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BASIC ELECTROMAGNETIC THEORY

This chapter presents basic electromagnetic theory, which includes a brief review of vector analysis that is essential for the mathematical treatment of electromagnetic fields, Maxwell's equations in both integral and differential forms that govern all electromagnetic phenomena, the Lorentz force law that relates electric and magnetic fields to measurable forces, constitutive relations that characterize the electromagnetic properties of a medium, boundary conditions at interfaces between different media and at perfectly conducting surfaces, the concepts of electromagnetic energy and power, the energy conservation law as expressed by Poynting's theorem, the concept of phasors for time-harmonic fields, and finally Maxwell's equations and Poynting's theorem in the complex form for time-harmonic fields. The presentation assumes that the reader has basic knowledge of vector calculus and electromagnetics at the undergraduate level [1–7].

1.1 REVIEW OF VECTOR ANALYSIS

We all know that both electric and magnetic fields are vectors since they have both a magnitude and a direction. Hence, the study of electromagnetic fields requires basic knowledge of vector analysis. The most useful concepts in vector analysis are those of divergence, curl, and gradient. In this section, we present definitions and related integral theorems for these quantities. This is followed by the introduction of a new method that can easily deal with various vector identities and the description of the Helmholtz decomposition theorem, which will be very useful for the study of Maxwell's equations.

Theory and Computation of Electromagnetic Fields, Second Edition. Jian-Ming Jin.

1.1.1 Vector Operations and Integral Theorems

Assume that $\mathbf{f}$ is a vector function,[1] a quantity whose magnitude and direction vary as functions of its position in space. The *divergence* of the vector function $\mathbf{f}$ is defined by the limit

$$\nabla \cdot \mathbf{f} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \left[\oiint_s \mathbf{f} \cdot \mathrm{d}\mathbf{s} \right] \tag{1.1.1}$$

where Δv denotes an infinitesimal volume and s denotes the closed surface of this volume. The differential surface $\mathrm{d}\mathbf{s}$ is normal to s and points outward. By applying Equation (1.1.1) to the differential volume constructed in rectangular, cylindrical, and spherical coordinates, we obtain the expressions of the divergence as

$$\nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{1.1.2}$$

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho} \frac{\partial (\rho f_\rho)}{\partial \rho} + \frac{\partial f_\phi}{\rho \partial \phi} + \frac{\partial f_z}{\partial z} \tag{1.1.3}$$

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(f_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial f_\phi}{\partial \phi} \tag{1.1.4}$$

in these three most important coordinate systems. It is important to remember that $\nabla \cdot \mathbf{f}$, a notation proposed by J. Willard Gibbs [8], is simply a mathematical notation for the divergence of $\mathbf{f}$. It should not be interpreted as the dot product between the operator ∇ and the vector $\mathbf{f}$; otherwise, mistakes can easily be made in the derivation of the expressions in cylindrical and spherical coordinates. Now, consider a finite volume denoted as V, which is enclosed by surface S. By dividing this volume into an infinite number of infinitesimal volumes, applying Equation (1.1.1) to each infinitesimal volume, and summing up the results, we obtain

$$\iiint_V \nabla \cdot \mathbf{f} \, \mathrm{d}V = \oiint_S \mathbf{f} \cdot \mathrm{d}\mathbf{S} \tag{1.1.5}$$

if the vector $\mathbf{f}$ and its first derivative are continuous in volume V as well as on its surface S. Equation (1.1.5) is known as the *divergence theorem* or *Gauss' theorem*, which is very useful in electromagnetics.

In addition to the divergence, another operation that quantifies the variation of a vector function is called the *curl*. The curl of the vector function $\mathbf{f}$ is defined by the limit

$$\nabla \times \mathbf{f} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \left[\oiint_s \mathrm{d}\mathbf{s} \times \mathbf{f} \right] \tag{1.1.6}$$

where Δv again denotes an infinitesimal volume enclosed by surface s. Again, we should remember that $\nabla \times \mathbf{f}$ is simply a mathematical notation for the curl of $\mathbf{f}$, and it should not

[1] All vectors are represented by boldfaced letters in this book. In contrast, a scalar quantity is represented by a nonboldfaced italic letter.

be interpreted as the cross-product between the operator ∇ and the vector $\mathbf{f}$. By applying Equation (1.1.6) to the differential volume constructed in rectangular, cylindrical, and spherical coordinates, we obtain the expressions of the curl as

$$\nabla \times \mathbf{f} = \hat{x}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{y}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{z}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \tag{1.1.7}$$

$$\nabla \times \mathbf{f} = \hat{\rho}\left(\frac{\partial f_z}{\rho \partial \phi} - \frac{\partial f_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \rho}\right) + \hat{z}\frac{1}{\rho}\left[\frac{\partial(\rho f_\phi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \phi}\right] \tag{1.1.8}$$

$$\begin{aligned}\nabla \times \mathbf{f} = {} & \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(f_\phi \sin\theta) - \frac{\partial f_\theta}{\partial \phi}\right] + \hat{\theta}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial f_r}{\partial \phi} - \frac{\partial}{\partial r}(r f_\phi)\right] \\ & + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial \theta}\right].\end{aligned} \tag{1.1.9}$$

Apparently, the curl itself is a vector that has a different magnitude and a different direction. Given a direction $\hat{a}$, the magnitude of the curl in this direction is given by

$$\hat{a} \cdot (\nabla \times \mathbf{f}) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\oint_c \mathbf{f} \cdot \mathrm{d}\mathbf{l}\right] \tag{1.1.10}$$

where Δs is an infinitesimal surface normal to $\hat{a}$ and c is a closed contour bounding Δs. The differential length $\mathrm{d}\mathbf{l}$ is tangential to the contour c, and its direction is related to that of $\hat{a}$ by the right-hand rule. Equation (1.1.10) can be derived by applying Equation (1.1.6) to an infinitesimal disk perpendicular to $\hat{a}$ with a vanishing thickness. Now, consider an open surface S bounded by a closed contour C. We can divide S into an infinite number of infinitesimal surfaces, then apply Equation (1.1.10) to each of the infinitesimal surfaces, and finally sum up the results to find

$$\iint_S (\nabla \times \mathbf{f}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{l} \tag{1.1.11}$$

if the vector $\mathbf{f}$ and its first derivative are continuous on surface S as well as along C. Equation (1.1.11) is known as *Stokes' theorem*, which is also very useful in the study of electromagnetics.

As we will see later, the divergence and curl are sufficient to characterize the variation of a vector function. The third useful operation in vector analysis is the *gradient*, which quantifies the variation of a scalar function. Let f be a scalar function of space. The gradient of this function is defined as

$$\nabla f = \lim_{\Delta v \to 0} \frac{1}{\Delta v}\left[\oiint_s f \, \mathrm{d}\mathbf{s}\right] \tag{1.1.12}$$

which is a vector. Its magnitude along a given direction $\hat{a}$ is given by

$$\hat{a} \cdot \nabla f = \frac{\partial f}{\partial a} \tag{1.1.13}$$

which can be derived by applying Equation (1.1.12) to an infinitesimal circular disk perpendicular to $\hat{a}$ with a vanishing radius and thickness. By applying Equation (1.1.12) to the differential volume constructed in rectangular, cylindrical, and spherical coordinates, we obtain the expressions of the gradient as

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} \tag{1.1.14}$$

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{\partial f}{\rho\partial \phi} + \hat{z}\frac{\partial f}{\partial z} \tag{1.1.15}$$

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{\partial f}{r\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \tag{1.1.16}$$

In vector analysis, another important operation is to take the divergence on the gradient of a function such as $\nabla \cdot (\nabla f)$. This operation is often referred to as the *Laplacian*, which is denoted as

$$\nabla^2 f = \nabla \cdot (\nabla f). \tag{1.1.17}$$

Its expressions in the three commonly used coordinates are given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{1.1.18}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{1.1.19}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{1.1.20}$$

1.1.2 Symbolic Vector Method

In vector analysis, we often have to manipulate vector expressions into different and yet equivalent forms. A difficulty in such a manipulation is that the operator ∇ cannot be treated rigorously as a vector. This difficulty can be alleviated by the introduction of the *symbolic vector method* [8]. This symbolic vector, denoted as $\tilde{\nabla}$, is defined as

$$T(\tilde{\nabla}) = \lim_{\Delta v \to 0}\frac{1}{\Delta v}\left[\oiint_s T(\hat{n})\,\mathrm{d}s\right] \tag{1.1.21}$$

where Δv denotes an infinitesimal volume, s denotes the closed surface of this volume, and $\hat{n}$ denotes the unit vector normal to the surface s and pointing outward, which is related to $\mathrm{d}\mathbf{s}$ by $\mathrm{d}\mathbf{s} = \hat{n}\,\mathrm{d}s$. The left-hand side of Equation (1.1.21), $T(\tilde{\nabla})$, represents an expression that contains the symbolic vector $\tilde{\nabla}$, such as $a\tilde{\nabla}$, $\mathbf{a}\cdot\tilde{\nabla}$, $\mathbf{a}\times\tilde{\nabla}$, and $\tilde{\nabla}\cdot(\mathbf{a}\times\mathbf{b})$. The integrand on the right-hand side, $T(\hat{n})$, represents the same expression with $\tilde{\nabla}$ being replaced by $\hat{n}$, so the corresponding expressions for the four aforementioned examples are $a\hat{n}$, $\mathbf{a}\cdot\hat{n}$, $\mathbf{a}\times\hat{n}$, and $\hat{n}\cdot(\mathbf{a}\times\mathbf{b})$.

Based on the definition given in Equation (1.1.21), we can show easily that

$$\tilde{\nabla}\cdot\mathbf{f} = \lim_{\Delta v \to 0}\frac{1}{\Delta v}\left[\oiint_s \hat{n}\cdot\mathbf{f}\,\mathrm{d}s\right] = \lim_{\Delta v \to 0}\frac{1}{\Delta v}\left[\oiint_s \mathbf{f}\cdot\hat{n}\,\mathrm{d}s\right] = \mathbf{f}\cdot\tilde{\nabla} \tag{1.1.22}$$

and similarly, $\tilde{\nabla} f = f\,\tilde{\nabla}$ and $\tilde{\nabla} \times \mathbf{f} = -\mathbf{f} \times \tilde{\nabla}$. This indicates clearly that $\tilde{\nabla}$ can be treated as a regular vector; hence, all valid vector manipulations and all algebraic identities are applicable to $\tilde{\nabla}$. However, by comparing Equation (1.1.21) to the definitions of the divergence, curl, and gradient, we also see that

$$\nabla \cdot \mathbf{f} = \tilde{\nabla} \cdot \mathbf{f} = \mathbf{f} \cdot \tilde{\nabla} \tag{1.1.23}$$

$$\nabla \times \mathbf{f} = \tilde{\nabla} \times \mathbf{f} = -\mathbf{f} \times \tilde{\nabla} \tag{1.1.24}$$

$$\nabla f = \tilde{\nabla} f = f\,\tilde{\nabla}. \tag{1.1.25}$$

These equations establish a relation between the symbolic vector $\tilde{\nabla}$ and the divergence, curl, and gradient operations. Given an expression that contains any of these operations, we can first convert it into an algebraic expression using Equations (1.1.23)–(1.1.25), then manipulate the algebraic expression using any of the valid algebraic identities, and finally convert the symbolic vector back to the divergence, curl, or gradient. For example, consider $\tilde{\nabla} \times (\tilde{\nabla} \times \mathbf{f})$. Since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, we have

$$\tilde{\nabla} \times (\tilde{\nabla} \times \mathbf{f}) = (\tilde{\nabla} \cdot \mathbf{f})\tilde{\nabla} - (\tilde{\nabla} \cdot \tilde{\nabla})\mathbf{f} = \tilde{\nabla}(\tilde{\nabla} \cdot \mathbf{f}) - \tilde{\nabla} \cdot (\tilde{\nabla}\mathbf{f}). \tag{1.1.26}$$

Applying Equations (1.1.23)–(1.1.25) and then Equation (1.1.17), we obtain a very useful identity

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}. \tag{1.1.27}$$

When a vector expression contains the symbolic vector $\tilde{\nabla}$ and two arbitrary functions, since $\tilde{\nabla}$ works on both functions, we can use the following chain rule to facilitate its manipulation:

$$T(\tilde{\nabla}, a, b) = T(\tilde{\nabla}_a, a, b) + T(\tilde{\nabla}_b, a, b) \tag{1.1.28}$$

where a and b represent two functions that can either be scalar or vector, $\tilde{\nabla}_a$ is the symbolic vector applying only to function a, and $\tilde{\nabla}_b$ applies only to function b. Equation (1.1.28) should not come as a surprise to anyone who is familiar with the following well-known differentiation formula:

$$\frac{\partial(ab)}{\partial x} = b\frac{\partial a}{\partial x} + a\frac{\partial b}{\partial x}. \tag{1.1.29}$$

To illustrate the application of Equation (1.1.28), we consider three examples. We first consider the expression $\nabla \cdot (a\mathbf{b})$. Using Equation (1.1.28), we find

$$\tilde{\nabla} \cdot (a\mathbf{b}) = \tilde{\nabla}_a \cdot (a\mathbf{b}) + \tilde{\nabla}_b \cdot (a\mathbf{b}) = (\tilde{\nabla}_a a) \cdot \mathbf{b} + a\tilde{\nabla}_b \cdot \mathbf{b}. \tag{1.1.30}$$

Since $\tilde{\nabla} \cdot (a\mathbf{b}) = \nabla \cdot (a\mathbf{b})$, $\tilde{\nabla}_a a = \nabla a$, and $\tilde{\nabla}_b \cdot \mathbf{b} = \nabla \cdot \mathbf{b}$, we obtain the vector identity

$$\nabla \cdot (a\mathbf{b}) = \mathbf{b} \cdot (\nabla a) + a\nabla \cdot \mathbf{b}. \tag{1.1.31}$$

As the second example, we consider $\nabla \times (a\mathbf{b})$. Using Equation (1.1.28), we find

$$\tilde{\nabla} \times (a\mathbf{b}) = \tilde{\nabla}_a \times (a\mathbf{b}) + \tilde{\nabla}_b \times (a\mathbf{b}) = (\tilde{\nabla}_a a) \times \mathbf{b} + a\tilde{\nabla}_b \times \mathbf{b} \tag{1.1.32}$$

which yields the vector identity

$$\nabla \times (a\mathbf{b}) = -\mathbf{b} \times \nabla a + a\nabla \times \mathbf{b}. \tag{1.1.33}$$

As the last example, we consider $\nabla \times (\mathbf{a} \times \mathbf{b})$. Using Equation (1.1.28) and the algebraic identity

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \tag{1.1.34}$$

we find

$$\begin{aligned}\tilde{\nabla} \times (\mathbf{a} \times \mathbf{b}) &= \tilde{\nabla}_a \times (\mathbf{a} \times \mathbf{b}) + \tilde{\nabla}_b \times (\mathbf{a} \times \mathbf{b}) \\ &= (\tilde{\nabla}_a \cdot \mathbf{b})\mathbf{a} - (\tilde{\nabla}_a \cdot \mathbf{a})\mathbf{b} + (\tilde{\nabla}_b \cdot \mathbf{b})\mathbf{a} - (\tilde{\nabla}_b \cdot \mathbf{a})\mathbf{b}\end{aligned} \tag{1.1.35}$$

which yields the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}\nabla \cdot \mathbf{a} + \mathbf{a}\nabla \cdot \mathbf{b} - (\mathbf{a} \cdot \nabla)\mathbf{b}. \tag{1.1.36}$$

These examples demonstrate the power of the symbolic vector in deriving various vector identities, which would otherwise be a rather tedious task.

Now, let us consider a finite volume V, which is enclosed by surface S. By dividing this volume into an infinite number of infinitesimal volumes, applying Equation (1.1.21) to each infinitesimal volume and summing up the results, we obtain

$$\iiint_V T(\tilde{\nabla})\, \mathrm{d}V = \oiint_S T(\hat{n})\, \mathrm{d}S \tag{1.1.37}$$

if the function involved in $T(\tilde{\nabla})$ is continuous within volume V. Equation (1.1.37) is referred to as the *generalized Gauss' theorem*, from which we can easily derive many integral theorems. For example, if we let $T(\tilde{\nabla}) = \tilde{\nabla} \cdot \mathbf{f} = \nabla \cdot \mathbf{f}$, we obtain the standard Gauss' theorem in Equation (1.1.5). If we let $T(\tilde{\nabla}) = \tilde{\nabla} \times \mathbf{f} = \nabla \times \mathbf{f}$, we obtain the so-called *curl theorem*

$$\iiint_V \nabla \times \mathbf{f}\, \mathrm{d}V = \oiint_S \mathrm{d}\mathbf{S} \times \mathbf{f} \tag{1.1.38}$$

from which we can also derive Stokes' theorem given in Equation (1.1.11) by applying it to a surface with a vanishing thickness.

EXAMPLE 1.1

Using the generalized Gauss' theorem, derive a new integral theorem

$$\iiint_V (\mathbf{b}\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\mathbf{b})\, \mathrm{d}V = \oiint_S (\hat{n} \cdot \mathbf{a})\mathbf{b}\, \mathrm{d}S.$$

Solution Based on the expression of the right-hand side, we let $T(\hat{n}) = (\hat{n} \cdot \mathbf{a})\mathbf{b}$. The corresponding symbolic expression is $T(\tilde{\nabla}) = (\tilde{\nabla} \cdot \mathbf{a})\mathbf{b}$, which can further be written as

$$T(\tilde{\nabla}) = (\tilde{\nabla}_a \cdot \mathbf{a})\mathbf{b} + (\tilde{\nabla}_b \cdot \mathbf{a})\mathbf{b} = (\tilde{\nabla}_a \cdot \mathbf{a})\mathbf{b} + (\mathbf{a} \cdot \tilde{\nabla}_b)\mathbf{b} = \mathbf{b}\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\mathbf{b}$$

where we have applied the chain rule in Equation (1.1.28) and the relationship between $\tilde{\nabla}$ and the divergence and gradient operations. The new integral theorem is then obtained by substituting the expressions of $T(\tilde{\nabla})$ and $T(\hat{n})$ into the generalized Gauss' theorem in Equation (1.1.37).

1.1.3 Helmholtz Decomposition Theorem

In vector analysis, there are two special vectors. One is called the *irrotational* vector, whose curl vanishes. Denoting this vector as $\mathbf{F}_\mathrm{i}$, we have

$$\nabla \times \mathbf{F}_\mathrm{i} = 0, \qquad \nabla \cdot \mathbf{F}_\mathrm{i} \neq 0. \tag{1.1.39}$$

Another special vector is called the *solenoidal* vector, whose divergence is zero. Denoting this vector as $\mathbf{F}_\mathrm{s}$, we have

$$\nabla \cdot \mathbf{F}_\mathrm{s} = 0, \qquad \nabla \times \mathbf{F}_\mathrm{s} \neq 0. \tag{1.1.40}$$

Using the symbolic vector method, we can easily prove the following two very important vector identities:

$$\nabla \times (\nabla \varphi) = 0 \tag{1.1.41}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{1.1.42}$$

These identities are valid for any continuous and differentiable scalar function φ and vector function $\mathbf{A}$. Clearly, $\nabla\varphi$ is an irrotational vector and $\nabla \times \mathbf{A}$ is a solenoidal vector.

Although a vector function can have a complicated variation, it can be shown that any smooth vector function $\mathbf{F}$ that vanishes at infinity can be decomposed into an irrotational and a solenoidal vector,

$$\mathbf{F} = \mathbf{F}_\mathrm{i} + \mathbf{F}_\mathrm{s}. \tag{1.1.43}$$

By taking the divergence and curl of Equation (1.1.43), respectively, we obtain

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_\mathrm{i}, \qquad \nabla \times \mathbf{F} = \nabla \times \mathbf{F}_\mathrm{s} \tag{1.1.44}$$

which clearly indicates that the solenoidal component is related to the curl of the function and the irrotational part is related to the divergence of the function. Therefore, once both the divergence and curl of a vector function are specified, the function is fully determined. This fact is known as the *Helmholtz decomposition theorem*.

1.1.4 Green's Theorems

From Gauss' theorem in Equation (1.1.5), we can derive some very useful integral theorems. If we substitute $\mathbf{f} = a\nabla b$ into Equation (1.1.5), where a and b are scalar functions, and apply a vector identity based on Equation (1.1.31), we obtain

$$\iiint_V (a\nabla^2 b + \nabla a \cdot \nabla b)\,\mathrm{d}V = \oiint_S a\frac{\partial b}{\partial n}\,\mathrm{d}S \tag{1.1.45}$$

which is called the *first scalar Green's theorem*. By exchanging the positions of a and b and subtracting the resulting equation from Equation (1.1.45), we obtain

$$\iiint_V (a\nabla^2 b - b\nabla^2 a)\,\mathrm{d}V = \oiint_S \left(a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right)\mathrm{d}S \tag{1.1.46}$$

which is known as the *second scalar Green's theorem*.

If we substitute $\mathbf{f} = \mathbf{a} \times \nabla \times \mathbf{b}$ into Equation (1.1.5), where both $\mathbf{a}$ and $\mathbf{b}$ are vector functions, and apply a vector identity, we obtain

$$\iiint_V [(\nabla \times \mathbf{a}) \cdot (\nabla \times \mathbf{b}) - \mathbf{a} \cdot (\nabla \times \nabla \times \mathbf{b})]\,\mathrm{d}V = \oiint_S (\mathbf{a} \times \nabla \times \mathbf{b}) \cdot \mathrm{d}\mathbf{S} \tag{1.1.47}$$

which is called the *first vector Green's theorem*. By switching the positions of $\mathbf{a}$ and $\mathbf{b}$ and subtracting the resulting equation from Equation (1.1.47), we obtain

$$\iiint_V [\mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \nabla \times \mathbf{b})]\,\mathrm{d}V = \oiint_S (\mathbf{a} \times \nabla \times \mathbf{b} - \mathbf{b} \times \nabla \times \mathbf{a}) \cdot \mathrm{d}\mathbf{S} \tag{1.1.48}$$

which is known as the *second vector Green's theorem*. Now, if we let $\mathbf{b} = \hat{b}b$, where $\hat{b}$ is an arbitrary constant unit vector and b is a scalar function, and then substitute it into Equation (1.1.48), we can obtain after some vector manipulations

$$\begin{aligned}\iiint_V &\left[b(\nabla \times \nabla \times \mathbf{a}) + \mathbf{a}\nabla^2 b + (\nabla \cdot \mathbf{a})\nabla b\right]\mathrm{d}V \\ &= \oiint_S [(\hat{n} \cdot \mathbf{a})\nabla b + (\hat{n} \times \mathbf{a}) \times \nabla b + (\hat{n} \times \nabla \times \mathbf{a})b]\,\mathrm{d}S\end{aligned} \tag{1.1.49}$$

which can be called the *scalar–vector Green's theorem*.

EXAMPLE 1.2

Derive the scalar–vector Green's theorem in Equation (1.1.49) from the second vector Green's theorem in Equation (1.1.48).

Solution By letting $\mathbf{b} = \hat{b}b$ with b as an arbitrary continuous scalar function and $\hat{b}$ as a constant unit vector pointing in an arbitrary direction, we have

$$\begin{aligned}\mathbf{a} \cdot (\nabla \times \nabla \times \mathbf{b}) &= \mathbf{a} \cdot [\nabla \times \nabla \times (\hat{b}b)] = \mathbf{a} \cdot [\nabla\nabla \cdot (\hat{b}b) - \nabla^2(\hat{b}b)] \\ &= \mathbf{a} \cdot [\nabla(\hat{b} \cdot \nabla b) - \hat{b}\nabla^2 b] = \mathbf{a} \cdot \nabla(\hat{b} \cdot \nabla b) - \hat{b} \cdot \mathbf{a}\nabla^2 b \\ &= \nabla \cdot [\mathbf{a}(\hat{b} \cdot \nabla b)] - \hat{b} \cdot (\nabla \cdot \mathbf{a})\nabla b - \hat{b} \cdot \mathbf{a}\nabla^2 b\end{aligned}$$

where we have applied the vector identities in Equations (1.1.27) and (1.1.31). Therefore, the integrand in the left-hand side of Equation (1.1.48) becomes

$$\begin{aligned}&\mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \nabla \times \mathbf{b}) \\ &\quad = \hat{b} \cdot [b(\nabla \times \nabla \times \mathbf{a}) + (\nabla \cdot \mathbf{a})\nabla b + \mathbf{a}\nabla^2 b] - \nabla \cdot [\mathbf{a}(\hat{b} \cdot \nabla b)].\end{aligned}$$

On the other hand, the integrand in the right-hand side of Equation (1.1.48) can be written as

$$(\mathbf{a} \times \nabla \times \mathbf{b} - \mathbf{b} \times \nabla \times \mathbf{a}) \cdot \hat{n} = [\mathbf{a} \times \nabla \times (\hat{b}b) - b\hat{b} \times \nabla \times \mathbf{a}] \cdot \hat{n}$$
$$= [\mathbf{a} \times (\nabla b \times \hat{b})] \cdot \hat{n} - b\hat{b} \cdot [(\nabla \times \mathbf{a}) \times \hat{n}]$$
$$= \hat{b} \cdot [(\hat{n} \times \mathbf{a}) \times \nabla b + (\hat{n} \times \nabla \times \mathbf{a})b]$$

where we have applied the vector identity in Equation (1.1.33) and the algebraic identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ repeatedly. Substituting the new expressions for the integrands into Equation (1.1.48) and applying Gauss' divergence theorem, we obtain

$$\hat{b} \cdot \iiint_V \left[b(\nabla \times \nabla \times \mathbf{a}) + \mathbf{a}\nabla^2 b + (\nabla \cdot \mathbf{a})\nabla b\right] \mathrm{d}V$$
$$= \hat{b} \cdot \oiint_S [(\hat{n} \cdot \mathbf{a})\nabla b + (\hat{n} \times \mathbf{a}) \times \nabla b + (\hat{n} \times \nabla \times \mathbf{a})b] \,\mathrm{d}S$$

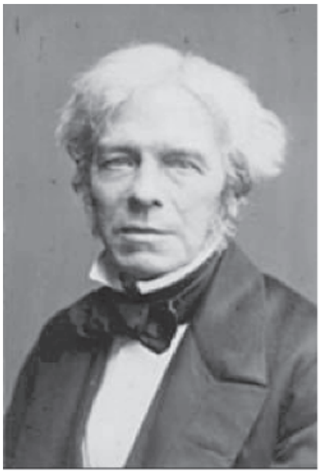
which becomes the scalar–vector Green's theorem in Equation (1.1.49) since $\hat{b}$ is an arbitrary constant unit vector.

1.2 MAXWELL'S EQUATIONS IN TERMS OF TOTAL CHARGES AND CURRENTS

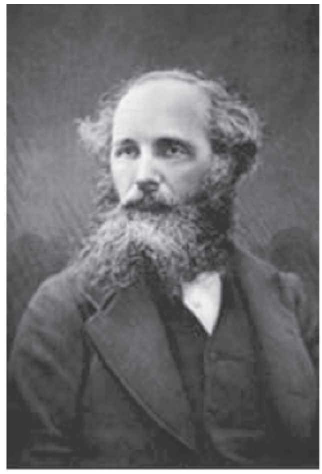
Maxwell's equations are a set of four mathematical equations that relate precisely the electric and magnetic fields to their sources, which are electric charges and currents. They were established by James Clerk Maxwell (1831–1879) [9, 10] based on the experimental discoveries of André Marie Ampère (1775–1836) and Michael Faraday (1791–1867) and a law for electricity by Carl Friedrich Gauss (1777–1855) and were reformulated into the vector form by Heinrich Hertz (1857–1894) [11] and Oliver Heaviside (1850–1925) [12]. Maxwell's equations can be expressed in both integral and differential forms. This section first presents Maxwell's equations in integral form as the fundamental postulates of electromagnetic theory and then derives Maxwell's equations in differential form for fields in a continuous medium, which are subsequently used to derive the current continuity condition. This is followed by a brief description of the Lorentz force law that relates the electric and magnetic fields to measurable forces.

André Marie Ampère (1775–1836)

Carl Friedrich Gauss (1777–1855)



Michael Faraday (1791–1867)



James Clerk Maxwell (1831–1879)

Heinrich Hertz (1857–1894)

Oliver Heaviside (1850–1925)

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1.2.1 Maxwell's Equations in Integral Form

Consider an open surface S bounded by a closed contour C. The first two Maxwell's equations are given by

$$\oint_C \mathscr{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathscr{B}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S} \tag{1.2.1}$$

$$\oint_C \mathscr{B}(\mathbf{r}, t) \cdot d\mathbf{l} = \epsilon_0 \mu_0 \frac{d}{dt} \iint_S \mathscr{E}(\mathbf{r}, t) \cdot d\mathbf{S} + \mu_0 \iint_S \mathscr{J}_{\text{total}}(\mathbf{r}, t) \cdot d\mathbf{S} \tag{1.2.2}$$

where

$$\begin{aligned} \mathscr{E} &= \text{electric field intensity (volts/meter)} \\ \mathscr{B} &= \text{magnetic flux density (webers/meter}^2) \\ \mathscr{J}_{\text{total}} &= \text{electric current density (amperes/meter}^2) \\ \epsilon_0 &= \text{permittivity of free space (farads/meter)} \\ \mu_0 &= \text{permeability of free space (henrys/meter).} \end{aligned}$$

The position vector **r** and time variable t are included explicitly to indicate that the associated quantities can be functions of position and time.[2] The subscript "total" in $\mathscr{J}_{\text{total}}$ is used to denote that this is the current density of total electric currents. In the MKS unit system, the numerical values for the free-space permittivity and permeability are

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \approx \frac{1}{36\pi} \times 10^{-9} \text{ F/m} \tag{1.2.3}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.} \tag{1.2.4}$$

Equation (1.2.1) is called *Faraday's induction law,* and Equation (1.2.2) is often called *Ampère's law* or the *Maxwell–Ampère law* because Maxwell augmented the original Ampère's law with the addition of the displacement current, the first term on the right-hand side. As we will see later, this term is very important because it predicts that electromagnetic fields can propagate as waves, which was experimentally verified by Hertz in 1887. Equations (1.2.1) and (1.2.2) indicate that a time-varying magnetic flux can generate an electric field, and an electric current and a time-varying electric field can generate a magnetic field.

In a folk song about Faraday's law as expressed in Equation (1.2.1), the author, Dr. Walter Fox Smith of Haverford College, elaborated eloquently its physical meaning and practical importance in a humorous manner. He wrote

Faraday's law of induction
The law of all sea and all land—
No lies, no deceit, no corruption
In this law so complete and so grand!

Our children will sing it in chorus—
"Circulation of vector cap E,"
Yes they'll sing as they march on before us,
"Equals negative d by dt
Of—
Magnetic flux through a surface,"
They'll conclude as we strike up the band.

[2] All instantaneous quantities are represented by cursive letters to distinguish them from time-invariant quantities.

We'll mark all our coins with our purpose—
"On Maxwell's equations we stand!"

It's Faraday's law of induction
That allows us to generate pow'r.
It gives voltage increase or reduction—
We could sing on and on for an hour!

By denoting the total current and total electric flux passing through the surface S as

$$\mathscr{I}(t) = \iint_S \mathscr{J}_{\text{total}}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S} \tag{1.2.5}$$

$$\phi_{\mathrm{E}}(t) = \iint_S \mathscr{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S} \tag{1.2.6}$$

the Maxwell–Ampère law in Equation (1.2.2) can also be written as

$$\oint_C \mathscr{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \mathscr{I} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \phi_{\mathrm{E}}. \tag{1.2.7}$$

In a folk song titled "Two great guys—one great law!" Smith described the development history of this law and the contributions by Ampère and Maxwell:

Mr. Ampère's magical, mystical, wonderful law!
Of Maxwell's equations, it is the longest and strangest of all!

On the left side, he wrote circulation
Of magnetic field, 'cause it was neat.
On the right-hand side of his equation—
Mu-naught I—he thought it was complete.

Decades later, Maxwell saw disaster,
Although he thought of Ampère as a saint—
In between the plates of a capacitor
The right side's zero, but the left side ain't!

To fix this problem, he added to the right side
Displacement current, a brand new quantity!
It started mu-naught eps'lon-naught and ended by
The time derivative of phi-sub-E.

And so to Maxwell the myst'ry was revealed—
He saw how light could move through empty space.
The changing B-field made the changing E-field,
And vice-a-versa, all at the perfect pace.

Next, consider a volume V enclosed by a surface S. The other two Maxwell's equations are given by

$$\oiint_S \mathscr{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \iiint_V \varrho_{\mathrm{e,total}}(\mathbf{r}, t)\, \mathrm{d}V \tag{1.2.8}$$

$$\oiint_S \mathscr{B}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S} = 0 \tag{1.2.9}$$

where $\varrho_{\mathrm{e,total}}$ denotes the electric charge density (coulombs/meter^3) in volume V. Again, the subscript "total" is used to denote that $\varrho_{\mathrm{e,total}}$ represents the density of total charges. Equation (1.2.8) is called *Gauss' law* and Equation (1.2.9) is called Gauss' law for the magnetic case. Clearly, Equation (1.2.9) indicates that the magnetic flux lines cannot be originated or terminated anywhere; they have to form closed loops. In contrast, the electric field lines, as indicated in Equation (1.2.8), can be originated from positive charges and terminated at negative charges.

By denoting the differential surface vector $\mathrm{d}\mathbf{S} = \hat{n}\,\mathrm{d}A$ and the total charge enclosed inside V as

$$\mathscr{Q}(t) = \iiint_V \varrho_{\mathrm{e,total}}(\mathbf{r}, t)\,\mathrm{d}V \tag{1.2.10}$$

Gauss' law in Equation (1.2.8) can be rewritten as

$$\oiint_S \mathscr{E} \cdot \hat{n}\,\mathrm{d}A = \frac{\mathscr{Q}}{\epsilon_0}. \tag{1.2.11}$$

This equation is the subject of another folk song by Smith, which says

Inside, outside, count the lines to tell—
If the charge is inside, there will be net flux as well.
If the charge is outside, be careful and you'll see
The goings in and goings out are equal perfectly.
If you wish to know the field precise,
And the charge is symmetric,
  you will find this law is nice—
Q upon a constant – eps'lon naught they say—
Equals closed surface integral of E dot n dA.

Equations (1.2.1), (1.2.2), (1.2.8), and (1.2.9) are usually referred to as Maxwell's equations in integral form. They are obtained directly from experiments and are valid everywhere for any case. They have been regarded as the fundamental postulates of electromagnetic theory ever since Maxwell formulated them over 140 years ago. The entire electromagnetic theory, valid from the static to the optical regimes and from subatomic to intergalactic length scales, is based on these four equations, as we will see repeatedly in this book.

EXAMPLE 1.3

Apply Equation (1.2.1) to a closed loop in a circuit that contains a resistor, a capacitor, an inductor, and a voltage source (Fig. 1.1) and derive Kirchhoff's voltage law.

Solution Assuming that all the components in the closed loop are connected with a perfectly conducting wire, along which the electric field vanishes, and the inductor is made of a solenoid of a conducting wire, the electric field along the loop is zero except across the resistor, the capacitor, and the voltage source. Therefore, the left-hand side of

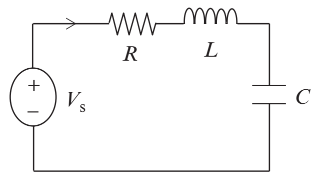

Figure 1.1 An *RLC* circuit with a voltage source.

Equation (1.2.1) becomes

$$\oint_C \boldsymbol{\mathscr{E}}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{l} = l_\mathrm{r}\mathscr{E}_\mathrm{r} + l_\mathrm{c}\mathscr{E}_\mathrm{c} - l_\mathrm{s}\mathscr{E}_\mathrm{s}$$

where l_r, l_c, and l_s denote the lengths of the resistor, the capacitor, and the voltage source, and $\mathscr{E}_\mathrm{r}$, $\mathscr{E}_\mathrm{c}$, and $\mathscr{E}_\mathrm{s}$ denote the electric fields along these components. The last term has a negative sign because the electric field in the source is opposite to the direction of the integration contour. Since $l_\mathrm{r}\mathscr{E}_\mathrm{r}$ represents the voltage drop across the resistor, which is denoted as $\mathscr{V}_\mathrm{r}$, we have

$$\oint_C \boldsymbol{\mathscr{E}}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{l} = \mathscr{V}_\mathrm{r} + \mathscr{V}_\mathrm{c} - \mathscr{V}_\mathrm{s}$$

where $\mathscr{V}_\mathrm{c}$ and $\mathscr{V}_\mathrm{s}$ represent the voltages across the capacitor and the source, respectively. If the solenoid has a length of ℓ and a cross-sectional area of s and is made of n turns, when it carries an electric current $\mathscr{I}$, the magnetic flux density inside the solenoid is $\mathscr{B} = \mu_0 n\mathscr{I}/\ell$. Hence, the right-hand side of Equation (1.2.1) becomes

$$-\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \boldsymbol{\mathscr{B}}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{S} = -\mu_0 \frac{n}{\ell}\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t} ns = -\mu_0 \frac{n^2 s}{\ell}\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}.$$

Since the inductance of the solenoid is given by $L = \mu_0 n^2 s/\ell$, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \boldsymbol{\mathscr{B}}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{S} = -L\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t} = -\mathscr{V}_\mathrm{i}$$

where $\mathscr{V}_\mathrm{i}$ denotes the voltage drop across the inductor. Rigorously speaking, we should also add the magnetic flux through the loop into this term, which would modify the value of L, but the expression would remain the same. Substituting the left- and right-hand terms derived earlier into Equation (1.2.1), we obtain

$$\mathscr{V}_\mathrm{r} + \mathscr{V}_\mathrm{c} + \mathscr{V}_\mathrm{i} - \mathscr{V}_\mathrm{s} = 0$$

which is Kirchhoff's voltage law. If the closed loop contains N components, Kirchhoff's voltage law can be expressed as

$$\sum_{i=1}^{N} \mathscr{V}_i = 0$$

which states that the sum of voltage drops along any closed loop in a circuit is always zero.

1.2.2 Maxwell's Equations in Differential Form

The integral-form Maxwell's equations are valid everywhere. Now, consider a point in a continuous medium. The fields at such a point should be continuous; therefore, we can use Stokes' and Gauss' theorems to convert Maxwell's equations in integral form into their counterparts in differential form. To be more specific, by applying Stokes' theorem to Equations (1.2.1) and (1.2.2) and using the fact that these equations are valid for any surface S, we obtain

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} \qquad \text{(Faraday's law)} \tag{1.2.12}$$

$$\nabla \times \mathscr{B} = \epsilon_0 \mu_0 \frac{\partial \mathscr{E}}{\partial t} + \mu_0 \mathscr{J}_{\text{total}} \qquad \text{(Maxwell–Ampère law)} \tag{1.2.13}$$

respectively. Here, we omit the position vector and time variable for the sake of brevity. By applying Gauss' theorem to Equations (1.2.8) and (1.2.9) and using the fact that these are valid for any volume V, we obtain

$$\nabla \cdot \mathscr{E} = \frac{\varrho_{\text{e,total}}}{\epsilon_0} \qquad \text{(Gauss' law)} \tag{1.2.14}$$

$$\nabla \cdot \mathscr{B} = 0 \qquad \text{(Gauss' law—magnetic)} \tag{1.2.15}$$

respectively. Equations (1.2.12) and (1.2.13) can also be obtained by shrinking the closed contour in Equations (1.2.1) and (1.2.2) to a point and then invoking the alternative definition of the curl given in Equation (1.1.10). Similarly, Equations (1.2.14) and (1.2.15) can also be obtained by shrinking the closed surface in Equations (1.2.8) and (1.2.9) to a point and then invoking the definition of the divergence in Equation (1.1.1). Therefore, Maxwell's equations in differential form describe the field behavior at a point in a continuous medium.

1.2.3 Current Continuity Equation

By taking the divergence of Equation (1.2.13) and applying the vector identity in Equation (1.1.42) and Gauss' law in Equation (1.2.14), we obtain

$$\nabla \cdot \mathscr{J}_{\text{total}} = -\frac{\partial \varrho_{\text{e,total}}}{\partial t}. \tag{1.2.16}$$

To understand the implication of this equation, we can simply integrate it over a finite volume and apply Gauss' theorem in Equation (1.1.5) to find

$$\oiint_S \mathscr{J}_{\text{total}} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \varrho_{\text{e,total}} \, \mathrm{d}V. \tag{1.2.17}$$

It is evident that the left-hand side represents the net current leaving the volume and the right-hand side represents the reduction rate of the total charge in the volume. As a result, this equation represents the *continuity of currents* or *conservation of charges*. Because of this continuity equation, the four Maxwell's equations are not independent for time-varying fields. This can be verified easily by taking the divergence of

Equations (1.2.12) and (1.2.13) and then applying Equations (1.2.16) and (1.1.42), respectively, which would yield Equations (1.2.14) and (1.2.15). This, however, does not hold for the static fields because for such a case the currents and charges are no longer related and the electric and magnetic fields are completely decoupled; hence, all four equations have to be considered.

EXAMPLE 1.4

Apply Equation (1.2.17) to a surface that encloses a node in a circuit and derive Kirchhoff's current law.

Solution Assuming that there are N branches of electric current connected to a node and there is no accumulation of electric charge at the node, we can apply Equation (1.2.17) to a mathematical surface that encloses the node to obtain

$$\oiint_S \mathcal{J}_{\text{total}} \cdot d\mathbf{S} = \sum_{i=1}^{N} \mathcal{I}_i = 0$$

where $\mathcal{I}_i$ denotes a signed electric current flowing away from the node. This is known as Kirchhoff's current law, which simply states that the total amount of current entering the node equals the total amount of current leaving the node.

1.2.4 The Lorentz Force Law

When a particle carrying electric charge q is placed in an electric field, it experiences a force given by $q\mathcal{E}$. When this charge is moving in a magnetic field, it experiences another force given by $q\boldsymbol{v} \times \mathcal{B}$, where $\boldsymbol{v}$ represents the velocity vector of the charge. Combining the two forces, we obtain the total force exerted on a charged particle as

$$\mathcal{F} = q(\mathcal{E} + \boldsymbol{v} \times \mathcal{B}) \tag{1.2.18}$$

which is known as the *Lorentz force law*. This law is useful for understanding the interaction between electromagnetic fields and matter, as we will discuss next. It is also the principle used in the design of many electrical devices such as electric motors, magnetrons, and particle accelerators.

1.3 CONSTITUTIVE RELATIONS

Maxwell's equations, as presented in the previous section, are valid in any kind of media. Since a medium has a significant effect on electromagnetic fields, we have to consider this effect in the study of electromagnetic fields. A medium affects electromagnetic fields through three phenomena—*electric polarization*, *magnetic polarization* or simply *magnetization*, and *electric conduction*. This section discusses these three phenomena and formulates a set of equations, known as *constitutive relations*, to account for the effect of a medium on electromagnetic fields. These constitutive relations are then used to classify media into various categories.

1.3.1 Electric Polarization

We first consider the effect of electric charges in a medium on electromagnetic fields. It is well known that a matter that makes up a medium is made of molecules, which consist of atoms. In an atom, there is a nucleus consisting of neutrons and protons. The neutrons are not charged, but the protons are positively charged. Surrounding a nucleus are negatively charged electrons, whose number equals the number of protons. These electrons are bound to the nucleus by the electric force, so they normally cannot break free; instead, they orbit around the nucleus at high speed. The center of the orbit coincides with the center of the protons so that an entire atom is electrically neutral. A molecule is made up of one or more atoms. For some molecules, the atoms are arranged such that the center of positive charges coincides with that of negative charges. This type of molecule is called a *nonpolar molecule,* and in such a case, the molecules and hence the matter that is made of nonpolar molecules appear electrically neutral. For some other molecules, the interaction between atoms creates a small displacement between the effective centers of positive and negative charges, thus creating a tiny electric dipole and generating a weak electric field. This type of molecule is called a *polar molecule*. However, since all polar molecules are randomly oriented, the effects of tiny electric dipoles cancel each other and the matter that is made of polar molecules is also electrically neutral.

The scenario described earlier changes drastically when an electric field is applied to the medium. According to the Lorentz force law, the applied electric field exerts a force on positive charges in the direction of the field, whereas it exerts a force on negative charges in the opposite direction. As a result, in both atoms and nonpolar molecules, the effective center of positive charges will be displaced from the effective center of negative charges, creating a tiny electric dipole in the direction of the electric field. (Here, we assume that the applied field is not strong enough to break the bound electrons loose from the nuclei. In such a case, the matter is often called a *dielectric*.) In the case of polar molecules, because of the Lorentz force, all the randomly oriented dipoles tend to line up with the applied electric field. When a large number of electric dipoles line in the same direction, the electric fields created by the dipoles add up and these electric fields are in the opposite direction to the applied field, resulting in a weaker total electric field in the medium. To quantify the effect of tiny dipoles, a vector quantity called the *dipole moment* is defined as

$$\boldsymbol{p} = q\boldsymbol{\ell} \tag{1.3.1}$$

where q denotes the charge and $\boldsymbol{\ell}$ denotes the vector pointing from the effective center of the negative charge to that of the positive charge. The sum of dipole moments per unit volume is then

$$\mathscr{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n_{\mathrm{p}}} \boldsymbol{p}_i \tag{1.3.2}$$

where n_{p} denotes the number of dipoles contained in Δv. The dipole moment density $\mathscr{P}$ is also called the *polarization intensity* or *polarization vector*.

When the dipole moment density is uniform, the positive charge of a dipole is completely canceled by the negative charge of the next dipole; hence, there is no net charge in the medium. However, when the dipole moment density is not uniform, the positive charge of a dipole cannot be completely canceled by the negative charge of the next dipole, resulting in a net charge at the point and hence a volume charge density. Based on the definition of divergence, this volume charge density is given by

$$\varrho_{\mathrm{e,b}} = -\nabla \cdot \mathscr{P} \tag{1.3.3}$$

where the subscript "b" is used to denote that this is the density of the bound charges. If the medium also contains free charges, the total charge density in the medium can then be expressed as

$$\varrho_{\mathrm{e,total}} = \varrho_{\mathrm{e,f}} + \varrho_{\mathrm{e,b}} = \varrho_{\mathrm{e,f}} - \nabla \cdot \mathscr{P} \tag{1.3.4}$$

where $\varrho_{\mathrm{e,f}}$ denotes the density of free electric charges. Substituting this expression into Equation (1.2.14), we obtain

$$\nabla \cdot (\epsilon_0 \mathscr{E} + \mathscr{P}) = \varrho_{\mathrm{e,f}}. \tag{1.3.5}$$

By defining a new quantity, called the *electric flux density*, as

$$\mathscr{D} = \epsilon_0 \mathscr{E} + \mathscr{P} \tag{1.3.6}$$

which has a unit of coulombs/meter^2, Equation (1.3.5) can be written as

$$\nabla \cdot \mathscr{D} = \varrho_{\mathrm{e,f}}. \tag{1.3.7}$$

This expression can be regarded as Gauss' law expressed in terms of free electric charges. In addition to the volume charge density, the electric polarization also produces an electric current when it changes in time. In view of the current continuity equation in Equation (1.2.16), the electric current density contributed by the electric polarization is

$$\mathscr{J}_{\mathrm{p}} = \frac{\partial \mathscr{P}}{\partial t}. \tag{1.3.8}$$

When this current is separated from the total current, Equation (1.2.13) can also be expressed in terms of $\mathscr{D}$ defined in Equation (1.3.6).

In most dielectric materials, the polarization intensity is usually proportional to the electric field:

$$\mathscr{P} = \epsilon_0 \chi_{\mathrm{e}} \mathscr{E} \tag{1.3.9}$$

where χ_{e} is called the *electric susceptibility*. Consequently, the electric flux density $\mathscr{D}$ is related to the electric field intensity $\mathscr{E}$ by

$$\mathscr{D} = \epsilon_0 (1 + \chi_{\mathrm{e}}) \mathscr{E} = \epsilon \mathscr{E} \tag{1.3.10}$$

where $\epsilon = \epsilon_0(1 + \chi_{\mathrm{e}})$ is called the *permittivity* of the dielectric. In engineering practice, we often use the relative permittivity, defined as $\epsilon_{\mathrm{r}} = \epsilon / \epsilon_0 = 1 + \chi_{\mathrm{e}}$, to help us memorize the value. Since χ_{e} is usually a positive number, ϵ_{r} is usually greater than 1. Equation (1.3.10) is called the *constitutive relation for the electric field*. In free space such as vacuum and air, the polarization intensity $\mathscr{P}$ either vanishes or is negligible; hence, the constitutive relation in Equation (1.3.10) becomes

$$\mathscr{D} = \epsilon_0 \mathscr{E}. \tag{1.3.11}$$

1.3.2 Magnetization

Next, we consider what happens when a magnetic field is applied to a medium. As mentioned earlier, electrons orbit the nucleus continuously in an atom. Such orbiting creates a tiny current loop, which generates a very weak magnetic field. Such a current loop can be quantified by a vector called the *magnetic dipole moment*, which is defined as

$$\boldsymbol{m} = I\boldsymbol{s} \tag{1.3.12}$$

where I denotes the current and $\boldsymbol{s}$ has a magnitude equal to the area of the current loop and a direction determined by the direction of the current flow via the right-hand rule. Quantum physics reveals that all electrons and protons rotate at high speed about their own axes, a motion called *spin*. Since electrons and protons are charged, such a rotation also creates current loops, which generate very weak magnetic fields and can be quantified by magnetic dipole moments as well. In the absence of any applied fields, the directions of all the magnetic dipoles are randomly oriented (except for those in a permanent magnet). As a result, the magnetic dipole moments cancel out macroscopically and the medium appears magnetically neutral. When a magnetic field is applied to the medium, the randomly oriented magnetic dipoles tend to align themselves either in the direction of the applied field or in the opposite direction. This produces an observable quantity called *magnetization intensity* or *magnetization vector* $\mathscr{M}$, which is defined as the sum of the magnetic dipole moments per unit volume,

$$\mathscr{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n_\mathrm{m}} \boldsymbol{m}_i \tag{1.3.13}$$

where n_m denotes the number of magnetic dipoles contained in Δv. This magnetization vector will either strengthen or weaken the total magnetic field.

When the magnetic dipole density is uniform, the electric current of a current loop is completely canceled by the current of the next current loop; hence, there is no net electric current in the medium. However, when the magnetic dipole density is not uniform, the electric current of a current loop cannot be canceled completely by the current of the next current loop, which then results in a net current at the point. Based on the definition of curl, the volume current density of this current is given by

$$\mathscr{J}_\mathrm{m} = \nabla \times \mathscr{M}. \tag{1.3.14}$$

Adding this current to the current due to the electric polarization and the free current, we have the total current in the medium

$$\mathscr{J}_\mathrm{total} = \mathscr{J}_\mathrm{p} + \mathscr{J}_\mathrm{m} + \mathscr{J}_\mathrm{f} = \frac{\partial \mathscr{P}}{\partial t} + \nabla \times \mathscr{M} + \mathscr{J}_\mathrm{f} \tag{1.3.15}$$

where $\mathscr{J}_\mathrm{f}$ denotes the density of the free electric current. Substituting this into Equation (1.2.13), we obtain

$$\nabla \times \left(\frac{\mathscr{B}}{\mu_0} - \mathscr{M} \right) = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}_\mathrm{f} \tag{1.3.16}$$

where we have also used Equation (1.3.6). By defining a new magnetic quantity, called the *magnetic field intensity*, as

$$\mathscr{H} = \frac{\mathscr{B}}{\mu_0} - \mathscr{M} \tag{1.3.17}$$

which has a unit of amperes/meter, Equation (1.3.16) can be written as

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}_{\mathrm{f}}. \tag{1.3.18}$$

This equation can be regarded as the Maxwell–Ampère law in terms of free electric currents. Note that there is no electric charge associated with $\mathscr{J}_{\mathrm{m}}$ since $\nabla \cdot \mathscr{J}_{\mathrm{m}} = \nabla \cdot (\nabla \times \mathscr{M}) \equiv 0$.

Equation (1.3.17) can also be written as

$$\mathscr{B} = \mu_0(\mathscr{H} + \mathscr{M}). \tag{1.3.19}$$

In most materials, the magnetization intensity is proportional to the magnetic field intensity:

$$\mathscr{M} = \chi_{\mathrm{m}}\mathscr{H} \tag{1.3.20}$$

where χ_{m} is called the *magnetic susceptibility*. In such a case, Equation (1.3.19) becomes

$$\mathscr{B} = \mu_0(1 + \chi_{\mathrm{m}})\mathscr{H} = \mu\mathscr{H} \tag{1.3.21}$$

where $\mu = \mu_0(1 + \chi_{\mathrm{m}})$ is called the *permeability* of the material. In engineering practice, we often use the relative permeability, defined as $\mu_{\mathrm{r}} = \mu/\mu_0 = 1 + \chi_{\mathrm{m}}$, to help us memorize the value. For most materials in reality, the magnetization is so small that $\mu_{\mathrm{r}} \approx 1$ and such materials are called *nonmagnetic*. Equation (1.3.21) is called the *constitutive relation for the magnetic field*. In free space such as vacuum and air, the magnetization intensity $\mathscr{M}$ either vanishes or is negligible; hence, the constitutive relation in Equation (1.3.21) is reduced to

$$\mathscr{B} = \mu_0\mathscr{H}. \tag{1.3.22}$$

1.3.3 Electric Conduction

In addition to the polarization and magnetization, a third phenomenon is called *conduction,* which happens in a medium containing free charges such as free electrons and ions. In the absence of any fields, these charges move in random directions so that they do not form electric currents macroscopically. However, when an electric field is applied to the medium, the free charges tend to flow either in the direction of the applied field or in the opposite direction depending on whether they are positively or negatively charged. As a result, they form electric currents, which are called *conduction currents*. In most materials, the current density of the conduction current is proportional to the electric field, which can be expressed as

$$\mathscr{J}_{\mathrm{c}} = \sigma\mathscr{E} \tag{1.3.23}$$

where σ is called the *conductivity* having a unit of siemens/meter. When the free charges such as electrons move in a medium, they collide with atomic lattices and their energy is dissipated and converted into heat. Hence, σ is also related to the dissipation of the energy. The conduction current can be regarded as a part of the free electric current.

1.3.4 Classification of Media

The preceding discussion indicates clearly that the electromagnetic properties of a medium are reflected in the following three constitutive relations:

$$\mathscr{D} = \epsilon\mathscr{E}, \qquad \mathscr{B} = \mu\mathscr{H}, \qquad \mathscr{J}_{\mathrm{c}} = \sigma\mathscr{E}. \tag{1.3.24}$$

Therefore, the three parameters ϵ, μ, and σ fully characterize the electromagnetic properties of a medium. Consequently, we can classify media based on the forms and values of these parameters.

Classification Based on the Spatial Dependence If any of ϵ, μ, or σ is a function of position in space, the medium is called *inhomogeneous* or *heterogeneous*. Otherwise, it is called a *homogeneous* medium, where $\nabla\epsilon = \nabla\mu = \nabla\sigma \equiv 0$. A homogeneous medium affects electromagnetic fields through the polarization current $\mathscr{J}_{\mathrm{p}}$ and the bound charges and currents on the surface of the medium.

Classification Based on the Time Dependence If any of ϵ, μ, or σ is a function of time, the medium is called *nonstationary*; otherwise, it is called *stationary.* Note that even if a medium is physically stationary, it can still be electrically nonstationary if its electromagnetic properties change with time.

Classification Based on the Directions of $\mathscr{D}$ and $\mathscr{B}$ If the direction of $\mathscr{D}$ is parallel to that of $\mathscr{E}$ and the direction of $\mathscr{B}$ is parallel to that of $\mathscr{H}$, the medium is called *isotropic*. Otherwise, it is called an *anisotropic* medium. For an anisotropic medium, the constitutive relations cannot be expressed in a simple form as in Equation (1.3.24). Instead, they have to be expressed as

$$\begin{bmatrix} \mathscr{D}_x \\ \mathscr{D}_y \\ \mathscr{D}_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} \mathscr{E}_x \\ \mathscr{E}_y \\ \mathscr{E}_z \end{bmatrix}, \qquad \begin{bmatrix} \mathscr{B}_x \\ \mathscr{B}_y \\ \mathscr{B}_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} \mathscr{H}_x \\ \mathscr{H}_y \\ \mathscr{H}_z \end{bmatrix} \tag{1.3.25}$$

which can be written compactly as

$$\mathscr{D} = \overline{\boldsymbol{\epsilon}} \cdot \mathscr{E}, \qquad \mathscr{B} = \overline{\boldsymbol{\mu}} \cdot \mathscr{H} \tag{1.3.26}$$

where $\overline{\boldsymbol{\epsilon}}$ and $\overline{\boldsymbol{\mu}}$ are called permittivity and permeability tensors.[3] When we discuss the reciprocity theorem, we will see that if these two tensors are symmetric, the medium is *reciprocal*; otherwise, it is *nonreciprocal*. A special case of general anisotropic media is crystals, which have a diagonal permittivity tensor,

$$\overline{\boldsymbol{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}. \tag{1.3.27}$$

In this case, if all three diagonal elements are different, the medium is called *biaxial*. If any two of the three are the same, the medium is called *uniaxial.* Of course, if all three elements

[3]A boldfaced letter with an overline denotes a tensor quantity.

are the same, the medium is *isotropic*. A further generalization of the anisotropic medium is the so-called *bianisotropic* medium, whose constitutive relations are given by

$$\mathscr{D} = \overline{\epsilon} \cdot \mathscr{E} + \overline{\xi} \cdot \mathscr{H}, \qquad \mathscr{B} = \overline{\mu} \cdot \mathscr{H} + \overline{\varsigma} \cdot \mathscr{E}. \tag{1.3.28}$$

When $\overline{\epsilon}$, $\overline{\mu}$, $\overline{\xi}$, and $\overline{\varsigma}$ reduce to scalars, the medium is called *bi-isotropic*. These kinds of materials are rare in nature, but they can be manufactured in laboratories.

Classification Based on the Field Dependence If any value of ϵ, μ, or σ depends on the field intensities $\mathscr{E}$ and $\mathscr{H}$, then the flux densities $\mathscr{D}$ and $\mathscr{B}$ and the conduction current density $\mathscr{J}_{\mathrm{c}}$ are no longer linear functions of $\mathscr{E}$ and $\mathscr{H}$. Such a medium is called *nonlinear*; otherwise, it is called *linear*. Nonlinear constitutive relations significantly complicate the study of the electromagnetic fields in the medium; nevertheless, nonlinear media do exist in nature even though their applications are not widespread.

Classification Based on the Frequency Dependence If any value of ϵ or μ depends on the frequency of the field such that $\epsilon = \epsilon(f)$ or $\mu = \mu(f)$, where f denotes the frequency, the medium is called *dispersive*; otherwise, it is called *nondispersive*. If a signal that contains multiple frequencies propagates in a dispersive medium, the shape of the signal will be distorted because different frequency components propagate at different speeds. Rigorously speaking, for a dispersive medium, the constitutive relations can no longer be written in the form of Equation (1.3.24). Because of the frequency dependence, they have to be written in terms of convolution:

$$\mathscr{D} = \epsilon_0 \mathscr{E} + \epsilon_0 \chi_{\mathrm{e}} * \mathscr{E} = \epsilon_0 \mathscr{E} + \epsilon_0 \int_{-\infty}^{t} \chi_{\mathrm{e}}(t-\tau) \mathscr{E}(\tau)\, \mathrm{d}\tau \tag{1.3.29}$$

$$\mathscr{B} = \mu_0 \mathscr{H} + \mu_0 \chi_{\mathrm{m}} * \mathscr{H} = \mu_0 \mathscr{H} + \mu_0 \int_{-\infty}^{t} \chi_{\mathrm{m}}(t-\tau) \mathscr{H}(\tau)\, \mathrm{d}\tau \tag{1.3.30}$$

where $*$ denotes the temporal convolution. The convolution is due to the fact that the medium cannot polarize and magnetize instantaneously in response to the applied field and, therefore, the polarization and magnetization vectors are related to the fields at previous times.

Classification Based on the Value of Conductivity In the static case, if $\sigma = 0$, the medium is called a *perfect dielectric* or *insulator*. On the other hand, if $\sigma \to \infty$, the medium is called a *perfect electric conductor*. In reality, there are no such things as perfect dielectrics or perfect conductors. But, in engineering practice, these are very useful concepts because the approximation of a very good conductor as a perfect conductor and the approximation of a good dielectric as a perfect dielectric can significantly simplify the analysis of electromagnetic problems. When σ has a nonnegligible finite value, the medium is called *lossy*. In the electrodynamic case, the conduction characterized by σ represents only one of the loss mechanisms. When a medium is exposed to a time-varying electromagnetic field, the polarization and magnetization can also cause losses, especially when the frequency of the field is very high. This is because the directions of time-varying electric and magnetic fields change rapidly, and, consequently, the electric and magnetic dipoles that follow the field directions change their directions as well. When these dipoles flip back and forth, the friction between the bound charges and dipoles causes energy dissipation (radiation of photons). This phenomenon can be described mathematically in the time domain as a damping term in the motion equation for the dipoles [13]; however, (and very fortunately) its description in the frequency domain is very simple. The Fourier transforms of the permittivity and permeability simply become two complex quantities with the imaginary parts representing the polarization and magnetization losses. Therefore, in addition to $\sigma = 0$, the imaginary parts of the permittivity and permeability must vanish for a medium to be a perfect dielectric.

Classification Based on the Value of Permeability As discussed earlier, when a magnetic field is applied to a medium, the randomly oriented magnetic dipoles tend to align themselves either in the direction of the applied field or in the opposite direction, producing a net magnetization intensity $\mathscr{M}$. When this net magnetization intensity is very small and its direction is opposite to the direction of the applied field, the magnetic susceptibility χ_{m} is a very small negative number and the relative permeability μ_{r} is slightly less than 1. This type of medium is called *diamagnetic.* When the net magnetization intensity is again very small but its direction is in the direction of the applied field, the magnetic susceptibility χ_{m} is a very small positive number and the relative permeability μ_{r} is slightly greater than 1. The medium is called *paramagnetic*. For both diamagnetic and paramagnetic media, the value of μ_{r} differs from 1 by any amount on the order of 10^{-4}. In most engineering applications, this difference can be neglected and μ_{r} can be practically approximated as $\mu_{\mathrm{r}} \approx 1.0$; hence, the medium can be considered as *nonmagnetic*. However, there is a type of medium in which the net magnetization intensity has a very large value and its direction is the same as that of the applied field, resulting in a large relative permeability μ_{r}. This type of medium is called *ferromagnetic*. Ferromagnetic materials usually have a high conductivity, and, hence, cannot sustain an appreciable electromagnetic field. There is yet another class of materials, called *ferrites,* which have a relatively large permeability and a very small conductivity at microwave frequencies. Because of this, ferrites find many applications in the design of microwave devices.

1.4 MAXWELL'S EQUATIONS IN TERMS OF FREE CHARGES AND CURRENTS

With the constitutive relations in Equation (1.3.24), Maxwell's equations in integral form can be written for $\mathscr{E}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{B}$ in terms of free charges and currents as

$$\oint_C \mathscr{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathscr{B} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Faraday's law)} \tag{1.4.1}$$

$$\oint_C \mathscr{H} \cdot \mathrm{d}\mathbf{l} = \frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathscr{D} \cdot \mathrm{d}\mathbf{S} + \iint_S \mathscr{J}_{\mathrm{f}} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Maxwell–Ampère law)} \tag{1.4.2}$$

$$\oiint_S \mathscr{D} \cdot \mathrm{d}\mathbf{S} = \iiint_V \varrho_{\mathrm{e,f}}\, \mathrm{d}V \qquad \text{(Gauss' law)} \tag{1.4.3}$$

$$\oiint_S \mathscr{B} \cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{(Gauss' law—magnetic).} \tag{1.4.4}$$

The free current $\mathscr{J}_{\mathrm{f}}$ includes the conduction current $\mathscr{J}_{\mathrm{c}} = \sigma\mathscr{E}$ and the current supplied by impressed sources.

Equations (1.4.1)–(1.4.4) are asymmetric because of the lack of magnetic currents and charges. Although magnetic currents and charges do not exist or have not been found so far in reality, the concepts of such currents and charges are useful because sometimes we can introduce equivalent magnetic currents and charges to simplify the analysis of some electromagnetic problems. By incorporating magnetic currents and charges, Equations (1.4.1) and (1.4.4) become

$$\oint_C \mathscr{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathscr{B} \cdot \mathrm{d}\mathbf{S} - \iint_S \mathscr{M}_\mathrm{f} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Faraday's law)} \qquad (1.4.5)$$

$$\oint\!\!\!\oint_S \mathscr{B} \cdot \mathrm{d}\mathbf{S} = \iiint_V \varrho_{\mathrm{m,f}}\, \mathrm{d}V \qquad \text{(Gauss' law—magnetic)} \qquad (1.4.6)$$

where $\mathscr{M}_\mathrm{f}$ denotes the free magnetic current density (volts/meter^2) and $\varrho_{\mathrm{m,f}}$ denotes the free magnetic charge density (webers/meter^3). With this modification, Maxwell's equations become more symmetric. The reader is cautioned not to confuse the magnetic current density $\mathscr{M}_\mathrm{f}$ with the magnetization intensity $\mathscr{M}$ used previously.

The corresponding Maxwell's equations in differential form for fields at a point in a continuous medium can be obtained by invoking Stokes' and Gauss' theorems. They can be written as

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} - \mathscr{M}_\mathrm{f} \qquad \text{(Faraday's law)} \qquad (1.4.7)$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}_\mathrm{f} \qquad \text{(Maxwell–Ampère law)} \qquad (1.4.8)$$

$$\nabla \cdot \mathscr{D} = \varrho_{\mathrm{e,f}} \qquad \text{(Gauss' law)} \qquad (1.4.9)$$

$$\nabla \cdot \mathscr{B} = \varrho_{\mathrm{m,f}} \qquad \text{(Gauss' law—magnetic).} \qquad (1.4.10)$$

The free charges and currents also satisfy the current continuity equations, which can be derived from Equations (1.4.7)–(1.4.10) by taking the divergence of Equations (1.4.7) and (1.4.8) and then applying the vector identity in Equation (1.1.42) and Gauss' laws in Equations (1.4.9) and (1.4.10). Their differential forms are given by

$$\nabla \cdot \mathscr{J}_\mathrm{f} = -\frac{\partial \varrho_{\mathrm{e,f}}}{\partial t} \qquad (1.4.11)$$

$$\nabla \cdot \mathscr{M}_\mathrm{f} = -\frac{\partial \varrho_{\mathrm{m,f}}}{\partial t}. \qquad (1.4.12)$$

The corresponding integral forms can be obtained by integrating these two equations over a finite volume and then applying Gauss' theorem in Equation (1.1.5), yielding

$$\oint\!\!\!\oint_S \mathscr{J}_\mathrm{f} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \varrho_{\mathrm{e,f}}\, \mathrm{d}V \qquad (1.4.13)$$

$$\oint\!\!\!\oint_S \mathscr{M}_\mathrm{f} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \varrho_{\mathrm{m,f}}\, \mathrm{d}V. \qquad (1.4.14)$$

Because of these continuity conditions, the four Maxwell's equations in Equations (1.4.7)–(1.4.10) are not independent for time-varying fields since Equations (1.4.9) and (1.4.10) can be derived from Equations (1.4.8) and (1.4.7), respectively.

Although Maxwell's equations for free charges and currents appear quite different from those for total charges and currents, both can be written uniformly in the form presented in this section with the charge and current densities defined based on the constitutive relations used. This is the subject of Problem 1.17. In engineering, we often prefer Maxwell's equations in terms of free charges and currents over the ones for total charges and currents because the total charges and currents are usually unknown before Maxwell's equations are solved, whereas the constitutive parameters ϵ, μ, and σ can usually be measured experimentally.

1.5 BOUNDARY CONDITIONS

The differential-form Maxwell's equations are valid at points in a continuous medium. They cannot be applied to discontinuous fields that may occur at interfaces between different media. Fortunately, we can employ Maxwell's equations in integral form to find the relations between the fields on the two sides of an interface. Such relations are called *boundary conditions*. The relationship between the integral-form Maxwell's equations and the differential-form Maxwell's equations and the boundary conditions is illustrated in Figure 1.2. In this section, we derive these boundary conditions using Maxwell's equations for free charges and currents. Hence, all the charge and current quantities used in this section are pertinent to free charges and currents.

Before deriving the boundary conditions, let us first introduce the concept of surface currents. So far, the current density $\mathscr{J}_{\rm f}$ is actually a volume current density, which is often simply called current density. It represents the amount of current passing through a unit area normal to the direction of the current flow. Now, imagine a current flow confined in a thin layer. If the total current is kept constant while the thickness of the layer is reduced to zero, the volume current density approaches infinity, which can no longer describe the current sheet. In this case, the current distribution can be described by the surface current density, which is a vector denoted as $\mathscr{J}_{\rm s}$. Its value represents the amount of current passing through a unit width normal to the direction of the current flow and has a unit of amperes/meter. The surface magnetic current density $\mathscr{M}_{\rm s}$ is defined similarly, which has a unit of volts/meter.

Now, let us consider an interface between two different media, and, for the sake of generality, a free surface current with a density of $\mathscr{J}_{\rm s}$ is assumed flowing on the interface. The normal unit vector $\hat{n}$ on the interface is defined to point from medium 1 to medium 2. To apply Equation (1.4.2), we construct a small rectangular frame with one of its sides in

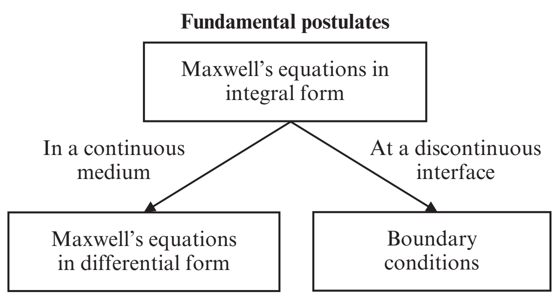

Figure 1.2 Relationship between Maxwell's equations in integral and differential forms and boundary conditions.

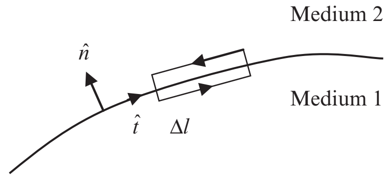

Figure 1.3 A rectangular frame across a discontinuous interface.

medium 1 and the other in medium 2, as illustrated in Figure 1.3. The length of the frame is Δl and the width Δt is vanishingly small. Applying Equation (1.4.2) to this frame and letting $\Delta t \to 0$, we have

$$\mathscr{H}_1 \cdot \hat{t}\Delta l - \mathscr{H}_2 \cdot \hat{t}\Delta l = \mathscr{J}_s \cdot (\hat{t} \times \hat{n})\Delta l \tag{1.5.1}$$

where $\hat{t}$ is a tangential unit vector as shown in Figure 1.3. Since the direction of $\hat{t}$ is not uniquely determined, it is desirable to remove it from Equation (1.5.1). For this, we rewrite $\hat{t}$ as $\hat{t} = \hat{n} \times (\hat{t} \times \hat{n})$ and employ the vector identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{1.5.2}$$

to find

$$(\hat{n} \times \mathscr{H}_2) \cdot (\hat{t} \times \hat{n}) - (\hat{n} \times \mathscr{H}_1) \cdot (\hat{t} \times \hat{n}) = \mathscr{J}_s \cdot (\hat{t} \times \hat{n}). \tag{1.5.3}$$

Since the orientation of $\hat{t}$ and, thus, $\hat{t} \times \hat{n}$, is arbitrary along the surface, we have

$$\hat{n} \times (\mathscr{H}_2 - \mathscr{H}_1) = \mathscr{J}_s \tag{1.5.4}$$

which indicates that the tangential component of the magnetic field intensity is discontinuous across an interface carrying a free surface electric current. By applying the same approach to Equation (1.4.5), we obtain another boundary condition

$$\hat{n} \times (\mathscr{E}_2 - \mathscr{E}_1) = -\mathscr{M}_s \tag{1.5.5}$$

showing a discontinuity in the tangential component of the electric field intensity across an interface carrying a free surface magnetic current. Since the magnetic current does not exist in reality, the tangential component of the electric field intensity is always continuous across any interfaces.

Next, we consider an interface between two different media and we assume a free surface charge distribution over the interface. The surface charge density is defined as the amount of charge over a unit area on the surface. To apply Equation (1.4.3), we construct a small pillbox with one of its faces in medium 1 and the other in medium 2, as illustrated in Figure 1.4. Each face of the pillbox has an area Δs and its thickness Δt is vanishingly small. Applying Equation (1.4.3) to this pillbox and letting $\Delta t \to 0$, we obtain

$$\mathscr{D}_{2n}\Delta s - \mathscr{D}_{1n}\Delta s = \varrho_{e,s}\Delta s \tag{1.5.6}$$

or

$$\hat{n} \cdot (\mathscr{D}_2 - \mathscr{D}_1) = \varrho_{e,s} \tag{1.5.7}$$

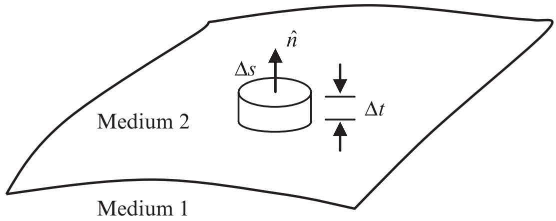

Figure 1.4 A pillbox across a discontinuous interface.

where $\varrho_{\mathrm{e,s}}$ denotes the surface electric charge density having a unit of coulombs/meter^2. This reveals that the normal component of the electric flux density is discontinuous across an interface carrying a free surface electric charge. By applying the same procedure to Equation (1.4.6), we obtain

$$\hat{n} \cdot (\mathscr{B}_2 - \mathscr{B}_1) = \varrho_{\mathrm{m,s}} \tag{1.5.8}$$

which shows that the normal component of the magnetic flux density is discontinuous across an interface carrying a free surface magnetic charge. Here, $\varrho_{\mathrm{m,s}}$ denotes the surface magnetic charge density and has a unit of webers/meter^2. However, since in reality the magnetic charges do not exist, the normal component of the magnetic flux density is always continuous across any interfaces.

Similar to the case for Maxwell's equations, the four boundary conditions in Equations (1.5.4), (1.5.5), (1.5.7), and (1.5.8) are not independent. When the first two are satisfied, the latter two are usually satisfied as well. Also note that unless one of the media is a perfect conductor, the electromagnetic fields usually cannot induce free surface charges or currents at the interface. Hence, the tangential component of the magnetic field intensity and the normal component of the electric flux density are continuous across an interface between two different media. However, when one of the media is a perfect conductor, the situation is different. A perfect conductor is a medium full of free charges. When an electromagnetic field is applied to this medium, the free charges, being pushed by the applied field, move themselves such that they produce an opposing field that completely cancels the applied field. This causes the formation of the surface currents and charges on the surface of a perfect conductor. If it is a *perfect electric conductor* (PEC), its surface can support a surface electric current and charge. If it is a *perfect magnetic conductor* (PMC), the surface can support a surface magnetic current and charge. Now, assuming that medium 1 is a PEC, the boundary conditions at the surface become

$$\hat{n} \times \mathscr{E} = 0 \tag{1.5.9}$$

$$\hat{n} \times \mathscr{H} = \mathscr{J}_{\mathrm{s}} \tag{1.5.10}$$

$$\hat{n} \cdot \mathscr{D} = \varrho_{\mathrm{e,s}} \tag{1.5.11}$$

$$\hat{n} \cdot \mathscr{B} = 0 \tag{1.5.12}$$

where the unit normal $\hat{n}$ points away from the conductor. As mentioned earlier, it is unnecessary to enforce all these conditions when solving an electromagnetic problem. It is usually sufficient to enforce either Equation (1.5.9) or (1.5.12) since the other two conditions involve the induced surface current and charge densities, which are usually unknown.

However, if the fields are known, Equations (1.5.10) and (1.5.11) provide a means to calculate the induced surface current and charge densities. The boundary conditions at the surface of a PMC can be deduced in a similar manner.

We wish to point out that the boundary conditions are as important as Maxwell's equations because they describe the field behavior across a discontinuous interface, whereas the differential-form Maxwell's equations describe the field behavior in a continuous medium, as illustrated clearly in Figure 1.2. Without boundary conditions, an electromagnetic problem is usually not completely defined and cannot be solved. Furthermore, understanding these boundary conditions can allow us to have a general idea about the field distribution in a given electromagnetic problem and help us to deal with the problem more effectively.

EXAMPLE 1.5

Derive corresponding boundary conditions from Maxwell's equations in terms of total currents and charges given in Equations (1.2.2) and (1.2.8) and compare to the boundary conditions in Equations (1.5.4) and (1.5.7), which are formulated for free currents and charges. Find the contributions of the electric polarization and magnetization vectors to the surface charges and surface currents.

Solution By applying Equation (1.2.2) to the contour shown in Figure 1.3 and following the same procedure as described there, we obtain the boundary condition

$$\hat{n} \times (\mathscr{B}_2 - \mathscr{B}_1) = \mu_0 \mathscr{J}_{\mathrm{s,total}}.$$

By substituting Equation (1.3.19) into this equation, we obtain

$$\hat{n} \times (\mathscr{H}_2 - \mathscr{H}_1) = \mathscr{J}_{\mathrm{s,total}} - \hat{n} \times (\mathscr{M}_2 - \mathscr{M}_1).$$

Comparing this with the boundary condition in Equation (1.5.4), we find that

$$\mathscr{J}_{\mathrm{s,total}} = \mathscr{J}_{\mathrm{s,f}} + \hat{n} \times \mathscr{M}_2 - \hat{n} \times \mathscr{M}_1$$

where $\mathscr{J}_{\mathrm{s}}$ in Equation (1.5.4) is denoted here as $\mathscr{J}_{\mathrm{s,f}}$ to emphasize that it represents a free surface current. This equation shows clearly that the magnetization in a medium produces a surface electric current $\mathscr{J}_{\mathrm{m,s}} = -\hat{n} \times \mathscr{M}$.

Similarly, by applying Equation (1.2.8) to the pillbox shown in Figure 1.4 and following the same procedure as described there, we obtain the boundary condition

$$\hat{n} \cdot (\mathscr{E}_2 - \mathscr{E}_1) = \frac{\varrho_{\mathrm{e,s,total}}}{\epsilon_0}.$$

On the other hand, substituting Equation (1.3.6) into the boundary condition in Equation (1.5.7) yields

$$\hat{n} \cdot (\mathscr{E}_2 - \mathscr{E}_1) = \frac{\varrho_{\mathrm{e,s,f}}}{\epsilon_0} - \frac{\hat{n} \cdot \mathscr{P}_2}{\epsilon_0} + \frac{\hat{n} \cdot \mathscr{P}_1}{\epsilon_0}$$

where $\varrho_{\mathrm{e,s}}$ in Equation (1.5.7) is denoted here as $\varrho_{\mathrm{e,s,f}}$ to emphasize that it represents a free surface charge. Comparing these two equations, we find that

$$\varrho_{\mathrm{e,s,total}} = \varrho_{\mathrm{e,s,f}} - \hat{n} \cdot \mathscr{P}_2 + \hat{n} \cdot \mathscr{P}_1$$

which clearly indicates that the electric polarization in a medium produces a bound surface electric charge with a density given by $\hat{n} \cdot \mathscr{P}$.

1.6 ENERGY, POWER, AND POYNTING'S THEOREM

Energy and power are two of the most fundamental quantities in physics. They play very important roles in electromagnetics as well. In this section, we start from Maxwell's equations and establish relations between electromagnetic fields and energy and power.

To start, we consider a medium characterized by permittivity ϵ, permeability μ, and conductivity σ. Maxwell's equations in Equations (1.4.7) and (1.4.8) in such a medium can be written as

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} - \mathscr{M}_\mathrm{i} \tag{1.6.1}$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \sigma\mathscr{E} + \mathscr{J}_\mathrm{i} \tag{1.6.2}$$

where $\mathscr{J}_\mathrm{i}$ and $\mathscr{M}_\mathrm{i}$ represent the actual source of the field and are often referred to as the *impressed currents*. In Equation (1.6.2), the total current is separated into the conduction current and the impressed current. By taking the dot product of Equation (1.6.1) with $\mathscr{H}$ and the dot product of Equation (1.6.2) with $\mathscr{E}$ and subtracting the latter from the former, we obtain

$$\mathscr{H} \cdot (\nabla \times \mathscr{E}) - \mathscr{E} \cdot (\nabla \times \mathscr{H}) = -\mathscr{E} \cdot \frac{\partial \mathscr{D}}{\partial t} - \mathscr{H} \cdot \frac{\partial \mathscr{B}}{\partial t} - \sigma\mathscr{E} \cdot \mathscr{E} - \mathscr{E} \cdot \mathscr{J}_\mathrm{i} - \mathscr{H} \cdot \mathscr{M}_\mathrm{i} \tag{1.6.3}$$

which can also be written as

$$\nabla \cdot (\mathscr{E} \times \mathscr{H}) + \mathscr{E} \cdot \frac{\partial \mathscr{D}}{\partial t} + \mathscr{H} \cdot \frac{\partial \mathscr{B}}{\partial t} + \sigma\mathscr{E} \cdot \mathscr{E} + \mathscr{E} \cdot \mathscr{J}_\mathrm{i} + \mathscr{H} \cdot \mathscr{M}_\mathrm{i} = 0 \tag{1.6.4}$$

using the vector identity $\nabla \cdot (\mathscr{E} \times \mathscr{H}) = \mathscr{H} \cdot (\nabla \times \mathscr{E}) - \mathscr{E} \cdot (\nabla \times \mathscr{H})$. To understand the physical meaning of this equation, we first integrate it over a finite volume, and by using Gauss' theorem, we obtain

$$\oiint_S (\mathscr{E} \times \mathscr{H}) \cdot \hat{n}\, \mathrm{d}S + \iiint_V \left(\mathscr{E} \cdot \frac{\partial \mathscr{D}}{\partial t} + \mathscr{H} \cdot \frac{\partial \mathscr{B}}{\partial t} + \sigma\mathscr{E} \cdot \mathscr{E} + \mathscr{E} \cdot \mathscr{J}_\mathrm{i} + \mathscr{H} \cdot \mathscr{M}_\mathrm{i} \right) \mathrm{d}V = 0 \tag{1.6.5}$$

where S is the surface enclosing V and $\hat{n}$ is the normal unit vector pointing outward. Next, we check the unit of each term. First, $\mathscr{E} \times \mathscr{H}$ has a unit volts/meter·amperes/meter = watts/meter^2, which is the unit of power flux density. A dot product with $\hat{n}$ and then integration over a closed surface S would yield total power passing through the surface, either entering or exiting. We denote this term as $\mathscr{P}_\mathrm{e}$:

$$\mathscr{P}_\mathrm{e} = \oiint_S (\mathscr{E} \times \mathscr{H}) \cdot \hat{n}\, \mathrm{d}S. \tag{1.6.6}$$

Second, we rewrite

$$\mathscr{E} \cdot \frac{\partial \mathscr{D}}{\partial t} = \frac{1}{2}\epsilon \frac{\partial \mathscr{E}^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon \mathscr{E}^2\right) = \frac{\partial w_{\mathrm{e}}}{\partial t} \tag{1.6.7}$$

where $w_{\mathrm{e}} = \frac{1}{2}\epsilon\mathscr{E}^2$. This quantity has a unit of farads/meter·(volts/meter)^2 = joules/meter^3, which represents energy density. Its integral over a volume would represent the total energy in the volume:

$$\mathscr{W}_{\mathrm{e}} = \iiint_V w_{\mathrm{e}}\, \mathrm{d}V = \frac{1}{2}\iiint_V \epsilon\mathscr{E}^2\, \mathrm{d}V. \tag{1.6.8}$$

Since this energy is associated with the electric field, it can be termed as the *electric energy*. Similarly, we find that

$$\mathscr{H} \cdot \frac{\partial \mathscr{B}}{\partial t} = \frac{1}{2}\mu \frac{\partial \mathscr{H}^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \mathscr{H}^2\right) = \frac{\partial w_{\mathrm{m}}}{\partial t} \tag{1.6.9}$$

where $w_{\mathrm{m}} = \frac{1}{2}\mu\mathscr{H}^2$ represents the magnetic energy density. Its integration over a volume represents the total *magnetic energy* in the volume:

$$\mathscr{W}_{\mathrm{m}} = \iiint_V w_{\mathrm{m}}\, \mathrm{d}V = \frac{1}{2}\iiint_V \mu\mathscr{H}^2\, \mathrm{d}V. \tag{1.6.10}$$

With these observations, we can now consider a special case, where the volume is lossless and does not contain any source. In such a case, Equation (1.6.5) can be written as

$$\mathscr{P}_{\mathrm{e}} = -\frac{\mathrm{d}(\mathscr{W}_{\mathrm{e}} + \mathscr{W}_{\mathrm{m}})}{\mathrm{d}t}. \tag{1.6.11}$$

The right-hand side represents the rate of decrease in the total energy in volume V. Based on energy conservation, the left-hand side must represent the power exiting through the surface of the volume.

With the aforementioned interpretations, we can readily find that

$$\mathscr{P}_{\mathrm{d}} = \iiint_V \sigma\mathscr{E} \cdot \mathscr{E}\, \mathrm{d}V = \iiint_V \sigma\mathscr{E}^2\, \mathrm{d}V \tag{1.6.12}$$

represents the power dissipated in the volume and

$$\mathscr{P}_{\mathrm{s}} = -\iiint_V (\mathscr{E} \cdot \mathscr{J}_{\mathrm{i}} + \mathscr{H} \cdot \mathscr{M}_{\mathrm{i}})\, \mathrm{d}V \tag{1.6.13}$$

represents the power supplied by the source. By using these notations, Equation (1.6.5) can be written as

$$\mathscr{P}_{\mathrm{s}} = \mathscr{P}_{\mathrm{e}} + \mathscr{P}_{\mathrm{d}} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{W}_{\mathrm{e}} + \mathscr{W}_{\mathrm{m}}) \tag{1.6.14}$$

which states that in a volume the supplied power must be equal to the sum of the exiting power, the dissipated power, and the rate of increase in the total energy in the volume. Obviously, Equation (1.6.14) is the statement of the *conservation of energy* for electromagnetic fields, which is also known as *Poynting's theorem*. Denote

$$\wp_e = \nabla \cdot (\mathscr{E} \times \mathscr{H}), \qquad \wp_d = \sigma \mathscr{E} \cdot \mathscr{E}, \qquad \wp_s = -(\mathscr{E} \cdot \mathscr{J}_i + \mathscr{H} \cdot \mathscr{M}_i). \tag{1.6.15}$$

Equation (1.6.4) can be written as

$$\wp_s = \wp_e + \wp_d + \frac{\partial}{\partial t}(w_e + w_m) \tag{1.6.16}$$

which is the statement of the conservation of energy in differential form. Equation (1.6.14) or (1.6.16) establishes a relation between five quantities. Knowing any four quantities, the remaining quantity can be calculated easily. This can be useful in a variety of applications where the desired quantity cannot be measured directly, but can be evaluated indirectly.

As illustrated earlier, $\mathscr{E} \times \mathscr{H}$ represents the power flux density in the direction determined by the cross-product. This quantity is named the *Poynting vector*, defined as

$$\mathscr{S} = \mathscr{E} \times \mathscr{H} \tag{1.6.17}$$

which indicates that once both the electric and magnetic fields are known at any point in space, the power flow density is determined and the power flow is perpendicular to the directions of the electric and magnetic fields. The directions of $\mathscr{E}$, $\mathscr{H}$, and $\mathscr{S}$ obey the right-hand rule.

1.7 TIME-HARMONIC FIELDS

Maxwell's equations in differential form represent a set of partial differential equations in four dimensions: three spatial dimensions and one in time. Such a mathematical problem is very difficult to deal with simply because of its high dimensionality. The complexity of the problem can be greatly reduced if the number of dimensions is lowered. Very fortunately, many problems in electrical engineering deal with time-harmonic fields—fields that oscillate at a single frequency. For such a time-harmonic field, the differentiation with time can be evaluated and the time variable can be eliminated, reducing Maxwell's equations to ones containing only three spatial variables. Since a non-time-harmonic field can be decomposed into many time-harmonic fields with different frequencies, the study of time-harmonic fields also enables the solution of general time-varying problems with the aid of the Fourier transform. This section introduces the concept of time-harmonic fields, derives its Maxwell's equations, and discusses the related energy and power.

1.7.1 Time-Harmonic Fields

When the currents, charges, and fields oscillate at a single frequency, each quantity can be expressed as a sinusoidal function with an amplitude and a phase. For example, the electric field can be written as

$$\mathscr{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) \cos[\omega t + \alpha(\mathbf{r})] \tag{1.7.1}$$

where $\mathbf{E}_0$ denotes the amplitude, α denotes the phase, and ω is the angular frequency. Using Euler's formula, this can be written as

$$\mathscr{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})\mathrm{Re}\left[\mathrm{e}^{\mathrm{j}\omega t + \mathrm{j}\alpha(\mathbf{r})}\right] = \mathrm{Re}\left[\mathbf{E}_0(\mathbf{r})\,\mathrm{e}^{\mathrm{j}\alpha(\mathbf{r})}\,\mathrm{e}^{\mathrm{j}\omega t}\right] \tag{1.7.2}$$

where $\mathrm{j} = \sqrt{-1}$ and Re stands for the real part. Now, define a complex quantity

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r})\,\mathrm{e}^{\mathrm{j}\alpha(\mathbf{r})} \tag{1.7.3}$$

which contains both the amplitude and phase of the field and is only a spatial function. Equation (1.7.2) can be written as

$$\mathscr{E}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r})\,\mathrm{e}^{\mathrm{j}\omega t}\right]. \tag{1.7.4}$$

The complex quantity defined in Equation (1.7.3) is called a *phasor*. By expressing each of the source and field quantities in the form of Equation (1.7.4) and substituting them into Equation (1.4.7), we obtain

$$\mathrm{Re}\left[\nabla \times \mathbf{E}\,\mathrm{e}^{\mathrm{j}\omega t}\right] = -\mathrm{Re}\left[\mathrm{j}\omega\mathbf{B}\,\mathrm{e}^{\mathrm{j}\omega t}\right] - \mathrm{Re}\left[\mathbf{M}_\mathrm{f}\,\mathrm{e}^{\mathrm{j}\omega t}\right]. \tag{1.7.5}$$

Since this is valid for any time variable t, we have

$$\nabla \times \mathbf{E} = -\mathrm{j}\omega\mathbf{B} - \mathbf{M}_\mathrm{f} \qquad \text{(Faraday's law)} \tag{1.7.6}$$

which no longer contains the time variable. It represents a partial differential equation in a three-dimensional space, whereas Equation (1.4.7) is an equation in a four-dimensional space. Applying the same procedure to other Maxwell's equations, we obtain

$$\nabla \times \mathbf{H} = \mathrm{j}\omega\mathbf{D} + \mathbf{J}_\mathrm{f} \qquad \text{(Maxwell–Ampère law)} \tag{1.7.7}$$

$$\nabla \cdot \mathbf{D} = \varrho_{\mathrm{e,f}} \qquad \text{(Gauss' law)} \tag{1.7.8}$$

$$\nabla \cdot \mathbf{B} = \varrho_{\mathrm{m,f}} \qquad \text{(Gauss' law—magnetic).} \tag{1.7.9}$$

Similarly, the continuity equations become

$$\nabla \cdot \mathbf{J}_\mathrm{f} = -\mathrm{j}\omega\varrho_{\mathrm{e,f}} \tag{1.7.10}$$

$$\nabla \cdot \mathbf{M}_\mathrm{f} = -\mathrm{j}\omega\varrho_{\mathrm{m,f}}. \tag{1.7.11}$$

Clearly, in this conversion, all one has to do is to replace the time derivative $\partial/\partial t$ with $\mathrm{j}\omega$. We can do the same to Maxwell's equations in integral form and boundary conditions to obtain the corresponding equations for phasors. In particular, the boundary conditions remain in the same form because they do not contain any time derivatives. Therefore, for time-harmonic fields, we only have to deal with Maxwell's equations in three dimensions. Once phasors are solved for, we can use expressions such as Equation (1.7.4) to obtain the corresponding instantaneous quantities.

Dealing with an electromagnetic problem in terms of frequency not only simplifies the problem itself but also makes the solution more interpretable. Many physical quantities of interest in electromagnetics are expressed as functions of frequency, instead of time. These quantities can be calculated directly from phasors; hence, solution to phasors is usually sufficient in many applications.

1.7.2 Fourier Transforms

The preceding introduction of the concept of phasors gives the impression that the solution to Maxwell's equations for phasors is restricted to time-harmonic fields. This is actually false. It is well known that an arbitrary function of time can be expressed as a Fourier integral,

$$\mathscr{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega \tag{1.7.12}$$

where $f(\omega)$ is called the *Fourier transform* of $\mathscr{f}(t)$ and is given by

$$f(\omega) = \int_{-\infty}^{\infty} \mathscr{f}(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t. \tag{1.7.13}$$

Accordingly, Equation (1.7.12) is called the *inverse Fourier transform*. This Fourier transform can be applied to all the source and field quantities. For example, the electric field can be written as

$$\mathscr{E}(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega)\mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega \tag{1.7.14}$$

where

$$\mathbf{E}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathscr{E}(\mathbf{r}, t)\mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t. \tag{1.7.15}$$

Substituting the Fourier transform for each of the source and field quantities into Equation (1.4.7), we obtain

$$\int_{-\infty}^{\infty} \nabla \times \mathbf{E}(\mathbf{r}, \omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega = -\int_{-\infty}^{\infty} [\,\mathrm{j}\omega \mathbf{B}(\mathbf{r}, \omega) + \mathbf{M}_\mathrm{f}(\mathbf{r}, \omega)]\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega. \tag{1.7.16}$$

Since this is valid for any time variable t, we obtain

$$\nabla \times \mathbf{E} = -\mathrm{j}\omega \mathbf{B} - \mathbf{M}_\mathrm{f} \tag{1.7.17}$$

which is identical to Equation (1.7.6). Applying this procedure to other Maxwell's equations and the continuity equations, we obtain the same equations as Equations (1.7.7)–(1.7.11). In other words, the Fourier transform of a quantity is equivalent to its phasor. Since the Fourier-transformed Maxwell's equations contain the angular frequency ω, we say that these equations are in the spectral domain or the frequency domain, whereas the original ones are in the time domain. Obviously, given an arbitrary time-varying source such as the electric current $\mathscr{J}_\mathrm{f}(\mathbf{r}, t)$, we can first find its Fourier transform $\mathbf{J}_\mathrm{f}(\mathbf{r}, \omega)$, then solve Maxwell's equations in Equations (1.7.6)–(1.7.9) for the transforms of the field quantities for all frequencies, and finally use the inverse Fourier transform such as Equation (1.7.14) to find their true values in the time domain.

The Fourier transform is a very important technique for research in all scientific and technical fields, especially in electrical engineering. On the surface, the Fourier transforms do not seem to satisfy causality because, for example, if we want to calculate the value of

$\mathscr{f}(t)$ at the instant t_0 using Equation (1.7.12), we have to know $f(\omega)$, whose calculation, according to Equation (1.7.13), requires the value of $\mathscr{f}(t)$ for all the time, including the future time with respect to t_0. A more careful examination, however, reveals that this is not true. To show this, we can first split the Fourier transform into two parts:

$$f(\omega) = \int_{-\infty}^{t_0} \mathscr{f}(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t + \int_{t_0+0}^{\infty} \mathscr{f}(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t \tag{1.7.18}$$

where $t_0 + 0 = t_0 + \varepsilon$ with $\varepsilon \to 0$. Obviously, the second integral contains the future value, $\mathscr{f}(t)$ with $t > t_0$. Now, substituting this into the inverse Fourier transform for calculating $\mathscr{f}(t_0)$, we have

$$\mathscr{f}(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{t_0} \mathscr{f}(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t\,\mathrm{e}^{\mathrm{j}\omega t_0}\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{t_0+0}^{\infty} \mathscr{f}(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t\,\mathrm{e}^{\mathrm{j}\omega t_0}\,\mathrm{d}\omega. \tag{1.7.19}$$

By exchanging the order of integration and using the Fourier transform of the delta function [14], we find that the second term of Equation (1.7.19) becomes

$$\frac{1}{2\pi}\int_{t_0+0}^{\infty} \mathscr{f}(t)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\omega(t_0-t)}\,\mathrm{d}\omega\,\mathrm{d}t = \int_{t_0+0}^{\infty} \delta(t_0 - t)\mathscr{f}(t)\,\mathrm{d}t = 0 \tag{1.7.20}$$

because t_0 is not included in the range of time integration. Hence, the future value, $\mathscr{f}(t)$ with $t > t_0$, makes no contribution to the calculation of $\mathscr{f}(t_0)$ and the Fourier transforms do not violate causality.

EXAMPLE 1.6

Consider a simple model of a dielectric medium made of molecules separated far enough so that their interactions can be ignored. Assume that the number of electrons in a unit volume is N_e and the frictional coefficient of electrons is δ. Find the electric susceptibility and permittivity of the dielectric.

Solution We can consider a system with an electron bound to a nucleus by the Coulomb force. The electron moves around the nucleus and forms an electron cloud, which can be modeled as a sphere of radius a with a charge of q_e ($q_\mathrm{e} = -1.602 \times 10^{-19}$ coulombs). When a time-harmonic electric field is applied to the dielectric, the center of the electron cloud is displaced from the nucleus with a distance $\boldsymbol{\ell}$ because of the Lorentz force $\mathscr{F}_\mathrm{L} = q_\mathrm{e}\boldsymbol{\mathscr{E}}$. Here we ignore the force from the accompanying magnetic field because it is much smaller than the electric force and we assume that the nucleus is stationary because it is much heavier than the electron. When the electron cloud center is displaced from the nucleus, the two are attracted by the Coulomb force, which can be found easily as $\mathscr{F}_\mathrm{c} = -q_\mathrm{e}^2\boldsymbol{\ell}/(4\pi\epsilon_0 a^3)$. The third force is the frictional force given by $\mathscr{F}_\mathrm{f} = -\delta m_\mathrm{e}\mathrm{d}\boldsymbol{\ell}/\mathrm{d}t$, where m_e denotes the mass of the electron ($m_\mathrm{e} = 9.109 \times 10^{-31}$ kg). With these three forces, the equation of motion for the electron becomes

$$m_\mathrm{e}\frac{\mathrm{d}^2\boldsymbol{\ell}}{\mathrm{d}t^2} = q_\mathrm{e}\boldsymbol{\mathscr{E}} - \kappa\boldsymbol{\ell} - \delta m_\mathrm{e}\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t}$$

where $\kappa = q_\mathrm{e}^2/(4\pi\epsilon_0 a^3)$. The phasor form of this equation is

$$(\mathrm{j}\omega)^2 m_\mathrm{e}\mathbf{l} = q_\mathrm{e}\mathbf{E} - \kappa\mathbf{l} - \mathrm{j}\omega\delta m_\mathrm{e}\mathbf{l}$$

which can be solved to give

$$\mathbf{l} = \frac{q_\mathrm{e}\mathbf{E}}{m_\mathrm{e}(\omega_0^2 - \omega^2 + \mathrm{j}\omega\delta)}$$

where $\omega_0 = \sqrt{\kappa/m_\mathrm{e}}$ and is called the *characteristic frequency of the electron*. The electric polarization vector is then given by

$$\mathbf{P} = N_\mathrm{e}q_\mathrm{e}\mathbf{l} = \frac{N_\mathrm{e}q_\mathrm{e}^2\mathbf{E}}{m_\mathrm{e}(\omega_0^2 - \omega^2 + \mathrm{j}\omega\delta)}.$$

Hence, the electric susceptibility is given by

$$\chi_\mathrm{e}(\omega) = \frac{N_\mathrm{e}q_\mathrm{e}^2}{\epsilon_0 m_\mathrm{e}(\omega_0^2 - \omega^2 + \mathrm{j}\omega\delta)}$$

and the relative permittivity is

$$\epsilon_\mathrm{r}(\omega) = 1 + \frac{N_\mathrm{e}q_\mathrm{e}^2}{\epsilon_0 m_\mathrm{e}(\omega_0^2 - \omega^2 + \mathrm{j}\omega\delta)}.$$

This is known as the *Lorentz model* of a dielectric medium.

1.7.3 Complex Power

Although for time-harmonic fields the instantaneous value of a field quantity is related to its phasor according to Equation (1.7.4), the same is not true for other quantities such as power and energy that involve the product of two field quantities. To see this, let us consider the product between two instantaneous quantities $\mathscr{A}(t)$ and $\mathscr{B}(t)$. We can easily find that

$$\begin{aligned}\mathscr{A}(t) \circ \mathscr{B}(t) &= \mathrm{Re}\left[\mathbf{A}\,\mathrm{e}^{\mathrm{j}\omega t}\right] \circ \mathrm{Re}\left[\mathbf{B}\,\mathrm{e}^{\mathrm{j}\omega t}\right] \\ &= \frac{1}{2}\mathrm{Re}\left[\mathbf{A} \circ \mathbf{B}^*\right] + \frac{1}{2}\mathrm{Re}\left[\mathbf{A} \circ \mathbf{B}\,\mathrm{e}^{\mathrm{j}2\omega t}\right]\end{aligned} \qquad (1.7.21)$$

where the circle denotes that the product can be either a dot or a cross-product and the star denotes the complex conjugate. If we take the time average over one cycle, we have

$$\overline{\mathscr{A}(t) \circ \mathscr{B}(t)} = \frac{1}{T}\int_0^T \mathscr{A}(t) \circ \mathscr{B}(t)\,\mathrm{d}t = \frac{1}{2}\mathrm{Re}\left[\mathbf{A} \circ \mathbf{B}^*\right] \qquad (1.7.22)$$

where $T = 2\pi/\omega$. Using this result, we can easily relate the complex field quantities to the time-average power and energy for time-harmonic fields. For example, taking the time average of the Poynting vector, we have

$$\overline{\mathscr{S}(t)} = \overline{\mathscr{E}(t) \times \mathscr{H}(t)} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E} \times \mathbf{H}^*\right]. \qquad (1.7.23)$$

By defining the complex Poynting vector as

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{1.7.24}$$

Equation (1.7.23) becomes $\overline{\mathscr{S}} = \mathrm{Re}(\mathbf{S})$. For another example, the time-average electric energy density becomes

$$\overline{w_\mathrm{e}(t)} = \frac{1}{2}\epsilon\overline{\mathscr{E}(t) \cdot \mathscr{E}(t)} = \frac{1}{4}\epsilon\mathrm{Re}\left[\mathbf{E} \cdot \mathbf{E}^*\right] = \frac{1}{4}\epsilon|\mathbf{E}|^2. \tag{1.7.25}$$

Now, let us consider the energy conservation law for time-harmonic fields. We take the time-average of Equation (1.6.16) to obtain

$$\overline{p_\mathrm{s}} = \overline{p_\mathrm{e}} + \overline{p_\mathrm{d}} + \overline{\frac{\partial w_\mathrm{e}}{\partial t}} + \overline{\frac{\partial w_\mathrm{m}}{\partial t}}. \tag{1.7.26}$$

Since from Equation (1.7.21), we can see that

$$w_\mathrm{e} = \frac{1}{2}\epsilon\mathscr{E}(t) \cdot \mathscr{E}(t) = \frac{1}{4}\epsilon\mathrm{Re}\left[\mathbf{E} \cdot \mathbf{E}^*\right] + \frac{1}{4}\epsilon\mathrm{Re}\left[\mathbf{E} \cdot \mathbf{E}\,\mathrm{e}^{\mathrm{j}2\omega t}\right] \tag{1.7.27}$$

and its time derivative is

$$\frac{\partial w_\mathrm{e}}{\partial t} = -\frac{\omega}{2}\epsilon\mathrm{Im}\left[\mathbf{E} \cdot \mathbf{E}\,\mathrm{e}^{\mathrm{j}2\omega t}\right] \tag{1.7.28}$$

we have $\overline{\partial w_\mathrm{e}/\partial t} = 0$ and, similarly, $\overline{\partial w_\mathrm{m}/\partial t} = 0$. This indicates that for time-harmonic fields, although the instantaneous energy density changes, the time average of the change vanishes. Hence, Equation (1.7.26) becomes

$$\overline{p_\mathrm{s}} = \overline{p_\mathrm{e}} + \overline{p_\mathrm{d}}. \tag{1.7.29}$$

It also follows that

$$\overline{\mathscr{P}_\mathrm{s}} = \overline{\mathscr{P}_\mathrm{e}} + \overline{\mathscr{P}_\mathrm{d}} \tag{1.7.30}$$

which is applicable to a finite volume. Equations (1.7.29) and (1.7.30) represent the energy conservation law in the time-average sense for time-harmonic fields.

The energy conservation law for time-harmonic fields can also be derived by following a procedure similar to the one described in Section 1.6. We start with the following two Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mathrm{j}\omega\mu\mathbf{H} - \mathbf{M}_\mathrm{i} \tag{1.7.31}$$

$$\nabla \times \mathbf{H} = \mathrm{j}\omega\epsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{J}_\mathrm{i}. \tag{1.7.32}$$

By taking the dot product of Equation (1.7.31) with $\mathbf{H}^*$ and the dot product of the complex conjugate of Equation (1.7.32) with $\mathbf{E}$, then subtracting the latter from the former, we obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -\mathrm{j}\omega\mu|\mathbf{H}|^2 + \mathrm{j}\omega\epsilon|\mathbf{E}|^2 - \sigma|\mathbf{E}|^2 - \mathbf{H}^* \cdot \mathbf{M}_\mathrm{i} - \mathbf{E} \cdot \mathbf{J}_\mathrm{i}^*. \tag{1.7.33}$$

Denoting

$$p_e = \frac{1}{2}\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) \tag{1.7.34}$$

$$p_d = \frac{1}{2}\sigma|\mathbf{E}|^2 \tag{1.7.35}$$

$$p_s = -\frac{1}{2}(\mathbf{H}^*\cdot\mathbf{M}_i + \mathbf{E}\cdot\mathbf{J}_i^*) \tag{1.7.36}$$

$$w_e = \frac{1}{4}\epsilon|\mathbf{E}|^2 \tag{1.7.37}$$

$$w_m = \frac{1}{4}\mu|\mathbf{H}|^2 \tag{1.7.38}$$

we can write Equation (1.7.33) as

$$p_s = p_e + p_d + j2\omega(w_m - w_e). \tag{1.7.39}$$

Integrating this over a finite volume and invoking Gauss' theorem yields its integral form

$$P_s = P_e + P_d + j2\omega(W_m - W_e) \tag{1.7.40}$$

where

$$P_e = \iiint_V p_e\,dV = \frac{1}{2}\oiint_S(\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{S} \tag{1.7.41}$$

$$P_d = \iiint_V p_d\,dV = \frac{1}{2}\iiint_V \sigma|\mathbf{E}|^2\,dV \tag{1.7.42}$$

$$P_s = \iiint_V p_s\,dV = -\frac{1}{2}\iiint_V(\mathbf{H}^*\cdot\mathbf{M}_i + \mathbf{E}\cdot\mathbf{J}_i^*)\,dV \tag{1.7.43}$$

$$W_e = \iiint_V w_e\,dV = \frac{1}{4}\iiint_V \epsilon|\mathbf{E}|^2\,dV \tag{1.7.44}$$

$$W_m = \iiint_V w_m\,dV = \frac{1}{4}\iiint_V \mu|\mathbf{H}|^2\,dV. \tag{1.7.45}$$

Here, P_e is called the complex exiting power, P_d the time-average dissipated power, P_s the complex supplied power, and W_e and W_m the time-average electric and magnetic energies, respectively.

Equations (1.7.39) and (1.7.40) are known as *Poynting's theorem for complex phasors*. Both are complex equations whose real parts yield

$$\mathrm{Re}(p_s) = \mathrm{Re}(p_e) + p_d \tag{1.7.46}$$

$$\mathrm{Re}(P_s) = \mathrm{Re}(P_e) + P_d \tag{1.7.47}$$

which are identical to Equations (1.7.29) and (1.7.30). However, if we take their imaginary parts, we obtain two more equations:

$$\mathrm{Im}(p_s) = \mathrm{Im}(p_e) + 2\omega(w_m - w_e) \tag{1.7.48}$$

$$\mathrm{Im}(P_s) = \mathrm{Im}(P_e) + 2\omega(W_m - W_e). \tag{1.7.49}$$

While the meaning of Equations (1.7.46) and (1.7.47) is very clear, the meaning of Equations (1.7.48) and (1.7.49) requires some explanation, which is attempted as follows.

From Maxwell's equations, it can be seen that the electric and magnetic fields can have a phase difference for a general time-harmonic field. The electric energy reaches its maximum value at some moments and the magnetic energy reaches its maximum at some other moments. To be more specific, within each cycle at one moment some of the magnetic energy converts into electric energy, and at another moment some of the electric energy converts into magnetic energy. This is analogous to what happens in an *LC*-circuit, where the energy stored in the inductor converts into the energy stored in the capacitor at one moment, and at another moment, the reverse happens. Now, if the maximum electric energy is not equal to, say greater than, the maximum magnetic energy in a volume, extra power is needed at the moment when the electric energy reaches its maximum value, and the same amount of power has to disappear at the other moment when the electric energy decreases and the magnetic energy reaches its maximum value. This extra power is called reactive power and, because of the power conservation, it can only come either from the source or from the power outside the volume. The source contribution is reflected by $\mathrm{Im}(P_s)$ and the external contribution is given by $\mathrm{Im}(P_e)$. Hence, $\mathrm{Im}(P_s)$ is related to the power generated by the source at one moment and then taken back at the other moment within a cycle. Similarly, $\mathrm{Im}(P_e)$ represents the power leaving the volume at one moment and then reentering at the other moment within a cycle. This reactive power does not show up in the time-average supplied power or exiting power because it takes two round trips within each cycle, but it is clearly reflected in the difference between the time-average electric and magnetic energies.

To understand the concept of reactive power better, let us consider the supplied power density for a time-harmonic source whose electric field and impressed current at a specific point are assumed to be

$$\mathbf{E} = \mathbf{E}_0\, e^{j\angle\mathbf{E}}, \qquad \mathbf{J}_i = \mathbf{J}_{i0}\, e^{j\angle\mathbf{J}_i} \tag{1.7.50}$$

where $\mathbf{E}_0$ and $\mathbf{J}_{i0}$ denote the amplitudes and $\angle\mathbf{E}$ and $\angle\mathbf{J}_i$ denote the phases of $\mathbf{E}$ and $\mathbf{J}_i$, respectively. The complex supplied power density at the point is then

$$p_s = -\frac{1}{2}\mathbf{E}\cdot\mathbf{J}_i^* = -\frac{1}{2}\mathbf{E}_0\cdot\mathbf{J}_{i0}\, e^{j(\angle\mathbf{E}-\angle\mathbf{J}_i)} \tag{1.7.51}$$

whose real and imaginary parts are

$$\mathrm{Re}(p_s) = -\frac{1}{2}\mathbf{E}_0\cdot\mathbf{J}_{i0}\cos(\angle\mathbf{E}-\angle\mathbf{J}_i) \tag{1.7.52}$$

$$\mathrm{Im}(p_s) = -\frac{1}{2}\mathbf{E}_0\cdot\mathbf{J}_{i0}\sin(\angle\mathbf{E}-\angle\mathbf{J}_i). \tag{1.7.53}$$

As discussed before, the real part represents the time-average supplied power density. To understand the imaginary part, let us examine the instantaneous power density

$$\mathscr{p}_s(t) = -\mathscr{E}(t)\cdot\mathscr{J}_i(t) = -\mathbf{E}_0\cos(\omega t+\angle\mathbf{E})\cdot\mathbf{J}_{i0}\cos(\omega t+\angle\mathbf{J}_i). \tag{1.7.54}$$

This expression can be rewritten as

$$\mathscr{p}_s(t) = -\frac{1}{2}\mathbf{E}_0 \cdot \mathbf{J}_{i0} \cos(\angle\mathbf{E} - \angle\mathbf{J}_i)\left[1 + \cos(2\omega t + 2\angle\mathbf{J}_i)\right]$$
$$+\frac{1}{2}\mathbf{E}_0 \cdot \mathbf{J}_{i0} \sin(\angle\mathbf{E} - \angle\mathbf{J}_i)\sin(2\omega t + 2\angle\mathbf{J}_i). \tag{1.7.55}$$

The first term contains $1 + \cos(2\omega t + 2\angle\mathbf{J}_i)$, which oscillates around one and is always positive. Its time average is the same as that in Equation (1.7.52). However, the second term contains $\sin(2\omega t + 2\angle\mathbf{J}_i)$, which oscillates around zero with either a positive or a negative value. It represents power generated at one moment and taken back at another moment and the time average is zero. This power is the reactive power mentioned earlier. Its peak value is the same as that in Equation (1.7.53). Hence, $\mathrm{Im}(p_s)$ represents the peak value of the reactive power density. The same interpretation can be made for the exiting power involved in Equations (1.7.48) and (1.7.49).

EXAMPLE 1.7

A metallic box having a dimension of $a \times b \times c$ is partially filled with a lossy material (Fig. 1.5). On its top surface, there is a slot having a dimension of $w \times l$. An electromagnetic wave of angular frequency ω is incident from the top and some of the electromagnetic energy enters the box. The electric and magnetic fields over the slot are measured to be

$$\mathbf{E} = \hat{y}E_0 \sin\frac{\pi x}{l}, \qquad \mathbf{H} = \hat{x}(\sqrt{3} + \mathrm{j})\frac{E_0}{2\eta}\sin\frac{\pi x}{l} \qquad 0 \le x \le l,\ 0 \le y \le w$$

where $\eta = 377\ \Omega$ and E_0 has a real value. Find the time-average power dissipated in the metallic box and the difference between the electric and magnetic energies in the box. Furthermore, find the instantaneous power entering the box.

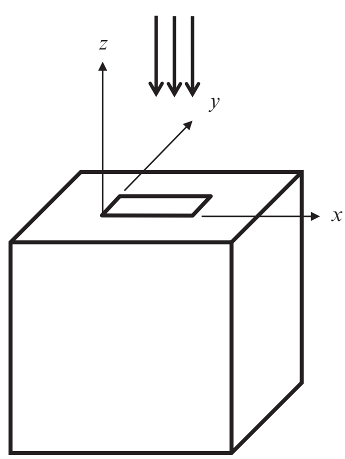

Figure 1.5 A conducting box with a slot.

Solution Given the electric and magnetic fields over the slot, we can calculate the complex power leaving the metallic box through the slot as

$$P_e = \frac{1}{2}\int_0^w \int_0^l (\mathbf{E}\times\mathbf{H}^*)\cdot\hat{z}\,dx\,dy = -(\sqrt{3}-j)\frac{wlE_0^2}{8\eta}.$$

The real part gives the time-average power leaving the box. Since its value is negative, it indicates that there is a time-average power entering the box, which is eventually dissipated. Hence, the time-average power dissipated in the box is

$$P_d = -\mathrm{Re}(P_e) = \frac{\sqrt{3}wlE_0^2}{8\eta}.$$

According to Equation (1.7.49), the difference between the electric and magnetic energies in the box is

$$W_e - W_m = \frac{1}{2\omega}\mathrm{Im}(P_e) = \frac{wlE_0^2}{16\omega\eta}.$$

Now we consider the instantaneous power entering the box. The instantaneous fields over the slot are

$$\mathscr{E} = \hat{y}E_0 \sin\frac{\pi x}{l}\cos\omega t, \qquad \mathscr{H} = \hat{x}\frac{E_0}{\eta}\sin\frac{\pi x}{l}\cos(\omega t + \pi/6).$$

Hence, the instantaneous power entering the box is

$$\mathscr{P}_{enter}(t) = \int_0^w \int_0^l (\mathscr{E}\times\mathscr{H})\cdot(-\hat{z})\,dx\,dy = \frac{wlE_0^2}{2\eta}\cos\omega t\cos(\omega t+\pi/6).$$

Evidently, because of a phase difference between the electric and magnetic fields, the instantaneous power does not always enter the box. In fact, within each period, there are two time intervals when the instantaneous power actually leaves the box.

1.7.4 Complex Permittivity and Permeability

As mentioned earlier, a medium can be lossy for a time-varying electromagnetic field because of the energy dissipation caused by the friction between bound charges and dipoles. The mathematical description of this loss is rather complicated in the time domain, but for time-harmonic fields, this loss translates into the imaginary part of a complex permittivity and/or a complex permeability. In this case, the relative permittivity and permeability can be written as

$$\epsilon_r = \epsilon_r' - j\epsilon_r'', \qquad \mu_r = \mu_r' - j\mu_r'' \tag{1.7.56}$$

where ϵ_r'' quantifies the dielectric loss and μ_r'' quantifies the magnetic loss. These two parameters are related to the electric loss tangent δ_e and the magnetic loss tangent δ_m by

$$\tan\delta_e = \frac{\epsilon_r''}{\epsilon_r'}, \qquad \tan\delta_m = \frac{\mu_r''}{\mu_r'} \tag{1.7.57}$$

which are commonly used in engineering practice. Since for a linear dispersive medium, $\epsilon(\omega)$ is the Fourier transform of $\epsilon = \epsilon_0 + \epsilon_0 \chi_e(t)$, its real and imaginary parts are related by Kramers–Krönig's relations [15, 16] given by

$$\epsilon'(\omega) = \epsilon_\infty + \frac{2}{\pi} \int_0^\infty \frac{z\epsilon''(z)}{z^2 - \omega^2}\, dz, \qquad \epsilon''(\omega) = -\frac{2\omega}{\pi} \int_0^\infty \frac{\epsilon'(z) - \epsilon_\infty}{z^2 - \omega^2}\, dz \quad (1.7.58)$$

where ϵ_∞ denotes the permittivity at an infinitely high frequency, which accounts for the contribution of the polarization that adapts instantaneously to the changes of the electric field in addition to ϵ_0. The integrals in Equation (1.7.58) are evaluated in the complex plane with the singular point $z = \omega$ excluded. Equation (1.7.58) is also called the causality condition since it is a direct result of $\chi_e(t)$ being a causal function. It indicates that the dielectric dispersion is always accompanied by dielectric loss and if one knows $\epsilon'(\omega)$ for the entire frequency spectrum, $\epsilon''(\omega)$ can be calculated and vice versa. A similar relation can be found between the magnetic dispersion and magnetic loss.

An additional benefit for dealing with time-harmonic fields in the frequency domain is that the conduction loss and the dielectric loss can often be combined in the analysis. This can be seen clearly by rewriting Equation (1.7.32) as

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{J}_i = j\omega\epsilon_0 \left[\epsilon_r' - j\left(\epsilon_r'' + \frac{\sigma}{\omega\epsilon_0}\right)\right]\mathbf{E} + \mathbf{J}_i. \qquad (1.7.59)$$

In this case, the electric loss tangent can be redefined as

$$\tan\delta_e = \left(\epsilon_r'' + \frac{\sigma}{\omega\epsilon_0}\right) \Big/ \epsilon_r' = \frac{\epsilon_r''}{\epsilon_r'} + \frac{\sigma}{\omega\epsilon_r'\epsilon_0} \qquad (1.7.60)$$

to include both the dielectric and conduction losses. Consequently, an effective ϵ_r'' can be defined to include the effect of σ, and conversely, an effective σ can be used to include the effect of ϵ_r''.

EXAMPLE 1.8

Plasma is an ionized gas consisting of negatively charged electrons and positively charged ions found naturally in the ionosphere. Both electrons and ions can move freely in the gas. Assume that the number of electrons in a unit volume is N_e and the collision frequency of electrons is ν. Find the effective permittivity of the plasma.

Solution Because ions are much heavier than electrons, we ignore the motion of ions and consider only the motion of electrons. When a time-harmonic electric field is applied to the plasma, it exerts a Lorentz force on an electron, which is given by $\mathscr{F} = q_e(\mathscr{E} + \boldsymbol{v} \times \mathscr{B})$, where q_e is the charge carried by an electron and $\mathscr{B}$ is the magnetic field accompanying the electric field $\mathscr{E}$. However, the value of $\boldsymbol{v} \times \mathscr{B}$ is much smaller than that of $\mathscr{E}$ so it can be neglected. Therefore, if we ignore the effect of the fields from ions and other electrons on the electron to be considered, which is a good assumption because the densities of both ions and electrons are quite low, the equation of motion for the electron is

$$m_e \frac{d\boldsymbol{v}}{dt} = q_e\mathscr{E} - m_e \nu \boldsymbol{v}$$

where m_e denotes the mass of an electron and the second term on the right-hand side represents the frictional force on the electron. In terms of phasors, this equation can be written as

$$j\omega m_e \mathbf{v} = q_e \mathbf{E} - m_e \nu \mathbf{v}$$

which yields

$$\mathbf{v} = \frac{q_e}{m_e(\nu + j\omega)} \mathbf{E}.$$

The electric current formed by the motion of the electrons is

$$\mathbf{J}_c = N_e q_e \mathbf{v} = \frac{N_e q_e^2}{m_e(\nu + j\omega)} \mathbf{E}$$

and when this is substituted into $\nabla \times \mathbf{H} = j\omega\epsilon_0 \mathbf{E} + \mathbf{J}_c = j\omega\epsilon_{\text{eff}} \mathbf{E}$, we obtain the effective permittivity as

$$\epsilon_{\text{eff}} = \epsilon_0 + \frac{N_e q_e^2}{j\omega m_e(\nu + j\omega)} = \epsilon_0 + \frac{\epsilon_0 \omega_p^2}{j\omega(\nu + j\omega)}$$

where $\omega_p = \sqrt{N_e q_e^2/\epsilon_0 m_e}$ and is called the *plasma frequency*. The permittivity of this form is known as the *Drude model*.

EXAMPLE 1.9

For a dispersive dielectric medium, show that its complex permittivity satisfies Kramers–Krönig's relations given in Equation (1.7.58).

Solution Consider the susceptibility function $\chi(t)$, which can be either an electric susceptibility function $\chi_e(t)$ or a magnetic susceptibility function $\chi_m(t)$. Since it is a causal function, that is, $\chi(t) = 0$ for $t < 0$, its Fourier transform can be written as

$$\chi(\omega) = \int_0^{\infty} \chi(t)\, e^{-j\omega t}\, dt$$

which is an analytic function of ω. Because $\chi(t)$ is a real function, the real part of $\chi(\omega)$, denoted as $\chi'(\omega)$, is an even function, and the imaginary part of $\chi(\omega)$, denoted as $\chi''(\omega)$, is an odd function; that is,

$$\chi'(-\omega) = \chi'(\omega), \qquad \chi''(-\omega) = -\chi''(\omega).$$

Now let us consider a closed contour integral in the complex plane ($z = z' + jz''$):

$$\oint_C \frac{\chi(z) - \chi_\infty}{z - \omega}\, dz$$

where C consists of the entire real axis except at $z = \omega$, where the contour is deviated to exclude the singular point and a lower half-circle with a radius approaching infinity to close the contour (Fig. 1.6), and $\chi_\infty = \chi(\omega \to \infty)$. Because the integrand is nonsingular inside C, according to the Cauchy integration theorem, this contour integral vanishes.

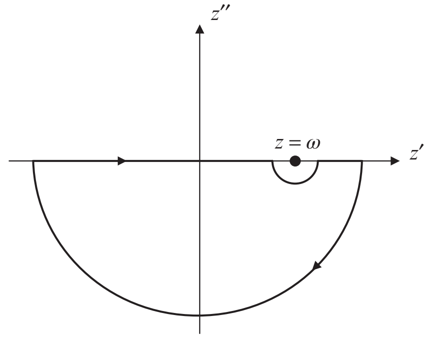

Figure 1.6 Closed contour of integration in the complex plane.

Furthermore, since $\chi(z) \to 0$ when $z'' \to -\infty$, the line integral over the lower half-circle vanishes. Therefore,

$$\left[\int_{-\infty}^{\omega-\varepsilon} \frac{\chi(z)-\chi_\infty}{z-\omega}\,\mathrm{d}z + \int_{\omega+\varepsilon}^{\infty} \frac{\chi(z)-\chi_\infty}{z-\omega}\,\mathrm{d}z\right] + \int_c \frac{\chi(z)-\chi_\infty}{z-\omega}\,\mathrm{d}z = 0$$

where c denotes a small half-circle with a radius of ε to exclude the singular point $z=\omega$. When $\varepsilon \to 0$, the integrals in the square brackets forms a principal-value integral and the integral over the small half-circle with a vanishing radius can be evaluated (by letting $z = \omega - \varepsilon\,\mathrm{e}^{\mathrm{j}\phi}$ and then integrating for ϕ from 0 to π), which yields

$$⨍_{-\infty}^{\infty} \frac{\chi(z)-\chi_\infty}{z-\omega}\,\mathrm{d}z = -\lim_{\varepsilon\to 0}\int_c \frac{\chi(z)-\chi_\infty}{z-\omega}\,\mathrm{d}z = -\mathrm{j}\pi[\chi(\omega)-\chi_\infty].$$

Taking the real and imaginary parts of this equation, we obtain

$$\chi'(\omega) = \chi_\infty + \frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{\chi''(z)}{z-\omega}\,\mathrm{d}z$$

$$\chi''(\omega) = -\frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{\chi'(z)-\chi_\infty}{z-\omega}\,\mathrm{d}z$$

where $\chi(\omega) = \chi'(\omega) - \mathrm{j}\chi''(\omega)$. By using the symmetric properties of $\chi'(\omega)$ and $\chi''(\omega)$, these equations can be written as

$$\chi'(\omega) = \chi_\infty + \frac{2}{\pi}⨍_{0}^{\infty} \frac{z\chi''(z)}{z^2-\omega^2}\,\mathrm{d}z$$

$$\chi''(\omega) = -\frac{2\omega}{\pi}⨍_{0}^{\infty} \frac{\chi'(z)-\chi_\infty}{z^2-\omega^2}\,\mathrm{d}z.$$

Substituting these into the relation $\epsilon(\omega) = \epsilon_0 + \epsilon_0\chi_{\mathrm{e}}(\omega)$, we obtain Kramers–Krönig's relations given in Equation (1.7.58), where $\epsilon_\infty = \epsilon_0 + \epsilon_0\chi_{\mathrm{e},\infty}$. Since the magnetic susceptibility function has the same property, the complex permeability satisfies the same Kramers–Krönig's relations. Also note that at an infinite frequency, since the polarization cannot adapt instantaneously to the changes of the electric field, we usually have $\chi_{\mathrm{e},\infty} = 0$ and hence $\epsilon_\infty = \epsilon_0$.

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PROBLEMS

1.1 Starting from the definition of the divergence in Equation (1.1.1), derive the expressions of the divergence in rectangular, cylindrical, and spherical coordinates as given in Equations (1.1.2)–(1.1.4). Furthermore, derive Gauss' theorem in Equation (1.1.5).

1.2 Derive the alternative definition of the curl in Equation (1.1.10) from the definition given in Equation (1.1.6). Furthermore, derive Stokes' theorem in Equation (1.1.11).

1.3 Starting from the definition of the gradient in Equation (1.1.12), derive its alternative definition in Equation (1.1.13).

1.4 Define R as the distance between point P located at (x, y, z) and point P' located at (x', y', z') and show that

$$\nabla\left(\frac{1}{R}\right) = -\frac{\mathbf{R}}{R^3}, \qquad \nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{R}}{R^3}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and ∇' operates on the primed variables.

1.5 Use the results obtained in Problem 1.4 and show that

$$\nabla \cdot \nabla\left(\frac{1}{R}\right) = -4\pi\delta(R)$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

1.6 Using the symbolic vector method, prove the following vector identities:

$$\mathbf{a} \times (\nabla \times \mathbf{b}) = (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot (\nabla \mathbf{b})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

Note that a quantity like $\nabla \mathbf{b}$ is called a dyad and a brief discussion on it can be found in Section 2.2.5.

1.7 Using the generalized Gauss' theorem, derive a new integral theorem

$$\iiint_V (\mathbf{b}\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \mathbf{b})\, \mathrm{d}V = \oiint_S (\hat{n} \cdot \mathbf{a})\mathbf{b}\, \mathrm{d}S.$$

1.8 Apply Green's theorems in Equations (1.1.45)–(1.1.48) to a vanishingly thin surface and derive the corresponding formulas that convert a surface integral to a contour integral.

1.9 The Helmholtz decomposition theorem presented in Section 1.1.3 can be stated more specifically as: A smooth vector function $\mathbf{F}(\mathbf{r})$ that vanishes at infinity can always be expressed as

$$\mathbf{F}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r})$$

where

$$\varphi(\mathbf{r}) = \frac{1}{4\pi} \iiint_V \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, \mathrm{d}V'$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \iiint_V \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, \mathrm{d}V'.$$

Prove this theorem.

1.10 A resistor can be considered as a conductive post having a finite conductivity σ, a length l, and a cross section s. Show that the total resistance is given by

$$R = \frac{l}{\sigma s}.$$

1.11 Three concentric conducting spherical shells have radii a, b, and c, and charges q_1, q_2, and q_3, respectively. Assume that $a < b < c$ (Fig. 1.7). What are the potentials on these spheres? If the innermost sphere is grounded (i.e., zero potential), what will be the change of the potential on the outmost sphere? (*Hint:* Find the potential inside, outside, and on a single spherical shell first.)

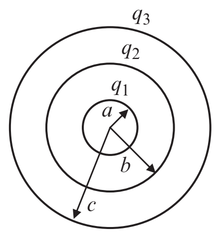

Figure 1.7 Three concentric conducting spherical shells.

1.12 An infinitely long cylindrical conductor of radius a has a hole of radius b whose axis is parallel to but offset by a distance d from the axis of the conductor (Fig. 1.8). Assume that a static current I uniformly distributed over the cross section flows along the conductor in the z-direction. What is the magnetic field on the axis of the hole? (*Hint:* Use superposition.)

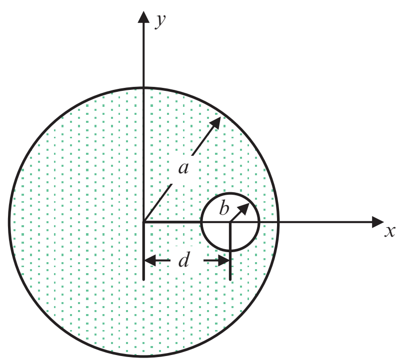

Figure 1.8 An infinitely long conductor with an offset hole.

1.13 A parallel-plate capacitor of width w and length l at a spacing d is connected to a battery of V volts. A dielectric slab of relative permittivity ϵ_r and thickness h ($h < d$) and having the same area $w \times l$ is inserted between the plates and placed on the bottom plate. Find the force on the top plate (neglect the edge effect).

1.14 A condenser consists of two parallel plates of width w and length l at a spacing d as shown in Figure 1.9. A dielectric slab of relative permittivity ϵ_r and of thickness d and the same area $w \times l$ is placed between the plates. Assume that the dielectric slab is pulled along its length from the plates so that a length x is left in-between the plates. (a) Show that if Q is the total charge on the plate, there is an electric force (neglect

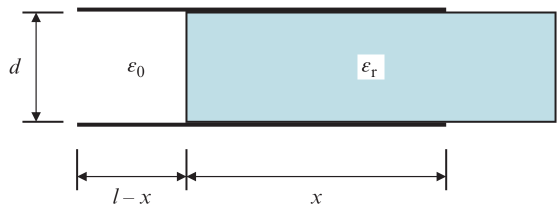

Figure 1.9 A dielectric slab inserted between two parallel plates.

the edge effect)

$$F = \frac{Q^2(\epsilon_r - 1)d}{2\epsilon_0 w[(l-x) + \epsilon_r x]^2}$$

pulling the slab back to its original position. (b) What is the charge over the dielectric slab portion x and also over the free-space portion $(l-x)$? (c) If the condenser is connected to a battery of V volts, what is the force? (*Hint:* Use the method of virtual work.)

1.15 The electric charge density is distributed symmetrically in a cylinder infinitely long in the z-direction. The charge density is given by the expression

$$\varrho_e(\rho) = \begin{cases} \varrho_0(\rho/b)^2 & \rho \le b \\ 0 & \rho > b \end{cases}$$

where ρ is the cylindrical coordinate, ϱ_0 is a constant, and b is the radius of the cylinder. (a) Using an appropriate Maxwell's equation in the integral form and the cylindrical symmetry, find expressions for the electric field in the region $\rho < b$ and the region $\rho > b$. (b) If a grounded metallic shell is added at $\rho = a$ $(a > b)$ such that the electric field $\mathbf{E} = 0$ for $\rho > a$, calculate the electric surface charge density $\varrho_{e,s}$ on the shell.

1.16 Consider two infinite planes parallel to the yz-plane (one at $x = 0$ and the other at $x = d$). The medium between the planes is characterized by permittivity ϵ_0 and permeability μ_0. The electric field between the planes is given by

$$\mathscr{E} = \hat{z} A \sin\frac{\pi x}{d} \cos\frac{\pi c t}{d}$$

where A is a constant and c is the wave velocity. Outside the planes both the electric and magnetic fields are zero. (a) Calculate the electric charge density distribution (volume if any and surface). (b) Calculate the magnetic field. (c) Calculate the electric current density distribution (volume if any and surface).

1.17 Even though Maxwell's equations can be expressed in terms of either total or free charges and currents, they can be more uniformly written as

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t}, \qquad \nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}$$

$$\nabla \cdot \mathscr{D} = \varrho_e, \qquad \nabla \cdot \mathscr{B} = 0.$$

The interpretation of the charge density ϱ_e and current density $\mathscr{J}$ depends on the constitutive relations used to relate $\mathscr{D}$ with $\mathscr{E}$ and $\mathscr{B}$ with $\mathscr{H}$. When the free-space constitutive relations $\mathscr{D} = \epsilon_0\mathscr{E}$ and $\mathscr{B} = \mu_0\mathscr{H}$ are used, the charge and current densities are those of total charges and currents, which include bound charges and currents. When the material constitutive relations $\mathscr{D} = \epsilon_0\mathscr{E} + \mathscr{P}$ and $\mathscr{B} = \mu_0(\mathscr{H} + \mathscr{M})$ are used, the charge and current densities are those of free charges and currents since the effects of bound charges and currents have already been included in the constitutive relations. Show that these two approaches are indeed equivalent by casting Maxwell's equations in terms of $\mathscr{E}$ and $\mathscr{B}$ for both cases.

1.18 Starting from Maxwell's equations in differential form in Equations (1.4.8) and (1.4.9) and boundary conditions in Equations (1.5.4) and (1.5.8), derive the corresponding Maxwell's equations in integral form in Equations (1.4.2) and (1.4.3) that are applicable to general cases that may contain arbitrary discontinuities (including the surface currents and charges).

1.19 Consider a thin sheet whose conductivity is σ and thickness is t ($t \to 0$). The product σt remains a constant as $t \to 0$. The sheet is placed in free space. (a) Find the relation between the tangential components of the electric fields on both sides of the sheet. (b) Find the relation between the tangential components of the magnetic fields on both sides of the sheet (in terms of the electric current density in the sheet). (c) Furthermore, find the relation between tangential electric and magnetic fields.

1.20 For the permittivity derived in Example 1.6 for a dielectric medium and the effective permittivity derived in Example 1.8 for a nonmagnetized plasma, find their electric susceptibility functions $\chi_e(t)$.

1.21 Consider a section of a rectangular waveguide shown in Figure 1.10. There is no source of any kind in this section of the waveguide. The transverse field components at $z = 0$ are given by

$$E_x = E_0 \sin\frac{\pi y}{b}, \qquad H_y = H_0(1 + \mathrm{j}) \sin\frac{\pi y}{b}$$

and those at $z = c$ are given by

$$E_x = \frac{E_0}{4} \sin\frac{\pi y}{b}, \qquad H_y = H_0 \sin\frac{\pi y}{b}$$

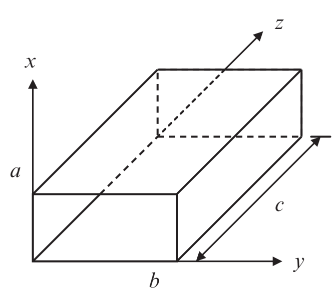

Figure 1.10 A section of a rectangular waveguide.

where E_0 and H_0 are real numbers. Find the time-average power dissipated in the waveguide. (Express your result in terms of E_0, H_0, a, and b.)

1.22 Suppose a filament of z-directed time-harmonic electric current of 5 A is impressed along the z-axis from $z = 0$ to $z = 1$ m and is completely enclosed in a perfectly conducting circular cavity filled with a lossy material (Fig. 1.11). If along the z-axis from $z = 0$ to $z = 1$ m the electric field is $\mathbf{E} = -\hat{z}\,(1 + \mathrm{j})$ V/m and the frequency is 1 kHz, determine the time-average power dissipated in the cavity and the difference between the time-average electric and magnetic energies within the cavity.

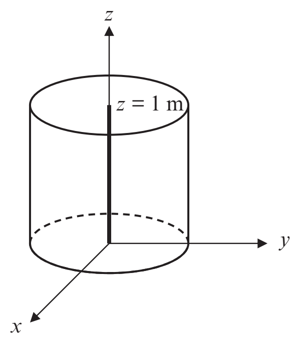

Figure 1.11 A filament of electric current placed in a circular cavity.

1.23 Figure 1.12 shows an open rectangular waveguide radiating into free space. The field at the opening is given by

$$E_y = E_0 \sin\frac{\pi x}{a}, \qquad H_x = -(1+\mathrm{j})\frac{E_0}{377}\sin\frac{\pi x}{a}.$$

Find the time-average power radiated into the free space. (Express your result in terms of E_0, a, and b.)

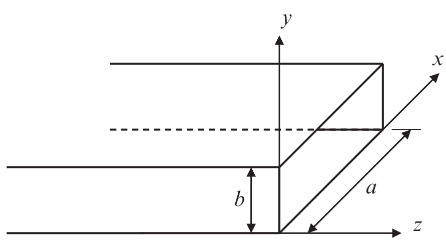

Figure 1.12 An open rectangular waveguide radiating into free space.

1.24 Consider a section of a coaxial waveguide shown in Figure 1.13. The length of the section is d and the coaxial waveguide is formed by two cylinders of radius a and b. The transverse time-harmonic field components at $z = 0$ are given by

$$\mathbf{E}|_{z=0} = \hat{\rho}\frac{A}{\rho}, \qquad \mathbf{H}|_{z=0} = \hat{\phi}\frac{B}{\rho}$$

and those at $z = d$ are given by

$$\mathbf{E}|_{z=d} = \hat{\rho}\frac{\mathrm{j}C}{\rho}, \qquad \mathbf{H}|_{z=d} = \hat{\phi}\frac{(D + \mathrm{j}E)}{\rho}$$

where A, B, C, D, and E are real numbers. (a) Find the time-average power dissipated or gained within the volume between the planes $z = 0$ and $z = d$. (b) What are the conditions on A, B, C, D, and E (or part of them) leading to (1) dissipated time-average power and (2) gained time-average power?

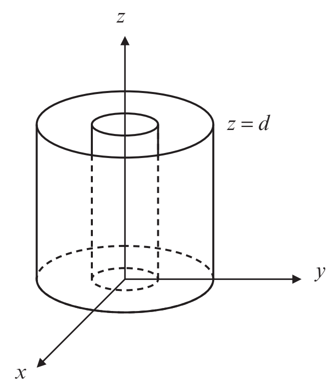

Figure 1.13 A section of a coaxial waveguide.

1.25 Show that the permittivity derived in Example 1.6 for a dielectric medium and the effective permittivity derived in Example 1.8 for a nonmagnetized plasma satisfy Kramers–Krönig's relations given in Equation (1.7.58).