1

RF Transmission Lines

1.1 Introduction

Transmission lines, in the form of cable and circuit interconnects, are essential components in RF and microwave systems. Furthermore, many distributed planar components rely on transmission line principles for their operation. This chapter will introduce the concepts of RF transmission along guided structures, and provide the foundations for the development of distributed components in subsequent chapters.

Four of the most common forms of RF and microwave transmission line are shown in Figure 1.1.

(i) *Coaxial cable* is an example of a shielded transmission line, in which the signal conductor is at the centre of a cylindrical conducting tube, with the intervening space filled with lossless dielectric. The dielectric is normally solid, although for higher-frequency applications it is often in the form of dielectric vanes so as to create a semi-air-spaced medium with lower transmission losses. A typical coaxial cable is flexible with an outer diameter around 5 mm, although much smaller diameters are available with 1 mm diameter cable being used for interconnections within millimetre-wave equipment. Also, for very high-frequency applications, the cable may have a rigid or semi-rigid construction. Further data on coaxial cables are provided in Appendix 1.A.

(ii) *Coplanar waveguide* (CPW), in which all the conductors are on the same side of the substrate, is also shown in Figure 1.1. This type of structure is very convenient for the mounting of active components, and also for providing isolation between signal tracks. Coplanar lines are widely used in compact integrated circuits for high-frequency applications. Further data on coplanar lines are given in Appendix 1.B.

(iii) *Waveguide*, formed from hollow metal tubes of rectangular or circular cross-section, is a traditional form of transmission line used for microwave frequencies above 1 GHz. For many circuit and interconnection applications, waveguide has been superseded by planar structures, and its use in modern RF and microwave systems is restricted to rather specialized applications. It is the only transmission line that can support the very high powers required in some transmitter applications. Another advantage of an air-filled metal waveguide is that it is a very low loss medium and therefore can be used to make very high-Q cavities, and this application is discussed in more detail in Chapter 3 in relation to dielectric measurements. A more recent application of traditional waveguides is in substrate integrated waveguide (SIW) structures for millimetre-wave applications, and this is explained in more detail in Chapter 4 in the context of emerging technologies. Further data on the theory of waveguides are given in Appendix 1.C.

(iv) *Microstrip* is the most common form of interconnection used in planar circuits for RF and microwave applications. As shown in Figure 1.1, it consists of a low-loss insulating substrate, with one side completely covered with a conductor to form a ground plane, and a signal track on the other side. Further data on microstrip are given in Appendix 1.D. This is a particularly important medium for high-frequency circuit design and so Chapter 2 is devoted to an in-depth discussion of microstrip and the associated design techniques.

1.2 Voltage, Current, and Impedance Relationships on a Transmission Line

In its simplest form, a transmission line can be viewed as a two-conductor structure with a go and return path for the current. For the purpose of analysis we may regard any transmission line as made up of a large number of very short lengths (δz), each of which can be represented by a lumped equivalent circuit, as shown in Figure 1.2. In the equivalent circuits, R and L

RF and Microwave Circuit Design: Theory and Applications, First Edition. Charles E. Free and Colin S. Aitchison.

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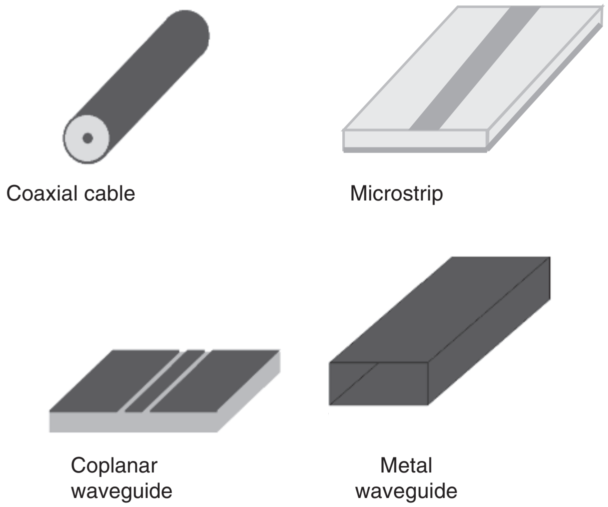

Figure 1.1 Common types of high-frequency transmission line.

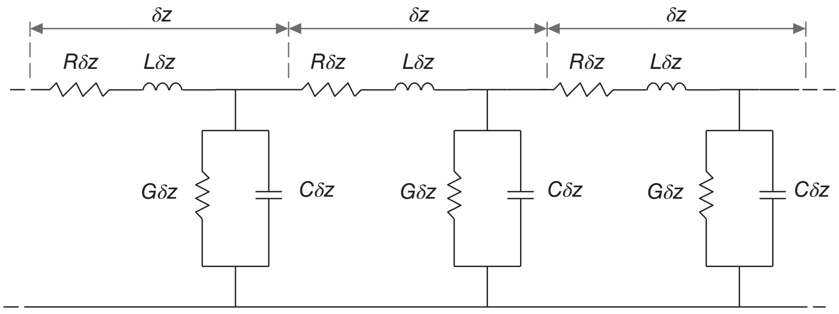

Figure 1.2 Representation of a transmission line in terms of lumped components.

represent the series resistance and inductance per unit length of the conductors, respectively, C represents the capacitance between the lines per unit length, and G is the parallel conductance per unit length, and represents the very high resistance of the insulating medium between the conductors.

It should be noted that it is legitimate to represent a continuous transmission line by the lumped equivalent circuit shown in Figure 1.2 providing that δz is small compared to a wavelength. R, L, G, and C are normally referred to as the primary line constants, and have the units of Ω/m, H/m, S/m, and F/m, respectively.

In order to establish relationships between the voltage and current on a transmission line we need first to specify a line excited by a sinusoidal voltage at the sending end whose angular frequency is ω. If we then let the voltage and current at some arbitrary point on the line be V and I, respectively, we can consider the effect on an elemental length at this point. The voltage drop across the elemental length will be δV and the parallel current will be δI, as shown in Figure 1.3.

Using standard AC circuit theory, we can relate the change in voltage, δV, to the components of the equivalent circuit as

$$\begin{aligned} -\delta V &= (R\delta z)I + (L\delta z)\frac{\partial I}{\partial t} \\ &= (R\delta z)I + (L\delta z)j\omega I, \end{aligned}$$

i.e.

$$\frac{\delta V}{\delta z} = -(R + j\omega L)I.$$

Considering the limit, as $\delta z \to 0$, $\dfrac{\delta V}{\delta z} \to \dfrac{dV}{dz}$ giving

$$\frac{dV}{dz} = -(R + j\omega L)I. \tag{1.1}$$

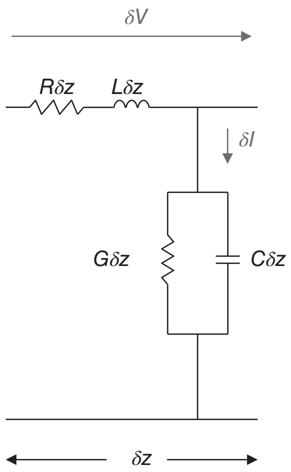

Figure 1.3 Equivalent circuit of an elemental length, δz, of a transmission line.

Considering the parallel current, δI, we have

$$\begin{aligned}-\delta I &= (G\delta z)V + (C\delta z)\frac{\partial V}{\partial t}\\ &= (G\delta z)V + (C\delta z)j\omega V,\end{aligned}$$

i.e.

$$\frac{\delta I}{\delta z} = -(G + j\omega C)V.$$

As $\delta z \to 0$, $\dfrac{\delta I}{\delta z} \to \dfrac{dI}{dz}$ giving

$$\frac{dI}{dz} = -(G + j\omega C)V. \tag{1.2}$$

Differentiating Eq. (1.1) with respect to time gives

$$\frac{d^2V}{dz^2} = -(R + j\omega L)\frac{dI}{dt}.$$

Substituting for $\dfrac{dI}{dz}$ from Eq. (1.2) gives

$$\frac{d^2V}{dz^2} = (R + j\omega L)(G + j\omega C)V,$$

which can be written as

$$\frac{d^2V}{dz^2} = \gamma^2 V, \tag{1.3}$$

where

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}. \tag{1.4}$$

Similarly

$$\frac{d^2I}{dz^2} = \gamma^2 I. \tag{1.5}$$

To determine the variation of V along the line, we have to solve the differential Eq. (1.3) for V. This is a second-order differential equation with a standard solution in the form

$$V = V_1 e^{-\gamma z} + V_2 e^{+\gamma z}. \tag{1.6}$$

The two terms on the right-hand side of Eq. (1.6) show how the peak amplitudes and phases of waves travelling in the forward and reverse directions vary with distance. The values of the amplitudes and phases of these waves are determined by the value of γ, which is defined as the propagation constant (this is considered in more detail in Section 1.3).

Differentiating the expression in Eq. (1.6) gives

$$\frac{dV}{dz} = -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{\gamma z}. \tag{1.7}$$

Combining Eqs. (1.7) and (1.1) gives

$$-(R + j\omega L)I = -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{\gamma z},$$

i.e.

$$I = \frac{\gamma}{(R + j\omega L)} V_1 e^{-\gamma z} - \frac{\gamma}{(R + j\omega L)} V_2 e^{\gamma z}. \tag{1.8}$$

Remembering that $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ we can rewrite Eq. (1.8) as

$$I = \sqrt{\frac{(G + j\omega C)}{(R + j\omega L)}} V_1 e^{-\gamma z} - \sqrt{\frac{(G + j\omega C)}{(R + j\omega L)}} V_2 e^{\gamma z}$$

or

$$I = \frac{V_1}{Z_O} e^{-\gamma z} - \frac{V_2}{Z_O} e^{\gamma z}, \tag{1.9}$$

where

$$Z_O = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}}. \tag{1.10}$$

The impedance, Z_O, is termed the characteristic impedance of the transmission line. Characteristic impedance is an important property of any transmission line and it is useful to have an appreciation of its physical significance. Theoretically, it is the ratio of the voltage to current at an arbitrary position on an infinitely long transmission line that supports a wave travelling in one direction. If the line is lossless, i.e. $R = 0$ and $G = 0$, then we see from Eq. (1.10) that $Z_O = \sqrt{L/C}$ and has a constant value that is independent of frequency. It follows that if such a line is terminated by an impedance equal to the characteristic impedance, there will be no reflections from the termination. Moreover, if a transmission line is terminated with its characteristic impedance, then the impedance at the input of the line will be equal to the characteristic impedance; under these conditions the line is said to be matched.

Considering the sending end of the line, i.e. $z = 0$, then from Eqs. (1.6) and (1.9) we obtain

$$\begin{aligned} V &= V_S = V_1 + V_2, \\ I &= I_S = \frac{V_1 - V_2}{Z_O}, \end{aligned} \tag{1.11}$$

where V_S and I_S are the voltage and current at the sending end of the line, respectively.

Rearranging Eq. (1.11) to obtain V_1 and V_2 gives:

$$\begin{aligned} V_1 &= \frac{V_S + Z_O I_S}{2}, \\ V_2 &= \frac{V_S - Z_O I_S}{2}. \end{aligned} \tag{1.12}$$

The voltage, V, and current, I, at any distance, z, along the transmission line can now be found in terms of the voltage and current at the sending end by substituting V_1 and V_2 from Eq. (1.12) into Eqs. (1.6) and (1.9) giving

$$\begin{aligned} V &= \frac{V_S + Z_O I_S}{2} e^{-\gamma z} + \frac{V_S - Z_O I_S}{2} e^{\gamma z} \\ &= V_S \left(\frac{e^{\gamma z} + e^{-\gamma z}}{2} \right) - I_S Z_O \left(\frac{e^{\gamma z} - e^{-\gamma z}}{2} \right). \end{aligned} \tag{1.13}$$

Equation (1.13) may be written in terms of hyperbolic functions as

$$V = V_S \cosh(\gamma z) - I_S Z_O \sinh(\gamma z). \tag{1.14}$$

Similarly,

$$I = I_S \cosh(\gamma z) - \frac{V_S}{Z_O} \sinh(\gamma z). \tag{1.15}$$

The impedance, Z_z, at any distance z from the sending end of the line can now be found by dividing Eq. (1.14) by Eq. (1.15) giving

$$Z_z = \frac{V}{I} = \left(\frac{Z_S \cosh(\gamma z) - Z_O \sinh(\gamma z)}{Z_O \cosh(\gamma z) - Z_S \sinh(\gamma z)}\right) Z_O, \tag{1.16}$$

where $Z_S = \frac{V_S}{I_S}$ is the impedance at the sending end of the line.

If we now consider a transmission line of finite length, l, terminated by an arbitrary impedance, Z_L, then $Z_z = Z_L$ when $z = l$, and Eq. (1.16) can be rewritten as

$$Z_L = \left(\frac{Z_S \cosh(\gamma l) - Z_O \sinh(\gamma l)}{Z_O \cosh(\gamma l) - Z_S \sinh(\gamma l)}\right) Z_O. \tag{1.17}$$

The input impedance, Z_{in}, of a transmission line terminated by an impedance, Z_L, can be found by rearranging Eq. (1.17) to give

$$Z_{in} = Z_S = \frac{Z_O(Z_L \cosh(\gamma l) + Z_O \sinh(\gamma l))}{(Z_O \cosh(\gamma l) + Z_L \sinh(\gamma l))}. \tag{1.18}$$

This is an important, but complicated, expression giving the input impedance, Z_{in}, of a transmission line terminated in Z_L in terms of the propagation constant, γ, the characteristic impedance, Z_O, and the line length, l. If the line is low loss, Eq. (1.18) can be significantly simplified, as is shown later in the chapter.

1.3 Propagation Constant

The propagation constant was introduced in Eq. (1.4). This constant determines the amplitude and phase of a wave propagating along a transmission line at a particular frequency, and may conveniently be expressed as

$$\gamma = \alpha + j\beta, \tag{1.19}$$

where α is the attenuation constant and β is the phase propagation constant. Considering the first term on the right-hand side of Eq. (1.6) we have

$$V_F = V_1 e^{-\gamma z} = V_1 e^{-(\alpha+j\beta)z} = V_1 e^{-\alpha z} e^{-j\beta z},$$

where V_F represents the voltage of the forward wave at a distance l along the line. The magnitude of this wave is given by

$$|V_F| = |V_1| e^{-\alpha z}.$$

Rearranging this equation gives

$$\alpha = -\log_e \left|\frac{V_F}{V_1}\right| = -\ln \left|\frac{V_F}{V_1}\right|. \tag{1.20}$$

Taking the natural logarithm of the ratio of two voltages gives the ratio in the units of Nepers; so α will have the units of Np/m (Nepers/metre). Although Nepers are not in common use as a unit in RF work, it is important to be able to convert a voltage ratio from Neper to the more usual power unit of dB.

Considering a voltage ratio, α, we have

$$\alpha_{\text{Np}} = \log_e \alpha \quad \Rightarrow \quad \alpha = e^{\alpha_{\text{Np}}}, \tag{1.21}$$

$$\alpha_{\text{dB}} = 20 \log_{10} \alpha \quad \Rightarrow \quad \alpha = 10^{\alpha_{\text{dB}}/20}. \tag{1.22}$$

Therefore,

$$e^{\alpha_{\text{Np}}} = 10^{\alpha_{\text{dB}}/20},$$

i.e.

$$\alpha_{\text{Np}} \log_e e = \frac{\alpha_{\text{dB}}}{20} \log_e 10,$$

$$\alpha_{\text{Np}} = 0.115 \times \alpha_{\text{dB}}, \tag{1.23}$$

i.e.

$$1 \ \text{Np} \equiv 8.686 \ \text{dB}. \tag{1.24}$$

The imaginary part of the propagation constant gives the transmission phase change experienced by the wave in travelling a distance, z. Since there are 2π radians in one wavelength, the phase propagation constant is always given by

$$\beta = \frac{2\pi}{\lambda}, \tag{1.25}$$

where λ is the wavelength along the line being considered. It follows from Eq. (1.25) that the phase propagation constant has the units of rad/m (radians/metre).

1.3.1 Dispersion

The foregoing theory describes the propagation along a transmission line at a single frequency. But since all information-carrying signals contain more than one frequency, it is important to know how the propagation characteristics of a line change with frequency.

If all the frequencies contained in a signal travel at the same velocity, the transmission line is said to be dispersionless. If this is not so, and if the phase velocity,[1] v_p, is a function of frequency, the transmission line will exhibit dispersion. If dispersion is present, a signal containing a number of frequency components, such as a voltage pulse, will become distorted as it propagates along the line, with the degree of distortion increasing with the distance of propagation.

A useful concept in determining the degree of dispersion is group velocity, v_g. We can explain this concept by considering the transmission of a signal which consists of a number of sinusoids, each having a different frequency and amplitude. These frequency components will combine to form a composite pattern, with a particular envelope. The group velocity is the velocity with which this envelope propagates along the transmission line. It can be shown that the group velocity is given by

$$v_g = \frac{\delta\omega}{\delta\beta}. \tag{1.26}$$

If this velocity is independent of frequency, then the line will be dispersionless and the phase relationships between the frequency components of the signal will be maintained.

The reciprocal of the group velocity is known as the group delay, and is the slope of the β–ω response at a particular frequency. If β is a linear function of frequency, then the β–ω response will be a straight line and the group delay will be constant, and independent of frequency. To avoid distortion it is important that the group delay is constant over the full frequency range of the signal being transmitted.

1.3.2 Amplitude Distortion

Amplitude distortion will occur if the attention constant, α, is a varying function of frequency, thus causing the frequency components of a complex signal to suffer different amplitude changes as the signal propagates. For no attenuation distortion to occur, we require $\frac{\delta\alpha}{\delta f} = 0$. Normally attenuation distortion is not significant for RF and microwave circuit interconnections, since these interconnections are short and deliberately designed to be low loss.

1.4 Lossless Transmission Lines

The majority of transmission lines encountered in RF and microwave circuits are both short and deliberately manufactured to have low dissipative losses. Consequently, it is useful to consider how the foregoing theory is modified by considering transmission lines to be lossless.

A lossless line will have $R = G = 0$, and Eq. (1.10) will be modified such that the characteristic impedance of the line is real and given by

$$Z_O = \sqrt{\frac{L}{C}}. \tag{1.27}$$

Also, since there are zero losses in the line, the attenuation coefficient, α, will be zero and the propagation coefficient will only represent the phase behaviour of the wave on the line, i.e.

$$\gamma = j\beta = j\omega\sqrt{LC} \tag{1.28}$$

1 The phase velocity is the velocity with which phase of a sinusoid is transmitted down the line. It is represented by v_p, where $v_p = \omega/\beta$. Phase velocity is discussed in more detail in Appendix 1.C.5, in relation to propagation through waveguides.

and

$$\beta = \omega\sqrt{LC}. \tag{1.29}$$

With $\gamma = j\beta$, the expression for the input impedance of a transmission line is also modified, and Eq. (1.18) becomes

$$Z_{in} = \frac{Z_O(Z_L \cosh(j\beta l) + Z_O \sinh(j\beta l))}{(Z_O \cosh(j\beta l) + Z_L \sinh(j\beta l))}. \tag{1.30}$$

Recalling that $\cosh(jx) = \cos(x)$ and $\sinh(jx) = j\sin(x)$, Eq. (1.30) can be rewritten as

$$Z_{in} = \frac{Z_O(Z_L \cos(\beta l) + jZ_O \sin(\beta l))}{(Z_O \cos(\beta l) + jZ_L \sin(\beta l))} \tag{1.31}$$

or

$$Z_{in} = \frac{Z_O(Z_L + jZ_O \tan(\beta l))}{(Z_O + jZ_L \tan(\beta l))}. \tag{1.32}$$

This is an important expression giving the input impedance, Z_{in}, of a loss-free transmission line terminated in Z_L in terms of the phase propagation constant, β, characteristic impedance, Z_O, and line length, l.

Example 1.1 The following line constants apply to a lossless transmission line operating at 100 MHz: $L = 0.5\ \mu\text{H/m}$, $C = 180\ \text{pF/m}$.

Determine:

(i) The characteristic impedance of the line.
(ii) The phase propagation constant.
(iii) The velocity of propagation on the line.
(iv) The phase change over a 20 cm length of the line.

Solution

(i) $Z_O = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.5 \times 10^{-6}}{180 \times 10^{-12}}}\Omega = 52.7\ \Omega,$

(ii) $\beta = \omega\sqrt{LC} = 2\pi \times 10^8 \times \sqrt{0.5 \times 10^{-6} \times 180 \times 10^{-12}}\ \text{rad/m} = 5.96\ \text{rad/m},$

(iii) $v_P = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times 10^{-6} \times 180 \times 10^{-12}}}\ \text{m/s} = 1.05 \times 10^8\ \text{m/s},$

(iv) $\phi = \beta l = 5.96 \times 0.2\ \text{rad} = 1.192\ \text{rad} \equiv 68.29°.$

Example 1.2 A particular lossless transmission line has a characteristic impedance of 75 Ω, and a phase constant of 4 rad/m. Determine the input impedance of a 30 cm length of the transmission line when it is terminated by an impedance of $(100 - j50)\ \Omega$.

Solution

$$\phi = \beta l = 4 \times 0.3\ \text{rad} = 01.2\ \text{rad} \equiv 68.7°.$$

Using Eq. (1.32):

$$\begin{aligned} Z_{in} &= \frac{Z_O(Z_L + jZ_O \tan(\beta l))}{(Z_O + jZ_L \tan(\beta l))} \\ &= \frac{75 \times (100 - j50 + j75 \times \tan(68.7°))}{75 + j(100 - j50) \times \tan(68.7°)}\ \Omega \\ &= 39.87\angle 3.26°\ \Omega \\ &\equiv (39.81 + j2.27)\ \Omega. \end{aligned}$$

1.5 Matched and Mismatched Transmission Lines

Using Eq. (1.32), we can establish the conditions for matching a lossless transmission line of characteristic impedance, Z_O, terminated in a load, Z_L:

(i) *Matched line*. If $Z_L = Z_O$, then $Z_{in} = Z_O$ and the line is described as being matched, and all of the energy travelling from the sending end will be absorbed by the load and there will be no reflected (reverse) wave.

(ii) *Totally mismatched line*. If the load impedance is replaced by either a short-circuit, or an open-circuit, the line is described as being totally mismatched and no energy will be dissipated in the termination. The input impedance is then entirely reactive.
With $Z_L = 0$ (i.e. a short-circuit) Eq. (1.32) becomes:

$$Z_{in} = Z_{S/C} = jZ_O \tan(\beta l). \tag{1.33}$$

With $Z_L = \infty$ (i.e. an open-circuit) Eq. (1.32) becomes:

$$Z_{in} = Z_{O/C} = \frac{Z_O}{j\tan(\beta l)} = -jZ_O \cot(\beta l). \tag{1.34}$$

(iii) *Partially mismatched line*. With an arbitrary value of load impedance, some of the incident energy will be reflected from the termination, giving rise to a standing wave as described in Section 1.6.

1.6 Waves on a Transmission Line

If $Z_L \neq Z_O$, a transmission line will be mismatched and some of the energy will be reflected from the load. Under these circumstances, the incident and reflected travelling waves will interact to form an interference pattern. Since the incident and reflected waves must be at the same frequency if the load is a passive impedance, the interference pattern will take the form of a standing wave, with the maxima and minima of the pattern in fixed positions. The distance between two adjacent maxima or minima must be $\lambda/2$, where λ is the wavelength on the line. A typical voltage standing wave pattern is shown in Figure 1.4.

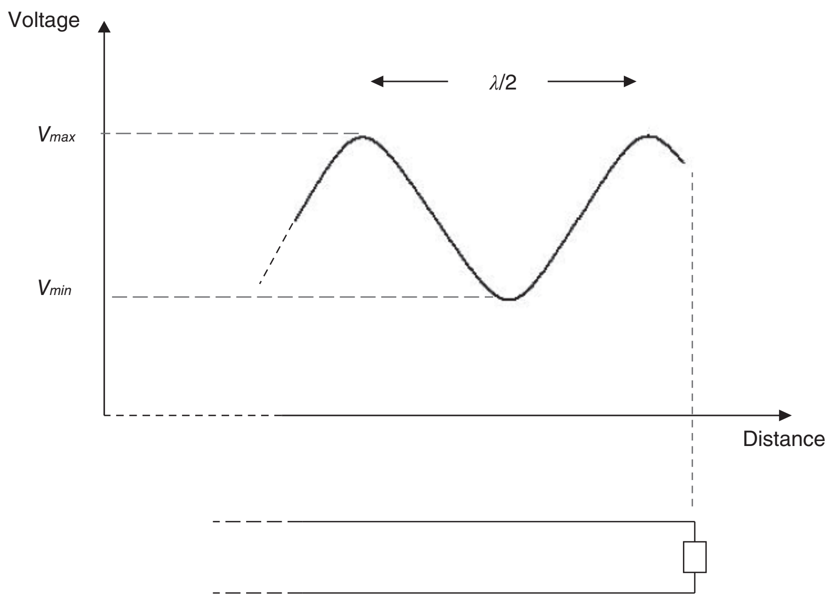

Figure 1.4 Standing voltage wave.

The voltage at the maximum point of the pattern will be $V_{max} = |V_F| + |V_R|$ and at the minimum point $V_{min} = |V_F| - |V_R|$, where V_F and V_R are the peak voltages of the forward and reflected waves, respectively. The degree of mismatch on a transmission can be specified by two parameters, namely the voltage standing wave ratio (VSWR) and the reflection coefficient, ρ. These two parameters are defined as follows.

VSWR is defined as

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{|V_F| + |V_R|}{|V_F| - |V_R|}. \tag{1.35}$$

Since we know that $V_R = 0$ for a matched load, and $V_R = V_F$ for a total mismatch where all the energy is reflected, we can deduce the range of VSWR values as

$$1 \leq (VSWR) \leq \infty. \tag{1.36}$$

When $Z_L = Z_O$, the VSWR is unity.

Reflection coefficient, ρ, is defined as

$$\rho = \frac{V_R}{V_F}. \tag{1.37}$$

It follows that the range of the magnitude of ρ is given by

$$0 \leq |\rho| \leq 1, \tag{1.38}$$

with zero being the best value, and unity corresponding to total reflection from the load. It should be noted that ρ is a complex quantity, giving both magnitude and phase information, and this is an essential parameter in the design of matching networks, which will be discussed later in the chapter.

Clearly, there must be some relationship between VSWR and reflection coefficient, since both parameters provide information about the degree of reflection from a load, and this relationship is shown in Eq. (1.39)

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{|V_F| + |V_R|}{|V_F| - |V_R|} = \frac{1 + \left|\frac{V_R}{V_F}\right|}{1 - \left|\frac{V_R}{V_F}\right|} = \frac{1 + |\rho|}{1 - |\rho|}. \tag{1.39}$$

Also, we can determine a relationship between reflection coefficient and impedance by first rewriting Eq. (1.11) in the form

$$\begin{aligned} V &= V_F + V_R \\ I &= \frac{V_F - V_R}{Z_O}. \end{aligned} \tag{1.40}$$

Then

$$Z = \frac{V}{I} = Z_O \frac{V_F + V_R}{V_F - V_R} = Z_O \frac{1 + \frac{V_R}{V_F}}{1 - \frac{V_R}{V_F}} = Z_O \frac{1 + \rho}{1 - \rho}. \tag{1.41}$$

Rearranging Eq. (1.41) gives

$$\rho = \frac{Z - Z_O}{Z + Z_O}. \tag{1.42}$$

Example 1.3 What is the VSWR corresponding to a reflection coefficient of $0.4\angle{-22°}$?

Solution

$$VSWR = \frac{1 + |\rho|}{1 - |\rho|} = \frac{1 + 0.4}{1 - 0.4} = \frac{1.4}{0.6} = 2.33.$$

1.7 The Smith Chart

The Smith chart, developed by J.B. Smith in 1935, is a graphical tool used in the design of RF and microwave circuits. Whilst a technique that involves graphical manoeuvres is liable to significant reading and plotting errors, the Smith chart is still useful in giving a quick visual appreciation of a circuit problem, which can subsequently be reworked using CAD to give precise design information.

One of the main modern applications of the Smith chart is in measuring instrumentation to display circuit parameters as a function of frequency, and the chart forms an essential part of the display in modern network analyzers (see Chapter 7).

1.7.1 Derivation of the Smith Chart

Equation (1.42) can be written in terms of normalized impedances as

$$\rho = \frac{z-1}{z+1}, \tag{1.43}$$

where z represents a normalized impedance, defined as

$$z = \frac{Z}{Z_O}, \tag{1.44}$$

where Z_O is the characteristic impedance of the line.

(*It should be noted that we have used the normal convention whereby* ***normalized*** *values of impedance, resistance, and reactance are represented by lower-case letters z, r, and x, respectively*.)

Writing $z = r + jx$, Eq. (1.43) can be expanded as

$$\rho = \frac{(r+jx)-1}{(r+jx)+1} = \frac{(r-1)+jx}{(r+1)+x}. \tag{1.45}$$

Since we know the reflection coefficient is a complex quantity, representing magnitude and phase, we can write ρ as

$$\rho = U + jV. \tag{1.46}$$

Combining Eqs. (1.45) and (1.46) gives

$$U + jV = \frac{(r-1)+jx}{(r+1)+jx}. \tag{1.47}$$

We can solve Eq. (1.47) by equating the real and imaginary parts, and after some laborious, but routine maths, we obtain

$$\left(U - \frac{r}{r+1}\right)^2 + V^2 = \left(\frac{1}{r+1}\right)^2 \tag{1.48}$$

and

$$(U-1)^2 + \left(V - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2. \tag{1.49}$$

It can be seen that Eqs. (1.48) and (1.49) are both in the form of equations representing circles in the $U - V$ plane.

For a particular value of r, Eq. (1.48) represents a circle of radius $\frac{1}{r+1}$ with a centre at $\left(\frac{r}{r+1}, 0\right)$. The value of r is therefore constant around any particular circle. These are termed constant normalized resistance circles and examples are drawn in Figure 1.5.

Similarly, for a particular value of x, Eq. (1.49) represents a circle of radius $\frac{1}{x}$ with centre at $\left(1, \frac{1}{x}\right)$. These are constant normalized reactance circles, and examples are drawn in Figure 1.6, where it should be noted that x can take both positive and negative values, since we can have both positive and negative reactance in a practical circuit. It can also be seen that the circle representing $x = 0$ has an infinite radius, and is therefore represented by a straight line coincident with the U-axis. For reference, the $r = 0$ circle has also been shown in Figure 1.6.

The sets of circles shown in Figures 1.5 and 1.6 can be combined onto a single $U - V$ plot, and this forms the Smith chart. A typical Smith chart is shown in Figure 1.7. Note that only reactance lines that lie within the $r = 0$ circle are shown, since normalized resistance values less than zero have no physical meaning. Note also that the resistance and reactance circles constituting the Smith chart are drawn in the polar diagram plane of the reflection coefficient, and concentric circles, centred on the $U = 0$ and $V = 0$ origin, represent values of constant reflection coefficient amplitude. These circles are not normally included on the Smith chart, but are drawn on by the user, as will be shown in subsequent examples.

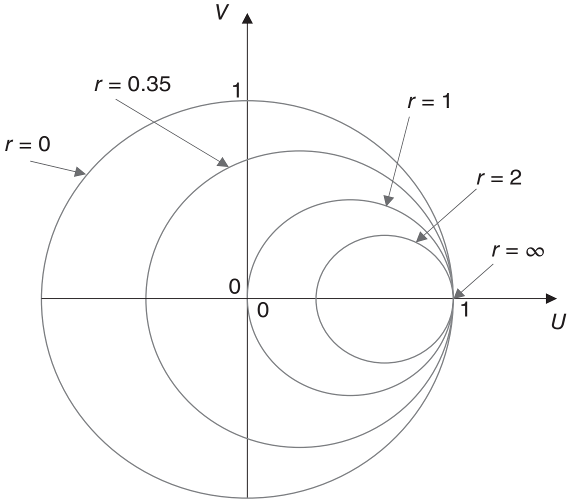

Figure 1.5 Normalized constant resistance circles.

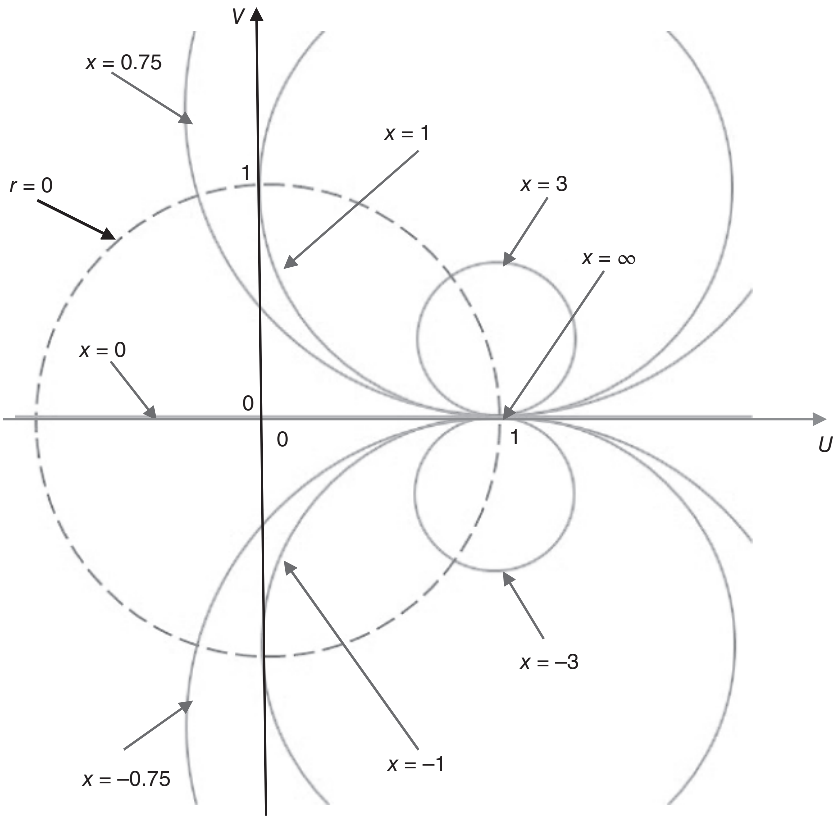

Figure 1.6 Normalized constant reactance circles.

The Smith chart shown in Figure 1.7 is an example of a commercially drawn chart, and it can be seen that in addition to the normalized resistance and reactance circles that have been discussed, a set of scales has been provided on the left-hand side. These scales are a useful aid for plotting radial distances on the chart, which correspond to particular values of VSWR and reflection coefficient. The use of these scales will be demonstrated in the solutions of worked examples later in the chapter.

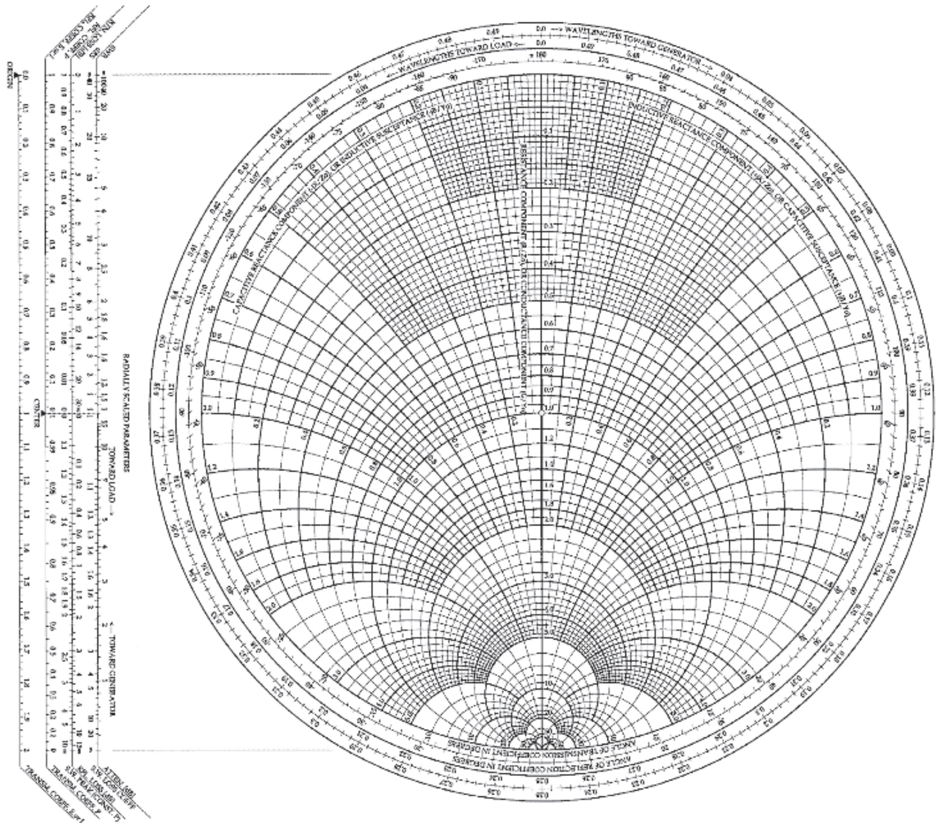

Figure 1.7 The Smith chart.

1.7.2 Properties of the Smith Chart

(i) An impedance point is plotted on the chart by locating the intersection of the appropriate resistance and reactance lines, remembering that the chart only displays normalized values.

(ii) Since the chart was drawn in the reflection plane, i.e. $\rho = U + jV$, points on circles that are plotted with their centre at the origin represent reflection coefficients of constant magnitude. These circles, which are not printed on the chart but need to be drawn by the user, are often referred to as constant VSWR circles (or usually just as VSWR circles). Moving around a constant VSWR circle corresponds to moving along a transmission line, thereby changing the angle of the reflection coefficient. However, the direction of rotation around a VSWR circle is important, since moving from the generator end of a transmission line towards the load will make the angle of the reflection coefficient more positive, and conversely moving from the load towards the generator will make the angle of the reflection coefficient more negative. To aid the user of the Smith chart, an annular scale showing the angle of the reflection coefficient is printed around the outside of the chart (the use of this scale is discussed in point (iii) and demonstrated in Example 1.4). We know from the earlier discussion of standing waves that the voltage pattern on a mismatched transmission line will repeat every half-wavelength, and therefore making a complete revolution from a given point on a VSWR circle, to return to the same point, must correspond to moving a distance $\lambda/2$ along the line. Appropriate wavelength scales are provided around the periphery of the chart. Note that there are two scales, denoting two different directions of movement. Distances on the Smith chart are always represented as electrical lengths, i.e. as fractions of a wavelength.

(iii) Reflection coefficients can be plotted directly on the chart. Radial distances correspond to the magnitude of the reflection coefficient on a linear scale, starting at 0 in the centre of the chart (the origin in the $U - V$ plane), and with a maximum of 1 at the maximum circumference. Some manufacturers of the Smith chart provide a reflection coefficient scale as an aid to plotting (Figure 1.7). Smith charts also contain circumferential scales corresponding to the angle of the reflection coefficient. So plotting a reflection coefficient point involves identifying the radial line through the appropriate angle, and then marking the required radial distance along this line.

(iv) A normalized impedance can be converted to a normalized admittance by rotating 180° around the chart. Once the normalized admittance has been plotted the resistance circles become conductance circles, and the reactance lines become susceptance lines.

The validity of this conversion from the impedance plane to the admittance plane can be established by first noting that a 180° rotation on the Smith chart corresponds to moving $\lambda/4$ along a transmission line; see the comment in point (ii). From Eq. (1.32) the impedance, $Z(l)$, at a distance l from the load is given by

$$\frac{Z(l)}{Z_O} = \frac{Z_L + jZ_O\tan(\beta l)}{Z_O + jZ_L\tan(\beta l)}. \tag{1.50}$$

Replacing l by $l + \lambda/4$, which corresponds to moving $\lambda/4$ along the transmission line we obtain

$$\frac{Z(l+\lambda/4)}{Z_O} = \frac{Z_L + jZ_O\tan(\beta(l+\lambda/4))}{Z_O + jZ_L\tan(\beta(l+\lambda/4))}$$

$$= \frac{Z_L + jZ_O\tan(\beta l + \beta\lambda/4)}{Z_O + jZ_L\tan(\beta l + \beta\lambda/4)},$$

i.e.

$$\frac{Z(l+\lambda/4)}{Z_O} = \frac{Z_L + jZ_O\tan(\beta l + \pi/2)}{Z_O + jZ_L\tan(\beta l + \pi/2)}. \tag{1.51}$$

Making use of the trigonometric relationship $\tan(x + \pi/2) = -\frac{1}{\tan(x)}$ we can rewrite Eq. (1.51) as

$$\frac{Z(l+\lambda/4)}{Z_O} = \frac{Z_L - jZ_O\frac{1}{\tan(\beta l)}}{Z_O - jZ_L\frac{1}{\tan(\beta l)}}. \tag{1.52}$$

Equation (1.52) can then be rearranged as

$$\frac{Z(l+\lambda/4)}{Z_O} = \frac{Z_O + jZ_L\tan(\beta l)}{Z_L + jZ_O\tan(\beta l)} = \frac{Z_O}{Z(l)}. \tag{1.53}$$

Thus, we see from Eq. (1.53) that the effect of moving a distance of $\lambda/4$ along the transmission line is to convert a normalized impedance into its reciprocal value, i.e. to convert a normalized impedance into a normalized admittance. Thus, any point which is plotted on the Smith chart as a normalized impedance can be converted directly to the equivalent normalized admittance by rotating 180° around the chart. This is a particularly useful technique when the chart is used in the analysis of circuits which involve a combination of series and parallel elements, as will be demonstrated in worked examples later in the chapter.

Some of the key features of the Smith chart are shown in Figures 1.8 through 1.13. Where points have been plotted on the chart to illustrate the principles involved, it should be appreciated that there will be plotting errors, as with any graphical technique. Consequently, where Smith charts have been used in this book to demonstrate RF design principles, readers should accept that precise data can only be obtained through use of appropriate CAD software.

Figure 1.8 shows examples of particular normalized constant resistance and reactance lines. The values of the resistance circles are shown on a vertical scale through the centre of the chart, and the values of the reactance line are shown on a scale around the periphery of the chart.

Impedance points are plotted on the chart by locating the intersection of the appropriate resistance and reactance lines. As an example, Figure 1.9 shows the position of the normalized impedance $0.3 + j0.6$, which is at the intersection of the 0.3 normalized resistance circle and the 0.6 normalized reactance line. Also shown in Figure 1.9 are impedance points of particular interest, namely the short-circuit, open-circuit, and matched impedance positions.

As mentioned earlier in the chapter, the Smith chart can also be used to plot and manipulate admittance data. In the admittance plane the ‘real’ circles printed on the chart become normalized conductance circles, and the ‘imaginary’ lines represent normalized susceptance. Figure 1.10 shows examples of admittances plotted on the chart. In the admittance plane the point $y = 0.3 + j0.6$ represents a normalized admittance with a normalized conductance of 0.3 and a normalized susceptance of 0.6.

VSWR circles were discussed in Section 1.7.2 (ii). These concentric circles can easily be plotted on the chart using the VSWR scale, which is one of the scales normally printed alongside the plotting area. The plot of a $VSWR = 4$ circle is shown in Figure 1.11, where the radius of the circle has been obtained from the VSWR scale. Note that on the scales printed on most Smith charts the VSWR scale is identified simply as the SWR scale.

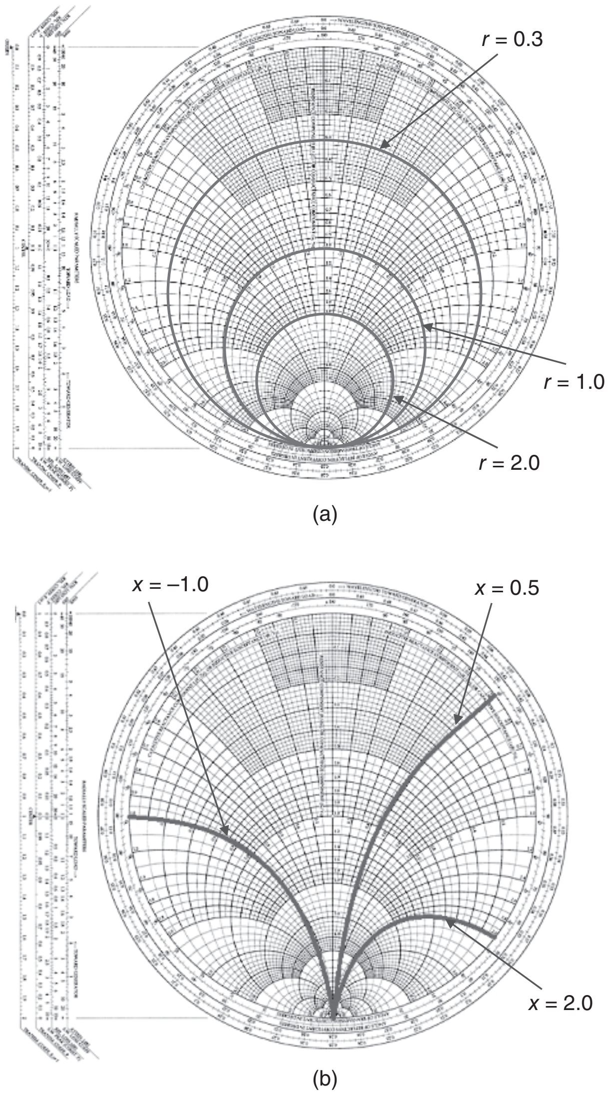

Figure 1.8 Examples of constant resistance lines (a) and reactance lines (b).

The procedure for plotting a reflection coefficient point on the chart is shown in Figure 1.12. In this case we show the plotting of point corresponding to a reflection coefficient $\rho = 0.7\angle 60°$. Using the reflection coefficient scale a concentric circle of radius 0.7 is first plotted. A radial line is then drawn from the centre of the chart to pass through the required angle (60° in this example) on the reflection coefficient scale, which is printed around the periphery of the plotting area. Where the radial line intersects the drawn circle gives the location of the required reflection coefficient.

The impedance at any point on a loss-free transmission line terminated with a particular load lies on a VSWR circle. Using the Smith chart it is straightforward matter to find the impedance at a given distance from the load; the procedure is illustrated in Figure 1.13. The normalized impedance, z_L, of the load is first plotted, and a VSWR circle is drawn through z_L. A radial line drawn from the centre of the chart through z_L establishes the position, s_1, of the load on the wavelength scale printed around the outside of the chart. The impedance at an electrical distance d from the load is then found by moving clockwise (load-to-generator) around the wavelength scale to a new position s_2, where $d = s_2 - s_1$. A radial line is then drawn from s_2 to the centre of the chart. Where this radial line intersects, the VSWR circle gives the normalized impedance a distance d from the load. It is important to note that when using the Smith chart distances can only be represented as electrical distances, i.e. the distance expressed as a fraction of a wavelength at the frequency being used, since on the chart we only know that one revolution corresponds to half of one wavelength measured along the line.

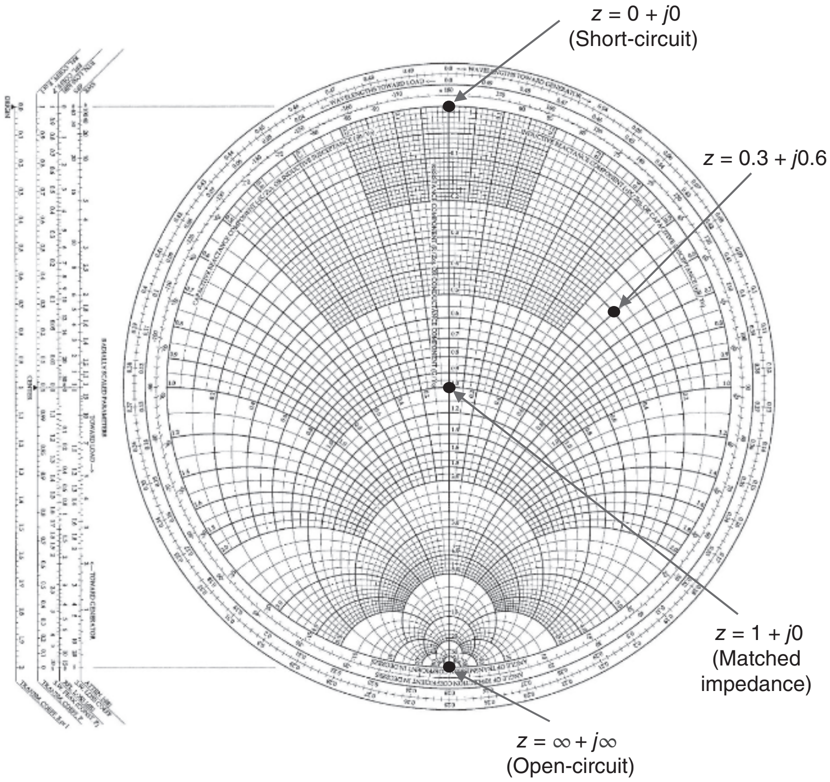

Figure 1.9 Examples of normalized impedance points.

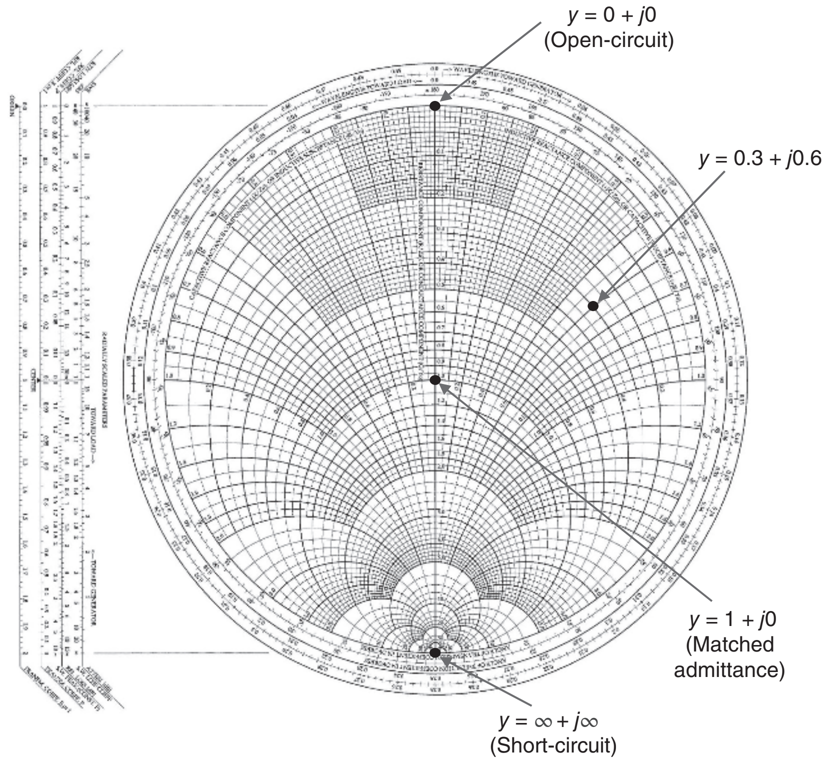

Figure 1.10 Examples of normalized admittance points.

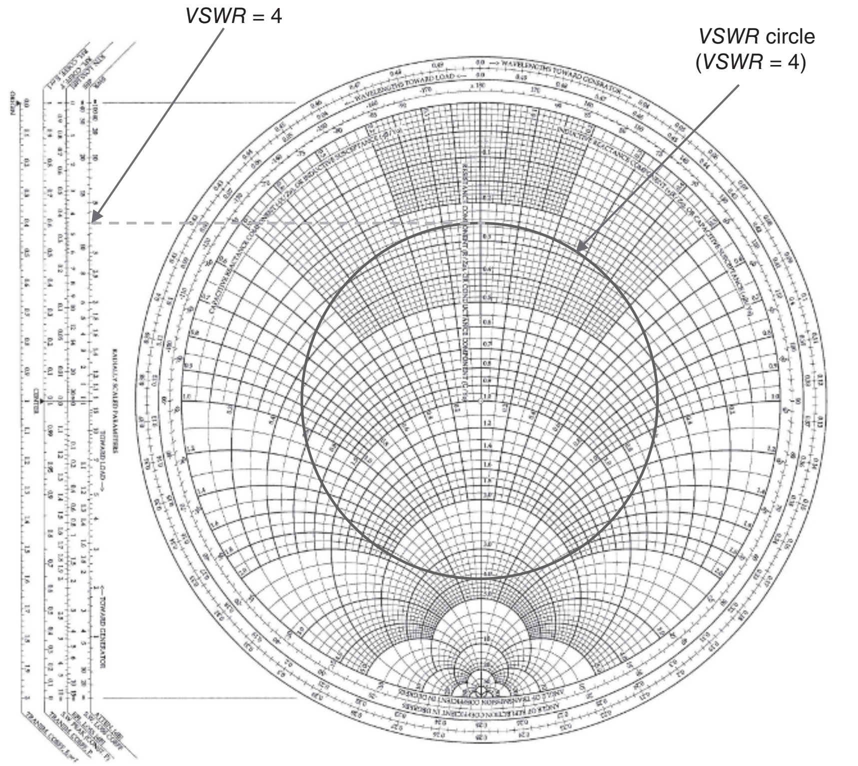

Figure 1.11 Plot of VSWR circle ($VSWR = 4$). (Note that the radius of the circle is obtained from the SWR scale.)

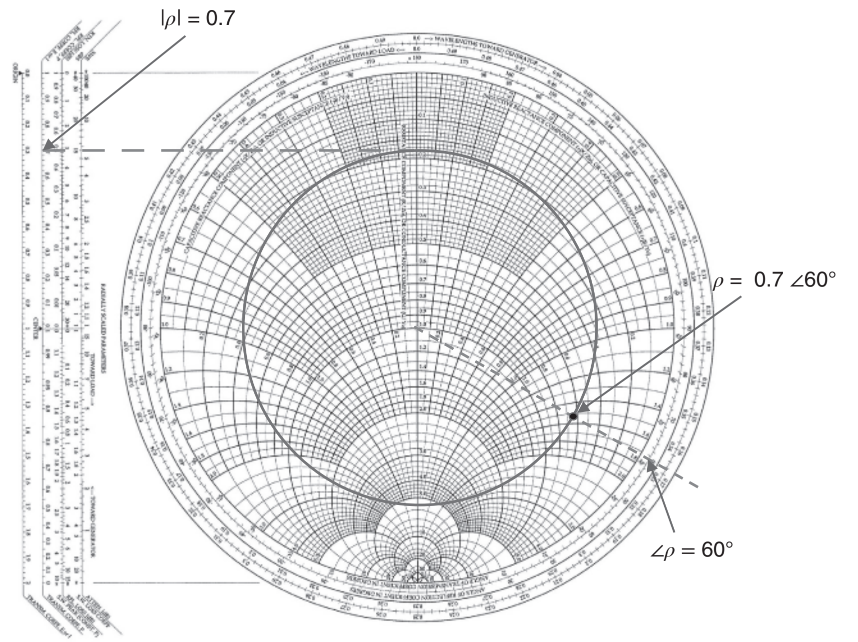

Figure 1.12 Plot of reflection coefficient point, $\rho = 0.7 \angle 60°$. (Note that the magnitude of the reflection coefficient point is obtained from the reflection coefficient scale.)

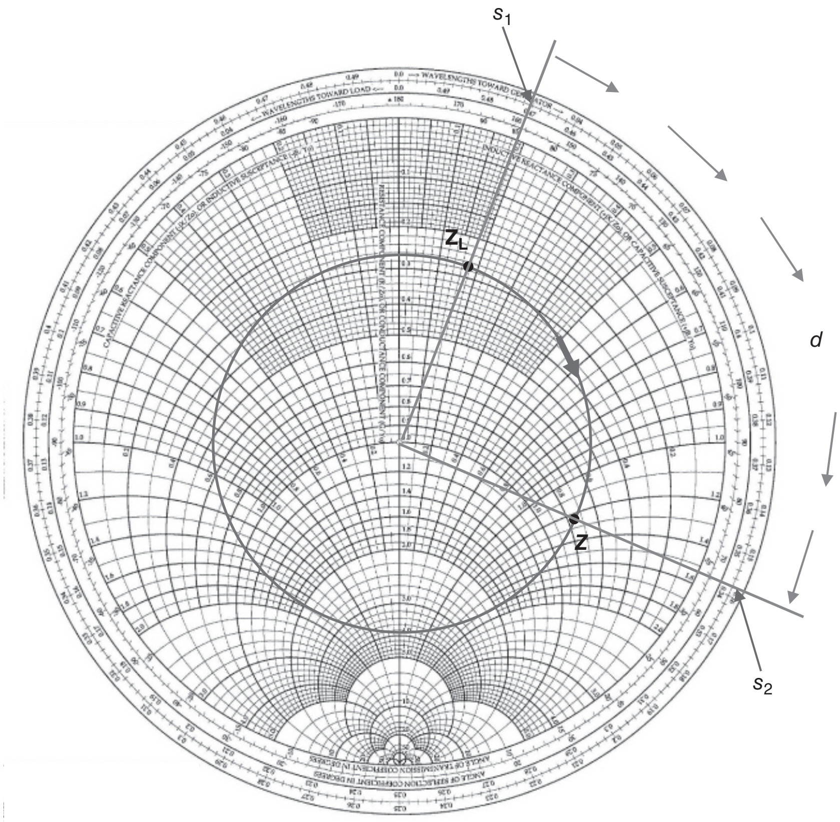

Figure 1.13 Impedance, Z, at a distance d in front of a load Z_L.

Example 1.4 A lossless transmission line having a characteristic impedance of 50 Ω is terminated by an impedance $(120 + j40)\ \Omega$.

Determine:

(i) The normalized impedance of the load.
(ii) The reflection coefficient of the load.
(iii) The VSWR on the line.

Solution

(i) $z_L = \dfrac{120 + j40}{50} = 2.4 + j0.8.$

(ii) and (iii)

Referring to the Smith chart shown in Figure 1.14: After plotting the normalized load impedance, and drawing the VSWR circle, we obtain

$$\rho = 0.46\angle 17^\circ.$$

$$VSWR = 2.7.$$

(Continued)

(Continued)

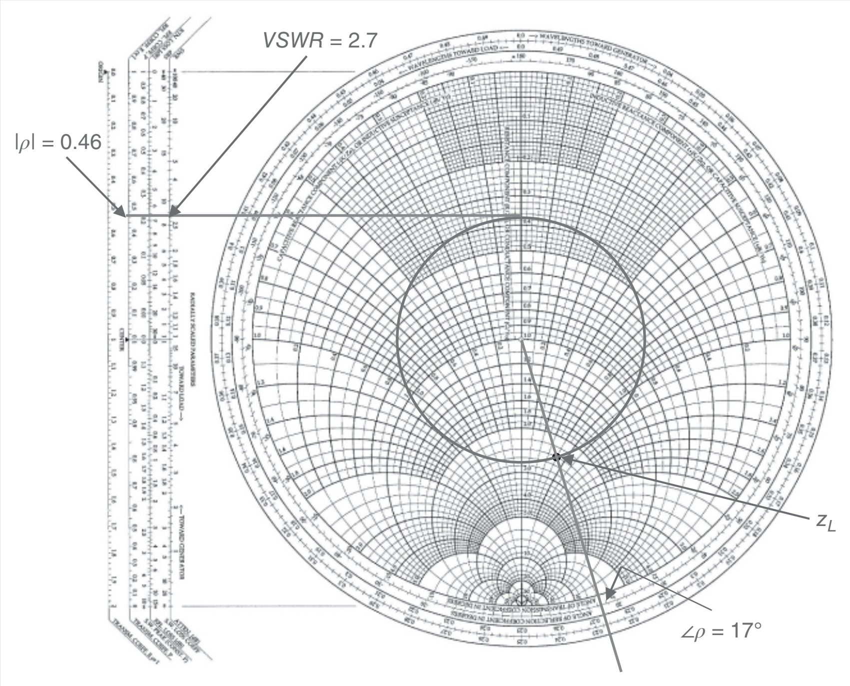

Figure 1.14 Smith chart solution for Example 1.4.

Example 1.5 A lossless transmission line having a characteristic impedance of 75 Ω is terminated by a load impedance $(18-j30)\ \Omega$. Determine the impedance on the line at a distance of 0.175λ from the load.

Solution

Normalized load impedance, $z_L = \dfrac{18-j30}{75} = 0.24 - j0.4.$

Moving 0.174λ clockwise around the VSWR circle from the plotted load impedance gives a point of intersection $z = 0.33 + j0.77$. Therefore, the required impedance is $Z = (0.33 + j0.77) \times 75\ \Omega = (24.75 + j57.75)\ \Omega$ (Figure 1.15).

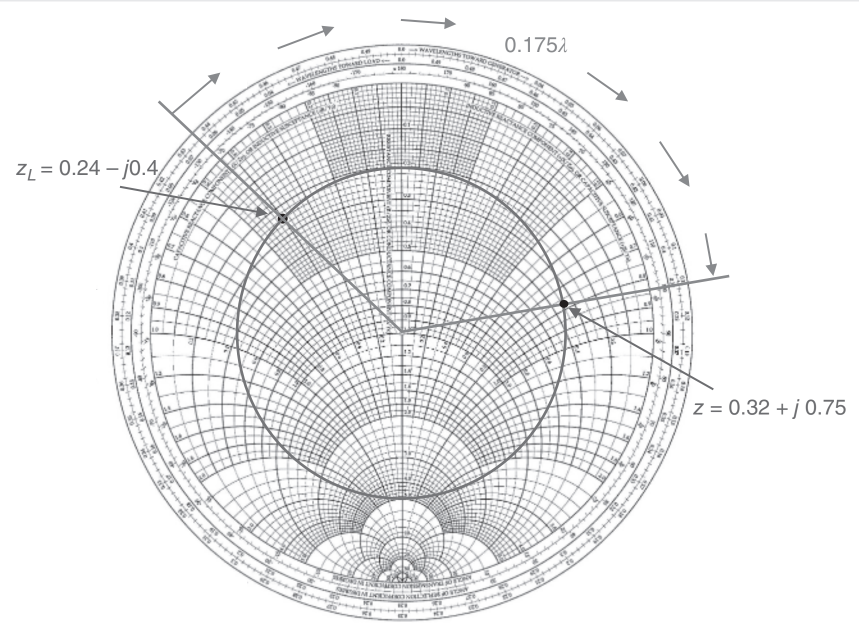

Figure 1.15 Smith chart solution for Example 1.5.

Example 1.6 A lossless transmission line having a characteristic impedance of 50 Ω is terminated by a load impedance (18.5 + j25.0) Ω. The velocity of propagation on the line is 2×10^8 m/s.

Determine:

(i) The admittance of the load.
(ii) The admittance 35 mm from the load at a frequency of 700 MHz.

Solution (Referring to Figure 1.16)

(i) $z_L = \dfrac{18.5 + j25.0}{50} = 0.37 + j0.50.$

Rotating through 180° on the Smith chart gives y_L:

$$y_L = 0.90 - j1.37 \quad \Rightarrow \quad Y_L = (0.90 - j1.37) \times \frac{1}{50} \ \text{S} = (18.0 - j27.4) \ \text{mS}.$$

(ii) $\lambda = \dfrac{2 \times 10^8}{700 \times 10^6}$ m = 285.7 mm $\quad \Rightarrow \quad$ 35 mm $= \dfrac{35}{285.7}\lambda = 0.123\lambda.$

Rotating 0.123λ (clockwise) around VSWR circle gives y_1:

$$y_1 = 0.32 - j0.27 \quad \Rightarrow \quad Y_1 = (0.32 - j0.27) \times \frac{1}{50} \ \text{S} = (6.4 - j5.4) \ \text{mS}.$$

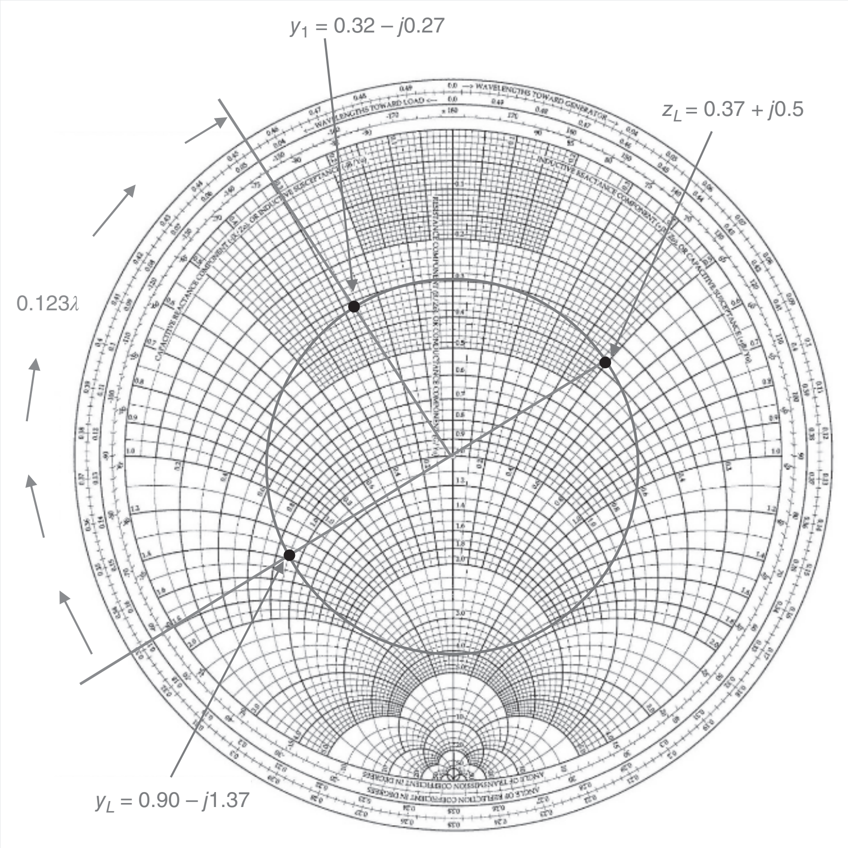

Figure 1.16 Smith chart solution for Example 1.6.

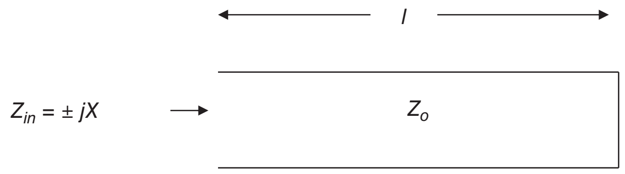

Figure 1.17 Schematic view of a transmission line stub.

1.8 Stubs

Stubs are short lengths of lossless transmission line that are terminated either in a short circuit or an open circuit. The input impedance of such stubs will be purely imaginary, as shown by Eqs. (1.33) and (1.34). Figure 1.17 shows a stub formed by a length of lossless transmission line of characteristic impedance, Z_O, terminated with a short circuit.

Thus, we can use a stub to create a positive or negative reactance (i.e. inductance or capacitance) of any magnitude, at a given frequency, purely by adjusting the length of the stub. Stubs provide very useful series or shunt matching elements at RF and microwave frequencies, where lumped reactive components suffer from unwanted parasitics, and where the physical geometry of lumped components may be incompatible with the circuit fabrication technology.

Example 1.7 A short-circuited stub, made from lossless 50 Ω transmission line, is to be used to create an inductance of 45 nH at a frequency of 820 MHz. If the velocity of propagation on the transmission line is 2.2×10^8 m/s, calculate the required length of the stub.

Solution
At 820 MHz:

$$450 \text{ nH} \quad \Rightarrow \quad jX_L = j(2\pi \times 820 \times 10^6 \times 45 \times 10^{-9})\ \Omega = j231.9\ \Omega,$$

$$\lambda = \frac{2.2 \times 10^8}{820 \times 10^6}\ \text{m} = 268.30\ \text{mm}.$$

Using Eq. (1.33):

$$Z_{in} = jZ_O \tan(\beta l)$$

$$j231.9 = j50 \tan\left(\frac{2\pi}{268.30} l\right),$$

$$\tan\left(\frac{2\pi}{268.30} l\right) = \frac{231.9}{50} = 4.64,$$

$$\frac{2\pi}{268.30} l = 77.84 \times \frac{\pi}{180},$$

$$l = 58.00\ \text{mm}.$$

Comment: The length of the stub can also be found using the Smith chart, as shown through Example 1.8.

Example 1.8 Repeat Example 1.7 using the Smith chart.

Solution
Normalizing the required input reactance gives:

$$jx_{in} = \frac{j231.9}{50} = j4.64 \text{ (from Example 1.7).}$$

Plotting $z = 0 + j4.64$ on the Smith chart and reading-off distance to $z_{S/C}$ gives $l = 0.216\lambda$ (Figure 1.18), i.e.

$$l = 0.216\lambda = 0.216 \times 268.3 \text{ mm} = 57.95 \text{ mm}.$$

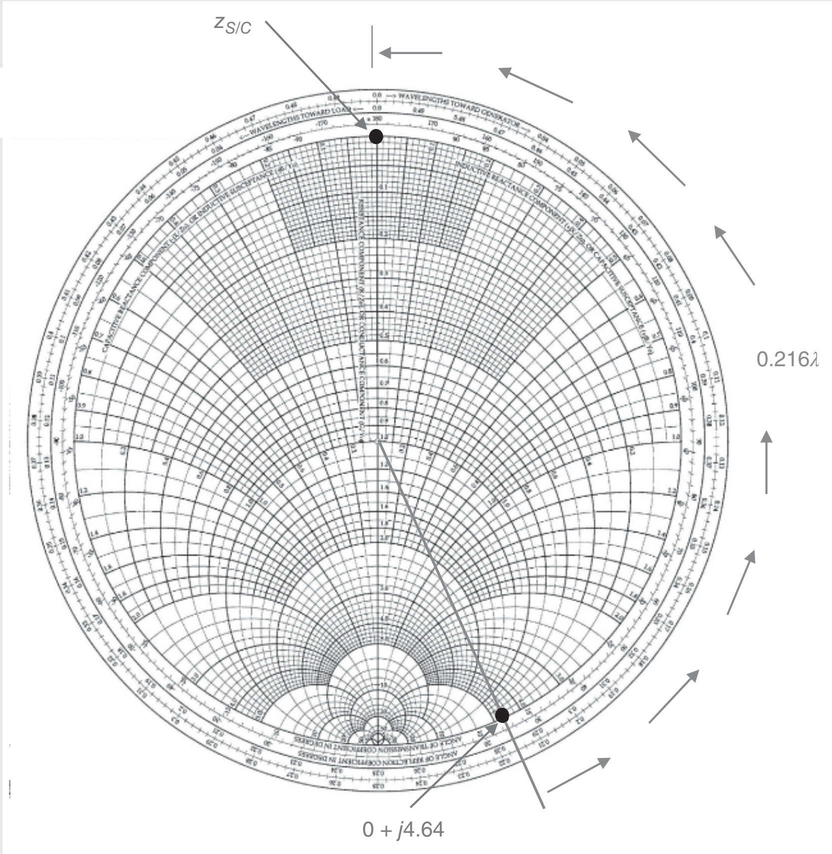

Figure 1.18 Smith chart solution for Example 1.8.

1.9 Distributed Matching Circuits

The primary function of a matching circuit is to create an impedance that will allow maximum power transfer from a source to a load. Matching networks are normally made from lossless distributed or lumped reactances. Distributed reactances are defined as those whose physical size is an appreciable fraction of a wavelength at the operating frequency, whereas the size of lumped reactances is small compared to a wavelength.

Figure 1.19 shows a load impedance, Z_L, connected to a source having an impedance Z_S, with an intervening matching network.

In order to ensure maximum power is transferred from the source to the load, the matching network must be designed so as to create an input impedance Z_{in}, which is the conjugate of the source impedance, i.e.

$$Z_{in} = Z_S^*. \tag{1.54}$$

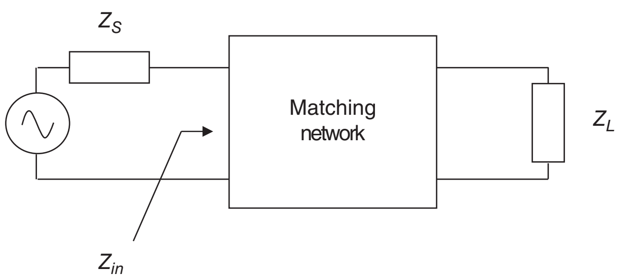

Figure 1.19 Matching network between source and load.

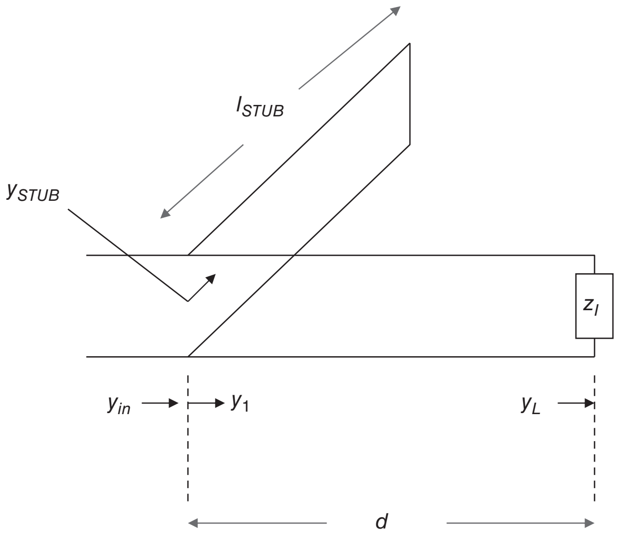

Figure 1.20 Single-stub tuner employing a short-circuited, shunt-connected stub.

A simple, effective distributed matching network can be made using lossless transmission lines. We will consider the special case where the source has a real impedance of Z_O at a specified frequency. Any complex load impedance can be matched to a real impedance by attaching a stub of the correct length at an appropriate point on a transmission line connecting the source to the load. This type of matching circuit is called a single-stub tuner (SST). The theory can be explained by reference to Figure 1.20, which shows a short-circuited shunt stub, of length l_{STUB} attached to the transmission line at a distance d from the load. In this example, the stub provides a matching susceptance across the line, and we have assumed that the characteristic impedance of the line and the stub are the same.

Referring to Figure 1.20:

(1) A shunt stub is attached at a point on the line where the normalized conductance is unity, because we can only use the stub to cancel the imaginary part of the line admittance. Let the normalized admittance at this point be y_1, so

$$y_1 = 1 \pm jb. \tag{1.55}$$

(2) The length of the stub is adjusted to create a normalized input susceptance that will cancel the residual susceptance on the main line, i.e. we need

$$y_{STUB} = \mp jb. \tag{1.56}$$

(3) Since the stub and the main transmission line are in parallel, the normalized input admittance at the input to the tuner will be the sum of the normalized stub admittance and the normalized admittance of the line

$$y_{in} = y_1 + y_{STUB} = (1 \pm jb) \mp jb = 1. \tag{1.57}$$

After de-normalizing we have $Y_{in} = Y_O$ and the line is therefore matched.

Notes on the SST:

(i) We work in terms of normalized admittance because stubs are normally connected in parallel with the main transmission line, and consequently we can directly sum admittances at the junction between the stub and the line.
(ii) The characteristic impedance of the stub is usually made equal to that of the main line, although this does not necessarily have to be so. In practical tuners there may be some advantage in having a stub of higher impedance than that of the main line, which means smaller dimensions, so as to minimize the physical discontinuity at the junction with the main line.
(iii) SSTs can employ short-circuited stubs or open-circuited stubs. Which type of stub is used in a particular design depends largely on practical fabrication issues. When using a coaxial cable it is normal to use short-circuited stubs because it is relatively easy to create a good short circuit at the end of the stub by simply using a metal disc to connect the centre conductor to the earthed sheath of the cable. If a coaxial line is left open at the end, the electromagnetic field will fringe

into space, causing two problems. Firstly, the fringing field at the open end will cause the line to behave electrically as if it is longer than the actual length. Secondly, there will be some radiation from the open end, thereby introducing loss into the tuner.

When using planar circuits, such as a microstrip, the stubs are normally open-circuited, because of the physical problems in making connections through the substrate to the ground plane so as to create a short circuit. Substrates for RF and microwave circuits are often made of ceramic, which is a very hard material, and making connections through the substrate can be difficult and expensive, and usually involves laser drilling of the substrate.

(iv) The SST can theoretically be used to match any load impedance to a source, but it suffers from the disadvantage that the position of the stub must be changed to match different load impedances. This disadvantage can be overcome by using a double-stub tuner in which there are two matching stubs with a fixed distance between them. However, a double-stub tuner with a particular spacing between the stubs cannot match all possible values of load impedance. A triple-stub tuner, with the three stubs having a fixed spacing, is required to match any value of load impedance, although the practical use of this type of tuner can be difficult. An informative discussion of the theory and design of multi-stub tuners is given by Collin [1].

Procedure for designing an SST employing a short-circuited, shunt-connected stub, using the Smith chart:

(i) Plot the normalized load impedance, and convert this to the normalized load admittance.
(ii) Draw the VSWR circle through the load admittance.
(iii) Traverse the VSWR circle clockwise (i.e. load-to-generator) from the load admittance to intersect the unity conductance circle. The distance traversed (using the wavelength scale) gives the value of d, as an electrical length. The point of intersection with the unit circle is y_1 and equal to $1 \pm jb$.
(iv) Plot y_{STUB} $(=0 \mp jb)$.
(v) Starting at y_{STUB} traverse the conductance circle counter-clockwise to $y_{S/C}$, the distance moved gives the electrical length of the stub, l_{STUB}. (The movement is counter-clockwise because as far as the stub is concerned we are moving from the generator end towards the short-circuited load.)
(vi) The design is completed by converting the electrical lengths of d and l_{STUB} into physical lengths using the appropriate guide wavelength.

Example 1.9 Design an SST that will match a load impedance $(80 - j65)\ \Omega$ to a $50\ \Omega$ source at a frequency of 1.3 GHz. The tuner is to employ a short-circuited stub connected in parallel with the main line. All of the cables used in the tuner have characteristic impedances of $50\ \Omega$, and a velocity of propagation of 2×10^8 m/s.

Solution

$$z_L = \frac{80 - j65}{50} = 1.6 - j1.3,$$

$$\lambda = \frac{2 \times 10^8}{1.3 \times 10^9}\ \text{m} = 153.85\ \text{mm}.$$

Referring to the Smith chart shown in Figure 1.21:

$$y_L = 0.38 + j0.31.$$

Rotating around the VSWR circle through y_L to intersect the unity conductance circle gives y_1:

$$y_1 = 1 + j1.14.$$

The distance moved gives d:

$$d = 0.166\lambda - 0.054\lambda = 0.112\lambda = 0.112 \times 153.85\ \text{mm} = 17.23\ \text{mm}.$$

For matching, we require $y_{STUB} = -j1.14$.
Rotating around the chart from y_{STUB} to $y_{S/C}$ gives the required length of the stub:

$$l_{STUB} = 0.250\lambda - 0.135\lambda = 0.115\lambda = 0.115 \times 153.85\ \text{mm} = 17.69\ \text{mm}.$$

(Continued)

(Continued)

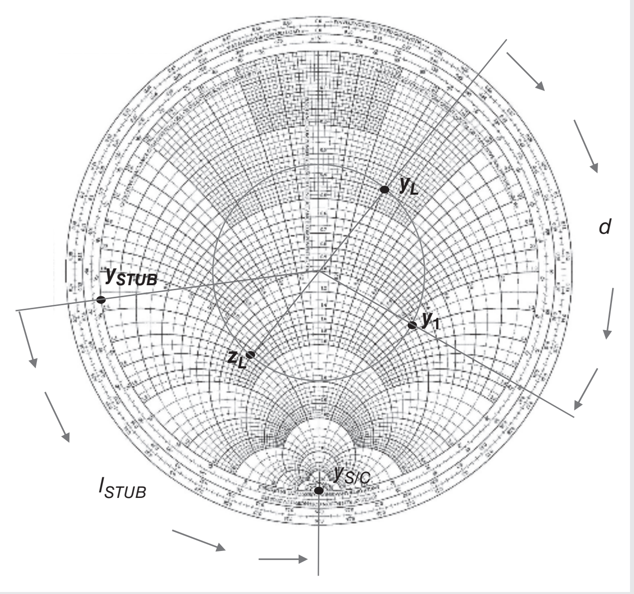

Figure 1.21 Smith chart solution for Example 1.9.

1.10 Manipulation of Lumped Impedances Using the Smith Chart

So far the discussion has focussed on the use of the Smith chart for situations involving transmission lines. Whilst the chart is often described as a transmission line calculator, it has many more general applications in electronics relating to the manipulation of impedance and admittance data.

Moving around the constant resistance line from any given impedance point on the chart corresponds to adding reactance to the impedance. Movements in the clockwise direction correspond to making the reactance more positive, and hence represent the addition of series inductance, whereas movements in the counter-clockwise direction correspond to making the reactance more negative, and therefore represent the addition of series capacitance. The effect of adding of inductance or capacitance to a given normalized impedance, z_1, is depicted in Figure 1.22.

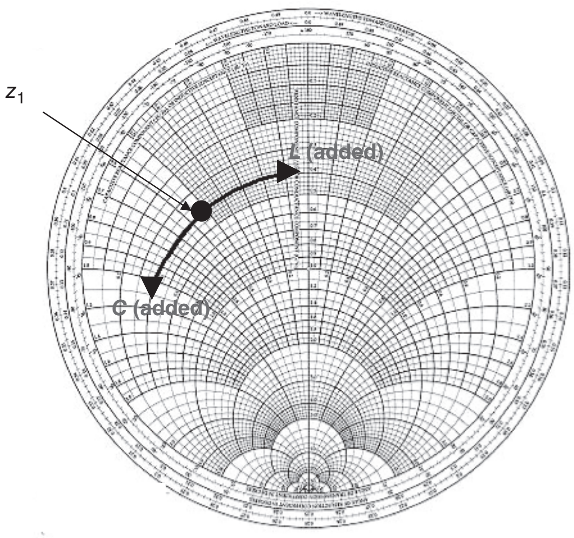

Figure 1.22 Effect of adding reactance to a normalized impedance, z_1.

Similarly, when working in the admittance plane, moving around the constant susceptance circles corresponds to adding the appropriate susceptance in shunt. In this case, clockwise movements represent the addition of capacitance, which has positive susceptance, and counter-clockwise movements the addition of inductance, which has negative susceptance.

Since we know that converting between impedance and admittance is very straightforward (a 180° rotation) on the Smith chart, it becomes very easy to analyze a network consisting of reactances connected in series and shunt. The technique for manipulating the data in such a network is demonstrated through Example 1.10.

Example 1.10 Using the Smith chart, determine the input impedance, Z_{in}, at 850 MHz of the network shown in Figure 1.23. The network consists of three lossless reactances, connected in a π-configuration, terminated by a load impedance $(20 - j15)\ \Omega$.

Firstly, the component values in the network must be normalized, so that data can be plotted on the Smith chart. For convenience, we will normalize with respect to 50 Ω.

Comment: When dealing with networks where there is no transmission line, and hence no characteristic impedance, we may normalize to any convenient value providing we de-normalize with respect to the same value. The normalization process in this case is merely a scaling operation to obtain a more convenient value to plot on the Smith chart.

$$z_L = \frac{20 - j15}{50} = 0.4 - j0.3.$$

Normalizing the reactance of the inductor:

$$5.15\ \text{nH}; j\frac{\omega L}{50} = j\frac{2\pi \times 850 \times 10^6 \times 5.15 \times 10^{-9}}{50} = j0.55.$$

Normalizing the susceptances of the capacitors:

$$6.93\ \text{pF:}\ j\frac{\omega C}{(1/50)} = j\omega C \times 50 = j2\pi \times 850 \times 10^6 \times 6.93 \times 10^{-12} \times 50 = j1.85,$$

$$4.87\ \text{pF:}\ j\frac{\omega C}{(1/50)} = j\omega C \times 50 = j2\pi \times 850 \times 10^6 \times 4.87 \times 10^{-12} \times 50 = j1.30.$$

The impedances and admittances at each of the junctions of the network are shown in Figure 1.24.

Working backwards from the load impedance, i.e. towards generator, we use the Smith chart to add the parallel components using admittance values, and add the series components using impedance values.

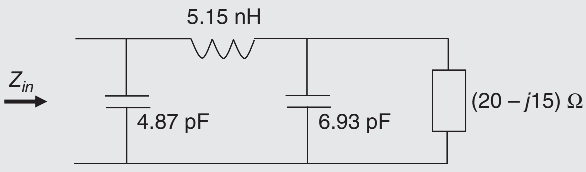

Figure 1.23 Circuit for Example 1.10.

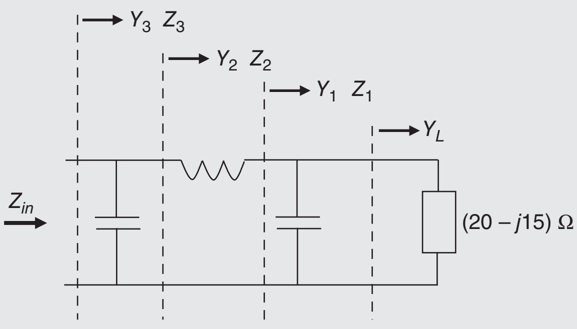

Figure 1.24 Nomenclature for solution for Example 1.10.

(Continued)

(Continued)

Step 1:	Plot z_L
Step 2:	Convert z_L to y_L
Step 3:	Move 1.85 units clockwise around the susceptance line from y_L. This gives the position of y_1. *Note that this movement represents the addition of the 6.93 pF capacitor; we move clockwise because a capacitor has positive susceptance.*
Step 4:	Convert y_1 to z_1
Step 5:	Move 0.55 units clockwise around the reactance line from z_1. This gives the position of z_2. *Note that this movement represents the addition of the 5.15 nH inductor; we move clockwise because an inductor has positive reactance.*
Step 6:	Convert z_2 to y_2
Step 7:	Move 1.30 units clockwise around the susceptance line from y_2. This gives the position of y_3. *Note that this movement represents the addition of the 4.87 pF capacitor; we move clockwise because a capacitor has positive susceptance.*
Step 8:	Convert y_3 to z_3

From the Smith chart (shown in Figure 1.25) we obtain:

$$z_3 = 0.36 + j0.37,$$

$$z_{in} \equiv z_3,$$

$$Z_{in} = (0.36 + j0.37) \times 50\ \ \Omega = (18.0 + j18.5)\ \ \Omega.$$

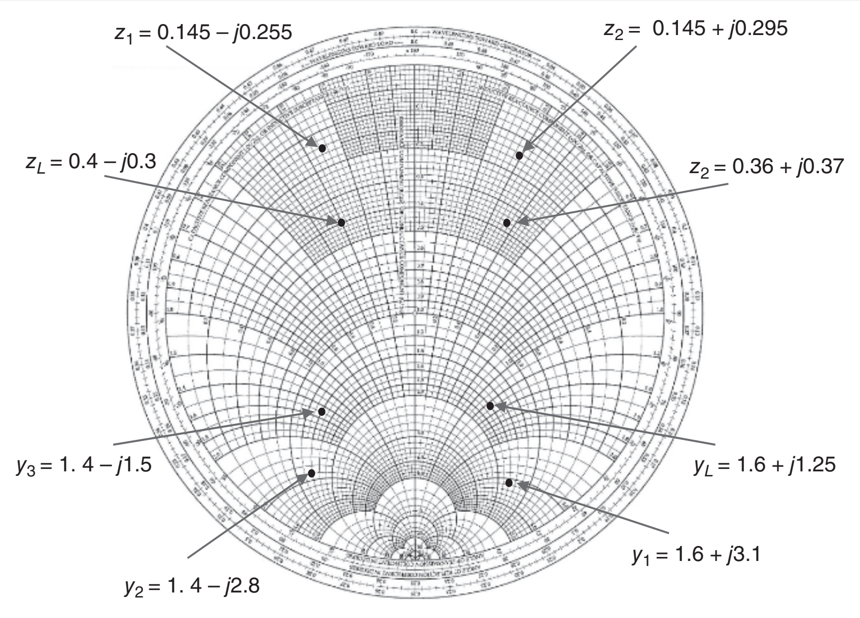

Figure 1.25 Smith chart solution for Example 1.10.

1.11 Lumped Impedance Matching

An alternative to the use of distributed transmission lines to create matching networks, as described in Section 1.9, is to use lumped elements in the form of lossless reactances. Normally, purely reactive elements are used in matching networks to avoid unnecessary dissipative losses. Although lumped elements suffer from unwanted parasitics at high frequencies, their use is essential in compact RF and microwave circuits (particularly integrated circuits) where size limitations prevent the use of transmission line elements, whose length may be a significant fraction of a wavelength. However, it must be remembered that the higher loss associated with lumped circuits means they tend to have lower Q-factors than distributed circuits.

1.11.1 Matching a Complex Load Impedance to a Real Source Impedance

A simple lumped-element matching network, consisting of two lossless reactances, one in series and the other in parallel, is shown in Figure 1.26, preceding a load whose impedance is Z_L. This network may be regarded as the lumped equivalent of the SST discussed in Section 1.9.

Theory: Let us consider that the matching network is to be used to match a load impedance, Z_L, to a source impedance of 50 Ω. The series reactance is used to create a normalized admittance, y_1, which has a real part of unity at the junction of the two reactances. The parallel reactance is then used to cancel out the normalized susceptance of y_1, thus making the normalized input admittance unity, and providing a match to the 50 Ω source.

The design procedure using the Smith chart can be deduced because we know three things concerning the impedance and admittance relationships:

(a) The real part of z_1 must be the same as the real part of z_L since these impedances differ only by the value of the series reactance, i.e.

$$z_1 = z_L \pm jx_S. \tag{1.58}$$

(b) The real part of y_1 must be the same as the real part of y_{in} since these admittances differ only by the value of the parallel susceptance, i.e.

$$y_{in} = y_1 \pm jb_P. \tag{1.59}$$

(c) For a match, the normalized input admittance must be of the form

$$y_{in} = 1 + j0. \tag{1.60}$$

Therefore, referring to the network configuration shown in Figure 1.26, the procedure using the Smith chart is:

(i) Plot the normalized load impedance, z_L.
(ii) Rotate the normalized unity resistance circle by 180°. *The reason for this construction is justified in step (iv).*
(iii) Traverse the constant resistance circle through z_L to intersect the rotated circle. The point of intersection is z_1, the movement from z_L to z_1 gives the value of the series component. *Note that there are two possible points of intersection on the rotated circle, giving rise to two possible vales of* z_1*, and hence two possible solutions.*
(iv) Move 180° from z_1 to find the position of y_1. *Note that the rotated circle was constructed to be the mirror-image of the normalized unity circle. Therefore, converting any normalized impedance point on this rotated circle to the equivalent normalized admittance will ensure that the normalized admittance lies on the unity conductance circle, thus satisfying the condition that the real part of* y_1 *must be unity.*
(v) Traverse the unity conductance circle from y_1 to the centre of the chart; this movement gives the value of the parallel component.

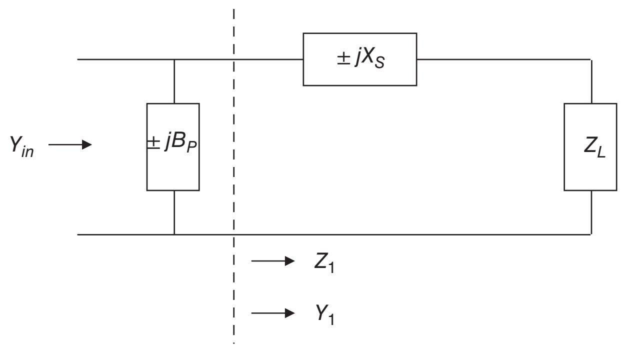

Figure 1.26 Lumped-element matching network.

Example 1.11 Design a lumped-element network that will match a load impedance $(20 - j40)\ \Omega$ to a $50\ \Omega$ source at a frequency of 2.4 GHz. The network is to be composed of two lossless reactances, with the configuration shown in Figure 1.26. Show that there are two possible solutions, and calculate the required reactive component values for each solution.

Solution

Normalizing the load impedance: $$z_L = \frac{20\ -\ j40}{50} = 0.4 - j0.8.$$

After rotating the unity resistance circle, and plotting the position of z_L, we see that the constant resistance circle through z_L will intersect the rotated circle at two points. These two points of intersection give rise to the two possible solutions.

First solution:

Consider the first intersection point, as shown in Figure 1.27.

We see that the addition of positive reactance is needed to move from z_L to z_1, the first point of intersection. Therefore, we need an inductance as the series component. After converting z_1 to the equivalent admittance, y_1, it can be seen that a negative susceptance is needed to move the admittance to the centre of the chart, which is the matched position. Thus, we need an inductor as the parallel component.

Using the data from the chart:

$$z_1 - z_L = (0.4 - j0.495) - (0.4 - j0.8) = j0.305$$

and

$$y_O - y_1 = 1.0 - (1.0 + j1.2) = -j1.2.$$

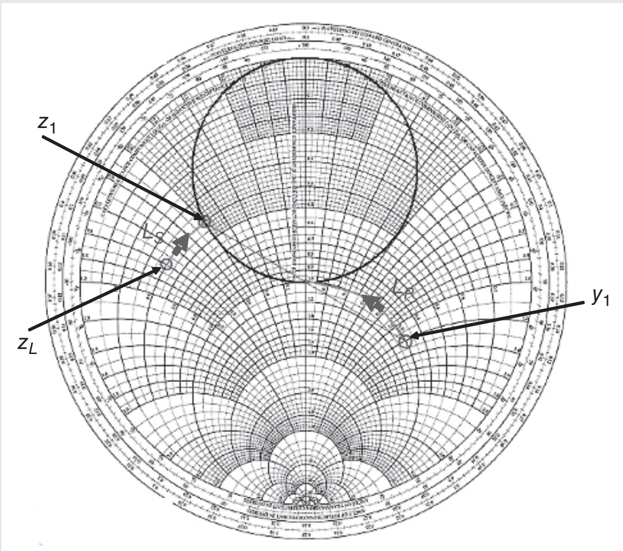

Figure 1.27 First Smith chart solution to Example 1.11.

Thus, we need a normalized series reactance of $j0.305$ and a normalized parallel susceptance of $-j1.2$,

$$\text{i.e.}\quad j0.305 = \frac{j\omega L_S}{50} \quad\Rightarrow\quad L_S = \frac{50 \times 0.305}{\omega} = \frac{50 \times 0.305}{2\pi \times 2.4 \times 10^9}\ \text{H} = 1.01\ \text{nH},$$

$$-j1.2 = -j\frac{1}{\omega L_P} \times 50 \quad\Rightarrow\quad L_P = \frac{50}{1.2 \times 2\pi \times 2.4 \times 10^9}\ \text{H} = 2.76\ \text{nH}.$$

Second solution:
Looking at Figure 1.28, we see that by using a larger series inductor there is a valid intersection point at z_2, leading to a second solution.

Using data from the chart:

$$z_2 - z_L = (0.4 + j0.495) - (0.4 - j0.8) = j1.295,$$

$$y_O - y_2 = 1.0 - (1.0 - j1.2) = j1.2.$$

Thus, for the second solution we need a normalized series reactance of $j1.295$, and a normalized parallel susceptance of $j1.2$,

$$\text{i.e.} \quad j1.295 = j\frac{\omega L_S}{50} \quad \Rightarrow \quad L_S = \frac{50 \times 1.295}{\omega} = \frac{50 \times 1.295}{2\pi \times 2.4 \times 10^9} \text{ H} = 4.29 \text{ nH},$$

$$j1.2 = j\omega C_P \times 50 \quad \Rightarrow \quad C_P = \frac{1.2}{2\pi \times 2.4 \times 10^9 \times 50} = 1.59 \text{ pF}.$$

Summary
The two possible matching networks that satisfy the requirements in Example 1.11 are shown in Figure 1.29.

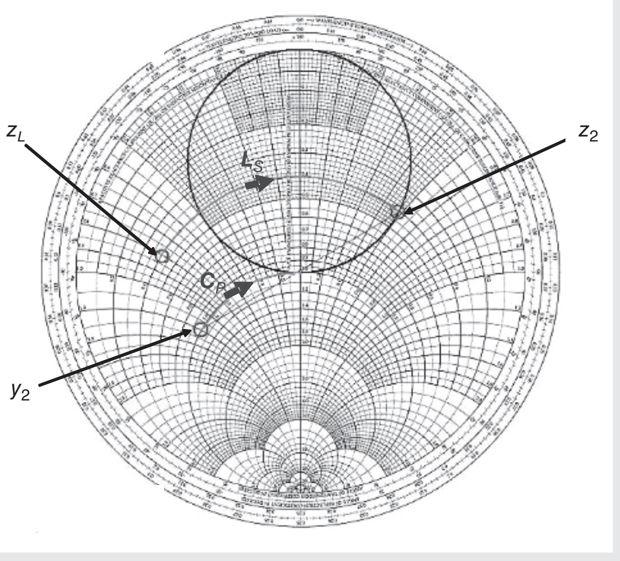

Figure 1.28 Second Smith chart solution to Example 1.11.

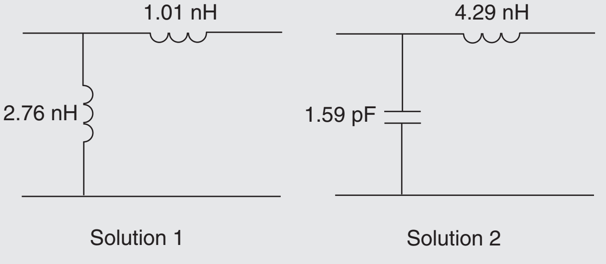

Figure 1.29 Matching networks for Example 1.11.

Example 1.12 The network of lossless reactances shown in Figure 1.26 is to be used to match a load impedance $(34 + j42f)\,\Omega$, where f is the frequency in GHz, to a 50 Ω source at a frequency of 2 GHz.

(i) Calculate the required values of the reactive components. Show that there are two solutions, and calculate the values of the reactances in both cases.
(ii) Choose one of the solutions found in part (i), and find the VSWR at the input to the network if the frequency is increased by 20%, and the component values are unchanged.

Solution

(i) Normalizing the load impedance (Figure 1.30): $z_L = \dfrac{34 + j(42 \times 2)}{50} = 0.68 + j1.68.$

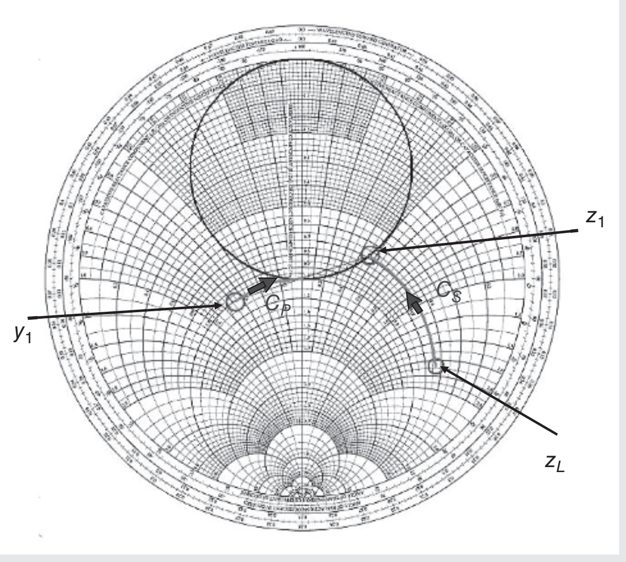

Figure 1.30 First solution to Example 1.12(i).

Using data from the Smith chart:

$$z_1 - z_L = (0.68 + j0.47) - (0.68 + j1.68) = -j1.21,$$

$$y_O - y_1 = 1.00 - (1.00 - j0.65) = j0.65.$$

Thus, we need a negative normalized series reactance of $-j1.21$ and a positive parallel normalized susceptance of $j0.65$,

$$\text{i.e.}\ -j1.21 = -j\frac{1}{\omega C_S} \times \frac{1}{50} \quad \Rightarrow \quad C_S = \frac{1}{1.21 \times \omega \times 50} = \frac{1}{1.21 \times 2\pi \times 2 \times 10^9 \times 50}\text{F} = 1.32\ \text{pF},$$

$$j0.65 = j\omega C_P \times 50 \quad \Rightarrow \quad C_P = \frac{0.65}{\omega \times 50} = \frac{0.65}{2\pi \times 2 \times 10^9 \times 50}\text{F} = 1.03\ \text{pF}.$$

Considering the second valid intersection point on the rotated circle gives the Smith chart solution shown in Figure 1.31.

Using data from the Smith chart:

$$z_2 - z_L = (0.68 - j0.47) - (0.68 + j1.68) = -j2.15,$$

$$y_O - y_2 = 1.00 - (1.00 + j0.65) = -j0.65.$$

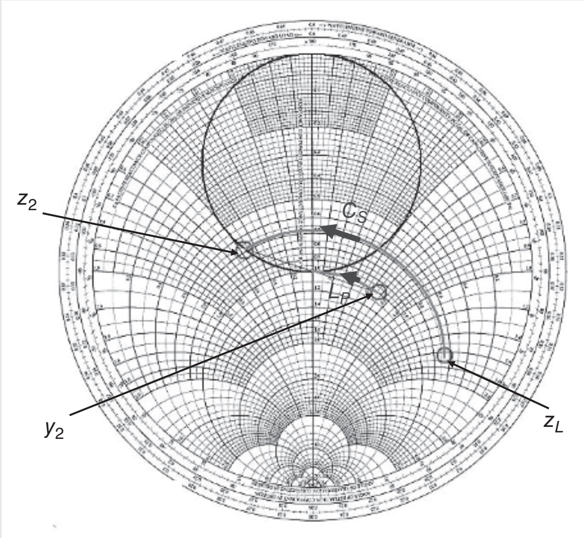

Figure 1.31 Second solution to Example 1.12(i).

Thus, we need negative series normalized reactance of $-j2.15$ and a negative parallel normalized susceptance of $-j0.65$,

$$\text{i.e.}\quad -j2.15 = -j\frac{1}{\omega C_S} \times \frac{1}{50} \quad \Rightarrow \quad C_S = \frac{1}{2.15 \times 50\omega} = \frac{1}{2.15 \times 2\pi \times 2 \times 10^9 \times 50}\text{F} = 0.74\ \text{pF},$$

$$-j0.65 = -j\frac{1}{\omega L_P} \times 50 \quad \Rightarrow \quad L_P = \frac{50}{0.65 \times \omega} = \frac{50}{0.65 \times 2\pi \times 2 \times 10^9}\text{H} = 6.12\ \text{nH}.$$

Summary

The two possible matching networks to satisfy the requirements in Example 1.12 are shown in Figure 1.32.

(ii) Choosing the *first* matching network from part (i):
New frequency: $f = 2.4$ GHz.
New value of load impedance: $Z_L^\dagger = (34 + j42 \times 2.4)\ \Omega = (34 + j100.8)\ \Omega$.
New value of normalized load impedance: $z_L^\dagger = \dfrac{34 + j100.8}{50} = 0.68 + j2.02$.
New value of normalized series reactance: $-j1.21 \times \dfrac{2}{2.4} = -j1.01$.
New value of normalized parallel susceptance: $j0.65 \times \dfrac{2.4}{2} = j0.78$.

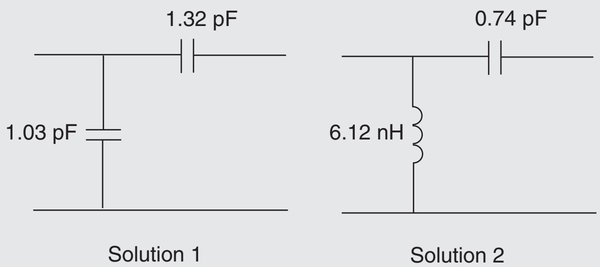

Figure 1.32 Matching networks for Example 1.12.

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The procedure on the Smith chart is as follows:

(1) Plot the new normalized load impedance, $z_L^\dagger$ (see Figure 1.33).
(2) Traverse the constant resistance line through $z_L^\dagger$ counter-clockwise 2.02 units (to represent the new series reactance) to give a new value of $z_1^\dagger$.
(3) Convert $z_1^\dagger$ to $y_1^\dagger$.
(4) Traverse the constant conductance circle through $y_1^\dagger$ clockwise 0.78 units (to represent the new parallel susceptance) to give a value of y_{in}.
(5) Measure the radial distance from the centre of the chart to y_{in} and use the appropriate scale to find the VSWR at the input to the network. (Note that an alternative method to measuring the radial distance would be to plot the VSWR circle through y_{in} and find the VSWR as in Figure 1.11.)
Data from the Smith chart:

$$z_1^\dagger = z_L^\dagger + (-j1.01) = 0.68 + j2.02 - j1.01 = 0.68 + j1.01,$$

$$y_1^\dagger = 0.48 - j0.68,$$

$$y_{in} = y_1^\dagger + j0.78 = 0.48 - j0.68 + j0.78 = 0.48 + j0.10.$$

$$VSWR = 2.1.$$

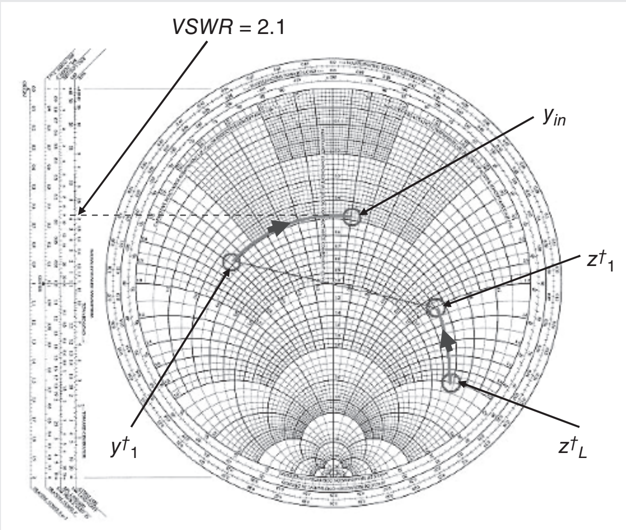

Figure 1.33 Solution to Example 1.12(ii).

Additional points to note about lumped-element matching:

(1) There are always two possible solutions for a particular network configuration (corresponding to the two possible points of intersection for Z_1). This gives the circuit designer an additional degree of design freedom in avoiding awkward-sized components.
(2) It will not be possible to achieve a match using the network configuration shown in Figure 1.26 if the value of the normalized load impedance lies within the unity resistance circle. In this case the configuration shown in Figure 1.34 must be used.

In the network shown in Figure 1.34, the parallel reactive element is used to create normalized impedance z_1 with a real part equal to unity. The imaginary part of z_1 is then cancelled using the series element of the matching network. The design procedure using the Smith chart is similar to that used for the circuit shown in Figure 1.26. The main difference is that we start by converting the load impedance into the equivalent normalized admittance, so that we can directly add the susceptance of the parallel element. As with the design of the earlier matching network, we need to draw a rotated

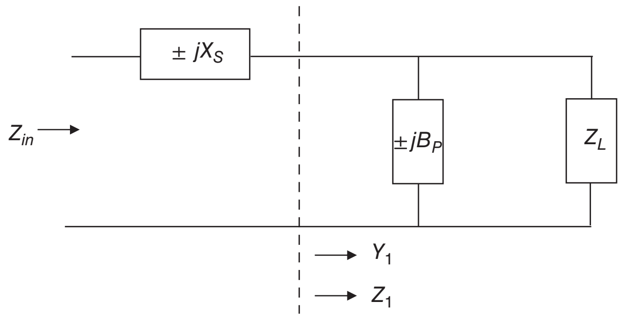

Figure 1.34 Lumped-element matching network 2.

unity circle on the Smith chart in order that we can convert y_1 to z_1 and ensure that the real part of z_1 is unity. The design procedure will be demonstrated through Example 1.13.

Example 1.13 Design a lumped-element network that will match a load impedance $(150 - j50)\ \Omega$ to a 50 Ω source at a frequency of 4 GHz. The network is to be composed of two lossless reactances, with the configuration shown in Figure 1.34. Show that there are two possible solutions, and calculate the required reactance values for each solution.

Solution

Normalizing the load impedance: $z_L = \dfrac{150 - j50}{50} = 3 - j1.$

Having plotted z_L and converted to y_L, we see from Figure 1.35 that there are two possible directions of movement around the constant conductance circle to intersect the rotated circle, giving rise to two possible solutions.

First solution:

Moving clockwise from y_L to intersect the rotated circle we find y_1 as shown in Figure 1.35.

Using data from the chart:

$$y_1 - y_L = (0.3 + j0.455) - (0.3 + j0.1) = j0.355,$$

$$z_O - z_1 = 1 - (1 - j1.55) = j1.55.$$

Thus, we need a positive parallel normalized susceptance of $j0.355$ and a positive series normalized reactance of $j1.55$,

$$\text{i.e. } j0.355 = j\omega C_P \times 50 \quad \Rightarrow \quad C_P = \frac{0.355}{50 \times \omega} = \frac{0.355}{50 \times 2\pi \times 4 \times 10^9}\text{F} = 0.28\ \text{pF},$$

$$j1.55 = \frac{j\omega L_S}{50} \quad \Rightarrow \quad L_S = \frac{1.55 \times 50}{\omega} = \frac{1.55 \times 50}{2\pi \times 4 \times 10^9}\text{H} = 3.08\ \text{nH}.$$

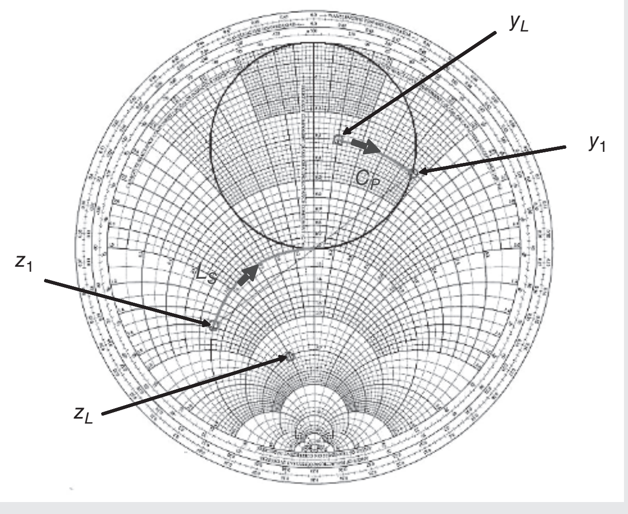

Figure 1.35 First solution to Example 1.13.

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Second solution:

Moving counter-clockwise from y_L we obtain y_2 as shown in Figure 1.36.

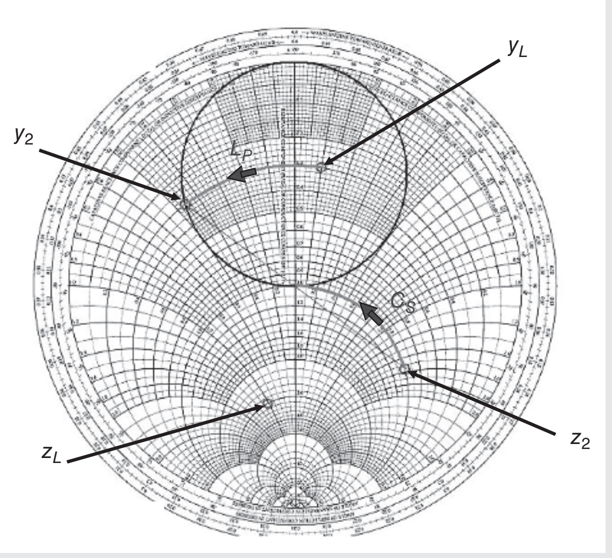

Figure 1.36 Second solution to Example 1.13.

Using data from the chart:

$$y_2 - y_L = (0.3 - j0.455) - (0.3 + j0.1) = -j0.555,$$

$$z_O - z_2 = 1 - (1 + j1.55) = -j1.55.$$

Thus, we need a negative parallel normalized susceptance of $-j0.555$ and a negative series normalized reactance of $-j1.55$,

$$\text{i.e.} \quad -j0.555 = -\frac{j}{\omega L_P} \times 50 \quad \Rightarrow \quad L_P = \frac{50}{0.555 \times \omega} = \frac{50}{0.555 \times 2\pi \times 4 \times 10^9}\text{H} = 3.58\ \text{nH},$$

$$-j1.55 = -\frac{j}{\omega C_S} \times \frac{1}{50} \quad \Rightarrow \quad C_S = \frac{1}{1.55 \times 2\pi \times 4 \times 10^9 \times 50}\text{F} = 0.51\ \text{pF}.$$

Summary

The two possible matching networks to satisfy the requirements in Example 1.13 are shown in Figure 1.37.

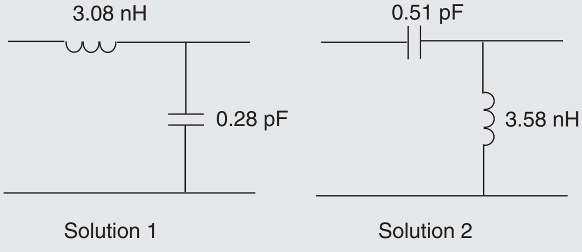

Figure 1.37 Matching networks for Example 1.13.

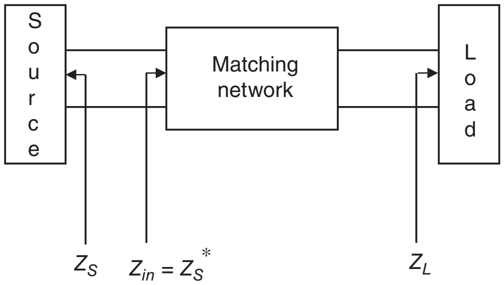

Figure 1.38 Matching complex impedances.

1.11.2 Matching a Complex Load Impedance to a Complex Source Impedance

In the previous section, we considered the matching of a complex load impedance to a real source. Whilst this is probably the most common situation encountered in practical designs, there may be situations, such as inter-stage matching in amplifiers, where we wish to match two complex impedances, i.e. match a complex load impedance to a complex source impedance. Such a match can be achieved at a single frequency by using two lossless reactances, as described in Section 1.11.1, but with a minor modification to the design technique using the Smith chart.

Suppose that we wish to match a complex source impedance, $Z_S = R_S + jX_S$, to a complex load impedance, $Z_L = R_L + jX_L$. The matching network must be designed to provide complex conjugate impedance matching between the source and the input to the network, i.e. the input impedance of the matching network must be equal to Z_S^*, as shown in Figure 1.38. (*Note that we have used the star notation to denote a complex conjugate quantity*.)

We will consider the matching network to have the same configuration as that shown in Figure 1.26. The design procedure on the Smith chart is similar to that previously discussed, with the exception that the series reactance must create a normalized admittance y_1 which has the same conductance as y_{in}. So the rotated circle must be formed by rotating the normalized conductance circle through z_L by 180°. The steps in the design are shown in Example 1.14. As with the previous examples of lumped element matching networks, there will be two valid solutions, corresponding to the two possible intersection points on the rotated circle.

Example 1.14 Design a lumped-element matching network to match a load impedance $(100 + j200)\ \Omega$ to a source whose impedance is $(25 + j130)\ \Omega$, at a frequency of 800 MHz. The matching network is to consist of two lossless reactances, connected in the configuration shown in Figure 1.26.

Solution

Normalizing the load impedance: $z_L = \dfrac{100 + j40}{50} = 2.0 + j0.8$.

Normalizing the source impedance: $z_S = \dfrac{25 + j130}{50} = 0.5 + j2.6$.

Note that the required normalized input impedance, z_{in}, of the matching network is given by

$$z_{in} = z_S^* = 0.5 - j2.6.$$

The initial steps in the design are to plot z_{in}, find the corresponding position of y_{in}, and rotate the constant conductance circle through z_L by 180°.

First solution:

It can be seen from Figure 1.39 that the normalized conductance line through y_{in} intersects the rotated circle at two points, leading to two possible solutions. The first solution will use the intersection point at y_1.

Using data from the Smith chart:

$$z_1 - z_L = (2.0 - j4.5) - (2.0 + 0.8) = -j5.3,$$

$$y_{in} - y_1 = (0.08 + j0.37) - (0.08 + j0.18) = j0.19.$$

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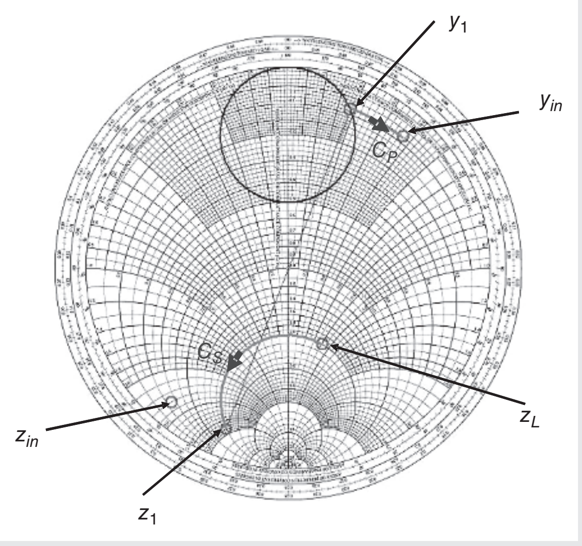

Figure 1.39 First solution to Example 1.14.

Thus, we need a negative series normalized reactance of $-j5.3$ and a positive parallel normalized susceptance of $j0.19$,

$$\text{i.e.}\quad -j5.3 = -j\frac{1}{\omega C_S}\times\frac{1}{50}\quad\Rightarrow\quad C_S = \frac{1}{5.3\times\omega\times 50} = \frac{1}{5.3\times 2\pi\times 800\times 10^6\times 50}\text{F} = 0.75\ \text{pF},$$

$$j0.19 = j\omega C_P\times 50\quad\Rightarrow\quad C_P = \frac{0.19}{\omega\times 50} = \frac{0.19}{2\pi\times 800\times 10^6\times 50}\text{F} = 0.76\ \text{pF}.$$

Second solution:
Choosing the second intersection point, y_2, gives the solution shown in Figure 1.40.

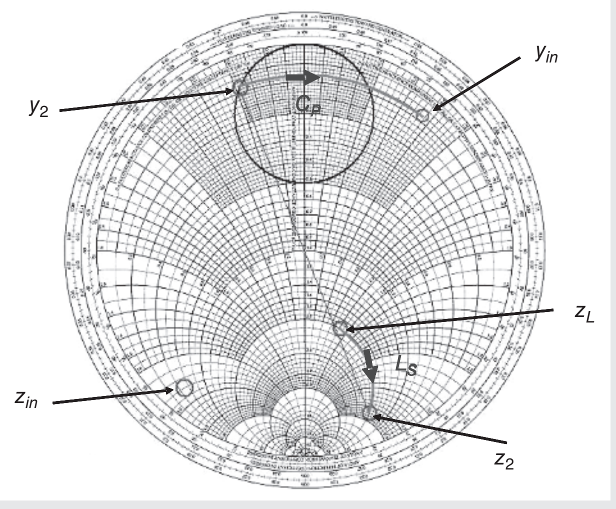

Figure 1.40 Second solution to Example 1.14.

From the Smith chart we obtain:

$$z_2 - z_L = (2.0 + j4.5) - (2.0 + j0.8) = j3.7,$$

$$y_{in} - y_2 = (0.08 + j0.37) - (0.08 - j0.18) = j0.55.$$

Thus, we need a positive series normalized reactance of $j3.7$ and a positive normalized parallel susceptance of $j0.55$,

$$\text{i.e.}\quad j3.7 = j\omega L_S\times\frac{1}{50}\quad\Rightarrow\quad L_S = \frac{3.7\times 50}{\omega} = \frac{3.7\times 50}{2\pi\times 800\times 10^6}\text{H} = 36.80\ \text{nH},$$

$$j0.55 = j\omega C_P \times 50 \quad \Rightarrow \quad C_P = \frac{0.55}{\omega \times 50} = \frac{0.55}{2\pi \times 800 \times 10^6 \times 50}\text{F} = 2.19 \text{ pF}.$$

Summary

The two possible matching networks to satisfy the requirements in Example 1.14 are shown in Figure 1.41.

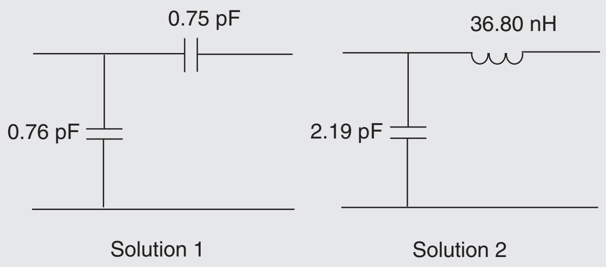

Figure 1.41 Matching networks for Example 1.14

1.12 Equivalent Lumped Circuit of a Lossless Transmission Line

One of the problems frequently encountered in RF and microwave circuits, whether in monolithic or hybrid formats, is the lack of space to implement distributed designs in which the dimensions of the circuit are appreciable fractions of a wavelength. A useful stratagem for overcoming this problem is to replace transmission line sections with their equivalent lumped circuits. It can be shown (Appendix 1.E) that a simple π-network of reactances exhibits the same electrical behaviour as a matched section of lossless transmission line. Figure 1.42 shows a transmission line and its equivalent circuit.

The values of the reactances shown in the equivalent circuit of Figure 1.42 can be calculated using the following expressions:

$$L = \frac{Z_O}{2\pi f}\sin(\theta) = \frac{Z_O}{2\pi f}\sin(\beta l), \tag{1.61}$$

$$C = \frac{1}{2\pi f Z_O}\tan\left(\frac{\theta}{2}\right) = \frac{1}{2\pi f Z_O}\tan\left(\frac{\beta l}{2}\right), \tag{1.62}$$

where Z_O is the characteristic impedance of the transmission line, f is the frequency of operation, and l is the length of transmission line.

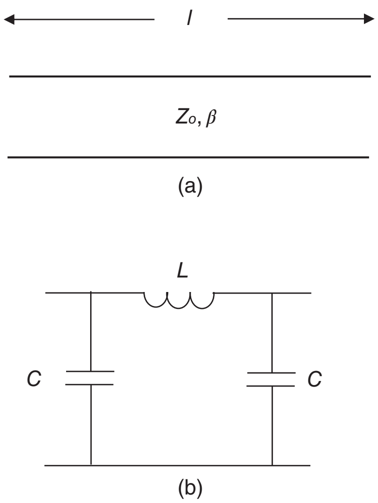

Figure 1.42 Equivalent lumped network of a transmission line: (a) length of lossless transmission line and (b) equivalent π-network.

For the particular case of a quarter-wavelength line, $\beta l = 90°$, and Eqs. (1.61) and (1.62) become

$$L = \frac{Z_O}{2\pi f}, \tag{1.63}$$

$$C = \frac{1}{2\pi f Z_O}. \tag{1.64}$$

1.13 Supplementary Problems

Note: (i) *All of the transmission lines specified in these problems are assumed to be lossless. Similarly, where reactances are specified in a question they are also assumed to be lossless.*

(ii) *It is intended that the problems be solved using the Smith chart. The numerical answers presented at the end of the book were obtained using Smith charts and it should be appreciated that these answers may contain small plotting errors.*

Q1.1 A 50 Ω transmission line is terminated by an impedance $(70 - j20)$ Ω. Determine the impedance at a distance of 0.35λ from the termination.

Q1.2 A 75 Ω transmission line is terminated by a $(150 + j40)$ Ω load. Determine the electrical length from the load to the nearest point where the voltage on the line is a maximum.

Q1.3 A 50 Ω transmission line is terminated by a load whose impedance is $(62 - j120)$ Ω. Determine the reflection coefficient of the load, and the VSWR on the line.

Q1.4 A 500 MHz generator having an output impedance of 50 Ω is to be connected to a 50 Ω load using a 2.5 m length of 50 Ω cable. If a 75 Ω cable is used by mistake, determine the reflection coefficient at the output of the generator. The velocity of propagation along the cable is 2.2×10^8 m/s.

Q1.5 An impedance terminating a 75 Ω transmission line causes a VSWR of 4.5 on the line when the frequency is 750 MHz. If the distance between the termination and the nearest voltage minimum on the line is 6 cm, determine the impedance of the termination, given that the velocity of propagation on the line is 2×10^8 m/s.

Q1.6 A transmission line that has a characteristic impedance of 100 Ω, is terminated by an impedance $(220 + j80)$ Ω. Determine the admittance of the termination and the electrical length (i.e. the length expressed as a fraction of a wavelength) between the load and the nearest point where the normalized conductance is unity.

Q1.7 A transmission line has a characteristic impedance of 50 Ω and a velocity of propagation of 2.2×10^8 m/s. The line is terminated by an impedance, Z_L. Determine the value of Z_L if the impedance on the line 0.75 m from the load is $(30 - j85)$ Ω when the frequency is 100 MHz.

Q1.8 A 75 Ω transmission line is terminated by an impedance $(60 - j95)$ Ω. Design an SST, using a short-circuited stub, which will match the terminating impedance to the line at a frequency of 500 MHz, given that the velocity of propagation on the line is 2.0×10^8 m/s. The stub is to have a characteristic impedance of 75 Ω and be connected in parallel with the main line.

Q1.9 Repeat Q1.8 using an open-circuited stub.

Q1.10 Repeat Q1.8 if both the transmission line and the stub have a characteristic impedance of 50 Ω.

Q1.11 A 4 nH inductor is required at a frequency of 800 MHz. The inductor is to be constructed using a length of short-circuited, 50 Ω transmission line on which the velocity of propagation is 2.2×10^8 m/s. Determine the required length of the transmission line.

Q1.12 A 10 pF capacitor at 1.5 GHz is to be made from a length of short-circuited, 75 Ω transmission line on which the velocity of propagation is 2.2×10^8 m/s. Find the required length of the transmission line.

Q1.13 Find the value of Z_{in} shown in Figure 1.43, at a frequency of 750 MHz.

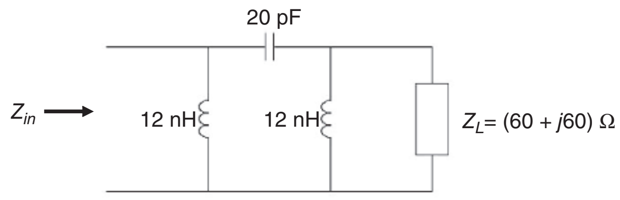

Figure 1.43 Circuit for Q1.13.

Q1.14 Find the value of Z_{in} shown in Figure 1.44, at a frequency of 2 GHz.

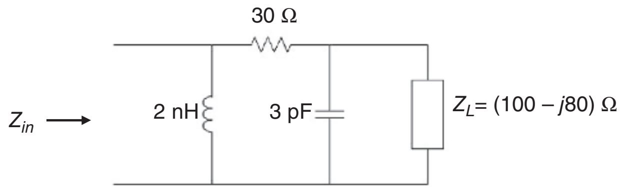

Figure 1.44 Circuit for Q1.14.

Q1.15 Find the value of Z_{in} shown in Figure 1.45, at a frequency of 400 MHz.

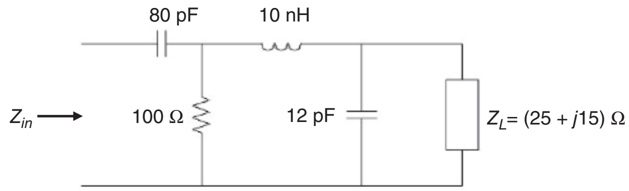

Figure 1.45 Circuit for Q1.15.

Q1.16 A matching network consisting of a lossless series reactance and a lossless shunt susceptance is shown in Figure 1.46. Determine the component values needed to match a load impedance $(10 - j15)$ Ω to a 50 Ω source at a frequency of 550 MHz. Show that there are two possible solutions and calculate the component values in both cases.

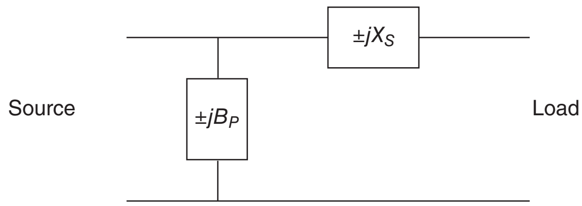

Figure 1.46 Circuit for Q1.16.

Q1.17 (i) A matching network consisting of a lossless series reactance and a lossless shunt susceptance is shown in Figure 1.47. The load impedance is $(120 - j10f)\ \Omega$, where f is the frequency in GHz. Find the component values needed to match the load impedance to a 50 Ω source at a frequency of 3 GHz. Show there are two possible solutions and find the component values in both cases.

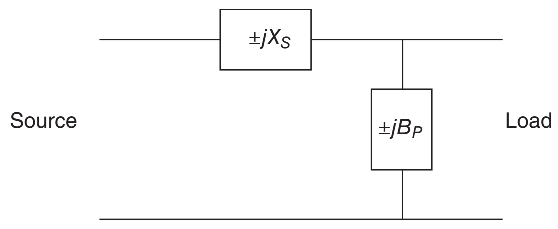

Figure 1.47 Circuit for Q1.17.

(ii) Suppose that the frequency decreases by 15%. Find the reflection coefficient at the input of each of the matching networks calculated in part (i).

Q1.18 A network is required to match a load impedance $(20 - j15)\ \Omega$ to a source whose impedance is $(25 - j35)\ \Omega$, at frequency of 1.5 GHz. Show that a network consisting of two inductors will provide a suitable match, and find the values of the inductors.

Appendix 1.A Coaxial Cable

1.A.1 Electromagnetic Field Patterns in Coaxial Cable

The electric and magnetic fields within a coaxial cable are shown in Figure 1.48. The electric field forms radial lines between the conductors, with the magnetic field forming closed loops around the centre conductor.

It can be seen that the electric and magnetic fields are orthogonal, and lie in a plane which is at 90° to the direction of propagation, i.e. in a plane that is at 90° to the axis of the cable. This is described as the transverse electromagnetic (TEM) mode of propagation.

1.A.2 Essential Properties of Coaxial Cables

The characteristic impedance, Z_O, of coaxial cable is given by [2]

$$Z_O = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{r_O}{r_i}\right), \tag{1.65}$$

where μ and ε are the permeability and permittivity respectively of the dielectric filling the cable, and r_O and r_i are the cable dimensions, as shown in Figure 1.49.

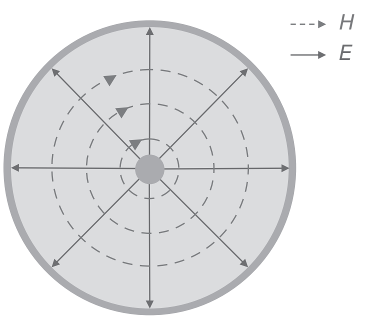

Figure 1.48 Electromagnetic field within a coaxial cable.

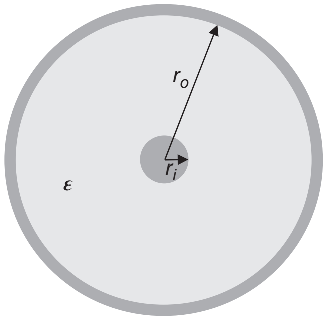

Figure 1.49 Nomenclature for a coaxial cable.

The propagation of a wave along coaxial cable depends on the value of the propagation constant (γ), which was defined in Section 1.3, and which provides information on the attenuation and the phase change per unit length in the cable. The attenuation is determined by the sum of the losses of the conductors and dielectric in the cable. The conductor loss at a frequency, f, for coaxial cable using copper conductors is given by [2] as

$$\alpha_C = \frac{9.5 \times 10^{-5} \times (r_O + r_i)\sqrt{f\varepsilon_r}}{r_O r_i \ln\left(\frac{r_O}{r_i}\right)} \quad \text{dB/unit length,} \tag{1.66}$$

where ε_r is the relative permittivity (dielectric constant) of the material filling the cable. The dielectric loss is given by [2] as

$$\alpha_d = 27.3\sqrt{\varepsilon_r}\frac{\tan\delta}{\lambda_O} \quad \text{dB/unit length,} \tag{1.67}$$

where $\tan\delta$ is the loss tangent of the cable, and λ_O is the free-space wavelength at the frequency of interest. Note that the loss tangent is a parameter used to specify loss in an insulating material, and this will be discussed in more detail in Chapter 3.

The velocity of propagation along a coaxial cable depends only on the permittivity of the dielectric filling the cable, and since the propagation is TEM the velocity is given simply by

$$v_p = \frac{c}{\sqrt{\varepsilon_r}}, \tag{1.68}$$

where ε_r is the relative permittivity (dielectric constant) of the material filling the cable, and c is the velocity of light.

Thus, at a given frequency, f, the propagation constant for coaxial cable can be written as

$$\gamma = \alpha + j\beta = (\alpha_c + \alpha_d) + j\frac{2\pi f}{c}\sqrt{\varepsilon_r}. \tag{1.69}$$

Another important parameter relating to coaxial cable is the cut-off wavelength, λ_c. This gives the frequency above which higher-order modes can propagate in a given size of coaxial cable. The cut-off wavelength is related to the physical parameters of the cable by [2]

$$\lambda_C = \pi\sqrt{\varepsilon_r}(r_O + r_i). \tag{1.70}$$

It is important to ensure that coaxial cable used for a particular application is working below the cut-off wavelengths of higher-order modes, so that a single unique mode is propagating, for two reasons:

(i) In order to calculate the precise phase change along a given length of coaxial cable, it is necessary to know that only one mode, with known propagation characteristics, is travelling along the line.
(ii) Where transitions are made from coaxial cable to another transmission medium, such as waveguide or microstrip, it is necessary to know the precise electromagnetic field pattern within the cable.

As the frequency of operation increases, the size of the cable must decrease. Typical coaxial cable used for RF work has a diameter around 6 mm, with a centre conductor having a diameter around 1 mm. Flexible cable with these dimensions is usually filled with a plastic dielectric, which has a relative permittivity of 2.3. So, using Eq. (1.70), the cut-off wavelength is approximately 17 mm, which corresponds to a cut-off frequency of 17.6 GHz, and this is the maximum frequency at which the cable should be used. If there was a requirement to use coaxial cable at millimetre-wave frequencies, say 50 GHz, the overall diameter of the cable would need to be less than around 2.5 mm. At 100 GHz, the coaxial cable and connectors used in millimetre-wave network analyzers are normally 1 mm in diameter.

Appendix 1.B Coplanar Waveguide

1.B.1 Structure of Coplanar Waveguide (CPW)

Two examples of coplanar transmission lines are illustrated in Figure 1.50. The principal feature of this type of line is that both the signal and grounded conductors are on the same side of the substrate. Conventional CPW lines have the ground plane extending to the edge of the substrate, but in many practical situations this is not possible and the ground plane is restricted to two wide strips on either side of the signal line. This latter type of structure is known as finite ground plane CPW.

1.B.2 Electromagnetic Field Distribution on a CPW Line

The *E*-field distribution of a coplanar line is depicted in Figure 1.51.

There are three practical points that should be noted about the distribution:

(i) There is a higher concentration of electric field within the substrate than above the substrate due to the substrate having a higher dielectric constant than air.
(ii) A significant proportion of the field will fringe into the air above the substrate, so that if the line is enclosed within a metallic package there must be adequate space above the line to avoid the fringing field coupling to the metal of the package.
(iii) If the substrate is thin there may also be fringing of the electromagnetic field into the air beneath the substrate, so again some caution would be needed in packaging the coplanar line.

The magnetic field on a coplanar line, not shown in Figure 1.51, forms closed loops around the signal tracks, and at frequencies up to the low microwave region, typically below 20 GHz, propagation can be regarded as quasi-TEM, which means

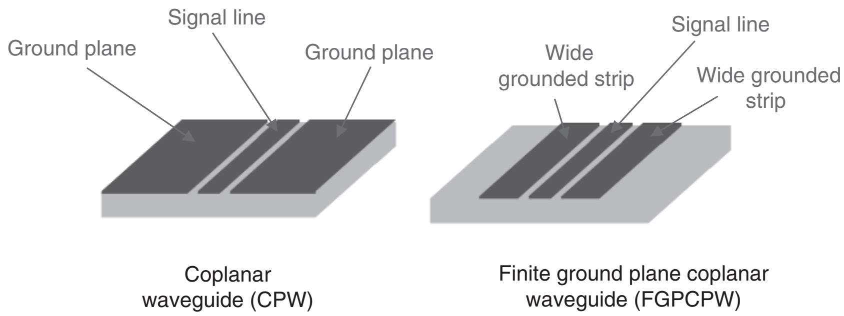

Figure 1.50 Coplanar waveguides.

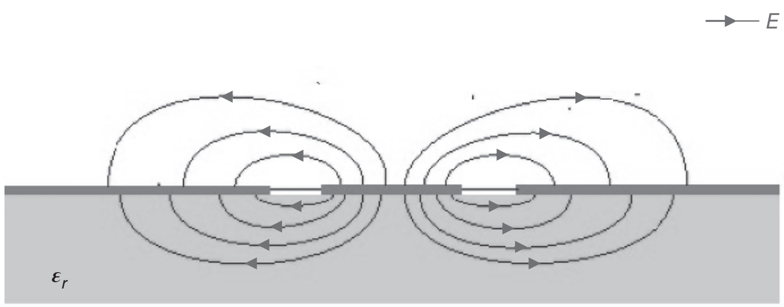

Figure 1.51 *E*-field distribution on a coplanar line.

that the electric and magnetic fields are directed entirely in a transverse plane perpendicular to the direction of propagation. At high microwave frequencies the propagation becomes non-TEM, since longitudinal components of the magnetic field exist. Collin [1] suggests that simple design formulae (as presented in Section A.1.2.3) based on quasi-TEM propagation may be used up to 50 GHz, if the CPW line dimensions are small compared to the wavelength, which is the case for monolithic microwave integrated circuits (MMIC) circuits. Where CPW lines are used at higher millimetre-wave frequencies, a full-wave analysis is required to accurately determine the propagation characteristics. Many CPW circuits operate in the millimetre-wave band above the cut-off frequency for surface-wave modes. However, Riaziat and colleagues [3] showed that for most MMIC applications, there was minimal interaction between CPW propagation and surface-wave modes.

1.B.3 Essential Properties of Coplanar (CPW) Lines

The velocity of propagation, v_P, along a *CPW* line is given by

$$v_P = \frac{c}{\sqrt{\varepsilon_{r,eff}^{CPW}}}, \tag{1.71}$$

where c is the velocity of light, and $\varepsilon_{r,eff}^{CPW}$ is the effective dielectric constant of the medium, and takes into account the dielectric constant of the substrate and the proportion of the electromagnetic field that fringes into the air above and below the substrate. The value of the effective dielectric constant is usually taken to be [3, 4]

$$\varepsilon_{r,eff}^{CPW} = \frac{\varepsilon_r + 1}{2}, \tag{1.72}$$

where ε_r is the dielectric constant of the substrate supporting the CPW line.

The characteristic impedance, Z_O, of a CPW line is given by [4]

$$Z_O = \frac{30\pi}{\sqrt{\varepsilon_{r,eff}^{CPW}}}\left[\frac{1}{\pi}\ln\left(\frac{2(1+\sqrt{k'})}{(1-\sqrt{k'})}\right)\right] \quad 0 \le k \le 0.71, \tag{1.73}$$

$$Z_O = \frac{30\pi}{\sqrt{\varepsilon_{r,eff}^{CPW}}}\left[\frac{1}{\pi}\ln\left(\frac{2(1+\sqrt{k})}{(1-\sqrt{k})}\right)\right]^{-1} \quad 0.71 \le k \le 1, \tag{1.74}$$

where

$$k = \frac{s}{s+2w}, \tag{1.75}$$

$$k' = \sqrt{1-k^2}, \tag{1.76}$$

and s and w are defined in Figure 1.52.

The loss in a CPW line is the sum of the conductor and dielectric losses. Collin [1] derives an expression for the total conductor loss in dB/unit length as

$$\alpha_c = \frac{A}{2Z_O}\left[\pi + \ln\left(\frac{4\pi s}{t}\right) - kB\right] + \frac{kA}{2Z_O}\left[\pi + \ln\left(\frac{4\pi(s+2w)}{t}\right) - \frac{B}{k}\right], \tag{1.77}$$

where

$$A = \frac{R_m}{4s(1-k^2)[K(k)]^2}, \tag{1.78}$$

$$B = \ln\left(\frac{1+k}{1-k}\right), \tag{1.79}$$

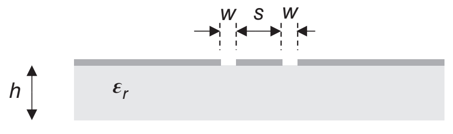

Figure 1.52 Nomenclature for a coplanar waveguide.

and

t = conductor thickness
R_m = conductor surface resistance
$K(k)$ = Complete elliptic integral[2] of first kind with a modulus k.

The dielectric loss in a CPW line in dB/unit length is given by [4]

$$\alpha_d = 2.73 \frac{\varepsilon_r}{\sqrt{\varepsilon_{r,eff}^{CPW}}} \frac{(\varepsilon_{r,eff}^{CPW} - 1)}{(\varepsilon_r - 1)} \frac{\tan\delta}{\lambda_O}, \tag{1.80}$$

where

ε_r = dielectric constant of substrate
$\tan\delta$ = loss tangent of substrate.

The propagation constant for CPW at a frequency f is then given by

$$\gamma = \alpha_c + \alpha_d + j\frac{2\pi f}{c}\sqrt{\varepsilon_{r,eff}^{CPW}}, \tag{1.81}$$

where α_c and α_d are found by substituting in Eqs. (1.77) through (1.80).

1.B.4 Summary of Key Points Relating to CPW Lines

(1) It is usually easier to mount packaged active components in hybrid RF circuits using CPW, compared to microstrip, because the components are often in fabricated in flat beam-lead packages which require one of more of the leads to be earthed. In microstrip, this is an awkward situation that requires the use of vertical connections (VIAs) to the ground plane.
(2) CPW are preferred for MMICs because the provision of ground planes on the same side of the substrate as the signal tracks gives good isolation between closely spaced signal lines.
(3) The symmetrical geometry of CPW permits easy implementation of transitions from microstrip or slotline; this allows flexibility in design, particularly for multilayer circuits.
(4) It is easy to convert from coaxial line to CPW, since both structures have symmetrical geometries with the signal line between ground planes. Thus, it is easy to mount coaxial connectors on packaged planar circuits that use CPW. Connectors mounted on RF and microwave packages are often a weak point in the design, and this will be discussed in more detail in Chapter 3, which deals with fabrication issues.
(5) Coplanar lines exhibit low dispersion, particularly for small line geometries. Jackson [5] showed that at 60 GHz the dimensions of CPW could be chosen to give better results than microstrip, in terms of both dispersion and conductor loss.
(6) The separation of the lines in coplanar structures is usually small, and this imposes limitations on the power handling capability, which is normally rather low.
(7) CPW lines generally need larger packaging enclosures than microstrip because of the greater fringing field; in CPW there may be fringing field from both the upper and lower surfaces of the structure.
(8) Electrical parameters such as characteristic impedance and velocity of propagation depend on a number of parameters, so it is less easy to use graphical design curves as can be done for microstrip (see Appendix 1.D and Chapter 2). The only practical design technique for CPW is to use appropriate CAD packages.
(9) In general, coplanar lines require relatively small line spacing to give useful values of characteristic impedance. For example, using 25 mil ($h = 0.025'' \equiv 0.635$ mm) thick alumina, which is a very common substrate for RF and microwave circuits, a line of characteristic impedance 50 Ω requires a signal track width (w) of 100 μm and a line spacing (s) of 90 μm. However, the range of practical line impedances with CPW structures tends to be greater than with microstrip.

2 The value of the complete elliptic integral with a given modulus k can be found from tables, or as an approximation [1] by summing the first three terms in the following series:

$$K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + \ldots\right) \qquad k \le 0.4.$$

Typical range of impedances:

Microstrip	15 Ω → 110 Ω
CPW	20 Ω → 250 Ω

(10) The small geometries of CPW or FGPCPW lines make them the preferred format for higher millimetre-wave frequency applications.

Appendix 1.C Metal Waveguide

1.C.1 Waveguide Principles

Waveguides are hollow metal tubes through which the microwave field patterns are propagated. The tubes are usually rectangular or circular in cross-section. Waveguides provide low loss transmission and are capable of handling high powers.

The walls of a rectangular waveguide are termed the broad and narrow walls, with their dimensions represented by a and b, as shown in Figure 1.53.

Previous mention was made of TEM waves; these are waves in which the electric and magnetic fields are entirely directed in a plane perpendicular to the direction of propagation. The transverse plane is considered to extend to infinity which means that the electric and magnetic fields have uniform density. A TEM wave is also described as plane wave. Such a wave cannot exist within a medium bounded by a metal conductor, because the boundary conditions would be violated at the metal surfaces. So a different pattern of electric and magnetic fields must exist within a metal waveguide; various different patterns can exist, and these are known as the waveguide modes.

1.C.2 Waveguide Propagation

The pattern of the electric and magnetic fields within a rectangular waveguide is often illustrated by considering the interference pattern due to the intersection of two plane waves in free space. Figure 1.54a shows two plane waves travelling at angles θ to the horizontal, with the electric field in a direction perpendicular to the paper. The solid lines represent lines of peak magnetic field, and the dotted lines represent lines of zero magnetic field. The peaks are separated by half the free-space wavelength. The electric field is shown using the normal convention that a circle represents a field line directed into the paper, and a solid dot a field line directed out of the paper.

The interference pattern resulting from the intersection of the two plane waves is shown in Figure 1.54b. From this pattern it can be seen that the magnetic field has formed closed loops, and that there are maxima and minima points in the electric field. Also, it can be seen that the electric fields are zero along the lines xx' and yy'. Thus conducting sheets, perpendicular to the plane of the diagram, could be placed along these two lines without disrupting the field pattern. This indicates the electromagnetic field pattern that would exist between the narrow walls of the rectangular waveguide. Metal plates forming the broad walls of the waveguide would be in an orthogonal plane, perpendicular to the electric field and parallel to the magnetic field, and would thus have no effect on the field pattern. So, as the wave propagates, the pattern of magnetic loops and the associated electric field travel inside the waveguide.

1.C.3 Rectangular Waveguide Modes

Many different patterns of electric and magnetic field can exist within a rectangular waveguide and these are called the waveguide modes. The modes are categorized into two types: *TE* modes in which the electric field lies entirely in the transverse plane, and *TM* modes in which the magnetic field lies entirely within the transverse plane. Subscripts are used to

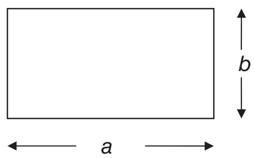

Figure 1.53 Cross-section of a rectangular waveguide.

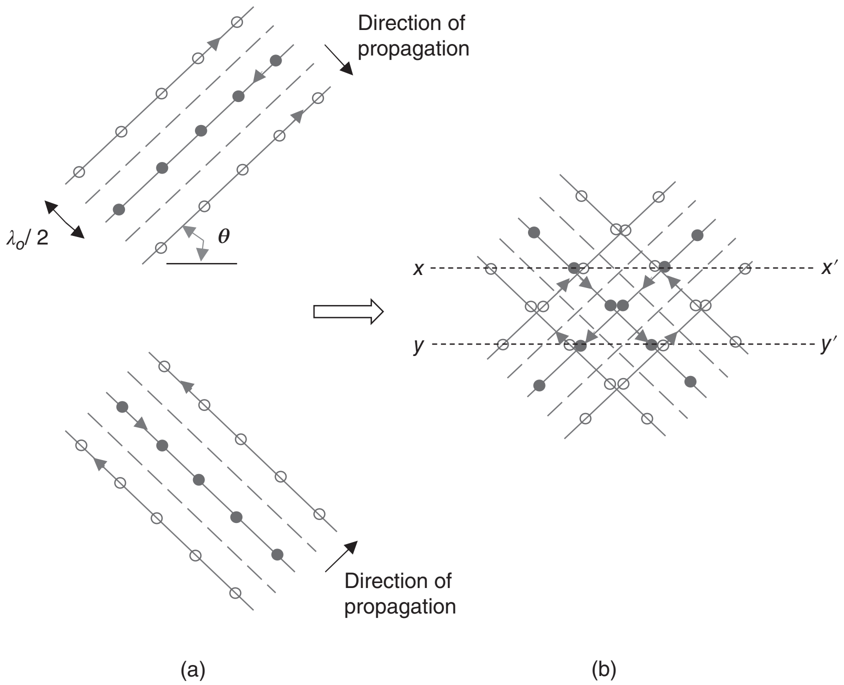

Figure 1.54 Intersection of two plane waves: (a) the individual waves; (b) the resulting interference pattern.

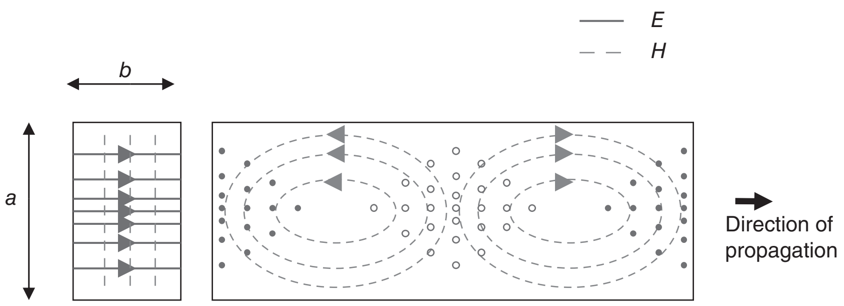

Figure 1.55 Sketch of electric and magnetic field patterns for TE_{10} mode in a rectangular waveguide.

identify particular modes. Thus, we have TE_{mn} and TM_{mn} modes. The first subscript, m, specifies the number of loops of variation of the field across the broad dimension, and the second subscript, n, gives the number of loops of variation of the field across the narrow dimension. In this context, a loop of variation refers to the magnitude of the particular field varying through zero–maximum–zero.

Thus, the electromagnetic field pattern we deduce from the intersection of plane waves, and shown in Figure 1.55, would be described as the TE_{10} mode, since there is one loop of variation of the E-field across the broad dimension, and the E-field is constant across the narrow dimension.

1.C.4 The Waveguide Equation

The waveguide equation is an expression relating the guide wavelength, λ_g, along the guide, the free-space wavelength, and the guide dimensions. The equation can be derived by considering the geometry of a single loop of magnetic field, as shown in Figure 1.56.

The line LN in Figure 1.56 represents a plane wave travelling at the velocity of light, c, at an angle θ to the axis of the waveguide. From the earlier consideration of the intersection of plane waves we know that the axial length of one magnetic loop will be $\lambda_g/2$, and it follows from the interference pattern of two plane waves that the distance MP must be $\lambda_O/2$, where

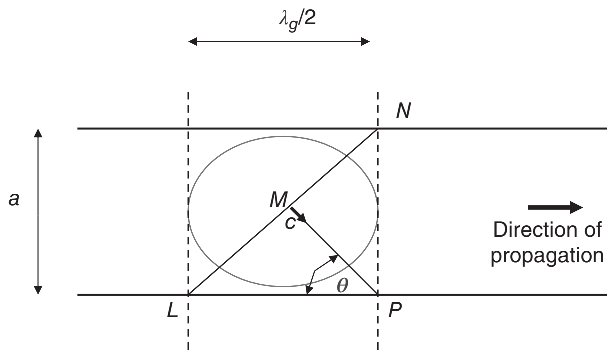

Figure 1.56 Single loop of magnetic field in a rectangular waveguide.

λ_O is the free-space wavelength. Considering the right-angled triangles LMP and MNP, we have

$$\cos\theta = \frac{\lambda_O/2}{\lambda_g/2} = \frac{\lambda_O}{\lambda_g} \quad \text{and} \quad \sin\theta = \frac{\lambda_O/2}{a} = \frac{\lambda_O}{2a}.$$

Now
$$\cos^2\theta + \sin^2\theta = 1.$$

Therefore,
$$\left(\frac{\lambda_O}{\lambda_g}\right)^2 + \left(\frac{\lambda_O}{2a}\right)^2 = 1,$$

i.e.
$$\frac{1}{\lambda_g^2} + \frac{1}{(2a)^2} = \frac{1}{\lambda_O^2}$$

or
$$\frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} = \frac{1}{\lambda_O^2}, \tag{1.82}$$

where $\lambda_c = 2a$ and is defined as the cut-off wavelength. The cut-off wavelength is the longest wavelength for normal propagation of the TE_{10} mode along a guide whose broad dimension is a; the corresponding frequency is the cut-off frequency, f_c. At frequencies less than f_c energy will only penetrate into the waveguide in the form of an evanescent mode, and will suffer very high attenuation. Thus, propagation is only possible in the TE_{10} mode at frequencies where the free-space wavelength is greater than $2a$.

1.C.5 Phase and Group Velocities

It is clear from Figure 1.56 that the time taken for point L to travel to point P must be the same as that for point M to travel to point P, i.e.

$$\frac{\lambda_g/2}{v_P} = \frac{\lambda_O/2}{c} \quad \Rightarrow \quad v_P = \frac{\lambda_g}{\lambda_O}c = \frac{c}{\cos\theta}, \tag{1.83}$$

where v_P is the velocity with which point L travels along the waveguide, and is termed the phase velocity. It is obvious from Eq. (1.83) that the phase velocity is greater than the velocity of light. But this is the velocity with which a particular point (or phase) of the electromagnetic pattern will travel, and not the velocity at which energy will travel. The velocity with which energy travels along the guide is the group velocity, v_g, and it can be seen from Figure 1.56 that this can be found by resolving c in the axial direction, i.e.

$$v_g = c \times \cos\theta. \tag{1.84}$$

Combining Eqs. (1.83) and (1.84) gives

$$v_P \times v_g = c^2. \tag{1.85}$$

1.C.6 Field Theory Analysis of Rectangular Waveguides

Whilst the explanation of waveguide propagation based on the intersection of two plane waves, as described in Section A.1.3.2, provides a useful pictorial view, particularly as an introduction to the topic, a more rigorous and comprehensive analysis can be obtained using classical field theory. In this approach, Maxwell's equations are solved using the boundary conditions set by the walls of the rectangular waveguide. A complete development of the theory is given in Collin [1], but it is useful here to state the principal results.

Based on field theory analysis [1], the components of the electric and magnetic fields for the TE modes in rectangular waveguide are given in Eq. (1.86) below, and are based on the coordinate system shown in Figure 1.57.

$$\begin{aligned}
E_x &= \frac{j\omega\mu n\pi}{bk_c^2} H_O \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
E_y &= -\frac{j\omega\mu m\pi}{ak_c^2} H_O \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
E_z &= 0,\\
H_x &= \frac{j\beta m\pi}{ak_c^2} H_O \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
H_y &= \frac{j\beta n\pi}{bk_c^2} H_O \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
H_z &= H_O \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},
\end{aligned} \tag{1.86}$$

where

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \tag{1.87}$$

and H_O is the magnetic field strength at the input to the waveguide.

Using Eq. (1.86) the field components for the TE_{10} mode are shown in Eq. (1.88).

$$\begin{aligned}
E_x &= 0,\\
E_y &= -\frac{j\omega\mu\pi}{ak_c^2} H_O \sin\left(\frac{\pi x}{a}\right)e^{j(\omega t-\beta z)},\\
E_z &= 0,\\
H_x &= \frac{j\beta\pi}{ak_c^2} H_O \sin\left(\frac{\pi x}{a}\right)e^{j(\omega t-\beta z)},\\
H_y &= 0,\\
H_z &= H_O \cos\left(\frac{\pi x}{a}\right)e^{j(\omega t-\beta z)}.
\end{aligned} \tag{1.88}$$

Through inspection of Eq. (1.88), we see that the field distribution agrees with that in Figure 1.55, which was deduced from the intersection of two plane waves. The E-field only has a component in the y-direction, and is zero at the waveguide walls ($x = 0$ and $x = a$) and is a maximum in the centre of the guide ($x = a/2$).

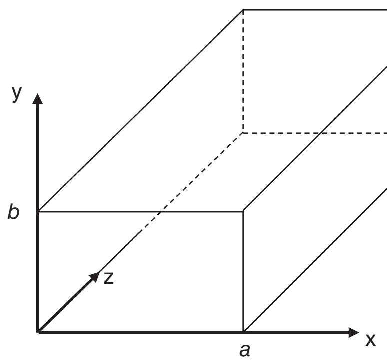

Figure 1.57 Coordinate system for electromagnetic field components.

For completeness the field components for the TM_{mn} modes are stated in Eq. (1.89), but these have very limited application in modern RF work.

$$
\begin{aligned}
E_x &= -\frac{j\beta m\pi}{ak_c^2}E_O\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
E_y &= -\frac{j\beta n\pi}{bk_c^2}E_O\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
E_z &= E_O\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
H_x &= \frac{j\omega\varepsilon n\pi}{bk_c^2}E_O\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
H_y &= -\frac{j\omega\varepsilon m\pi}{ak_c^2}E_O\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{j(\omega t-\beta z)},\\
H_z &= 0,
\end{aligned}
\tag{1.89}
$$

where E_O is the electric field strength at the input to the waveguide.

1.C.7 Waveguide Impedance

The impedance, Z_O, of an electromagnetic wave propagating in free space is given by

$$Z_O = \left|\frac{E}{H}\right| = \sqrt{\frac{\mu_O}{\varepsilon_O}} = 120\pi = 377\ \Omega. \tag{1.90}$$

In a waveguide, the free-space impedance is modified by the presence of the waveguide walls. For the TE_{10} mode the waveguide impedance will be

$$Z_g = \left|\frac{E_y}{H_x}\right|. \tag{1.91}$$

Substituting from Eq. (1.88) into Eq. (1.91) gives

$$
\begin{aligned}
Z_g &= \frac{\omega\mu\pi H_O/ak_c^2}{\beta\pi H_O/ak_c^2} = \frac{\omega\mu}{\beta} = \frac{2\pi f\mu}{2\pi/\lambda_G} = f\mu\lambda_g\\
&= \frac{c}{\lambda_O}\mu\lambda_g = \frac{1}{\sqrt{\mu\varepsilon}\lambda_O}\mu\lambda_g = \sqrt{\frac{\mu}{\varepsilon}}\frac{\lambda_g}{\lambda_O},
\end{aligned}
$$

i.e.

$$Z_g = Z_O\frac{\lambda_G}{\lambda_O}. \tag{1.92}$$

1.C.8 Higher-Order Rectangular Waveguide Modes

The previous theory has focused on one particular mode, namely the TE_{10} mode. This mode is often described as the dominant mode; it is the mode which propagates at the lowest frequency for a given size of waveguide, and there will be a frequency range over which it is the only mode that can propagate. Thus, the TE_{10} mode is the most useful for practical waveguide transmission and devices. The cut-off wavelength for a particular mode can be found using Eq. (1.93) below:

$$\lambda_c = \frac{1}{\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}}. \tag{1.93}$$

The corresponding cut-off frequencies are found from Eq. (1.94) below:

$$f_c = c\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}. \tag{1.94}$$

Table 1.1 Designations for a rectangular waveguide.

WG designation	Recommended operating range (GHz)	$f_{c\,(TE10)}$ (GHz)
WG90	8.20–12.40	6.56
WG28	26.50–40.00	21.08
WG10	75.00–110.00	59.01
WG5	140.00–220.00	115.750

1.C.9 Waveguide Attenuation

Typical variations of attenuation (α) as a function of frequency for modes in rectangular waveguide are sketched in Figure 1.58. Two particular modes are sketched, namely the TE_{10} mode and the TE_{11} mode, which is the higher-order mode with the closest cut-off frequency to that of the dominant mode.

It can be seen from Figure 1.58 that the attenuation increases sharply as the frequency approaches the cut-off value. Above cut-off, the attenuation slowly increases with frequency due to dissipative losses in the waveguide walls; the dissipative losses increase with increasing frequency due to the skin effect, which will be discussed in more detail in Chapter 2. It is also apparent from Figure 1.58 that there is a frequency region where only the TE_{10} mode will propagate, and within this region Δf indicates those frequencies where the attenuation is minimum. Thus, we can regard a waveguide of a particular size as having an operational bandwidth indicated by Δf.

1.C.10 Sizes of Rectangular Waveguide and Waveguide Designation

It is clear from the previous discussion that waveguide size is a crucial issue, and that for operation using the dominant mode the required size will decrease as the frequency increases. The sizes of rectangular waveguide are designated using the WG notation, and some examples of commonly used waveguides are given in Table 1.1. In the table the recommended operating range for the TE_{10} mode corresponds to Δf in Figure 1.58.

1.C.11 Circular Waveguide

The propagation of high-frequency electromagnetic waves along waveguides having a circular cross-section involves similar principles to those already discussed for rectangular guides. The main differences are that the analysis uses spherical

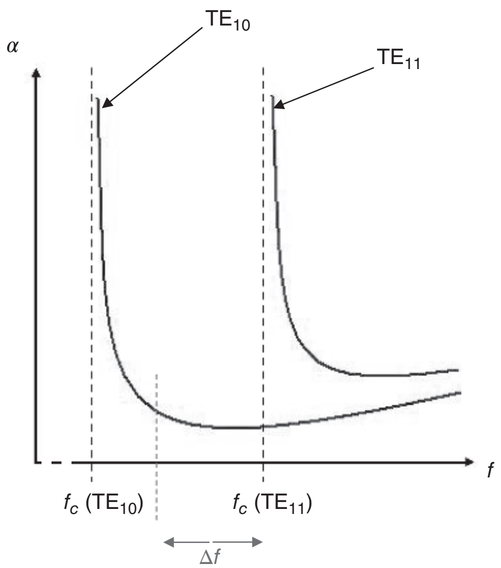

Figure 1.58 Variation of attenuation for rectangular waveguide modes.

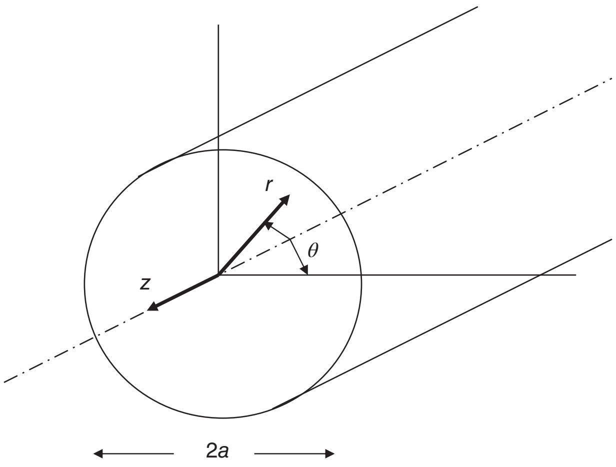

Figure 1.59 Coordinate system for a circular waveguide.

polar coordinates, to be compatible with the geometry of the guide, and that solutions to Maxwell's equations are found in terms of Bessel functions. The modes of propagation in circular waveguide can be classified as TE_{nm} and TM_{nm}, where the subscript m refers to the loops of variation of the field across the radius of the guide, and the subscript n to the double loops (full sinusoid) of variation in the circumferential direction. The coordinate system for circular waveguides is shown in Figure 1.59.

The fields for the TE_{nm} modes in circular waveguide are stated in Eq. (1.95) below:

$$\begin{aligned}
E_r &= -\frac{\omega\mu n}{rk_c^2} H_O J_n(k_c r)\cos(n\theta)e^{j(\omega t-\beta z)},\\
E_\theta &= \frac{j\omega\mu}{k_c} H_O J_n'(k_c r)\cos(n\theta)e^{j(\omega t-\beta z)},\\
E_z &= 0,\\
H_r &= -\frac{j\beta}{k_c} H_O J_n'(k_c r)\cos(n\theta)e^{j(\omega t-\beta z)},\\
H_\theta &= -\frac{\beta n}{rk_c^2} H_O J_n(k_c r)\cos(n\theta)e^{j(\omega t-\beta z)},\\
H_z &= H_O J_n(k_c r)\cos(n\theta)e^{j(\omega t-\beta z)}.
\end{aligned} \tag{1.95}$$

Note that a function of the form $J_n(x)$ is called a Bessel function of the first kind, of order m, and argument x; the values of the function are normally found from tables [6]. The amplitude of the Bessel function oscillates between positive and negative values, and decays as the argument increases. There will be certain values of the argument which make the function zero, and these are called the roots of the function. The differential of a Bessel function of the first kind is written as $J'_n(x)$.

The mode of particular interest in circular waveguide is TE_{01} because it exhibits the unusual property of decreasing attenuation with increasing frequency. The field pattern for the TE_{01} mode is sketched in Figure 1.60, in which it can be seen that the E-field forms closed loops around the axis of the guide, with the magnetic field forming closed loops in axial planes through the centre of the guide. This means that the current will flow in the circumferential direction in the waveguide walls.

The symmetry of the field and current about the central axis means that the mode is also useful for forming rotating waveguide joints, since a joint in circular TE_{01} waveguide will not significantly disturb the current flow. But it is the low loss property of the TE_{01} mode which is its most attractive feature, and this mode can be used for low-loss feeder connections to antennas and for constructing very high-Q cavities, and this latter application will be discussed in more detail in Chapter 3 in the context of the measurement of dielectrics. It should be noted that whilst the TE_{01} mode has useful practical properties, it is not the dominant mode in circular waveguide and caution must be taken when exciting the TE_{01} mode not to generate unwanted modes. The dominant mode in circular waveguide is TE_{11}, and the attenuation characteristic for this mode is shown in Figure 1.61.

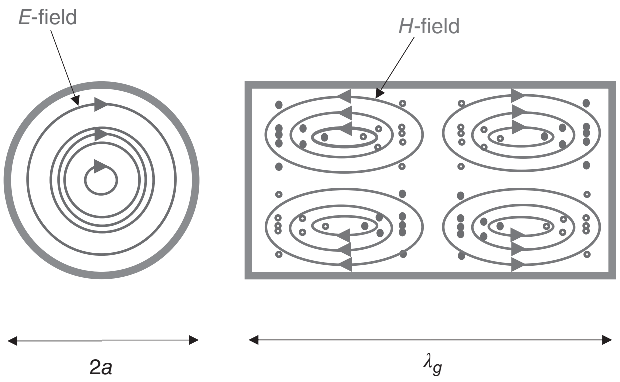

Figure 1.60 Sketch of electric and magnetic field patterns for TE_{01} mode in a circular waveguide.

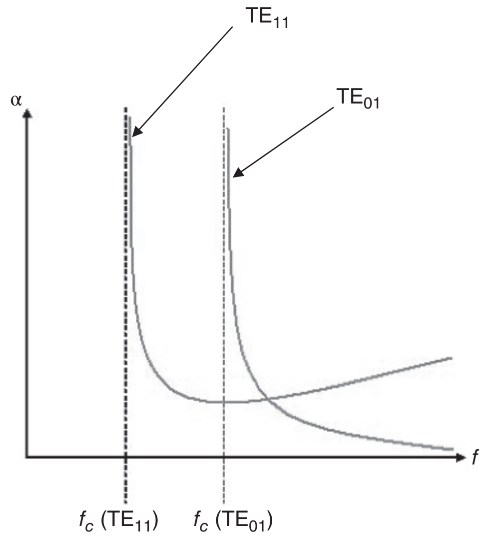

Figure 1.61 Attenuation curves of typical modes in a circular waveguide.

We know that for any mode in metallic waveguide the tangential component of electric field must be zero at the waveguide walls, and considering the expression for E_θ in Eq. (1.95) this means $J'_n(k_c r) = 0$ when $r = a$. Therefore, for the TE_{01} mode, where $n = 0$, we require $J'_0(k_c a) = 0$. From tables of Bessel functions the first root of $J'_0(x)$ occurs when $x = 3.83$.

Karbowiak [7] showed that the cut-off wavelength for the TE_{01} mode in circular waveguide of radius a is given by

$$\lambda_c = \frac{2\pi a}{x}. \tag{1.96}$$

It then follows that the cut-off frequency for the TE_{01} mode is

$$f_c = \frac{c}{\lambda_c}. \tag{1.97}$$

As an example, for a 50 mm diameter circular waveguide

$$\lambda_c = \frac{2\pi \times 25}{3.83} \text{ mm} = 41.02 \text{ mm},$$

$$f_c = \frac{3 \times 10^8}{41.02 \times 10^{-3}} \text{ Hz} = 7.31 \text{ GHz}.$$

Karbowiak [7] also showed that the attenuation of the TE_{01} mode (in dB/m) for a frequency, f_O, which is well above the cut-off value is given by

$$\alpha = 17.37 \frac{F^2 R_m}{2a}, \tag{1.98}$$

where
$F = f_c/f_O$.
R_m = surface resistance of the metal waveguide.

It is clear from Eq. (1.98) that the attenuation of the TE_{01} mode decreases with increasing frequency, and also that increasing the diameter of the waveguide will decrease the attenuation at any given frequency.

Appendix 1.D Microstrip

The cross-section of a microstrip line is shown in Figure 1.62.

The line consists of a low-loss insulating substrate, having a relative permittivity ε_r, with conductor covering the lower surface to form a ground plane, and a signal track on the upper surface. The key parameters of a microstrip line are the substrate relative permittivity (ε_r), the substrate thickness (h), and the track width (w). It is these three parameters that determine the characteristic impedance (Z_O) of the line and the velocity of propagation (v_p) along the line. The track thickness need only be considered if the track is exceptionally thick, typically more than 10 μm.

The characteristic impedance of a microstrip line is related to the geometric parameters of the line by the following expressions [8].

For $\frac{w}{h} < 3.3$:

$$Z_O = \frac{119.9}{\sqrt{2(\varepsilon_r + 1)}} \ln\left(4\frac{h}{w} + \sqrt{16\left(\frac{h}{w}\right)^2 + 2}\right). \tag{1.99}$$

For $\frac{w}{h} > 3.3$:

$$Z_O = \frac{119.9\pi}{2\sqrt{\varepsilon_r}}\chi, \tag{1.100}$$

where

$$\chi = \frac{w}{2h} + \frac{\ln 4}{\pi} + \frac{\ln(e\pi^2/16)}{2\pi}\left(\frac{\varepsilon_r - 1}{\varepsilon_r^2}\right) + \frac{\varepsilon_r + 1}{2\pi\varepsilon_r}\left[\ln\left(\frac{\pi e}{2}\right) + \ln\left(\frac{w}{2h} + 0.94\right)\right] \tag{1.101}$$

and $e = 2.718$.

Microstrip is an open structure and some of the electromagnetic field fringes into the air above the microstrip line. Since the propagating wave is not entirely within the substrate we need to specify an effective relative permittivity ($\varepsilon_{r,eff}^{MSTRIP}$) for the propagation medium, where

$$1 \leq \varepsilon_{r,eff}^{MSTRIP} \leq \varepsilon_r. \tag{1.102}$$

The effective relative permittivity of a microstrip line having a characteristic impedance Z_O can be found from the following expression [8]

$$\varepsilon_{r,eff}^{MSTRIP} = \frac{\varepsilon_r + 1}{2}\left[1 + \frac{29.98}{Z_O}\left(\frac{2}{\varepsilon_r + 1}\right)\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right)\left(\ln\left[\frac{\pi}{2}\right] + \frac{1}{\varepsilon_r}\ln\left[\frac{4}{\pi}\right]\right)\right]^{-2}, \tag{1.103}$$

where ε_r is the relative permittivity of substrate.

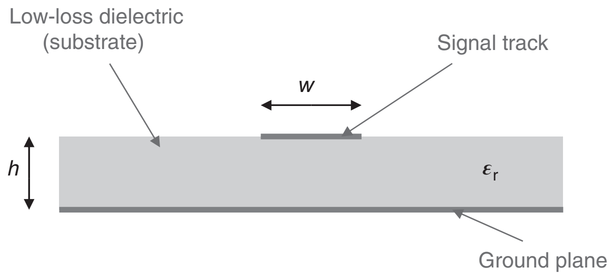

Figure 1.62 Cross-section of a microstrip line.

Once the effective relative permittivity of the substrate is known the velocity of propagation along the microstrip line can be found from

$$v_p = \frac{c}{\sqrt{\varepsilon_{r,eff}^{MSTRIP}}}, \tag{1.104}$$

where c is the velocity of light (3×10^8 m/s).

The wavelength along the microstrip line (also known as the substrate wavelength) at a given frequency (f) will be

$$\lambda_s = \frac{v_p}{f} = \frac{c}{f\sqrt{\varepsilon_{r,eff}^{MSTRIP}}}. \tag{1.105}$$

Hammerstad and Bekkadal [9] give the following widely used expression for the conductor loss, α_c, in microstrip

$$\alpha_c = \frac{0.072\sqrt{f}}{wZ_O}\left(1 + \frac{2}{\pi}\tan^{-1}[1.4(\Delta R_m \sigma)^2]\right) \ \text{dB/unit length}, \tag{1.106}$$

where

f = working frequency
w = microstrip track width
Z_O = characteristic impedance of microstrip line
Δ = RMS surface roughness of metal
R_m = surface resistance of metal
σ = conductivity of metal

The dielectric loss in a microstrip line can be calculated from the expression given by Gupta [4] as

$$\alpha_d = 27.3\frac{\varepsilon_r(\varepsilon_{r,eff}^{MSTRIP} - 1)\tan\delta}{\sqrt{\varepsilon_{r,eff}^{MSTRIP}}(\varepsilon_r - 1)\lambda_O} \quad \text{dB/unit length}, \tag{1.107}$$

where

ε_r = relative permittivity of substrate
$\varepsilon_{r,eff}^{MSTRIP}$ = effective relative permittivity of substrate
λ_O = free-space wavelength at the working frequency
$\tan\delta$ = loss tangent of substrate (see Chapter 3).

The propagation constant, γ, for a particular microstrip line can then be found from

$$\gamma = \alpha_c + \alpha_d + j\frac{2\pi f}{c}\sqrt{\varepsilon_{r,eff}^{MSTRIP}}, \tag{1.108}$$

by substituting the appropriate values of α_c, α_d, and $\varepsilon_{r,eff}^{MSTRIP}$ for the given line.

Appendix 1.E Equivalent Lumped Circuit Representation of a Transmission Line

The equivalence between a two-port network and a length of lossless transmission line can be deduced using transmission, or *ABCD*, parameters. These parameters are defined in Chapter 5, where network parameters are discussed in more detail.

A two-port network comprising a π-network of lossless admittances, as shown in Figure 1.63, can be represented [8] by the *ABCD* matrix given in Eq. (1.109).

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\pi\text{-network}} = \begin{bmatrix} 1 + \dfrac{Y_2}{Y_3} & \dfrac{1}{Y_3} \\ Y_1 + Y_2 + \dfrac{Y_1 Y_2}{Y_3} & 1 + \dfrac{Y_1}{Y_3} \end{bmatrix}. \tag{1.109}$$

We know from Section 1.2 that a lossless transmission line will have series inductance and shunt capacitance, and so a representative equivalent π-network for an electrically short length is shown in Figure 1.64.

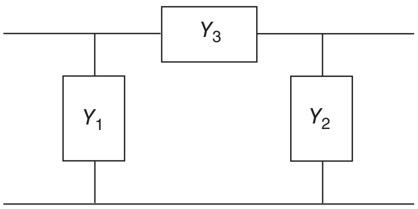

Figure 1.63 π-Network of admittances.

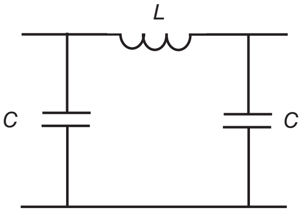

Figure 1.64 π-Network representing an electrically short transmission line.

Comparing Figures 1.63 and 1.64 we have:

$$Y_1 = Y_2 = j\omega C,$$
$$Y_3 = \frac{1}{j\omega L} = -j\frac{1}{\omega L}. \tag{1.110}$$

The *ABCD* matrix representing the circuit shown in Figure 1.64 will then be as shown in Eq. (1.111).

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\textit{equivalent } \pi\textit{-network}} = \begin{bmatrix} 1-\omega^2 LC & j\omega L \\ j(2\omega C - \omega L[\omega C]^2) & 1-\omega^2 LC \end{bmatrix}. \tag{1.111}$$

A length of lossless transmission line, having a characteristic impedance Z_O and a propagation constant β, and shown in Figure 1.65, can also be represented by an *ABCD* matrix [10], and this is given in Eq. (1.112).

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\textit{Tx line}} = \begin{bmatrix} \cos\beta l & jZ_O \sin\beta l \\ jY_O \sin\beta l & \cos\beta l \end{bmatrix}. \tag{1.112}$$

If the electrical performance of the π-network shown in Figure 1.64 is to be equivalent to that of a short transmission line, the *ABCD* matrix in Eq. (1.111) must be identical to that in Eq. (1.112). Equating the matrix elements in these two equations gives

$$\cos\beta l = 1-\omega^2 LC,$$
$$jZ_O \sin\beta l = j\omega L,$$
$$jY_O \sin\beta l = j(2\omega C - \omega L[\omega C]^2). \tag{1.113}$$

Rearranging the terms in Eq. (1.113) gives

$$L = \frac{Z_O}{\omega}\sin\beta l,$$
$$C = \frac{1}{\omega Z_O}\tan\left(\frac{\beta l}{2}\right). \tag{1.114}$$

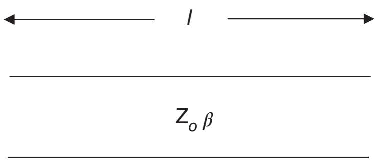

Figure 1.65 Length of a lossless transmission line.

Equation (1.114) can be used to calculate the equivalent lumped circuit elements of a lossless transmission line of characteristic impedance, Z_O, and phase constant, β, at a frequency $\omega/2\pi$.

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