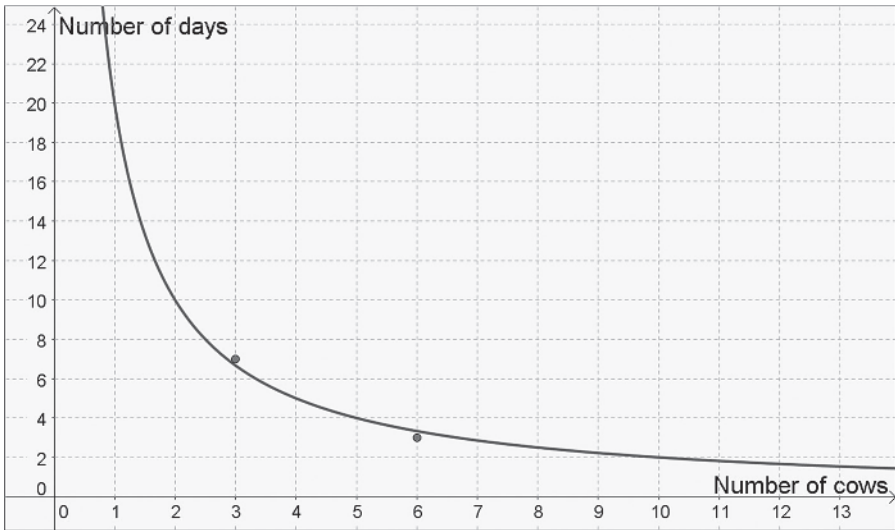


FIGURE 1.8 Nonlinear model based on the concept of cow days.

1.3 A BIT OF CHEMISTRY

A laboratory assistant has a large container with a 20% hydrochloric acid solution and a small container with 300 grams of 43% hydrochloric acid solution as shown in Figure 1.9. The assistant would like to dilute the strong solution until it is 28% acid. How much of the 20% hydrochloric acid should the laboratory assistant use?

Try to solve the problem in different ways and try to construct or derive a general solution model that shows how much of a solution you need to add to m grams of $b\%$ solution in order to get $c\%$ solution.

Proposed Solution 1

We start with an algebraic solution. In problems with a lot of information it is generally a good idea to organize the data. First, introduce a variable. Let x be the amount of added 20% hydrochloric acid solution, measured in grams. Always be clear about what your variables stand for and what units they are measured in.

Once you have done that, you can now represent the data as a bullet list. You have to determine how much acid and solution you have in the small container, before and after the dilution:

- Amount of hydrochloric acid before: 43% of 300 grams = 129 grams
- Amount hydrochloric acid after: 129 grams + 20% of x grams
- Total amount of solution before: 300 grams
- Total amount of solution after: 300 grams + x grams

These values are well prepared to be presented in a table, such as that shown in Table 1.2. The table setup will help you organize your data and your modeling



FIGURE 1.9 Solutions. Photo: Flickr, user usehung (CC BY 2.0) <https://www.flickr.com/photos/13639809@N03/2295905317/sizes/o/>.

TABLE 1.2 Organizing Your Data in a Table

| Grams | Before | After |
|--------------------------|--------|--------------|
| Pure hydrochloric acid | 129 | $129 + 0.2x$ |
| Total amount of solution | 300 | $300 + x$ |

strategies. The solution process becomes easier if you focus on one part of the problem at a time, and using a table to organize problems like these helps you in doing this.

Finally use the only bit of information that you haven't yet used to set up your equation. The new solution should be 28%; in other words, 28% of your new total acid should be hydrochloric acid. The equation that we use to solve the problem takes the form

$$0.28 \cdot (300 + x) = 129 + 0.2x, \quad \text{which yields } x = 562.5 \text{ grams}$$

Equations of this type are often viewed as difficult to set up as well as difficult to solve, but letting students practice on problems like these often gives a well-needed boost to their general problem-solving skills.

So 562.5 grams of the 20% solution need to be added to the small container in order to get 28% strength for the final solution.

To derive a general model, you need to substitute the parameters a , b , c , and m into suitable places in the equation above to get

$$c \cdot (m + x) = b \cdot m + a \cdot x, \quad \text{which yields the solution}$$

$$x = m \cdot \frac{b - c}{c - a}$$

Proposed Solution 2

Now you will make a dynamic model in GeoGebra where you will vary a slider until you get the right concentration for the solution. In order to make a general model, begin by creating five sliders: a , b , c , m , and x_0 . If you wish, you can simply create these as numbers first, and then show them as sliders by clicking on the white bullet to the left of their algebraic representations in the Algebra window. Define a , b , and c to vary between 0 and 1 with a step of 0.01, while m and x_0 will vary between 0 and 600 with a step of 1. Then adjust the sliders to see the values of the current problem. All of these settings can be found in the properties of the numbers, on the **Slider** tab.

Next adjust the window settings so that the graphic area shows 0–1000 along the x -axis and 0–1 along the y -axis. Since every calculation of the amount of hydrochloric acid is a multiplication of a percentage and a total mass, you could represent the amount of hydrochloric acid as the area of a rectangle, with the percentage as the

height along the y -axis and the total mass as the width on the x -axis. Use the following commands:

| | |
|----------------------|-----------------------------------|
| $O = (0, 0)$ | Origin |
| $A = (0, a)$ | |
| $B = (0, b)$ | A, B, and C represent percentages |
| $C = (0, c)$ | |
| $M = (m, 0)$ | |
| $X = (x_0, 0)$ | M, T, and X represent |
| $T = (m + x_0, 0)$ | masses, in grams |
| $H_1 = (x_0, a)$ | |
| $H_2 = (m, b)$ | H1, H2, and H3 are |
| $H_3 = (m + x_0, c)$ | vertices of the rectangle |

After that, create the rectangles with the following command:

```
Added = Polygon [O, A, H_1, X]
Existing = Polygon [O, B, H_2, M]
Total = Polygon [O, C, H_3, T]
```

The result can be seen in Figure 1.10. By changing the value of x_0 , you can search for the value when the sum of the Existing and the Added rectangles' area

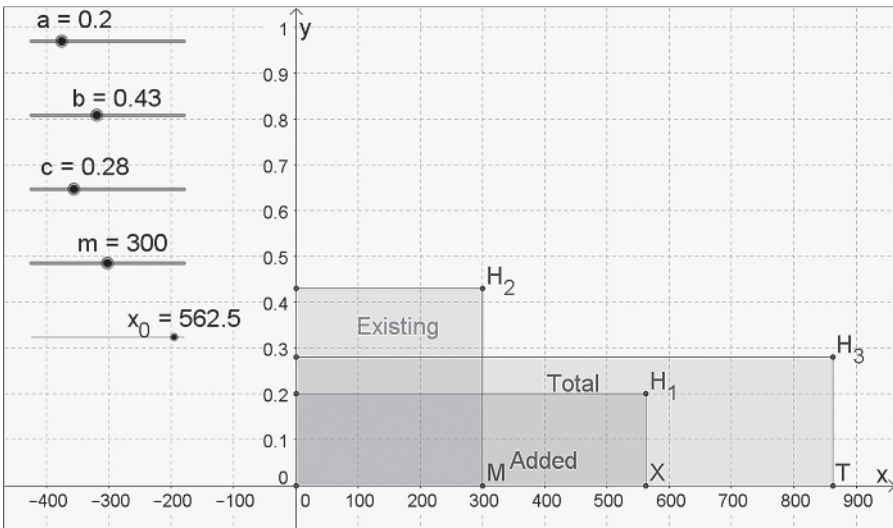


FIGURE 1.10 Dynamic graphic model of the chemical mixture.


Proposed Solution 3

The previous solution suggestion was visual and pedagogical but could just as easily be obtained on a calculator. Therefore you will now build a calculator in GeoGebra where you may enter values and receive answers.

Start a new session of GeoGebra by pressing **Ctrl-N**. Type

$$x_0 = m (b-c) / (c-a)$$

in the command field, and let GeoGebra create sliders for a , b , c , and m . Since it is hard to reach exact values with a slider, you will now represent the other parameters with input boxes instead. First you need to adjust some properties for a , b , c , and m . Right click on any of them in the Algebra window and select **Properties**, on the **Sliders** tab. In the left-hand pane you will find the properties dialogue with all objects represented in an object list. You should **Ctrl-Click** and select a number of different objects all at the same time. Enter **Min = 0** and **Max = 1000** for m and **Max = 1** for a , b , and c . The standard values **Min = -5** and **Max = 5** that GeoGebra uses for sliders are normally not ideal. So hide the sliders by clicking the blue buttons to the left of the numbers in the Algebra view.

You should create an input box for each of your four parameters by clicking on the Input Box tool  and then in the Graphics area to show where you want them placed, often far to the left, one above the other. In the dialogue shown in Figure 1.12, you can enter a text label (normally the variable name) and the variable to be connected to the input field.

Adjust the boxes according to the grid after all boxes have been created. You can also chose to set the input field length to 5 in the **Input Box Length**, Fix Objects in the **Basic** tab and set text size to **Medium** in the **Text** tab in the properties dialogue. Last, drag x_0 from the Algebra window to the graphics area to create a text box. Select the text size to be **Medium**. This can be done in the style bar. Then hide the grid and the axis, and also, if you want to, close the Algebra window. Figure 1.13 shows a typical result.

This calculator calculates x_0 for any parametric value. But if you know how much you should add and want to calculate the concentration it will get, how do you do that?

Start by taking away the text box showing the value for x_0 and create an input field for x_0 instead, just like you did for the other parameters. Make sure that you change

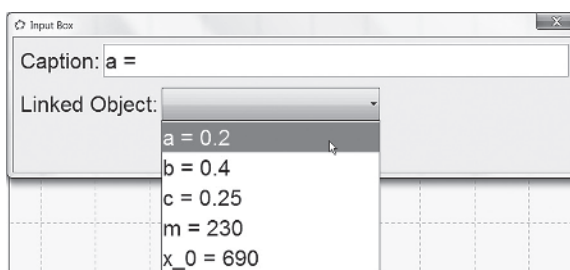


FIGURE 1.12 Input box dialogue.

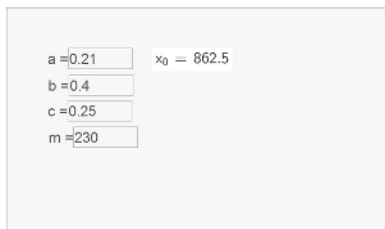


FIGURE 1.13 A simple calculator.

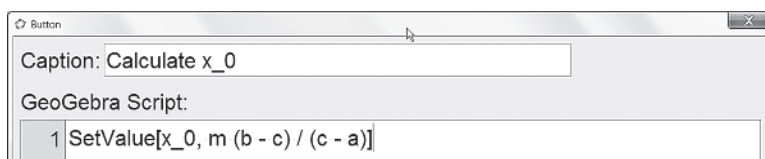


FIGURE 1.14 Button dialogue.

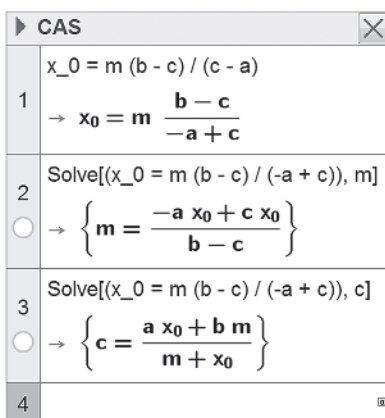



FIGURE 1.15 CAS window used to help you solve for variables in a formula.

text size, length for the field, and also lock this field. Enter an arbitrary value for x_0 so that the formula will disappear.

You can then create a command button by clicking on the Button tool  and then somewhere in the graphics field. Fill in the dialogue box as in Figure 1.14.

The command `SetValue[x_0, m (b - c) / (c - a)]` calculates the value of the formula and updates x_0 to this value. It then creates four more buttons, one for each parameter. The formula that you should use may be found in different ways. You could use manual algebraic manipulations on paper, or you could open up a new window in GeoGebra with **Ctrl-N**, next open up the CAS windows with **Ctrl-Shift-K**, and last, write `x_0 = m (b - c) / (c - a)` in line 1. In line 2 you write `Solve[$1, m]`. In this command `$1` is a reference for the equation in line 1. Figure 1.15 shows this process.

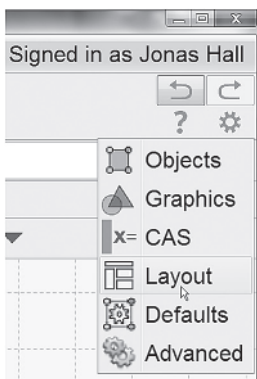


FIGURE 1.16 Where to find the layout settings.

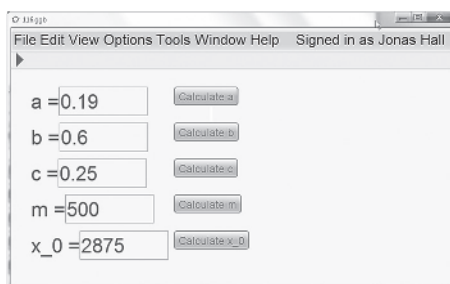


FIGURE 1.17 A “Solver” tool in GeoGebra.

When you have entered your formulas using the command buttons, you could—if you like—disable the visibility of the tool field and the input field by clicking on the property gear wheel in the top right corner and selecting **Layout** as shown in Figure 1.16.

In Figure 1.17 you can see the finished “Solver.” Similar solvers can be created for other problems, and if you upload your solver to GeoGebraTube, it may be run as an app on a phone or a tablet.

1.4 SYDNEY HARBOR BRIDGE

Figure 1.18 and Figure 1.19 show the famous Sydney Harbor Bridge. If the bridge heads are 89 meters high, how long is the bridge between the bridge heads?

Proposed Solution

This is a common situation. You have a photograph where you know one distance and you want to find another distance. As long as these distances are at the same distance from and close to perpendicular to the viewer, a linear proportional model can be



FIGURE 1.18 Sydney Harbor Bridge. By Yun Huang Yong via Flickr, CC BY-SA 2.0. <https://www.flickr.com/photos/goosmurf/3001997390/sizes/o/>.



FIGURE 1.19 One of the bridge heads stands prominent in the mist behind a ferry. Photo: Hpeterswald/Wikimedia Commons. https://commons.wikimedia.org/wiki/File:Sydney_Ferry_Queenscliff_1.jpg.

TABLE 1.3 Two by Two Table for Organizing Proportional Data

| | Unknown | Known |
|-------------|---------|-------|
| Reality (m) | x | 89 |
| Photo (px) | 1403 | 175 |

TABLE 1.4 Proportionality Is Invariant under Rotation and Mirroring Operations

| | Reality (m) | Photo (px) |
|---------|-------------|------------|
| Known | 89 | 175 |
| Unknown | x | 1403 |

used. By measuring the distances in the photo, you will get the remaining information you need. Assuming proportionality yields

$$\frac{\text{Unknown distance (real)}}{\text{Unknown distance (image)}} = \frac{\text{Known distance (real)}}{\text{Known distance (image)}}$$


In this simple model you can ignore that perception issues might cause the proportionality to break. Proportionality can also be visualized through a simple 2×2 table or box. If the unknown distance is x and you measure the known distances in the photo through some computer program to 175 pixels and to 1,403 pixels, you can then set up your data in Table 1.3.

One interesting property of Table 1.3 is that it can be rotated and mirrored and still be valid. You may, for example, turn it 90 degrees counterclockwise to get Table 1.4.

It is useful to show this with the table written on a bit of cardboard that can be physically flipped and rotated. Any of these configurations may be set up as an equation. In the example above you will get the equation:

$$\frac{89}{x} = \frac{175}{1403}, \text{ which may be written without denominators as the crosswise product}$$

$$175x = 89.1403, \text{ and so } x = 715.5 \text{ meters.}$$

However, there are several ways to measure the distances in the photo. One way is to measure directly the distance in the photo using a ruler. The photo can be imported into GeoGebra with the Image tool , and then you can place points on the photo to measure the distances. If you want more precision, there is a small free program called *Digitize It* that gives much better precision for measuring distances using an image or photo.

In Figure 1.20 is an example of how several points' positions are captured by the software and put in the table to the left. Note the small magnifying glass window floating between the table and the image. You can download this program for free at http://download.cnet.com/Digitize-It/3000-20415_4-10489008.html.

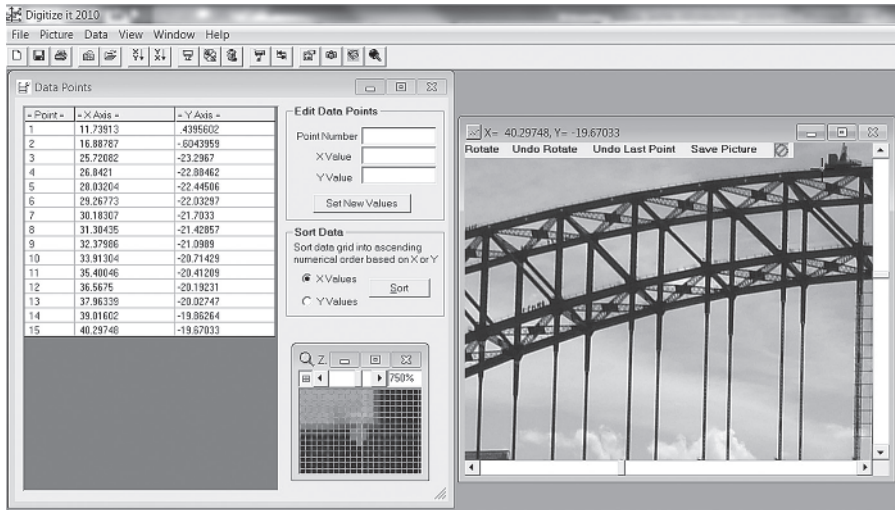


FIGURE 1.20 *Digitize It*, a program for digitalization of photos.

Many students today can use cameras in smart phones, tablets, and the like, to capture interesting situations or the objects themselves. It is best to zoom in and take a photo from far away to minimize nonlinear effects. One student could hold a meter stick and another jump as high as possible. How high was the jump? How tall is the student who holds the stick? Proportionality will give the answer.

1.5 PERSPECTIVE

In Figure 1.21 you can see the same girl photographed five different times. The perspective makes her look bigger when she is closer to you and smaller when she is far away.

If an object is far away enough, so that you experience it under a viewing angle that is less than about 30° , the product of the subject’s size and distance will be approximately constant. Another way to express the same relation is to say that the size is inversely proportional to the distance.

If the distance to the girl at her closest instance is 3 meters, how far away was she at the most distant instance?

Proposed Solution


Start a new window in GeoGebra with **Ctrl-N**. Import your image by using the Image tool  and next click on, for example, the origin. Then right-click on the photo and select **Properties**, the **Position** tab, and lock the image’s both lower corners in the coordinate system as shown in Figure 1.22.



FIGURE 1.21 Perspective run. Photo: Nevit Dilmen (original work) CC-BY-SA-3.0 via Wikimedia Commons. http://commons.wikiwithia.org/wiki/File:08913-Perspective_Run.jpg.

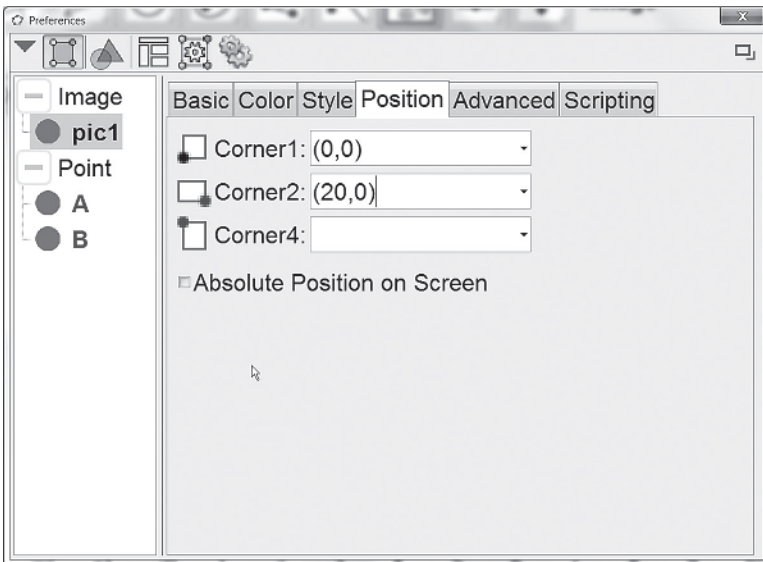


FIGURE 1.22 Lock the image in the coordinate system.

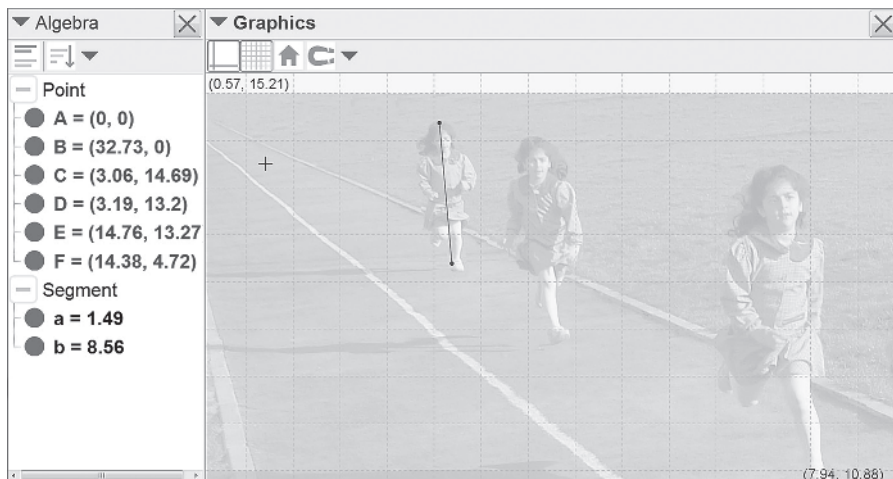




FIGURE 1.23 Measuring the girl's height in GeoGebra.

In the **Basics** tab you select **Background image**, and under the **Color** tab you adjust the opacity to 25% in order to more easily be able to see the coordinate system.

Measure the girl's height by using the Segment tool . Click on the girl's head and heel both when she is at her closest and when she is at her farthest. The lengths of the segments will be shown in the Algebra window in Figure 1.23.

If you would like to fine-tune the position of the points, first press **Escape** to activate the Select and Move tool ; then right-click on the word **Point** in the Algebra window and select **Properties**, the **Algebra** tab, and set the step length **0.001** for all points at the same time. Next you select a point and use the arrow keys to move it.

You may also zoom in on the image as you place the points on the girl farthest away. If you want to stick with the proportions 1:1 between the x - and y -axes, it is best to zoom by right-clicking on an empty part of the graphic area and then selecting **Zoom > 400%**. That done, drag the graphic area to pan in order to find the girl again.

If the product of the distance d and the size s is constant, then $d_1 \cdot s_1 = d_2 \cdot s_2$. In this case $d_1 = 3$ m, $s_1 = 8.56$, and $s_2 = 1.49$; the result is $d_2 = 17.2$ m. Let the students practice this by photographing each other in groups of three and checking the distances with long tape measures.

1.6 LAKE ERIE'S AREA

Try to estimate the size or the area of Lake Erie from Figure 1.24 or a similar map from Google Earth in several different ways. Also try to estimate the error in the method you choose and show your work clearly.

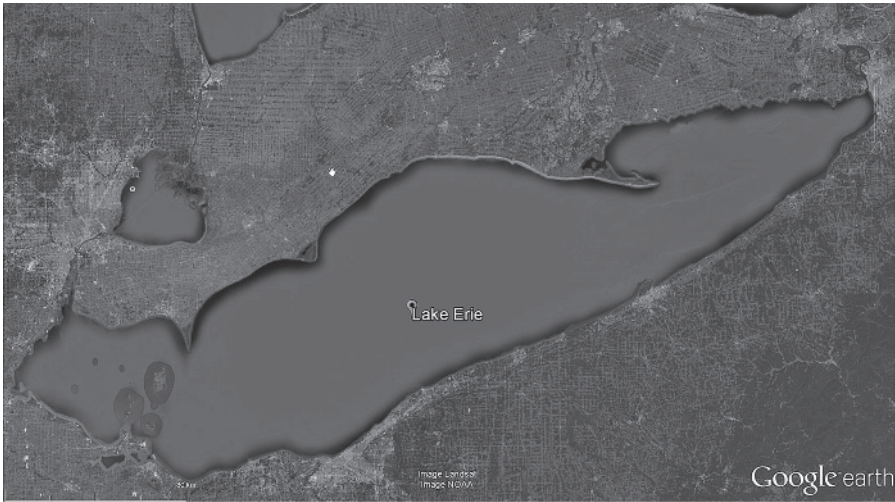


FIGURE 1.24 Lake Erie, Google Earth. Image Data from Landsat and NOAA.

Proposed Solution 1

The classical solution to cut out the image and estimate the weight only works if you have a very accurate scale available. Enlarge the image so that it almost fills an 8 × 11 inch sheet of paper. Cut out the lake. The surface will be something like 20 × 10 cm. Ordinary printer paper weighs about 80 grams/m², so a square 10 cm across has a weight of 0.8 grams. Lake Erie should then have a weight of about 1.6 grams. With a weighing scale that can handle a resolution of 0.1 grams or maybe even 0.01 grams, the accuracy of the paper weight will be crucial. It may therefore be necessary to calibrate the weight of a sheet of paper in this particular batch and by a particular brand.


Assuming that an A4 sheet paper weighs in at 4.83 grams and the cutout of Lake Erie weighs 0.94 grams, this will yield a surface weight of 4.83 grams/(21.0 cm × 29.7 cm) = 0.007744 grams/cm², from which the cutout's area is found to be 0.94 grams/0.007744 grams/cm² = 121.4 cm². You can wait until later to do any rounding.

Once you have an estimate of how large Lake Erie is, you can transform this to square kilometers. Assuming the 50 km tick mark to be 3.55 cm in the image, the distance scale can be estimated to 1 cm in the image = 50/3.55 = 14.08 km in reality, which give an area scale where 1 cm² on the image is 33.71² = 198.4 km² in reality.



With an area of 121.4 cm² you can get a value for the area of Lake Erie: $A = 121.4 \text{ cm}^2 \cdot 198.4 \text{ km}^2/\text{cm}^2 \approx 24,000 \text{ km}^2$. How accurate is this estimate? Which of your measurements has limited the accuracy most? How can you improve the accuracy? How well does your cutout conform to the actual shape of the lake?


Note also how units are handled together with the scales. This is an important skill for students to learn.

Proposed Solution 2

You can take the image or a screen dump of it and import it into GeoGebra in order to measure the area there. Import an image by selecting the Image tool  and clicking somewhere down to the left in the graphics area. You then get a pop-up dialogue where you can select an image. You may also drop an image in the graphics area.

Two points are provided to control the image, and you can set the coordinates of B to be (10, 0). Then right-click on the image and select **Object Properties**; check the **Background image** on the **Basic** tab and the image will be placed behind the axis and the grid.

Now you can start placing points along the coast line. It might be a good idea to select **Options>Labeling>No new objects** from the menu before you start. When you have placed your points, you should select the Create list tool  and then drag a rectangle around all points to create a list with these points. Last, create a polygon from all the points using the command **Polygon[list1]**. The area of the polygon is approximately 16.08 area units as shown in Figure 1.25. You could also have used the Polygon tool  directly.

Now deal with the scale: zoom in on the scale as shown in Figure 1.26 and measure the 80 km tick to 2.04 length units by placing segment  over the mark. One length unit = $80/2.04 = 39.216$ km and one area unit is $39.216^2 = 1537.9$ km². The area of the lake is $A = 16.08$ a.u. $\cdot 1537.9$ km²/a.u. $\approx 24\,730$ km².

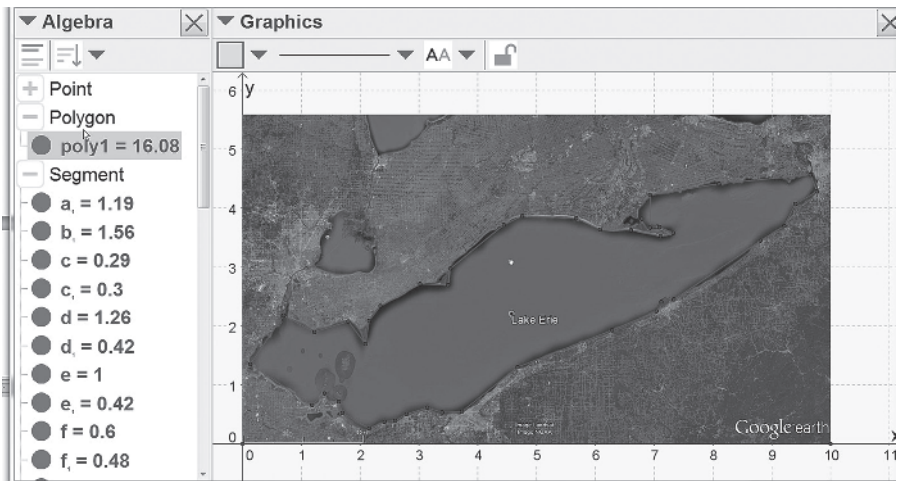


FIGURE 1.25 Estimating the area of the lake by a polygon.

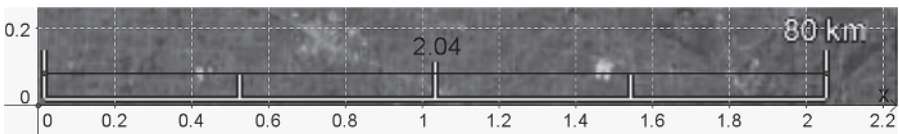
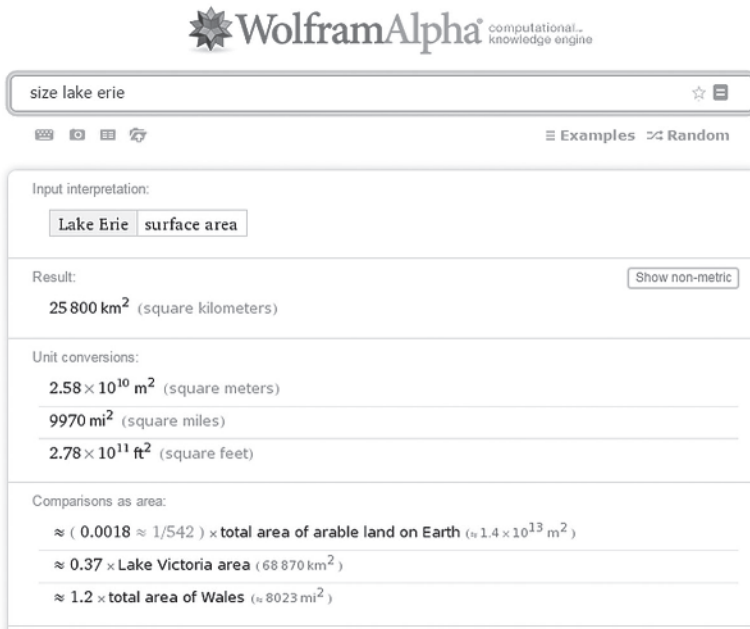


FIGURE 1.26 A segment used to measure a distance.



WolframAlpha computational knowledge engine

size lake erie

Examples Random

Input interpretation:

Lake Erie surface area

Result: Show non-metric

25 800 km² (square kilometers)

Unit conversions:

2.58 × 10¹⁰ m² (square meters)

9970 mi² (square miles)

2.78 × 10¹¹ ft² (square feet)

Comparisons as area:

≈ (0.0018 ≈ 1/542) × total area of arable land on Earth (≈ 1.4 × 10¹³ m²)

≈ 0.37 × Lake Victoria area (68 870 km²)

≈ 1.2 × total area of Wales (≈ 8023 mi²)

FIGURE 1.27 Wolfram Alpha can provide you with geographical data.



FIGURE 1.28 Lorain Lighthouse. Photo: Rona Proudfoot (CC BY-SA 2.0). <https://www.flickr.com/photos/ronnie44052/2420422668/in/photostream/>.

The error can be estimated by fine-tuning a few of the point's positions, noting the effect of this on scale and area. To fine-tune a point's position, open the properties dialogue box and then click on the **Algebra** tab and change the value of step length to 0.001. Thereafter you may fine-tune the position of the point by using the arrow keys. If you hold down **Shift** at the same time as you use the arrow keys, you will increase the accuracy 10 times and if you hold down **Ctrl**, you will lose accuracy by a factor 10, jumping faster.

Proposed Solution 3

Wikipedia, referring to the US Environmental Protection Agency, claims that Lake Erie has a surface area of 25,667 km². NOAA claims it to be 25,655 km², and Wolfram Alpha, using several sources, says in Figure 1.27 that the area is 25,800 km². Most likely these sources define the lake area differently, so let's say we are satisfied with 25,700 km². The Lorain lighthouse on the south coast of Lake Erie is depicted in Figure 1.28.

1.7 ZEBRA CROSSING

In Figure 1.29 you see people traversing a huge zebra crossing. You can assume that the white lines and the distances between them are the same everywhere and that the ground is flat. The camera that took the photo is located some distance behind the closest line and placed some distance above the ground.

Try to determine where the camera is placed, both in terms of height and in terms of depth, in relation to the first visible white line. Also try to determine how tall the single man with necktie and briefcase is, if the man with blue scarf and briefcase is 170 cm tall.



FIGURE 1.29 Zebra crossing. Photo: Flickr, user MrHayata (CC BY-SA 2.0). <https://www.flickr.com/photos/mrhayata/6789615983/sizes/o/>.

Proposed Solution 1

In Figure 1.30 you see a figure of what it might look like from the side the crossing when the photo was taken. You can see sections consisting of white lines and black ground that are represented by the segments labeled with d . To the left in Figure 1.30, or at the bottom of Figure 1.29, there is a small segment called d_0 that is not a whole section.

The points between these sections are projected on to the image plane to create an image. We now drastically simplify the situation and assume that the image plane is perpendicular to the ground plane in order to make calculations less difficult. But, in making this assumption, we introduce an error that we hope will not become too large. It is, however, better to have a simple model than no model at all. In general, always begin with a model sufficiently simple in order to get started!

In Figure 1.31 we concentrate on the incoming ray for just one of the section boundaries. Proportionality in triangles KCB and POB then gives

$$\frac{h}{a + d_0 + n \cdot d} = \frac{p_n}{d_0 + n \cdot d}$$

Here n is the serial number for the white line in the zebra crossing and p_n is the horizontal distance from the camera to the photo's bottom edge.

The proportionality can be rewritten in the following way by solving for p :

$$p_n = \frac{h \cdot (d_0 + n \cdot d)}{a + d_0 + n \cdot d}$$

where n is an independent variable and the numbers h , a , d , and d_0 are unknown numbers that need to be measured.

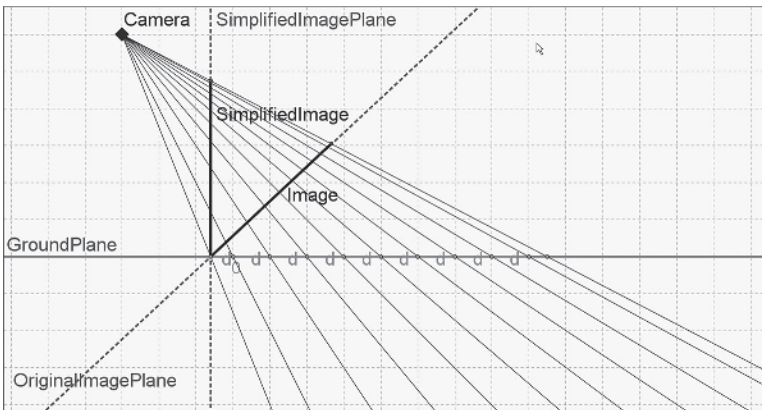


FIGURE 1.30 Camera photographing people strolling over Zebra crossing.

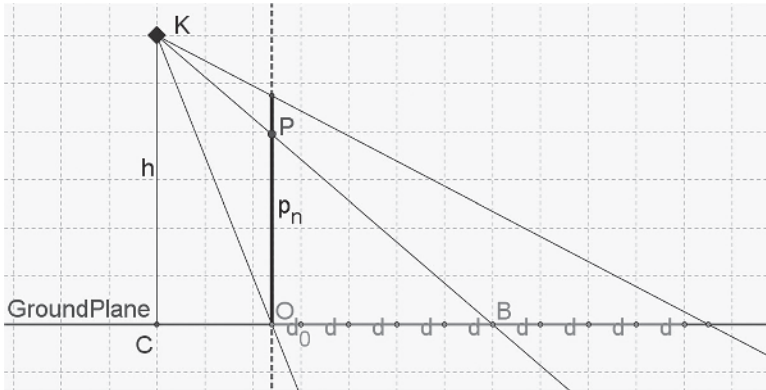


FIGURE 1.31 Triangles KCB and POB are uniform.

Start a new window in GeoGebra with **Ctrl-N**, and import your image by selecting the Image tool ; then click on, for example, the origin.

On the **Basic** tab of the image’s properties dialogue, you should select **Background image** and on the **Color** tab you should set the opacity to 25% so that you can see the coordinate system.

Now zoom in at the image and start measuring the values. Measure the y-coordinate for the upper part of the white lines to the left of the image and enter the values into the spreadsheet that you open by pressing **Ctrl-Shift-S**.

The values can be seen in the spreadsheet in Figure 1.32. The first white lines upper edge corresponds to $n = 1$, and so forth. The x-coordinate is simply the number of the white line in this model.

To visualize these points in a graph, open up Graphics Area 2 by pressing **Ctrl-Shift-2**. Click once in the Graphics 2 area to make it active. Select the values (but not the headings) in the spreadsheet; right click on the selection and choose **Create...>List with points**. The points will appear in Graphics Area 2.

With this method, the image planes’ x-coordinate will not be exactly 0, since the bottom edge of the image not is aligned with a white lines upper edge. So assume that the image plane is at where the x-coordinate = w . Then w is used in roughly the same way as d_0 . Create the slider w to go from -5 to 5 with an increment of 0.001 .

Now recreate the light rays. Light ray n comes from the ground at the point $(n, 0)$ and goes through the vertical image plane in the point (w, p) . Create these rays by typing

$$\text{Ray}[(A2, 0), (w, B2)]$$

in the spreadsheet cell C1 and copy this cell downward by dragging the fill handle. If you now adjust the value of w , you can make all rays go through the same point. That must be the position for the camera, as is clearly visible in Figure 1.32.

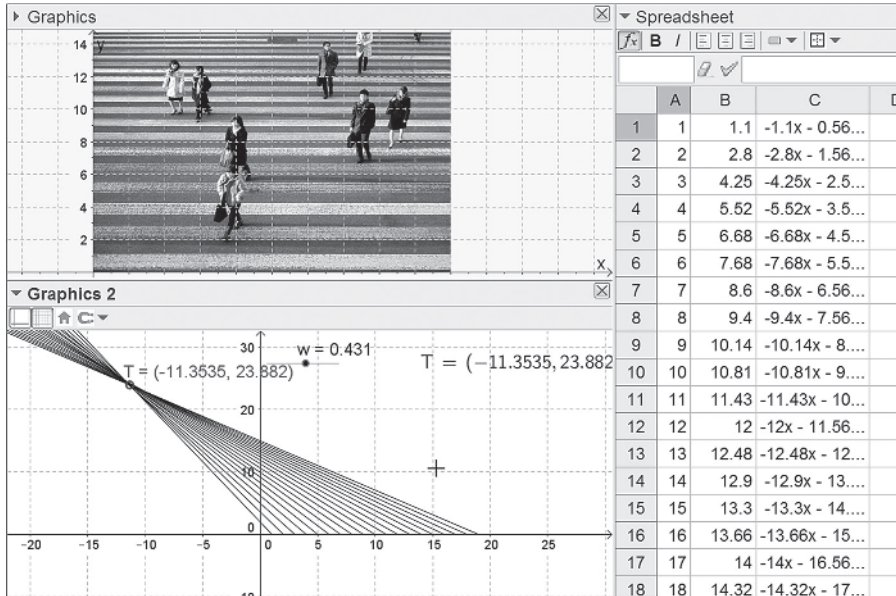



FIGURE 1.32 Distances from the bottom of the image to the upper edge of the white lines as shown in the spreadsheet.

You have now found the relative position of the camera, but you have not yet used any real units; therefore you do not know the scale for your figure. The only true measurement you have is the height of the man with the blue scarf who is 170 centimeters tall.

By zooming into the image and creating points at the man's feet and head, you can get the points' coordinates. In Figure 1.33 you can find the p -values, or y -coordinates, and so create the rays L2 and L3 by typing

$$\mathbf{L2=Ray[K, (w, y(P2))]} \quad \text{and} \quad \mathbf{L3=Ray[K, (w, y(P3))]}$$

After that, create a line perpendicular to the x -axis going through the intersection of L2 and the x -axis with the Perpendicular Line tool  or with the command

$$\mathbf{PerpendicularLine[Intersect[L2, xAxis], xAxis]}$$

If you do not see a specific construction in Graphics 2, it may have happened that it has been created in Graphics 1 instead. If so, just find the object in the Algebra window, right-click on it, and select **Properties**; click on the **Advanced** tab in the dialogue shown in Figure 1.34, and change the graphics area to where you want the object to be shown.

The intersection between the blue line and the ray L3 creates a point that corresponds to the head and the segment between head and foot has a length of 6.1391 length units. This corresponds to 1.70 m, which means that the scale will be that one

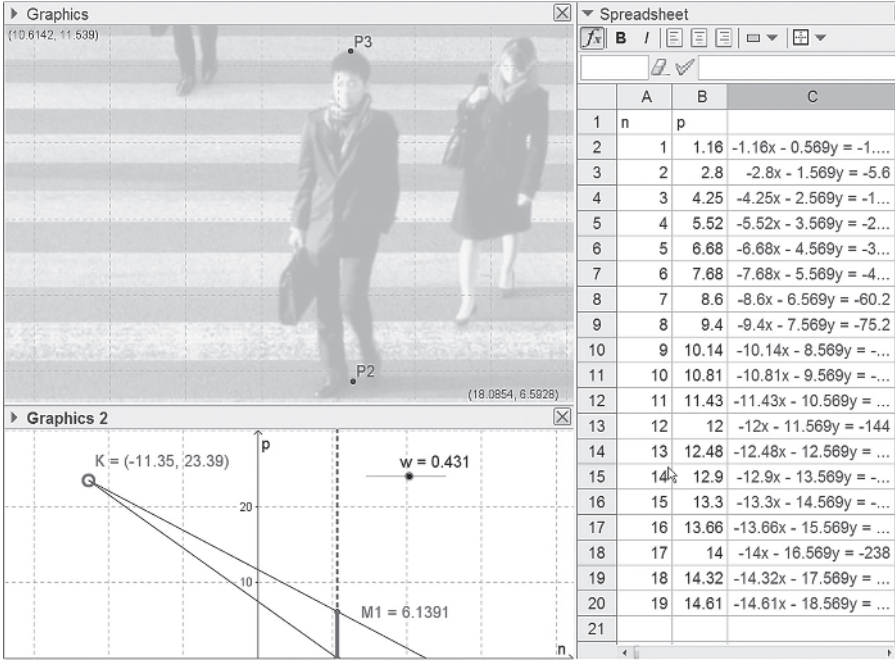


FIGURE 1.33 Measuring a man’s length to find the scale.

unit is equal to 0.2769 meter. If you use this to calculate the position of the camera, you will find that the camera is 3.1 meters behind the lower image edge and about 6.6 meters above the ground.

If you recalculate the construction for the man with the necktie, you find that his height is 6.2499 length units, which also corresponds to 1.73 m. With the accuracy you have now and the uncertainties in the measurements, due to the two men walking far apart when the photograph was taken, you would probably be hard pressed to justify saying anything other than that the two men are roughly the same height, with only a hint of the second man being taller.

Proposed Solution 2

You probably noticed that you never used the proportionality that you initially set out to do in the previous solution. You could have done the geometrical construction at the initial image plane, but that construction would have been a little more complicated than the one you did, since you used the coordinates for the point in a simpler way. This happens sometimes. Problem solving led you into different and unexpected paths where you probably did not plan to go.

Another path you might have followed, using proportionality, and shown in Figure 1.35, is to find the values for your parameters through regression analysis.

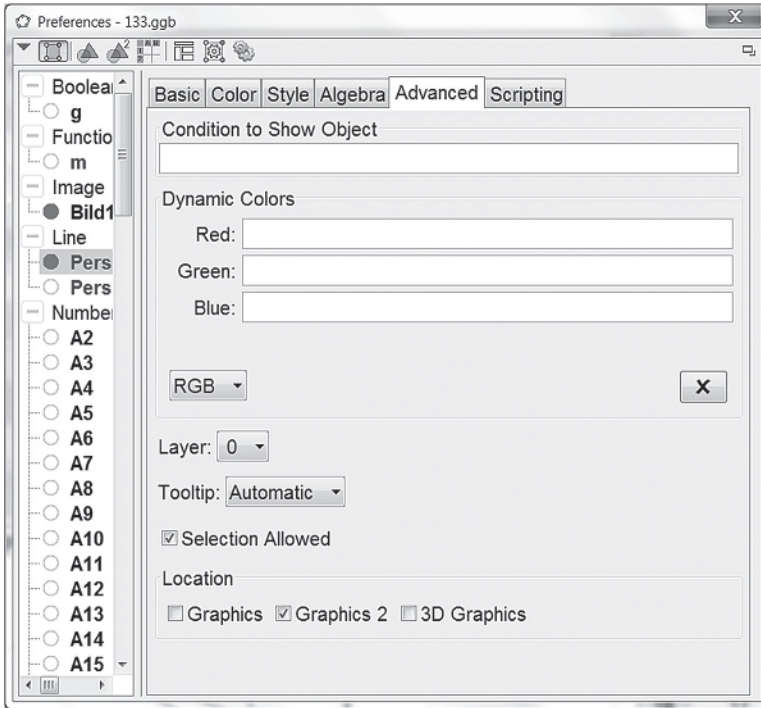


FIGURE 1.34 Window to change the graphics area you want the object to be visible in.

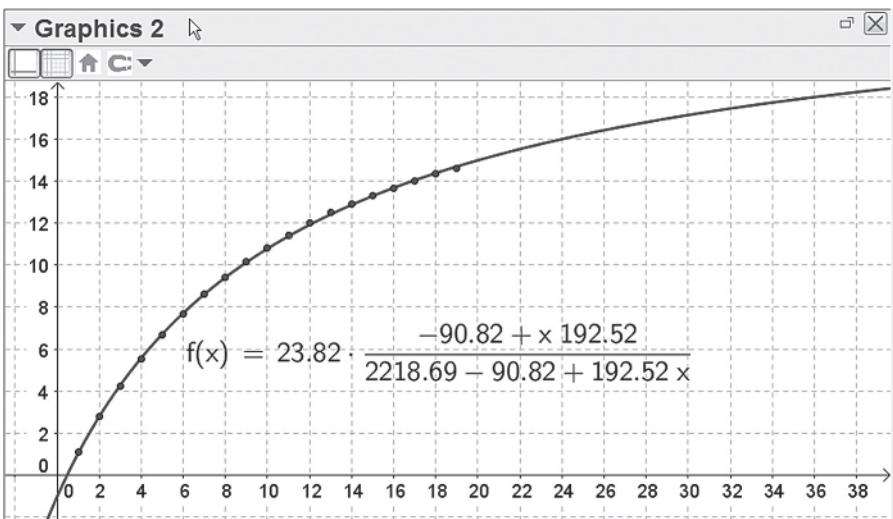


FIGURE 1.35 Regression used to find answers.

Through the definition of the numbers h , a , d , and d_0 together with a model function

$$m(x) = h (d_0 + x d) / (a + d_0 + d x)$$

you can use the command

Fit[list1,m]

to create a function that is fitted to the data points. The values for h , a , d , and d_0 should be reasonably accurate, so that the function $m(x)$ at least has the correct form compared with the values for the data points. Using this method you get $h=23.82$, $a=2218.7$, $d=192.5$, and $d_0 = -90.82$. For images where the camera's direction is more or less parallel with the ground plane, you can use this function to determine the height scale for different objects at different distances from the camera.

The model function is of the type $(a \cdot x + b)/(c \cdot x + d)$, which has a horizontal asymptote at $y = a/c$. This corresponds to $y = h \cdot d/d = h$, or in other words the graph flattens out at $y = 23.8$. This corresponds to the situation where the camera is positioned parallel with the ground plane, so the camera can see all the infinitely many white lines below the horizon.

You can identify other relations as well, namely the graph intercepts the x -axis when $x = -d_0/d = 0.47$, which you can compare with $w = 0.43$ length units. The vertical asymptote here is $-(a + d_0)/d = -11.05$ length units, corresponding to the camera's position in the x -direction, as compared with the -11.35 length units you got earlier. To derive the scale, you have to do the same as you did in the earlier proposed solution with similar results.

These two different methods gave similar results, and the discrepancies are a measure of the size of the errors you make in your models. This quite complicated problem and the two suggested attempts to solve it is probably beyond most high school courses of mathematics in terms of complexity but is included in this introductory chapter as an example of a task well suited for longer coursework in modeling.

1.8 THE SECURITY CASE

A metal workshop has received an order for a security case. This case will have a square bottom and is required to have a volume of exactly 900 liters $= 0.9 \text{ m}^3$. The material for the sides and the top cost 20 USD per square meter and the material for the bottom, which needs to be thicker, is 40 USD per square meter. What dimensions should the box have in order for the material costs to be as low as possible?

When you have a first solution to this problem, you should attempt to improve on your model, using Figure 1.36 as inspiration.



FIGURE 1.36 Type of security locker case. Photo: HenryWBee (own work) CC-BY-SA-3.0 via Wikimedia Commons. https://commons.wikimedia.org/wiki/File:HK_CWB_Victoria_Park_Sport_Court_Locker_Cages.JPG.

Proposed Solution 1

A routine algebraic solution to this typical optimization problem could look something like this:

Assume the sides of the square bottom to be s meters and the height to be h meters. Then $s^2 \cdot h = 0.9$ (cubic meters), leading to $h = 0.9/s^2$. The cost for the material can then be divided into the following parts:

$$\begin{aligned} \text{Top:} & \quad 20 \cdot s^2 \\ \text{Sides:} & \quad 4 \cdot 20 \cdot s \cdot h = 80s \cdot (0.9/s^2) = 72/s \\ \text{Bottom:} & \quad 40 \cdot s^2 \end{aligned}$$

The total cost sums to $C(s) = 60s^2 + 72/s$.

It is obvious that there exists a minimum because the function is increasing both as it approaches $0+$ ($72/s$ dominates) and for large values of x ($60s^2$ dominates). The minimum is found by setting $C'(s) = 0$, leading to the equation $120s - 72/s^2 = 0$, which can be reduced to $s^3 = 0.6$. So $s \approx 0.843$ meter, giving a height of 1.265 meter and a total cost of 128 USD.

Proposed Solution 2

GeoGebra is an excellent tool for modeling situations like these. You can then solve the problem without the use of derivatives. Begin by noting that $h = 0.9/s^2$, and let $x = s$ and $y = h$. You might want to draw a graph, but you should instead make a geometric

model of the locker case. Because the case has a square bottom, you can make a reasonable model using two dimensions just by constructing a scale figure of one side.

Start by creating a point on the x -axis. You can do this using the point tool. Just click on \cdot^A and then on the x -axis. Next press **Escape** to avoid producing more points by mistake. You could rename the point by right-clicking on it or by simply typing “P” **immediately** after creating it, before pressing **Escape**. Alternatively you could create the point by typing

$$P = \text{Point}[xAxis]$$

in the Input bar. The P will locate itself at the origin, but it can be dragged anywhere on the x -axis. Your work will be easier if you create s and h , so type in

$$s = x(P) \text{ and } h = 0.9/s^2$$

where the function $x(P)$ gives the x -coordinate of the point P. Then create another three points by typing

$$O = (0, 0) \text{ and } Q = (0, h) \text{ and } R = (s, h) \text{ and finally } \text{Polygon}[O, P, R, Q]$$

You can now drag P and see how the case changes its appearance, as indicated in Figure 1.37. Calculate the total material cost by typing

$$c = 600s^2 + 720/s$$

then drag the result from the Algebra window to the graphics view. You can now simply drag P to find the minimum cost. By selecting P, you can control it using the

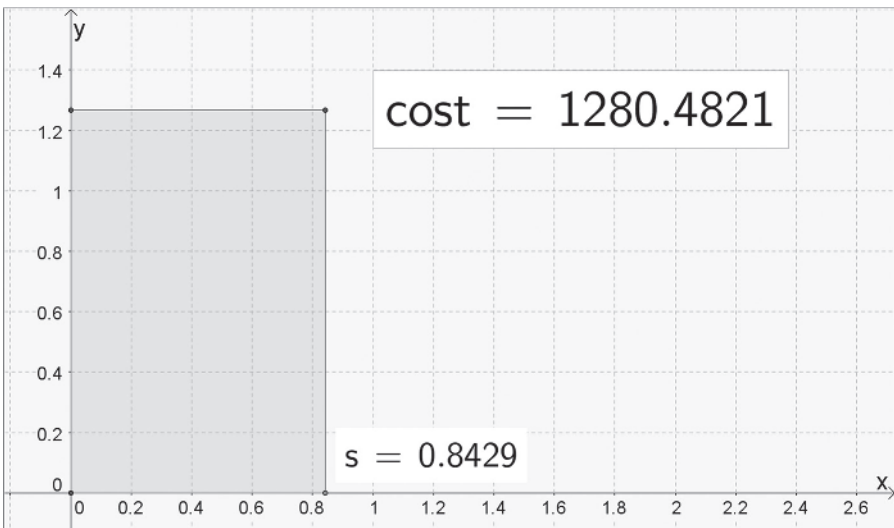


FIGURE 1.37 Dynamic model of the locker case.

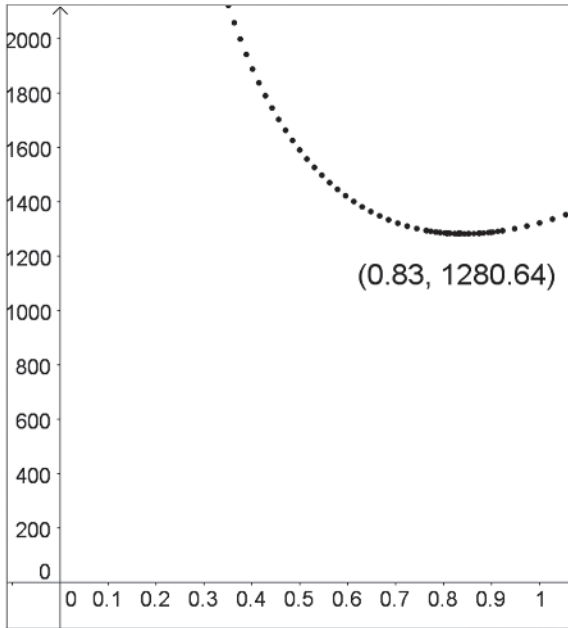


FIGURE 1.38 Plot of cost c against side s .

arrow keys. The step size is set to 1 for points by default, but this can be changed in the properties of the point. On the **Algebra** tab you set the **Increment** to be 0.001.

You have now built a dynamic simulation of the locker case. In Figure 1.38, Graphics view 2 is open where a point (s, c) , having its **Trace On**, has been created. Moving P will now result in a trace, clearly showing the minimum.

1.9 PERSONAL MEASUREMENTS

Measure the length of your right foot, the width of your left hand between your extended thumb and little finger, your weight, your body length, and your foot length. Also write down whether you are female or male. Collect these data from all students in your class. Try to find relationships between some of these measurement values. With five measured values you can use two of them in 10 different ways. You may for these 10 different investigations select to study all females, all males, or the whole class, altogether 30 different combinations. If you work two and two, it will be enough to do two investigations per group. What do you find?

1.10 HEIGHT OF THE BODY

Measure your body height, your leg length, the height of your belly button above the floor, the length of your arm from your armpit to your middle fingertip. Then do the same measurement with 10 of your friends. What relationships can be found? Is it true

that the length relates to the belly button according to the golden section, 1.618 ...? What about arm length and the length of the forearm? Also control if there is a relationship between length and shoe size.

1.11 LAMP POLE

Photograph lamp poles or fence poles with equal distances on a straight road. Carefully note from where you are taking the photo and the distance between the poles. Then measure the number of pixels from the edge to of the photo to every pole. Calculate the actual distance to every pole and make a diagram of the relationship between your two measurements. Can you find a point in the photo that is exactly 143 meters from your camera?

1.12 THE SKYSCRAPER

At <http://www.flickr.com/photos/aaronescobar/2178776113/> there is a photo of a skyscraper in Miami where you cannot see the ground level. Is there a way for you to measure the number of floors in the building directly from the photo?

1.13 THE FENCE

At <http://www.flickr.com/photos/lesphotosdejerome/3899807799/> there is a photo of a woman at a fence. Is it possible for you to cut out the woman, make her twice as large, and place her at the right position in the photo according to her new size?

1.14 THE CORRIDOR

At <http://www.flickr.com/photos/ell-r-brown/5128688426/> there is a photo of a corridor with stone slabs and wall panels. Make an assumption regarding the distance to the first stone slab and the closest wall panel and then determine the length of the corridor and the length of the left wall.

1.15 BIRD FEEDERS

An ordinary cylindrical bird feeder with dual eating holes and perches at the bottom, two-thirds down and one-third down takes 8 hours to half empty. How full was it two hours after it was filled? When will it be completely empty?

If you were to construct a bird feeder with the same number of holes and perches that had the property that it was half emptied in exactly half the time it is completely empty, how would you construct it, meaning where would you place the holes and perches?

1.16 GOLF

An average golf player is out practicing how to play toward green. He is particularly interested in one specific part of the amazing golf course of the club in which he is a member. The green is small and circular with only about a 5 meter radius and is situated on the top of a high-rise hotel about 30 meters higher than the surrounding flat fairway.

Any golfer who wants to go for the green has to shoot the ball about 90 meters in such a way that it stays on the green. The way the shot should be executed depends on the wind. If there was no wind, a good shot toward green should be steep and short, with a good backspin to make the ball stop. Nevertheless, there is almost always a steady headwind at this part of the golf course when the balls and the players reach the fairway close to the hotel. So the player wants to have a low pinch that will stop dead in the wind above the platform and just drop down on the green. This will require a really strong backspin to make the ball “balloon.”

Golf shots are difficult to model, but it can be done. One freeware program is called TrajectoWare Drive 1.0 and can be downloaded (2.5 MB) by registering at <http://www.trajectoware.com/>.

Another possibility is to download the CDF player (690 MB) from <http://demonstrations.wolfram.com/download-cdf-player.html> and then investigate the Wolfram Demonstration Projects.

Trajectory of a golf ball at <http://demonstrations.wolfram.com/TrajectoryOfAGolfShot/>

Flight of a golf ball at <http://demonstrations.wolfram.com/FlightOfAGolfBall/>

Study either or several of these simulation models of golf shots. Find reasonable parameters that will enable the golfer to succeed with the shot. Take measurements of the trajectories and plot the points in GeoGebra.