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Preliminaries from differential topology and microlocal analysis

Here we collect some facts concerning manifolds of jets, spaces of smooth maps and transversality, as well as some material from microlocal analysis. A special emphasis is given to the definition and main properties of the generalized bicharacteristics of the wave operator and the corresponding generalized geodesics.

1.1 Spaces of jets and transversality theorems

We begin with the notion of transversality, manifolds of jets and spaces of smooth maps. The reader is referred to Golubitsky and Guillemin [GG] or Hirsch [Hir] for a detailed presentation of this material.

In this book **smooth** means C^∞.

Let X and Y be smooth manifolds and let $f : X \longrightarrow Y$ be a smooth map. Given $x \in X$, we will denote by $T_x f$ the *tangent map* of f at x. Sometimes we will use the notation $d_x f = T_x f$. If $\mathrm{rank}(T_x f) = \dim(X) \leq \dim(Y)$ (resp. $\mathrm{rank}(T_x f) = \dim(Y) \leq \dim(X)$), then f is called an *immersion* (resp. a *submersion*) at x. Let W be a smooth submanifold of Y. We will say that f is *transversal* to W at $x \in X$, and will denote this by $f \pitchfork_x W$, if either $f(x) \notin W$ or $f(x) \in W$ and $\mathrm{Im}(T_x f) + T_{f(x)} W = T_{f(x)} Y$. Here for every $y \in W$ we identify $T_y W$ with its image under the map $T_y i : T_y W \longrightarrow T_y W$, where $i : W \longrightarrow Y$ is the inclusion. Clearly, if f is a submersion at x, then $f \pitchfork W$ for every submanifold W of Y. If $Z \subset X$ and $f \pitchfork W$

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for every $x \in Z$, we will say that f is transversal to W on Z. Finally, if f is transversal to W on the whole X, we will say that f is transversal to W and write $f \pitchfork W$.

The next proposition contains a basic property of transversality that will be used several times throughout.

Proposition 1.1.1: *Let $f : X \longrightarrow Y$ be a smooth map, and let W be a smooth submanifold of Y such that $f \pitchfork W$. Then $f^{-1}(W)$ is a smooth submanifold of X with*

$$\text{codim } (f^{-1}(W)) = \text{codim } (W). \tag{1.1}$$

In particular:

(a) if $\dim(X) < \text{codim } (W)$, then $f^{-1}(W) = \emptyset$, that is $f(X) \cap W = \emptyset$.

(b) if $\dim(X) = \text{codim } (W)$, then $f^{-1}(W)$ consists of isolated points in X.

Consequently, if f is a submersion, then for every submanifold W of Y, $f^{-1}(W)$ is a submanifold of X with (1.1). Thus, in this case, $f^{-1}(y)$ is a submanifold of X of codimension equal to $\dim(Y)$ for every $y \in Y$.

Let again X and Y be smooth manifolds and let $x \in X$. Given two smooth maps $f, g : X \longrightarrow Y$, we will write $f \sim_x g$ if $d_x f = d_x g$. For an integer $k \geq 2$, we will write $f \sim_x^k g$ if for the smooth maps $df, dg : TX \longrightarrow TY$, we have $df \sim_\xi^{k-1} dg$ for every $\xi \in T_x X$. In this way by induction one defines the relation $f \sim_x^k g$ for all integers $k \geq 1$. Fix for a moment $x \in X$ and $y \in Y$. Denote by $J_k(X, Y)_{x,y}$ the family of all equivalence classes of smooth maps $f : X \longrightarrow Y$ with $f(x) = y$ with respect to the equivalence relation $\sim_x^k$. Define the *space of k-jets* by

$$J^k(X, Y) = \bigcup_{(x,y) \in X \times Y} J^k(X, Y)_{x,y}.$$

So, for each k-jet $\sigma \in J^k(X, Y)$, there exist $x \in X$ and $y \in Y$ with $\sigma \in J^k(X, Y)_{x,y}$. We set $\alpha(\sigma) = x$ and $\beta(\sigma) = y$, thus obtaining maps

$$\alpha : J^k(X, Y) \longrightarrow X, \quad \beta : J^k(X, Y) \longrightarrow Y, \tag{1.2}$$

called the *source* and the *target* map, respectively. Given an arbitrary smooth $f : X \longrightarrow Y$, let

$$j^k f : X \longrightarrow J^k(X, Y) \tag{1.3}$$

be the map assigning to every $x \in X$ the equivalence class $j^k f(x)$ of f in $J^k(X, Y)_{x,f(x)}$.

There is a natural structure of a smooth manifold on $J^k(X, Y)$ for every k. We refer the reader to [GG] or [Hir] for its description and main properties. Let us only mention that with respect to this structure for every smooth map f the maps (1.2) and (1.3) are also smooth.

For a non-empty set A and an integer $s \geq 1$, define

$$A^{(s)} = \{(a_1, \dots, a_s) \in A^s : a_i \neq a_j, 1 \leq i < j \leq s\}.$$

Note that if A is a topological space, then $A^{(s)}$ is an open (dense) subset of the product space A^s. If $f : A \longrightarrow B$ is an arbitrary map, define $f^s : A^s \longrightarrow B^s$ by

$$f^s(a_1, \dots, a_s) = (f(a_1), \dots, f(a_s)),$$

Let X and Y be smooth manifolds, let s and k be natural numbers and let $\alpha^s : (J^k(X,Y))^s \longrightarrow X^s$. The open submanifold

$$J^k_s(X,Y) = (\alpha^s)^{-1}(X^{(s)})$$

of $(J^k(X,Y))^s$ is called an *s-fold k-jet bundle*. For a smooth $f : X \longrightarrow Y$, define the smooth map

$$j^k_s f : X^{(s)} \longrightarrow J^k_s(X,Y)$$

by

$$j^k_s f(x_1, \dots, x_s) = (j^k f(x_1), \dots, j^k f(x_s)).$$

We will now define the Whitney C^k topology on *the space* $C^\infty(X,Y)$ *of all smooth maps* from X into Y. Let $k \geq 0$ be an integer and let U be an open subset of $J^k(X,Y)$. Set

$$M(U) = \{f \in C^\infty(X,Y) : j^k f(X) \subset U\}.$$

The family $\{M(U)\}_U$, where U runs over the open subsets of $J^k(X,Y)$, is the basis for a topology on $C^\infty(X,Y)$, called *the Whitney* C^k *topology*. The supremum of all Whitney C^k topologies for $k \geq 0$ is called *the Whitney* C^∞ *topology*. It follows from this definition that $f_n \to f$ as $n \to \infty$ in the C^∞ topology if $f_n \to f$ in the C^k topology for all $k \geq 0$. Note that if X is not compact (and $\dim(Y) > 0$), then any of the C^k topologies (including the case $k = \infty$) does not satisfy the first axiom of countability, and therefore is not metrizable. On the other hand, if X is compact, then all C^k topologies on $C^\infty(X,Y)$ are metrizable with complete metrics.

In this book we always consider $C^\infty(X,Y)$ with the Whitney C^∞ topology. An important fact about these spaces, which will be often used in what follows, is that whenever X and Y are smooth manifolds, the space $C^\infty(X,Y)$ is a Baire topological space. Recall that a subset R of a topological space Z is called *residual* in Z if R contains a countable intersection of open dense subsets of Z. If every residual subset of Z is dense in it, then Z is called a *Baire space*.

In some of the next chapters we will consider spaces of the form $C^\infty(X,\mathbb{R}^n)$, X being a smooth submanifold of $\mathbb{R}^n$ for some $n \geq 2$. Let us note that these spaces have a natural structure of Frechet spaces. Moreover, if X is compact, then $C^\infty(X,\mathbb{R}^n)$ has a natural structure of a Banach space. Denote by

$$\mathbf{C}(X) = C^\infty_{emb}(X,\mathbb{R}^n)$$

the *subset of* $C^\infty(X, \mathbb{R}^n)$ *consisting of all smooth embeddings* $X \longrightarrow \mathbb{R}^n$. Then $\mathbf{C}(X)$ is open in $C^\infty(X, \mathbb{R}^n)$ (cf. Chapter II in [Hir]), and therefore it is a Baire topological space with respect to the topology induced by $C^\infty(X, \mathbb{R}^n)$. Finally, notice that for compact X the space $\mathbf{C}(X)$ has a natural structure of a Banach manifold. We refer the reader to [Lang] for the definition of Banach manifolds and their main properties.

The following theorem is known as the *multijet transversality theorem* and will be used many times later in this book.

Theorem 1.1.2: *Let X and Y be smooth manifolds, let k and s be natural numbers and let W be a smooth submanifold of $J_s^k(X, Y)$. Then*

$$T_W = \{F \in C^\infty(X, Y) : j_s^k F \pitchfork W\}$$

is a residual subset of $C^\infty(X, Y)$. Moreover, if W is compact, then T_W is open in $C^\infty(X, Y)$.

For $s = 1$, this theorem coincides with Thom's transversality theorem.

We conclude this section with a special case of the Abraham transversality theorem which will be used in Chapter 6. Now by a smooth manifold we mean a smooth Banach manifold of finite or infinite dimension (cf. [Lang]).

Let $\mathcal{A}$, X and Y be smooth manifolds, and let

$$\rho : \mathcal{A} \longrightarrow C^\infty(X, Y) \tag{1.4}$$

be a map, $\mathcal{A} \ni a \mapsto \rho_a$. Define

$$\mathrm{ev}_\rho : \mathcal{A} \times X \longrightarrow Y \tag{1.5}$$

by $\mathrm{ev}_\rho(a, x) = \rho + a(x)$.

The next theorem is a special case of Abraham's transversality theorem (see [AbR]).

Theorem 1.1.3: *Let ρ have the form (1.4) and let W be a smooth submanifold of Y.*

(a) If the map (1.5) is C^1 and K is a compact subset of X, then

$$\mathcal{A}_{K,W} = \{a \in \mathcal{A} : \rho_a \pitchfork_x W, x \in K\}$$

is an open subset of $\mathcal{A}$.

(b) Let $\dim(X) = n < \infty$, $\mathrm{codim}\ (W) = q < \infty$ *and let r be a natural number with $r > n - q$. Suppose that the manifolds $\mathcal{A}$, X and Y satisfy the second axiom of countability, the map (1.5) is C^r and* $\mathrm{ev}_\rho \pitchfork W$. *Then*

$$\mathcal{A}_W = \{a \in \mathcal{A} : \rho_a \pitchfork W\}$$

is a residual subset of $\mathcal{A}$.

1.2 Generalized bicharacteristics

Our aim in this section is to define the generalized bicharacteristics of the *wave operator*

$$\Box = \partial_t^2 - \Delta_x$$

and to present their main properties which will be used throughout the book. Here we use the notation from Section 24 in [H3]. In what follows Ω is a closed domain in $\mathbb{R}^{n+1}$ with a smooth boundary $\partial\Omega$.

Given a point on $\partial\Omega$, we choose local normal coordinates

$$x = (x_1, \ldots, x_{n+1}), \quad \xi = (\xi_1, \ldots, \xi_{n+1})$$

in $T^*(\mathbb{R}^{n+1})$ about it such that the boundary $\partial\Omega$ is given by $x_1 = 0$ and Ω is locally defined by $x_1 \geq 0$. We assume that the coordinates ξ_i are those dual to x_i. The coordinates x, ξ can be chosen so that the *principal symbol* of $\Box$ has the form

$$p(x, \xi) = \xi_1^2 - r(x, \xi'),$$

where

$$x' = (x_2, \ldots, x_{n+1}), \quad \xi' = (\xi_2, \ldots, \xi_{n+1}),$$

and $r(x, \xi')$ is homogeneous of order 2 in ξ'. Introduce the sets

$$\begin{aligned}
\Sigma &= \{(x, \xi) \in T^*\mathbb{R}^{n+1} \setminus \{0\} : p(x, \xi) = 0\}, \\
\Sigma_0 &= \{(x, \xi) \in T^*\mathbb{R}^{n+1} : x_1 > 0\}, \\
H &= \{(x, \xi) \in \Sigma : x_1 = 0, r(0, x', \xi') > 0\}, \\
G &= \{(x, \xi) \in \Sigma : x_1 = 0, r(0, x', \xi') = 0\}.
\end{aligned}$$

The sets Σ, H and G are called the *characteristic set*, the *hyperbolic set* and the *glancing set*, respectively. Let

$$r_0(x', \xi') = r(0, x', \xi'), \quad r_1(x', \xi') = \frac{\partial r}{\partial x_1}(0, x', \xi').$$

The *diffractive* and the *gliding* sets are defined by

$$\begin{aligned}
G_d &= \{(x, \xi) \in G : r_1(x', \xi') > 0\}, \\
G_g &= \{(x, \xi) \in G : r_1(x', \xi') < 0\},
\end{aligned}$$

respectively.

Next, consider the Hamiltonian vector fields

$$H_p = \sum_{j=1}^{n+1} \left(\frac{\partial p}{\partial \xi_j} \cdot \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \cdot \frac{\partial}{\partial \xi_j} \right),$$

$$H_{r_0} = \sum_{j=1}^{n+1} \left(\frac{\partial r_0}{\partial \xi_j} \cdot \frac{\partial}{\partial x_j} - \frac{\partial r_0}{\partial x_j} \cdot \frac{\partial}{\partial \xi_j} \right).$$

Notice that $d_\xi p(x, \xi) \neq 0$ on Σ and $d_{\xi'} r_0(x, \xi') \neq 0$ on G, so H_p and H_{r_0} are not radial on Σ and G, respectively. Next, introduce the sets

$$G^k = \{(x, \xi) \in G : r_1 = H_{r_0}(r_1) = \ \dots \ = H_{r_0}^{k-3}(r_1) = 0\}, \quad k \geq 3,$$

$$G^\infty = \bigcap_{k=3}^{\infty} G^k.$$

The above definitions are independent of the choice of local coordinates. Let us mention that if $\partial\Omega$ is given locally by $\varphi = 0$ and Ω by $\varphi > 0$, φ being a smooth function, then

$$H = \{(x, \xi) \in T^*(\mathbb{R} \times \Omega) : p(x, \xi) = 0,\ H_p\varphi(x, \xi) \neq 0\},$$

$$G = \{(x, \xi) \in T^*(\mathbb{R} \times \Omega) : p(x, \xi) = 0,\ H_p\varphi(x, \xi) = 0\},$$

$$G_d = \{(x, \xi) \in G : H_p^2\varphi(x, \xi) > 0\},$$

$$G_g = \{(x, \xi) \in G : H_p^2\varphi(x, \xi) < 0\},$$

$$G^k = \{(x, \xi) \in G : H_p^j\varphi(x, \xi) = 0,\ 0 \leq j < k\}.$$

We define the generalized bicharacteristics of $\Box$ using the special coordinates (x, ξ) chosen above.

Definition 1.2.1: Let I be an open interval in $\mathbb{R}$. A curve

$$\gamma : I \longrightarrow \Sigma \tag{1.6}$$

is called a *generalized bicharacteristic* of $\Box$ if there exists a discrete subset B of I such that the following conditions hold:

(i) If $\gamma(t_0) \in \Sigma_0 \cup G_d$ for some $t_0 \in I \setminus B$, then γ is differentiable at t_0 and

$$\frac{d}{dt}\gamma(t_0) = H_p(\gamma(t_0)).$$

(ii) If $\gamma(t_0) \in G \setminus G_d$ for some $t_0 \in I \setminus B$, then $\gamma(t) = (x_1(t), x'(t), \xi_1(t), \xi'(t))$ is differentiable at t_0 and

$$\frac{dx_1}{dt}(t_0) = \frac{d\xi_1}{dt}(t_0) = 0, \quad \frac{d}{dt}(x'(t), \xi'(t))_{|t=t_0} = H_{r_0}(\gamma(t_0)).$$

(iii) If $t_0 \in B$, then $\gamma(t_0) \in \Sigma_0$ for all $t \neq t_0$, $t \in I$, with $|t - t_0|$ sufficiently small. Moreover, for $\xi_1^{\pm}(x', \xi') = \pm\sqrt{r_0(x', \xi')}$, we have

$$\lim_{t \to t_0, \pm(t - t_0) > 0} \gamma(t) = (0, x'(t_0), \xi_1^{\pm}(x'(t_0)), \xi'(t_0)) \in H.$$

This definition does not depend on the choice of the local coordinates. Note that when $\partial\Omega$ is given by $\varphi = 0$ and Ω by $\varphi > 0$, then the condition (ii) means that if $\gamma(t_0) \in G \setminus G_d$, then

$$\frac{d\gamma}{dt}(\gamma(t_0)) = H_p^G(\gamma(t_0)),$$

where

$$H_p^G = H_p + \frac{H_p^2 \varphi}{H_\varphi^2 p}\, H_\varphi$$

is the so-called *glancing vector field* on G.

It follows from the above definition that if (1.6) is a generalized bicharacteristic, the functions $x(t)$, $\xi'(t)$, $|\xi_1(t)|$ are continuous on I, while $\xi_1(t)$ has jump discontinuities at any $t \in B$. The functions $x'(t)$ and $\xi'(t)$ are continuously differentiable on I and

$$\frac{dx'}{dt} = -\frac{\partial r}{\partial \xi'}, \quad \frac{d\xi'}{dt} = \frac{\partial r}{\partial x'}. \tag{1.7}$$

Moreover, for $t \in B$, $x_1(t)$ admits left and right derivatives

$$\frac{d^{\pm} x_1}{dt}(t) = \lim_{\epsilon \to +0} \pm \frac{x_1(t \pm \epsilon) - x_1(t)}{\epsilon} = 2\xi_1(t \pm 0). \tag{1.8}$$

The function $\xi_1(t)$ also has a left derivative and a right derivative. For $\gamma(t) \notin G_g$, we have

$$\frac{d^{\pm}\xi_1}{dt}(t) = \lim_{\epsilon \to +0} \pm \frac{\xi_1(t \pm \epsilon) - \xi_1(t)}{\epsilon} = \frac{\partial r}{\partial x_1}(x(t), \xi'(t)), \tag{1.9}$$

while $\frac{d^{\pm}\xi_1}{dt}(t) = 0$ for $\gamma(t) \in G_g$. Thus, if $\gamma(t)$ remains in a compact set, then the functions $x(t), \xi'(t), \xi_1^2(t)$ and $x_1(t)\xi_1(t)$ satisfy a uniform Lipschitz condition. For the left and right derivatives of $|\xi_1(t)|$, one gets

$$\left| \frac{d^{\pm}|\xi_1(t)|}{dt} \right| \leq \left| \frac{\partial r}{\partial x_1}(x(t), \xi'(t)) \right|. \tag{1.10}$$

Melrose and Sjöstrand [MS2] (see also Section 24 in [H3]) showed that for each $z_0 \in \Sigma$, there exists a generalized bicharacteristic (1.6) of $\Box$ with $\gamma(t_0) = z_0$ for some $t_0 \in I$. Since the vector fields H_p and H_p^G are not radial on Σ and G, respectively, such a bicharacteristic γ can be extended for all $t \in \mathbb{R}$. However, in general, γ is not unique. We refer the reader to [Tay] or [H3] for examples demonstrating this.

For $\rho \in \Sigma$, denote by $C_t(\rho)$ the set of those $\mu \in \Sigma$ such that there exists a generalized bicharacteristic (1.6) with $0, t \in I$, $\gamma(0) = \rho$ and $\gamma(t) = \mu$. In many cases $C_t(\rho)$ is related to a uniquely determined bicharacteristic γ. In the general case it is convenient to introduce the following.

Definition 1.2.2: A generalized bicharacteristic $\gamma : \mathbb{R} \longrightarrow \Sigma$ of $\Box$ is called *uniquely extendible* if for each $t \in \mathbb{R}$, the only generalized bicharacteristics (up to a change of parameter) passing through $\gamma(t)$ is γ. That is, for $\rho = \gamma(0)$, we have $C_t(\rho) = \{\gamma(t)\}$ for all $t \in \mathbb{R}$.

It was proved by Melrose and Sjöstrand [MS1] that if $\mathrm{Im}(\gamma) \subset \Sigma \setminus G^\infty$, then γ is uniquely extendible. If $z_0 = \gamma(t_0) \in H$ for some $t_0 \in B$, then $\gamma(t)$ meets $\partial\Omega$ transversally at $x(t_0)$ and (iii) holds. For $z_0 \in \Sigma_0 \cup G_d$ we have $\gamma(t) \in \Sigma_0$ for $|t - t_0|$ small enough, while in the case $z_0 \in G_g$ for small $|t - t_0|$, $\gamma(t)$ coincides with the gliding ray

$$\gamma_0(t) = (0, x'(t), 0, \xi'(t)), \tag{1.11}$$

where $(x'(t), \xi(t))$ is a null bicharacterstic of the Hamiltonian vector field H_{r_0}.

To discuss the local uniqueness of generalized bicharacteristics, let $\gamma(t) = (x(t), \xi(t))$ be such a bicharacteristic and let $y'(t), \eta'(t))$ be the solution of the problem

$$\begin{cases} \dfrac{dy'}{dt}(t) = \dfrac{\partial r_0}{\partial \xi'}(y'(t), \eta'(t)), \\ \dfrac{d\eta'}{dt}(t) = -\dfrac{\partial r_0}{\partial x'}(y'(t), \eta'(t)), \\ y'(0) = x'(0), \;\; \eta'(0) = \xi'(0). \end{cases} \tag{1.12}$$

Then setting $e(t) = r_1(y'(t), \eta'(t))$, we have the following local description of γ.

Proposition 1.2.3: *Let $\gamma(0) \in G^3$. If $e(t)$ increases for small $t > 0$, then for such t the bicharateristic $\gamma(t)$ is a trajectory of H_p. If $e(t)$ decreases for $0 \le t \le T$, then for such t, $\gamma(t)$ is a gliding ray of the form (1.11).*

A proof of this proposition and some other properties of generalized bicharacteristics can be found in Section 24.3 in [H3].

It should be mentioned that for $k \ge 3$ and $\gamma(0) \in G^k \setminus G^{k+1}$, we have

$$e(t) = \frac{1}{2(k-2)!} H_p^k \varphi(\gamma(0))\, t^{k-2} + \mathrm{O}(t^{k-1}),$$

therefore the sign of $H_p^k \varphi(\gamma(0))$ determines the local behaviour of $e(t)$.

Corollary 1.2.4: *In each of the following cases, every generalized bicharacteristic of $\Box$ is uniquely extendible:*

(a) the boundary $\partial\Omega$ is a real analytic manifold;

(b) there are no points $y \in \partial\Omega$ at which the normal curvature of $\partial\Omega$ vanished of infinite order in some direction $\xi \in T_y(\partial\Omega)$;

(c) $\partial\Omega$ is given locally by $\varphi = 0$ and

$$H_p^2\varphi(z) \leq 0 \tag{1.13}$$

for every $z \in G$. If $\partial\Omega$ is locally convex in the domain of φ, then (1.13) holds.

Proof: In the case (a) the symbols $r_0(x', \xi')$ and $r_1(x', \xi')$ are real analytic, so the solution $(y'(t), \eta'(t))$ of (1.12) is analytic in t. Consequently, the function $e(t)$ is analytic and we can use its Taylor expansion in order to apply Proposition 1.2.3.

In the case (c), using the special coordinates x, ξ, and combining (1.13) with (1.9), we get $\frac{d^{\pm}\xi_1}{dt}(t) \geq 0$. On the other hand, if $\xi_1(t)$ has a jump at $\gamma(t) \in H$, then this jump is equal to $2r_0(x'(t), \xi'(t)) > 0$. Thus, the function $\xi_1(t)$ is increasing. If $e(t) = 0$ for $0 \leq t \leq t_0$, we get $x_1(t) = \xi_1(t) = 0$ for such t, so $\{\gamma(t) : 0 \leq t \leq t_1\}$ is a gliding ray. Assume that there exists a sequence $t_k \searrow 0$ such that $e(t_k) \neq 0$ for all $k \geq 1$. Then $\xi_1(t) > 0$ for all sufficiently small $t > 0$. Now (1.8) shows that $x_1(t)$ is increasing for such t, therefore there is $t' > 0$ such that $\{\gamma(t) : 0 \leq t \leq t'\}$ coincides with a trajectory of H_p.

Let $p = \sum_{j=1}^{n} \xi_j^2 - \xi_{n+1}^2$ and let φ depend on $x_1, \ldots, x_n$ only. Then

$$(H_p^2\varphi)(x, \xi) = 4 \sum_{i,j=1}^{n} \frac{\partial^2\varphi}{\partial x_i\, \partial x_j}(x)\xi_i\xi_j,$$

and if the boundary $\partial\Omega$ is locally convex, we obtain (1.13).

Finally, in the case (b), for each $x \in \partial\Omega$ there exists a multi-index α, depending on x, such that $(\partial^\alpha\varphi)(x) \neq 0$. This implies $G^\infty = \emptyset$, which completes the proof. ■

According to Lemma 6.1.2, in the generic case discussed in Chapter 6 the assumption (b) is always satisfied.

Let $Q = \Omega \times \mathbb{R}$. We will again use the coordinates $x = (x_1, \ldots, x_{n+1})$, this time denoting the last coordinate by t, that is $t = x_{n+1}$. For $x \in \partial Q = \partial\Omega \times \mathbb{R}$, let $N_x(\partial Q)$ be the space of covectors $\xi \in T_x^*Q$ vanishing on $T_x(\partial Q)$. Define the equivalence relation $\sim$ on T^*Q by $(x, \xi) \sim (y, \eta)$ if and only if either $x = y \in Q \setminus \partial Q$ and $\xi = \eta$, or $x = y \in \partial Q$ and $\xi - \eta \in N_x(\partial Q)$. Then $T^*Q/\sim$ can be naturally identified over ∂Q with $T^*(\partial Q)$. Consider the map

$$\sim : T^*Q \ni (x, \xi) \mapsto (x, \xi_{|T_x(\partial Q)}) \in T^*(\partial Q),$$

defined as the identity on $T^*(Q \setminus \partial Q)$. Then $\widetilde{\Sigma} = \Sigma_b$ is called the *compressed characteristic set*, while the image $\tilde{\gamma}$ of a bicharacteristic γ under $\sim$ is called a *compressed generalized bicharacteristic*. Clearly $\tilde{\gamma}$ is a continuous curve in Σ_b.

Given $\rho = (x, \xi)$, $\mu = (y, \eta) \in T^*Q$, denote by $d(\rho, \mu)$ the standard Euclidean distance between ρ and μ. For $\rho, \mu \in \Sigma$ define

$$D(\rho, \mu) = \inf_{\nu', \nu'' \in \Sigma, \nu \sim \nu''} (\min\{d(\rho, \mu), d(\rho, \nu') + d(\nu'', \mu)\}).$$

Clearly, $D(\rho, \mu) = 0$ if and only if $\rho \sim \mu$, and $D(\rho, \mu) = D(\rho', \mu')$ provided $\rho \sim \rho'$ and $\mu \sim \mu'$. It is easy to check that D is symmetric and satisfies the triangle inequality. Thus, D is a pseudo-metric on Σ, which induces a metric on Σ_b.

For the next lemma we assume that I is a closed non-trivial interval in $\mathbb{R}$, $(y_0, \eta_0) \in \Sigma$ and Γ is a neighbourhood of (y_0, η_0) in Q.

Lemma 1.2.5: *There exists a constant $C_0 > 0$ depending only on Γ and I such that for every generalized bicharacteristic $\gamma : I \longrightarrow \Sigma \cap \gamma$ we have*

$$D(\gamma(t), \gamma(s)) \leq C_0 |t - s|$$

for all $t, s \in I$.

Proof: It is enough to consider the case when $|t - s|$ is small. Then we can use the local coordinates introduced earlier. From (1.7), (1.8) and (1.10), we get

$$|x(t) - x(s)| + |\xi'(t) - \xi'(s)| \leq C_1 |t - s|, \quad ||\xi_1(t)| - |\xi_1(s)|| \leq C_1 |t - s|,$$

where $C_1 > 0$ is a constant independent of t and s. Thus, if $\xi_1(t) = 0$ or $\xi_1(s) = 0$ we get $|\xi_1(t) - \xi_1(s)| \leq C_1 |t - s|$. The latter holds also in the case when $\gamma(t') \notin \partial\Omega$ for all $t' \in (t, s)$. Consequently, $D(\gamma(t), \gamma(s)) \leq C_2 |t - s|$ whenever either $\xi_1(t)\xi_1(s) = 0$ or $\gamma(t') \in \partial\Omega$ only for finitely many $t' \in (t, s)$.

Assume that there are infinitely many $t' \in (t, s)$ such that $\gamma(t')$ is a reflection point of γ. Then there exists $u \in [t, s]$ with $\gamma(u) \in G$. Hence,

$$D(\gamma(t), \gamma(u)) \leq C_2 |t - u|, \quad D(\gamma(u), \gamma(s)) \leq C_2 |u - s|,$$

and using the triangle inequality for D, we complete the proof of the assertion. ■

The next lemma shows that any sequence of generalized bicharacteristics has a subsequence that is convergent on a given compact interval.

Lemma 1.2.6: *Let $I = [a, b]$ be a compact interval in $\mathbb{R}$, let K be a compact subset of Σ and let $\gamma^{(k)}(t) = (x^{(k)}(t), \xi^{(k)}(t)) : I \longrightarrow K \subset \Sigma$ be a generalized bicharacteristic of $\Box$ for every natural number k. Then there exists an infinite sequence $k_1 < k_2 < \dots$ of natural numbers and a generalized bicharacteristic $\gamma(t) = (x(t), \xi(t)) : I \longrightarrow \Sigma$ such that*

$$\lim_{m \to \infty} D(\gamma^{(k_m)}(t), \gamma(t)) = 0 \tag{1.14}$$

for all $t \in I$.

Proof: Using local coordinates, we see that the derivatives of $(x^{(k)})'(t)$ and $(\xi^{(k)})'(t)$ and the left and right derivatives of $x_1^{(k)}(t)$ and $\xi_1^{(k)}(t)$ are uniformly bounded for $t \in I$ and $k \geq 1$. Hence the maps $x^{(k)}(t)$, $(\xi^{(k)})'(t)$, $x_1^{(k)}(t)\xi_1^{(k)}(t)$ and $(\xi_1^{(k)}(t))^2$ are uniformly Lipschitz, which implies that there exists an infinite sequence $k_1 < k_2 < \ldots$ of natural numbers such that the sequences $x^{(k_m)}(t)$, $(\xi^{(k_m)})'(t)$, $(\xi_1^{(k_m)}(t))^2$ and $x_1^{(k_m)}(t), \xi_1^{(k_m)}(t)$ are uniformly convergent for $t \in I$. It now follows from Proposition 24.3.12 in [H3] that there exists a generalized bicharacteristic $\gamma(t) : I \to \Sigma$ of $\Box$ such hat

$$\lim_{m \longrightarrow \infty} \gamma^{(k_m)}(t) = \gamma(t) \tag{1.15}$$

for all $t \in I$ with $\gamma(t) \notin H$.

Let $t' \in I$ be such that $\gamma(t')$ is a reflection point of γ. Then there exists a sequence $t_j \to t'$ with $\gamma(t_j) \in \Sigma_0 \cup G$ for all j. Thus,

$$\begin{aligned} &D(\gamma^{(k_m)}(t'), \gamma(t')) \\ &\leq D(\gamma^{(k_m)}(t'), \gamma^{(k_m)}(t_j)) + D(\gamma(t_j), \gamma(t')) + D(\gamma^{(k_m)}(t_j), \gamma(t_j)). \end{aligned}$$

By Lemma 1.2.5, the first two terms in the right-hand side can be estimated uniformly with respect to m, while for the third term we can use (1.15). Taking j and m sufficiently large, we obtain (1.14), which proves the lemma. ■

In what follows we will use local coordinates $(t, x) \in \mathbb{R} \times \Omega$ and the corresponding local coordinates $(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \Omega)$. In these coordinates the principal symbol p of $\Box$ has the form

$$p(x, \tau, \xi) = \xi_1^2 - q_2(x, \xi') - \tau^2,$$

where $\xi' = (\xi_2, \ldots, \xi_n)$ and $q_2(x, \xi')$ is homogeneous of order 2 in ξ'. Consequently, the vector fields H_p and H_p^G do not involve derivatives with respect to τ, so by Definition 1.2.1, the variable τ remains constant along each generalized bicharacteristic. This makes it possible to parametrize every generalized bicharacteristic by the time t.

Given $(y, \eta) \in T^*(\Omega) \setminus \{0\}$, consider the points

$$\mu_\pm = (0, y, \mp|\eta|, \eta) \in \Sigma.$$

Assume that locally $\partial\Omega$ is given by $x_1 = 0$ and Ω by $x_1 \geq 0$. Let μ_+be a hyperbolic point and let $\xi_1^\pm(y', \eta)$ be the different real roots of the equation

$$p(0, y', |\eta|, z, \eta') = 0$$

with respect to z. Denote by γ the generalized bicharacteristic parameterized by a parameter s such that

$$\lim_{s \searrow 0} \gamma(s) = \mu_+.$$

Then $\tau = -|\eta| < 0$ along γ and the time t increases when s increases. Such a bicharacteristic will be called *forward*. For the right derivative of $x_1(t)$ we get

$$\frac{d^+x_1}{dt} = \frac{d^+x_1/ds}{dt/ds} = \frac{\xi_1(+0)}{-\tau} > 0,$$

since for small $t > 0$, $\gamma(t)$ enters the interior of Ω and $x_1(t) > 0$. Therefore, setting

$$\xi_1^{\pm}(y', \eta) = \pm\sqrt{|\eta|^2 + q_2(0, y', \eta')},$$

we find

$$\lim_{s \searrow 0} \xi_1(s) = \xi_1^+(y', \eta).$$

In the case $\mu_+ \in G$ it may happen that there exist several forward bicharacteristic passing through μ_+. Denote by C_+ the set of those

$$(t, x, y; \tau, \xi, \eta) \in T^*(\mathbb{R} \times \Omega \times \Omega) \setminus \{0\}$$

such that $\tau = -|\xi| = -|\eta|$ and (t, x, τ, ξ) and $(0, y, \tau, \eta)$ lie on forward generalized bicharacteristics of $\Box$. In a similar way we define C_- using a *backward bicharacteristic*, determined as the forward ones replacing μ_+ by μ_-. The set $C = C_+ \cup C_-$ is called the *bicharacteristic relation* of $\Box$. If $\mu = (0, y, \tau, \eta) \in H$ and $\tau < 0$ (resp. $\tau > 0$), we will say that μ is a reflection point of a forward (resp. backward) bicharacteristic. Similarly, if $\rho = (t, x, \tau, \xi) \in H$, then ρ is a reflection point of a generalized bicharacteristic passing through $(0, y, \tau, \eta)$, and, working in local coordinates as before, the sign of τ determines uniquely $\xi_1(t + 0)$. The sets $C_\pm$ and C are homogeneous with respect to (τ, ξ, η), that is $(t, x, y, \tau, \xi, \eta) \in C_\pm$ implies $(t, x, y, s\tau, s\xi, s\eta) \in C_\pm$ for all $s \in \mathbb{R}^+$.

Lemma 1.2.7: *The sets $C_\pm$ are closed in $T^*(\mathbb{R} \times \Omega \times \Omega) \setminus \{0\}$.*

Proof: Since C_+ is homogeneous, it is sufficient to show that if

$$C_+ \ni z_k = (t_k, x_k, y_k, -1, \xi_k, \eta_k), \quad |\xi_k| = |\eta_k| = 1$$

for all $k \geq 1$ and there exists

$$\lim_{k \to \infty} z_k = z_0 = (t_0, x_0, y_0, -1, \xi_0, \eta_0),$$

then $z_0 \in C_+$. Let $\gamma^{(k)}(t)$ be a generalized bicharacteristic of $\Box$ such that $(t_k, x_k, -1, \xi_k)$ and $(0, y_k, -1, \eta_k)$ lie on $\mathrm{Im}(\gamma^{(k)})$. If one of these points belongs to H, we consider it as a reflection point of $\gamma^{(k)}$, according to the above-mentioned convention by suitably choosing $\xi_1^{(k)}(t)$. Assume $|t_k| \leq T$. Then there exists a compact set $K \subset \Sigma$ such that $\gamma^{(k)}(t) \in K$ for all $|t| \leq T$, so we can apply the argument in the proof of Lemma 1.2.6. Consequently, there exists an infinite sequence $k_1 < k_2 < \dots$ of natural numbers and a generalized bicharacteristic γ satisfying (1.14) and (1.15). Then for the Euclidean distance d we find

$$d(\gamma^{(k_m)}(t_{k_m}), \gamma(t_0)) \leq d(\gamma^{(k_m)}(t_{k_m}), \gamma^{k_m)}(t_0)) + d(\gamma^{(k_m)}(t_0), \gamma(t_0)).$$

If $\gamma(t_0) \in \Sigma_0 \cup G$, according to (1.15) and the continuity of $x(t)$, $\xi'(t)$ and $|\xi_1(t)|$, we get

$$d(\gamma^{(k_m)}(t_{k_m}), \gamma(t_0)) \to 0 \tag{1.16}$$

as $m \to \infty$, which shows that $z_0 \in C_+$. If $\gamma(t_0) \in H$, then by our convention, $\xi_1(t+0)$ and $\xi_1^{(k_m)}(t+0)$ have the same sign for large m, which implies $z_0 \in C_+$.

Therefore, C_+ is closed. In the same way one proves that C_- is closed as well. ■

Using C_+ we now define the so-called *generalized Hamiltonian flow* $\mathcal{F}_t$ of $\Box$; it is sometimes called the *broken Hamiltonian flow*. Given $(y, \eta) \in T^*\Omega \setminus \{0\}$, set

$$\mathcal{F}_t(y, \eta) = \{(x, \xi) \in T^*\Omega \setminus \{0\} : (t, x, y, -|\eta|, \xi, \eta) \in C_+\}.$$

In general, $\mathcal{F}_t(y, \eta)$ is not a one-point set. Nevertheless, setting

$$\mathcal{F}_t(V) = \{\mathcal{F}_t(y, \eta) : (y, \eta) \in V\}$$

for $V \subset T^*\Omega \setminus \{0\}$, we have the group property

$$\mathcal{F}_{t+s}(y, \eta) = \mathcal{F}_t(\mathcal{F}_s(y, \eta)).$$

The flow generated by C_- is $\mathcal{F}_t(y, -\eta)$.

Let $\partial\Omega$ be locally given by $x_1 = 0$ and let

$$p(x, \tau, \xi) = \xi_1^2 - q_2(x, \xi') - \tau^2$$

be the principal symbol of $\Box$. A point

$$\sigma = (t, x', \tau, \xi') \in T^*(\mathbb{R} \times \partial\Omega) \setminus \{0\}$$

is called *hyperbolic* (resp. *glancing*) for $\Box$ if the equation

$$p(0, x', \tau, \xi_1, \xi') = 0 \tag{1.17}$$

with respect to ξ_1 has two different real roots (resp. a double real root). These definitions are invariant with respect to the choice of the local coordinates. If (1.17) has no real roots, then σ is called an *elliptic* point. Clearly, the set of hyperbolic points is open in $T^*(\mathbb{R} \times \partial\Omega)$, while that of the glancing points is closed.

Let $\pi : T^*(\mathbb{R} \times \Omega) \longrightarrow \Omega$ be the *natural projection*, $\pi(t, x, \tau, \xi) = x$.

Definition 1.2.8: A continuous curve $g : [a, b] \longrightarrow \Omega$ is called a *generalized geodesic* in Ω if there exists a generalized bicharacteristic $\gamma : [a, b] \longrightarrow \Sigma$ such that

$$g(t) = \pi(\gamma(t)), \quad t \in [a, b]. \tag{1.18}$$

Notice that, in general, a generalized geodesic is not uniquely determined by a point on it and the corresponding direction. If the generalized bicharacteristic γ with (1.18) satisfies

$$\gamma(t) \in \Sigma_0 \cup H, \quad t \in [a, b],$$

we will say that g (or $\text{Im}(g)$) is a *reflecting ray* in Ω. Two special kinds of such rays will be studied in detail in Chapter 2. One of them is defined as follows.

Definition 1.2.9: A point $(x, \xi) \in T^*\Omega \setminus \{0\}$ is called *periodic* with *period* $T \neq 0$ if

$$(T, x, x, \pm|\xi|, \xi, \xi) \in C.$$

A generalized bicharacteristic $\gamma(t) = (t, x(t), \tau, \xi(t)) \in \Sigma$, $t \in \mathbb{R}$, will be called *periodic* with *period* $T \neq 0$ if for each $t \in \mathbb{R}$ the point $(x(t), \xi(t))$ is periodic with period T. The projections on Ω of the periodic generalized bicharacteristics of $\Box$ are called *periodic generalized geodesics*.

Notice that if $(T, x, x, -|\xi|, \xi, \xi) \in C_+$, then $(T, x, x, |\xi|, -\xi, -\xi) \in C_-$, since we can change the orientation on the bicharacteristic passing through $(0, x, -|\xi|, \xi)$. A uniquely extendible bicharacteristic γ is periodic provided $\text{Im}(\gamma)$ contains a periodic point. If T is the period of a generalized geodesic g, then $|T|$ coincides with the standard length of the curve $\text{Im}(g)$.

Let $\mathcal{L}_\Omega$ be the *set of all periodic generalized geodesics* in Ω. For $g \in \mathcal{L}_\Omega$ we denote by T_g the *length* of $\text{Im}(g)$. We call *length spectrum* the following set

$$L_\Omega = \{T_g : g \in \mathcal{L}_\Omega\}.$$

Lemma 1.2.10: *The set L_Ω is closed in $\mathbb{R}$ and $0 \notin L_\Omega$.*

Proof: Consider a convergent sequence $\{T_k\}$ of elements of L_Ω converging to some $T_0 \in \mathbb{R}$ as $k \to \infty$. Then for every $k \geq 1$ there exists a generalized bicharacteristic $\gamma^{(k)}$ of $\Box$ with period T_k passing through a point of the form $(0, x_k, -1, \xi_k)$. If $T_0 \neq 0$, choosing a subsequence as in the proof of Lemma 1.2.7, we obtain $T_0 \in L_\Omega$.

It remains to show that the case $T_0 = 0$ is impossible. Assume $T_0 = 0$. Passing to an appropriate subsequence, we may assume that there exists $\lim_{k\to\infty}(x_k, \xi_k) = (x_0, \xi_0)$ and for every t there exists

$$\lim_{k\to\infty} \gamma^{(k)}(t) = \lim_{k\to\infty} (t, x^{(k)}(t), -1, \xi^{(k)}(t)) = \gamma_0(t) = (t, x_0(t), -1, \xi_0(t)),$$

provided $\gamma_0(t) \notin H$ and $|t| \leq T$. If x_0 is in the interior of Ω, then x_k is also in the interior of Ω for large k. Then for such k, $x^{(k)}(t)$ is in the interior of Ω for sufficiently small $t > 0$, which is a contradiction. If there exists t' with $|t'| \leq T$ and $x_0(t')$ in the interior of Ω, then we get a contradiction by the same argument.

It remains to consider the case when $\gamma_0(t) \in G$ for all $t \in [-T, T]$. Then for such t, $\gamma_0(t) = (x_0(t), \xi_0(t))$ is an integral curve of the glancing vector field H_p^G. Since the latter is not radial, $\gamma_0(t)$ has no stationary points for $t \in [-T, T]$. Given a small neighbourhood U of x_0 in $\partial\Omega$, there exist δ_0, δ_1 such that $0 < \delta_0 < \delta_1 \leq T$ and $x_0(t) \notin U$ for $\delta_0 \leq |t| \leq \delta_1$. Since $x^{(k)}(t) \to x_0(t)$ as $k \to \infty$ uniformly for $|t| \leq T$, for sufficiently large k there exists a natural number m_k with

$$\delta_0 \leq m_k T_k \leq \delta_1, \quad x^{(k)}(T_k) = x^{(k)}(m_k T_k).$$

Then $x_0 = \lim_{k\to\infty} x_k = \lim_{k\to\infty} x^{(k)}(T_k) \notin U$, which is a contradiction. This proves that $T_0 \neq 0$ and this completes the proof of the proposition. ■

1.3 Wave front sets of distributions

In this section we collect some basic facts concerning wave fronts of distributions. For more details, we refer the reader to the books of Hörmander [Hl], [H3].

Let X be an open subset of $\mathbb{R}^n$ and let $\mathcal{D}'(X)$ be the *space of all distributions* on X. The *singular support* $\operatorname{sing\,supp}(u)$ of $u \in \mathcal{D}'(X)$ is a closed subset of X such that if $x_0 \notin \operatorname{sing\,supp}(u)$ there exists an open neighbourhood U of x_0 in X and a smooth function $f \in C^\infty(U)$ such that

$$\langle u, \varphi \rangle = \int f(x)\varphi(x)\, dx, \quad \varphi \in C_0^\infty(U).$$

For a more precise analysis of $\operatorname{sing\,supp}(u)$, it is useful to consider the directions $\xi \in \mathbb{R}^n \setminus \{0\}$ along which the *Fourier transform* $\widehat{\varphi u}(\xi)$ of the distribution $\varphi u \in \mathcal{E}'(X)$ is not rapidly decreasing, provided $\varphi \in C_0^\infty(X)$ and $\operatorname{supp}(\varphi) \cap \operatorname{sing\,supp}(u) \neq \emptyset$.

Definition 1.3.1: Let $u \in \mathcal{D}'(X)$ and let $\mathcal{O}$ be the set of all $(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus \{0\}$ for which there exists an open neighbourhood U of x_0 in X and an open conic neighbourhood V of ξ_0 in $\mathbb{R}^n$ so that for $\varphi \in C_0^\infty(U)$ and $\xi \in V$ we have

$$|\widehat{\varphi u}(\xi)| \leq C_m(1 + |\xi|)^{-m}, \quad m \in \mathbb{N}.$$

The closed subset

$$WF(u) = (X \times \mathbb{R}^n) \setminus \{0\}$$

of $X \times \mathbb{R}^n \setminus \{0\}$ is called the *wave front set* of u.

It is easy to see that $WF(u)$ is a conic subset of $X \times \mathbb{R}^n \setminus \{0\}$ with the property

$$\pi(WF(u)) = \text{sing supp}(u),$$

where $\pi : X \times \mathbb{R}^n \longrightarrow X$ is the natural projection.

For our aims in Chapter 3 we will describe the wave front sets of distributions given by oscillatory integrals. Such integrals have the form

$$\int e^{\mathbf{i}\varphi(x,\theta)} a(x,\theta)\, d\theta. \tag{1.19}$$

Here the *phase* $\varphi(x,\theta)$ is a C^∞ real-valued function, defined for $(x,\theta) \in \Gamma \subset X \times (\mathbb{R}^N \setminus \{0\})$, and Γ is an open conic set, i.e. $(x,\theta) \in \Gamma$ implies $(x, t\theta) \in \Gamma$ for all $t > 0$. We assume that φ has the properties:

$$\varphi(x, t\theta) = t\, \varphi(x,\theta), \quad (x,\theta) \in \Gamma, t > 0,$$

$$d_{x,\theta}\varphi(x,\theta) \neq 0, \quad (x,\theta) \in \Gamma.$$

The *amplitude* $a(x,\theta)$ belongs to the class of symbols $S^m(X \times \mathbb{R}^N)$, formed by C^∞ functions on $X \times \mathbb{R}^N$ such that for each compact $K \subset X$ and all multi-indices α, β, we have

$$|\partial^\alpha \partial^\beta a(x,\theta)| \leq C_{\alpha,\beta,K}\ (1 + |\theta|)^{m-|\beta|}, \quad x \in K, \quad \theta \in \mathbb{R}^N. \tag{1.20}$$

We endow $S^m(X \times \mathbb{R}^N)$ with the topology defined by the semi-norms

$$p_{\alpha,\beta,j}(a) = \sup_{x \in K_j, \theta \in \mathbb{R}^N} (1 + |\theta|)^{-m+|\beta|}\ |\partial^\alpha \partial^\beta a(x,\theta)|,$$

where $\{K_j\}$ is an increasing sequence of compact sets with $\cup_{j=1}^\infty K_j = X$.

Let $F \subset \Gamma \cup (X \times \{0\})$ be a closed cone and let $\text{supp}\,(a) \subset F$. For $\psi \in C_0^\infty(X)$ we will now define the integral

$$\int e^{\mathbf{i}\varphi(x,\theta)} a(x,\theta)\ \psi(x)\ dx\ d\theta$$

to obtain a distribution in $\mathcal{D}'(X)$. To do this, we need a regularization, since the integral in θ is not convergent for $m > -N$.

Choose a function $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi(\theta) = 1$ for $|\theta| \leq 1$ and $\chi(\theta) = 0$ for $|\theta| \geq 2$. For $0 < \epsilon \leq 1$, the functions $\chi(\epsilon\theta)$ form a bounded set in $S^0(X \times \mathbb{R}^N)$. Then the functions $a_\epsilon = a(x,\theta)\chi(\epsilon\theta)$ also form a bounded set in $S^0(X \times \mathbb{R}^N)$ and

$$a_\epsilon \to a \in S^{m'}(X \times \mathbb{R}^N)$$

as $\epsilon \to 0$ for each $m' > m$.

Consider the operator

$$L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{N} b_j \frac{\partial}{\partial \theta_j} + \chi$$

with

$$a_j = -\mathbf{i}(1-\chi)\kappa^{-1}\varphi_{x_j}, \quad b_j = -\mathbf{i}(1-\chi)\kappa^{-1}|\theta|^2\varphi_{\theta_j},$$

and $\kappa = |\varphi_x|^2 + |\theta|^2|\varphi_\theta|^2$. For each compact set $K \subset X$ we have

$$\kappa(x,\theta) \geq \delta_K|\theta|^2, \quad x \in K, \quad (x,\theta) \in \Gamma,$$

where $\delta_K > 0$ depends on K only. Clearly

$$L(e^{\mathbf{i}\varphi}) = e^{\mathbf{i}\varphi},$$

and the operator ${}^tL$ formally adjoint to L has the form

$${}^tL = -\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} - \sum_{j=1}^{N} b_j \frac{\partial}{\partial \theta_j} + c$$

with

$$a_j \in S^{-1}(X \times \mathbb{R}^N), \; b_j \in S^0(X \times \mathbb{R}^N), \; c \in S^{-1}(X \times \mathbb{R}^N).$$

The operator ${}^t(L)^k$ is a continuous map of S^m onto S^{m-k}. Define the linear map $I_{\varphi,a} : C_0^\infty(X) \to \mathbb{R}$ by

$$\begin{aligned} I_{\varphi,a}(\psi) &= \lim_{\epsilon\to 0} \int\int e^{\mathbf{i}\varphi(x,\theta)} a(x,\theta)\chi(\epsilon\theta)\psi(x)\; dx\; d\theta \\ &= \lim_{\epsilon\to 0} \int\int e^{\mathbf{i}\varphi(x,\theta)} ({}^tL)^k \;[a(x,\theta)\chi(\epsilon\theta)\psi(x)]\; dx\; d\theta. \end{aligned} \tag{1.21}$$

For $m-k < -N$ the integral on the right-hand side of (1.21) is absolutely convergent, and it is easy to see that $I_{\varphi,a}$ becomes a distribution in $\mathcal{D}'(X)$. Thus, we obtain the following.

Proposition 1.3.2: *Let $\varphi(x,\theta)$ and $a(x,\theta)$ be as above. Then the oscillatory integral (1.19) defines a distribution $I_{\varphi,a}$ given by (1.21).*

We are now going to describe the set $WF(I_{\varphi,a})$.

Theorem 1.3.3: *We have*

$$WF(I_{\varphi,a}) \subset \{(x, \varphi_x(x,\theta)) : (x,\theta) \in F, \;\; \varphi_\theta(x,\theta) = 0\}. \tag{1.22}$$

Proof: Let $f \in C_0^\infty(X)$. Then the Fourier transform

$$\widehat{fI_{\varphi,a}}(\xi) = \int\int e^{\mathbf{i}(\varphi(x,\theta)-\langle x,\xi\rangle)}a(x,\theta)f(x)\,dx\,d\theta$$

is expressed by an oscillatory integral. Let V be a closed cone in $\mathbb{R}^N$ such that

$$V \cap \{\varphi_x(x,\theta) : (x,\theta) \in F,\ x \in \text{supp}\,(f),\ \varphi_\theta(x,\theta) = 0\} = \emptyset.$$

By compactness, there exists $\delta > 0$ such that

$$\mu = |\xi - \varphi_x(x,\theta)|^2 + |\theta|^2|\varphi_\theta(x,\theta)|^2 \geq \delta(|\theta| + |\xi|)^2 \tag{1.23}$$

for $(x,\theta) \in F$, $x \in \text{supp}\,(f)$ and $\xi \in V$. To obtain (1.23) it suffices to observe that if the latter conditions are satisfied, then the left-hand side of (1.23) is positive and then use the homogeneity with respect to (θ,ξ). As above, consider the operator

$$L = \sum_{j=1}^n a_j\frac{\partial}{\partial x_j} + \sum_{j=1}^N b_j\frac{\partial}{\partial\theta_j} + \chi$$

with

$$a_j = -\frac{\mathbf{i}(1-\chi)}{\mu}(\varphi_{x_j} - \xi_j), \quad b_j = -\frac{\mathbf{i}(1-\chi)}{\mu}|\theta|^2\varphi_{\theta_j}.$$

Then

$$\widehat{fI_{\varphi,a}}(\xi) = \lim_{\epsilon\to 0}\int\int e^{\mathbf{i}(\varphi(x,\theta)-\langle x,\xi\rangle)}({}^tL)^k[a(x,\theta)\chi(\epsilon\theta)f(x)]\,dx\,d\theta,$$

and applying (1.23), we conclude that

$$|\widehat{fI_{\varphi,a}}(\xi)| \leq C_N(1+|\xi|)^{-N}, \quad \xi \in V.$$

This implies (1.22). ∎

For asymptotics of oscillatory integrals depending on a parameter $\lambda \in \mathbb{R}$ we have the following.

Lemma 1.3.4: *Let $u \in \mathcal{D}'(X)$, $f \in C_0^\infty(X)$ and let $\varphi \in C_0^\infty(X)$ be a real-valued function. Assume*

$$WF(u) \cap \{(x,\varphi_x) : x \in \text{supp}\,(f)\} = \emptyset.$$

Then for each $m \in \mathbb{N}$ we have

$$|\langle u, f(x)e^{\mathbf{i}\lambda\varphi(x)}\rangle| \leq C_m(1+|\lambda|)^{-m}, \quad \lambda \in \mathbb{R}.$$

Proof: Choosing a finite partition of unity, we can restrict our attention to the case $u \in \mathcal{E}'(X)$. Set

$$\Sigma_f = \{\xi \in \mathbb{R}^n \setminus \{0\} : \exists x \in \text{supp}\,(f) \text{ with } (x,\xi) \in WF(u)\}.$$

Then

$$\begin{aligned}\langle u, f(x)e^{\mathbf{i}\lambda\varphi(x)}\rangle &= (2\pi)^{-n}\int\int e^{\mathbf{i}(\langle x,\xi\rangle - \lambda\varphi(x))} f(x)\hat{u}(\xi)\;dx\;d\xi \\ &= \int_X\int_W + \int_X\int_{\mathbb{R}^n\setminus W} = I_1(\lambda) + I_2(\lambda).\end{aligned}$$

Here W is a closed conic set such that $\Sigma_f \subset W$,

$$W \cap \{\varphi_x(x) : x \in \text{supp}\,(f)\} = \emptyset,$$

and $I_1(\lambda)$ is interpreted as an oscillatory integral. For $x \in \text{supp}\,(f)$ and $\xi \in W$ we have

$$|\xi - \lambda\varphi_x(x)| \geq \delta(|\xi| + |\lambda|), \quad \lambda \in \mathbb{R},$$

with $\delta > 0$. Using the same argument as in the proof of Theorem 1.3.3, we see that $I_1(\lambda) = O(|\lambda|^{-m})$ for all $m \in \mathbb{N}$. For $I_2(\lambda)$ we use the fact that if $\xi \in \mathbb{R}^n \setminus W$ and $\text{supp}\,(u) \cap \text{supp}\,(f) \neq \emptyset$, then $\hat{u}(\xi)$ is rapidly decreasing. This proves the assertion. ■

Now let $\Gamma \subset X \times \mathbb{R}^n \setminus \{0\}$ be a closed conic set. Set

$$\mathcal{D}'_\Gamma(X) = \{u \in \mathcal{D}'(X) : WF(u) \subset \Gamma\}.$$

Using an argument similar to that in the proof of Lemma 1.3.4, it is easy to see that $u \in \mathcal{D}'_\Gamma(X)$ if and only if for each $\varphi \in C_0^\infty(X)$ and each closed cone $V \subset \mathbb{R}^n$ with

$$(\text{supp}\,(\varphi) \times V) \cap \Gamma = \emptyset \tag{1.24}$$

we have

$$\sup_{\xi \in V} |\xi|^m |\widehat{\varphi u}(\xi)| < \infty, \quad m \in \mathbb{N}.$$

This makes it possible to introduce the following.

Definition 1.3.5: Let $\{u_j\}_j \subset \mathcal{D}'_\Gamma(X)$ and let $u \in \mathcal{D}'_\Gamma(X)$. We will say that the sequence $\{u_j\}$ converges to u in $\mathcal{D}'_\Gamma(X)$ if:

(a) $u_j \to u$ weakly in $\mathcal{D}'(X)$,

(b) $\sup_{j\in\mathbb{N}}\sup_{\xi\in V}\, |\xi|^m|\widehat{\varphi u_j}(\xi)| < \infty$ for every $m \in \mathbb{N}$, every $\varphi \in C_0^\infty(X)$ and every closed cone V satisfying (1.24).

For every $u \in \mathcal{D}'_\gamma(X)$ there exists a sequence $\{u_j\} \subset C_0^\infty(X)$ converging to u in $\mathcal{D}'_\Gamma(X)$. To prove this, consider two sequences $\chi_j, \varphi_j \in C_0^\infty(X)$ such that $\chi_j = 1$ on K_j, $\varphi_j \geq 0$, $\int \varphi_j(x)\, dx = 1$ and $\text{supp}\,(\chi_j) + \text{supp}\,(\varphi_j) \subset X$. Then

$$u_j = \varphi_j * \chi_j u \in C_0^\infty(X)$$

and $u_j \to u$ in $\mathcal{D}'(X)$. Moreover, the condition (b) also holds, so $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$.

For our aims in Chapter 3 we need to justify some operations on distributions (see [Hl] for more details). For convenience of the reader we list these properties, including only one proof of these – namely that of the existence of the pull-back f^*. We use the notation from [Hl].

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets and let $f : X \longrightarrow Y$ be a smooth map. Consider a closed cone $\Gamma \subset Y \times \mathbb{R}^m \setminus \{0\}$ and set

$$N_f = \{(f(x), \eta) \in Y \times \mathbb{R}^n : {}^t f'(x)\eta = 0\},$$
$$f^*(\Gamma) = \{(x, {}^t f'(x)\eta : (f(x), \eta) \in \Gamma\}.$$

For $u \in C_0^\infty(Y)$, consider the map

$$(f^*u)(x) = u(f(x)).$$

Theorem 1.3.6: *Let $N_f \cap \Gamma = \emptyset$. Then the map f^*u can be extended uniquely on the space $\mathcal{D}'_\Gamma(Y)$ such that*

$$WF(f^*u) \subset f^*\Gamma. \tag{1.25}$$

Proof: Using a partition of unity, we may consider only the case when X and Y are small open neighbourhoods of $x_0 \in X$ and $y_0 \in Y$, respectively. Set

$$\Gamma_y = \{\eta : (y, \eta) \in \Gamma\}.$$

Choose a small compact neighbourhood X_0 of x_0 and a closed conic neighbourhood V of Γ_{y_0} so that

$${}^t f'(x)\eta \neq 0 \;\text{ for } x \in X_0, \eta \in V.$$

Next, choose a small compact neighbourhood Y_0 of y_0 with $\Gamma_y \subset V$ for all $y \in Y_0$.

Now let $\chi \in C_0^\infty(X_0)$ and let $\{u_j\}_j \subset C_0^\infty(Y)$ be a sequence such that $u_j \to u$ in $\mathcal{D}'_\Gamma(Y)$. Choosing $\varphi \in C_0^\infty(Y_0)$ with $\varphi = 1$ on $f(X_0)$, we have

$$\langle f^*u_j, \chi\rangle = \langle f^*(\varphi u_j), \chi\rangle = (2\pi)^{-m} \int \widehat{\varphi u_j}(\eta) I_\chi \, d\eta = \int_V + \int_{\mathbb{R}^m \setminus V} = I_1 + I_2,$$

where

$$I_\chi(\eta) = \int e^{\mathbf{i}\langle f(x), \eta\rangle} \chi(x)\, dx.$$

For $x \in \text{supp}\,(\chi)$ and $\eta \in V$ we obtain

$$|\nabla_x \langle f(x), \eta\rangle| \geq \delta|\eta|, \quad \delta > 0.$$

Using the operator

$$L = \frac{-\mathbf{i}}{|\nabla_x \langle f(x), \eta \rangle|^2} \sum_{j=1}^{n} \partial_{x_j} (\langle f(x), \eta \rangle) \frac{\partial}{\partial x_j},$$

we integrate by parts in $I_\chi(\eta)$ and get

$$|I_\chi(\eta)| \leq C_p (1 + |\eta|)^{-p}, \quad \eta \in V,$$

for all $p \in \mathbb{N}$. On the other hand, there exists $M > 0$ such that

$$|\widehat{\varphi u_j}(\eta)| \leq C(1 + |\eta|)^{-M}, \ j \in \mathbb{N}.$$

Thus, I_1 is absolutely convergent, and we can consider the limit as $j \to \infty$. To deal with I_2, notice that $(\mathrm{supp}\varphi \setminus V) \cap \Gamma = \emptyset$. For $\eta \notin V$, (b) yields the estimates

$$|\widehat{\varphi u_j}(\eta)| \leq C'_p (1 + |\eta|)^{-p}, \ p \in \mathbb{N}, \tag{1.26}$$

uniformly with respect to j. Thus, we can let $j \to \infty$ in I_2.

To establish (1.25), replace $\chi(x)$ by $\chi(x) e^{-\mathbf{i}\langle x, \xi \rangle}$ and write

$$I_\chi(\eta, \epsilon) = (2\pi)^{-n} \int e^{\mathbf{i}\langle f(x), \eta \rangle - \mathbf{i}\langle x, \xi \rangle} \chi(x) \, dx.$$

Choose a small open conic neighbourhood W of the set

$$\{\xi = {}^t f'(x_0)\eta : (f(x_0), \eta) \in \Gamma\}$$

so that $x \in X_0$ and $\eta \in V$ imply ${}^t f'(x)\eta \in W$. As above, for $x \in X_0$, $\eta \in V$ and $\xi \notin W$ we deduce the estimate

$$|\xi - {}^t f'(x)\eta| \geq \delta(|\xi| + |\eta|), \quad \delta > 0.$$

For such ξ and η we integrate by parts in $I_\chi(\eta, \epsilon)$ and obtain

$$|I_\chi(\eta, \epsilon)| \leq C''_p (1 + |\xi| + |\eta|)^{-p}, \quad p \in \mathbb{N}.$$

For $\eta \notin V$, $\xi \notin W$ we choose a function $\psi(\xi) \in C_0^\infty(\mathbb{R})$ with $\psi(\xi) = 1$ for $|\xi| \leq 1$, and consider the operator

$$L = -\mathbf{i}(1 - \psi(\xi))|\xi|^{-2} \left\langle \xi, \frac{\partial}{\partial x} \right\rangle + \psi(x).$$

Then $L(e^{\mathbf{i}\langle x, \xi \rangle}) = e^{\mathbf{i}\langle x, \xi \rangle}$, and, as in the previous case, for $\eta \notin V$ and $\xi \notin W$, we get the estimates

$$|I_\chi(\eta, \epsilon)| \leq C_p (1 + |\eta|)^p (1 + |\xi|)^{-p}, \quad p \in \mathbb{N}.$$

Combining these estimates with (1.26), we obtain

$$|\chi(\widehat{f^* u_j})(\xi)| \leq C_N (1 + |\xi|)^{-N}$$

for $\xi \notin W$, where the constant C_N does not depend on j. Letting $j \to \infty$ proves (1.25). ∎

By an easy modification of the above-mentioned argument, one proves the following modification of Theorem 1.3.6 for distributions depending on a parameter.

Corollary 1.3.7: *Let Z be a compact subset of $\mathbb{R}^p$ and let*

$$Z \ni z \mapsto (u, \cdot, z) \in \mathcal{D}'_\Gamma(Y)$$

be a continuous map. Under the assumptions of Theorem 1.3.6, the map

$$Z \ni z \mapsto f^*(u, \cdot, z) \in \mathcal{D}'_{f^*\Gamma}(X)$$

is continuous.

Next, consider a linear continuous map

$$\mathcal{K} : C_0^\infty(Y) \longrightarrow \mathcal{D}'(X).$$

By Schwartz's theorem (cf. Theorem 5.2.1 in [Hl]), there exists a distribution $K \in \mathcal{D}'(X \times Y)$, called the *kernel* of $\mathcal{K}$, such that

$$\langle K, \varphi(x) \otimes \psi(y) \rangle = \langle (\mathcal{K}\psi)(x), \varphi(x) \rangle$$

for all $\varphi \in C_0^\infty(X)$ and $\psi \in C_0^\infty(Y)$. $WF(K)$ will be called the *wave front set* of $\mathcal{K}$. Set

$$WF'(K) = \{(x, y, \xi, \eta) : (x, y, \xi, -\eta) \in WF(K)\},$$
$$WF(K)_X = \{(x, \xi) : (x, y, \xi, 0) \in WF(K) \text{ for some } y \in Y\},$$
$$WF'(K)_Y = \{(y, \eta) : (x, y, 0, \eta) \in WF'(K) \text{ for some } x \in X\},$$

and consider the composition

$$WF'(K) \circ WF(u) = \{(x, \xi) : \exists (y, \eta) \in WF(u) \text{ with } (x, y, \xi, \eta) \in WF'(K)\}.$$

The following two results will also be necessary for Chapter 3. Their proofs can be found in Section 8.2 of [Hl].

Theorem 1.3.8: *For $\psi \in C_0^\infty(Y)$ we have*

$$WF(\mathcal{K}\psi) \subset \{(x, \xi) : (x, y, \xi, 0) \in WF(K) \text{ for some } y \in \text{supp}\,(\psi)\}.$$

Theorem 1.3.9: *There exists a unique extension of $\mathcal{K}$ on the set*

$$\{u \in \mathcal{E}'(Y) : WF(u) \cap WF'(K)_Y = \emptyset\}$$

such that for each compact $M \subset Y$ *and each closed conic set* Γ *with* $\Gamma \cap WF'(K)_Y = \emptyset$ *the map*

$$\mathcal{E}'(M) \cap \mathcal{D}'_\Gamma(Y) \ni u \mapsto \mathcal{K}u \in \mathcal{D}'(X)$$

is continuous. Moreover, the inclusion

$$WF(\mathcal{K}u) \subset WF(K)_X \cup WF'(K) \circ WF(u)$$

holds.

The wave front of $u \in \mathcal{D}'(X)$ can be described by means of the characteristic set of pseudo-differential operators on X. Denote by $L^m(X)$ the *class of all pseudo-differential operators* (*PDO*) *in* X *of order* m. If $x(x, \xi) \in S^m(X \times \mathbb{R}^n)$ is the symbol of $A \in L^m(X)$, then the oscillatory integral

$$K_A(x, \eta) = (2\pi)^{-n} \int e^{\mathbf{i}\langle x-y,\xi\rangle} a(x, \xi)\, d\xi$$

determines the kernel of A and $WF(A) = WF(K_A)$. The operator $A \in L^m(X)$ is called *properly supported* if for each compact $K \subset X$ there exists another compact $K' \subset X$ so that supp $(u) \subset K$ implies supp $(Au) \subset K'$, and if $u = 0$ on K', then $Au = 0$ on K. A point $(x_0, \xi_0) \in T^*X \setminus \{0\}$ is called *non-characteristic* for a properly supported PDO $A \in L^m(X)$ if there exists a properly supported PDO $B \in L^{-m}(X)$ so that

$$(x_0, \xi_0) \notin WF(AB - Id) \cup WF(BA - Id).$$

In this case A is called *elliptic* at (x_0, ξ_0).

Proposition 1.3.10: *If there exists a properly supported PDO* $A \in L^m(X)$, *elliptic at* (x_0, ξ_0), *such that* $Au \in C^\infty(X)$, *then* $(x_0, \xi_0) \notin WF(u)$.

The reader may consult Section 18 in [Hl] for the main properties of PDOs and for a proof of the above-mentioned proposition.

1.4 Boundary problems for the wave operator

Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n, n \geq 2$ with C^∞ smooth compact boundary $\partial\Omega$. Consider the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \ \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = u_0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u|_{t<t_0} = 0. \end{cases} \tag{1.27}$$

Here the trace $u|_{(t,x)\in\mathbb{R}\times\partial\Omega}$ exists, since the boundary $\mathbb{R} \times \partial\Omega$ is not characteristic for the operator $\Box = \partial_t^2 - \Delta_x$. For the existence of a solution of (1.27) we refer to

[H3], Section 24. In particular, we have the following result proved in [H3], Theorem 24.1.1.

Theorem 1.4.1: *Let $f \in H_s^{loc}(\mathbb{R} \times \Omega^\circ)$, $u_0 \in H_{s+1}^{loc}(\mathbb{R} \times \partial\Omega)$ with $s \geq 0$. Assume that f and u_0 vanish for $t < t_0$. Then there exists an unique solution $u \in H_{s+1}^{loc}(\Omega^\circ)$ of (1.27).*

We may apply the above theorem when Ω is a bounded domain in $\mathbb{R}^n$, as well as in the case when $\Omega = \mathbb{R}^n \setminus \bar{K}$, K being a bounded non-empty open obstacle with smooth boundary.

To study the singularities of the solution of the Dirichlet problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \ \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = u_0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.28}$$

we need to introduce the wave front set $WF_b(u)$. Let $Q = \mathbb{R} \times \Omega$. Consider the space $\widetilde{T}^*(Q) = T^*(Q^\circ) \cup T^*(\partial Q)$ of equivalence classes in T^*Q with respect to the equivalence relation $\sim$ defined in Section 1.2. It will be called the *compressed cotangent bundle* of Q. For the solution u of (1.27) we can define the *generalized wave front set* $WF_b(u) \subset \widetilde{T}^*Q \setminus \{0\}$ in such a way that

$$WF_b(u)|_{T^*(Q^\circ)} = WF(u_{|Q^\circ}),$$

and

$$WF_b(u)|_{T^*(\partial Q)} \subset \Sigma_b.$$

(See Section 1.2 for the definition of Σ_b.) For this purpose, as in Section 1.2, introduce local coordinates (x_1, x'), $x' = (x_2, \ldots, x_n, x_{n+1})$, $x_{n+1} = t$ in Q so that ∂Q is locally given by $x_1 = 0$. Let (ξ_1, ξ') be the dual coordinates to (x_1, x').

Now define $WF_b(u)|_{T^*(\partial Q)}$ as the subset of $T^*(\partial Q) \setminus \{0\}$, the complement of which consists of all $(x_0', \xi_0') \in T^*(\partial Q) \setminus \{0\}$ such that there exists a PDO $B(x, D')$, depending smoothly on x_1, elliptic at $(0, x_0, \xi_0')$, and such that $B(x, D_{x'})u \in C^\infty(Q)$. This definition does not depend on the choice of the local coordinates.

In Q° the set $WF(u) \setminus WF(f)$ is contained in the characteristic set Σ and it is propagating along the bicharacteristics of $\Box$ which are rays. For simplicity assume that $f \in C^\infty(Q^\circ)$. The singularities of the solution $u|_Q$ of (1.28) can be described by means of $WF_b(u)$. The simplest case is when $(0, x', \xi') \in H$ is a hyperbolic point. Then if $(0, x', \xi') \in (WF_b(u) \cap H) \setminus WF(u_0)$, the outgoing and incoming bicharacteristics issued from this point are included in $WF_b(u)$ over a small neighbourhood of $(0, x_0', \xi_0')$. If $(0, x_0', \xi_0') \in G$ is a gliding point, the situation is more complicated and we must consider the generalized compressed bicharacteristics of $\Box$ issued from this point. The following result was proved by Melrose and Sjöstrand [MS2] (see also Section 24 and Theorem 24.5.3 in [H3]).

Theorem 1.4.2: *Let $u \in \mathcal{D}'(Q)$ be a solution of problem (1.28) with $f \in C^\infty(Q)$ and $u_0 \in \mathcal{D}'(\partial\Omega)$ and let*

$$\hat{z} \in (WF_b(u) \setminus WF(u_0)) \cap \{(x, \xi) \in \widetilde{T}^*Q : x_{n+1} = t > t_0\}.$$

Then $\hat{z}$ is either a characteristic point in Σ_0 or a point in $T^*(\partial\Omega) \cap \Sigma = H \cup G$, and there exists a maximal compressed generalized bicharacteristics $\tilde{\gamma}(\sigma) = (x(\sigma), \xi(\sigma))$ *of $\Box$, passing through $\hat{z}$ and staying in $WF_b(u)$ as long as $t(\sigma) = x_{n+1}(\sigma) > t_0$.*

One can also describe the singularities of a boundary problem with non-homogeneous boundary condition

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \text{ in } \mathbb{R} \times \Omega^\circ, \\ u = g \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.29}$$

with $f = 0, g = 0$ for $t < t_0$. In this situation we have the following result established in [MS2] (see Theorem 6.14).

Theorem 1.4.3: *Let u be a solution of (1.29) and let $f \in C^\infty$. Then $WF_b(u)$ is a complete union of the generalized half-bicharacteristics issued from $WF(g)$.*

Here half-bicharacteristics means that we consider these bicharacteristics γ for which the time increases when we move along γ.

The same results hold for the boundary problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = f \text{ in } \mathbb{R} \times \Omega^\circ, \\ (\partial_\nu + \alpha(x))u = u_0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.30}$$

where ∂_ν is the derivative with respect to a normal field of $\partial\Omega$ and $\alpha(x)$ is a C^∞ function on $\partial\Omega$. For $\alpha(x) = 0$ we have the Neumann problem, while for $\alpha(x) \neq 0$ we obtain the Robin problem.

1.5 Notes

The results in Section 1.1 can be found with detailed proofs in [GG] and [Hir]. In Section 1.2 we follow [MS1], [MS2] and [H3]. Lemma 1.2.5 is proved in [MS1], while Lemmas 1.2.6, 1.2.7 and 1.2.10 can be found in [H3]. The results in Section 1.3 concerning wave front sets of distributions and operators are due to Hörmander [Hl], [H3]. The definition of generalized wave front set $WF_b(u)$ was introduced by Melrose and Sjöstrand [MS1]. Theorem 1.3.11 was established in [MS1], [MS2]. We refer the reader to Section 24 in [H3] for more details concerning the generalized bicharacteristics and the propagation of singularities for the Dirichlet problem.