1

Introduction

The ongoing development of the fractional calculus and the related development of fractional differential equations have created new opportunities and new challenges. The need for a generalized exponential function applicable to fractional-order differential equations has given rise to new functions. In the traditional integer-order calculus, the role of the exponential function and the trigonometric functions is central to the solution of linear ordinary differential equations. Such a supporting structure is also needed for the fractional calculus and fractional differential equations.

The purpose of this book is the development of the fractional trigonometries and hyperboletries that generalize the traditional trigonometric and hyperbolic functions based on generalizations of the common exponential function. The fundamental idea is that through the development of the fractional calculus, which generalizes the integer-order calculus, generalizations of the exponential function have been developed. The exponential function in the integer-order calculus provides the basis for the solution of linear fractional differential equations. Also, it may be thought of as the basis of the trigonometry.

A high-level summary of the flow of the development of the book is given in Figure 1.1. The generalized exponential functions that we use, the F-function and the R-function, are fractional eigenfunctions; that is, they return themselves on fractional differintegration. The F-function is the solution to the fundamental fractional differential equation

$$ {}_0d_t^q x(t) + ax(t) = \delta(t) $$

when driven by a unit impulse. The R-function, $R_{q,v}(a,t)$, generalizes the F-function by including its integrals and derivatives as well. First, we show that these functions provide solutions to the fundamental fractional-order differential equation. Then, we explore the properties of the generalized exponential functions and develop some properties of the functions that will aid in the development and understanding of the fractional trigonometries and hyperboletries. This development follows a few mathematical preliminaries.

The R_1, R_2, and R_3 trigonometries along with the R_1 hyperboletry are developed by replacing a and t in the R-function with various combinations of real and purely imaginary variables. Based on the newly defined functions, a variety of basic properties and identities are determined. Furthermore, the Laplace transforms of the functions are determined and the fractional derivatives and fractional integrals of the functions elucidated.

The following chapters develop an overarching fractional trigonometry called the fractional meta-trigonometry that contains all of the fractional trigonometries shown in Figure 1.1 and infinitely many more. This is accomplished by replacing a and t in the R-function with general complex variables. We find that the fractional trigonometric functions lead to a generalization

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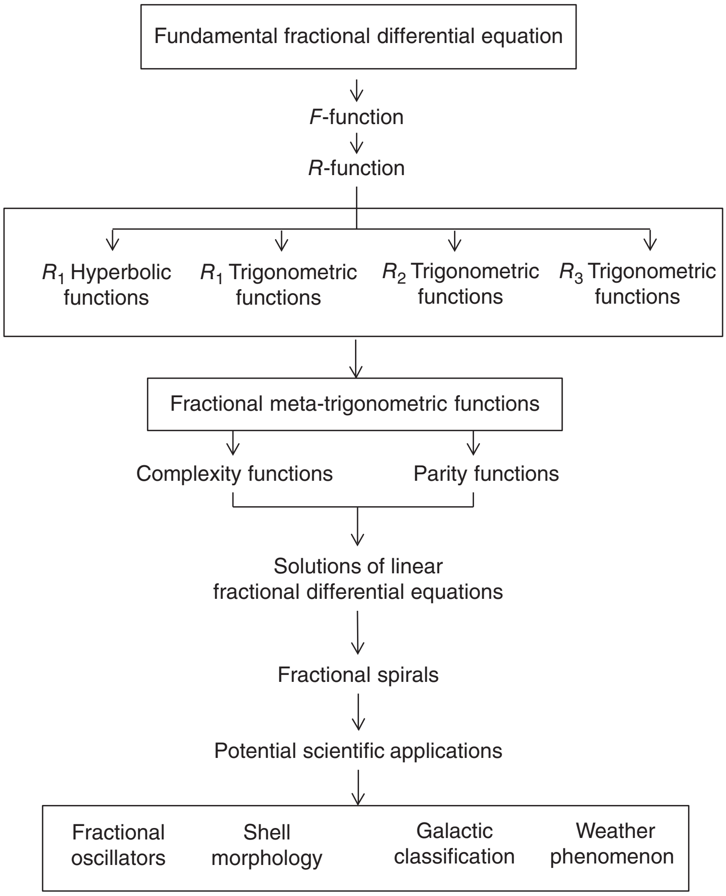

Figure 1.1 Overview of the development of the fractional trigonometry and its applications.

of the circular functions, which we have called the *fractional spiral functions*. These functions appear to model various natural phenomena, and preliminary applications of these functions to the properties of fractional oscillators, sea shells, galaxies, and more are explored. An important aspect of this modeling is that we can infer from the spiral functions the underlying fractional differential equations describing the phenomena, which is demonstrated. More importantly, the new fractional functions provide the solutions to classes of linear fractional differential equations.

1.1 Background

Because of the close association of the fractional calculus and the fractional trigonometry to be developed, we present here a brief introduction to the concepts of the fractional calculus for the reader who is unfamiliar with the area.

Several important textbooks have been written that are extremely helpful to someone entering the field. Perhaps the most useful from the engineering and scientific viewpoint, are the textbooks by Oldham and Spanier, "*The Fractional Calculus*" [104], and by Igor Podluby entitled "*Fractional Differential Equations*" [109]. A more mathematically oriented treatment is given in the book by Miller and Ross [95]. An encyclopedic reference volume written by Samko et al. [114] has also been published. Furthermore, a very good engineering book has been written by Oustaloup [105] and is available in French and Bush [19].

There are a growing number of physical systems whose behavior can be compactly described using fractional differential equations theory. Areas include long lines, electrochemical processes, diffusion, dielectric polarization, noise, viscoelasticity, chaos, creep, rheology, capacitors, batteries, heat conduction, percolation, cylindrical waves, cylindrical diffusion, water through a weir notch, Boussinesq shallow water waves, financial systems, biological systems, semiconductors, control systems, electrical machinery, and more.

1.2 The Fractional Integral and Derivative

The first question we need to address is "just what is a fractional derivative?" There are two separate but equivalent definitions of the fractional differintegral (Oldham and Spanier [104]), known as the Grünwald definition and the Riemann–Liouville definition. We present the Grünwald definition first, as it most recognizably generalizes the standard calculus. We then follow with the Riemann–Liouville definition as it is most easily used in practice.

1.2.1 Grünwald Definition

The Grünwald definition of the fractional-order differintegral is essentially a generalization of the derivative definition that most of us learned in introductory calculus, namely

$$\left.\frac{d^q f(t)}{[d(t-a)]^q}\right|_{GRUN} \equiv \lim_{N\to\infty} \frac{\left(\frac{t-a}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(t - j\left(\frac{t-a}{N}\right)\right), \tag{1.1}$$

or in a slightly more familiar form

$$\left.\frac{d^q f(t)}{[d(t-a)]^q}\right|_{GRUN} \equiv \lim_{N\to\infty} \sum_{j=0}^{N-1} b_j(q) \frac{f(t - j\Delta t_N)}{(\Delta t_N)^q}, \tag{1.2}$$

where

$$\Delta t_N = \frac{t-a}{N}, \quad b_j(q) = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}.$$

In this definition, q is not limited to the integers and may be any real or complex number, and a is the starting time of the fractional differintegration, not to be confused with a in the differential equation in the introduction. Also, $q > 0$ defines differentiation, and $q < 0$ integration. Furthermore, $\Gamma(\circ)$ is the gamma function, or the generalized factorial function. It basically acts as a calibration constant here to properly interpolate the operators for values of q between the integers. In terms of notation, Oldham and Spanier [104] provide a development of equation (1.2) and generalize the fractional differintegral as

$$\frac{d^q x(t)}{[d(t-a)]^q}, \tag{1.3}$$

where it should be noticed that the differential in the denominator starts at some time a, and ends at a final time t. Thus, we see that the fractional derivative is defined on an interval and is no longer a local operator except for integer orders. Interestingly, the gamma functions force the series to terminate with a finite number of terms whenever q is any integer greater than or equal to zero, which represent the usual integer-order derivatives. When q is a negative integer, this series also contains the single and multiple integrals as well (which have always had infinite memory). The important aspect to be recognized is that there exists an uncountable infinity of fractional derivatives and integrals between the integers. The Grünwald definition is also equivalent to the more often used Riemann–Liouville definition, which is discussed in the following section.

1.2.2 Riemann–Liouville Definition

The Riemann–Liouville definition is easiest to present for fractional integrals first, and then generalize that to the fractional derivatives. The qth-order integral is defined as (see, e.g., Oldham and Spanier [104], Podlubny [109])

$$ {}_a d_t^{-q} x(t) \equiv \frac{d^{-q} x(t)}{[d(t-a)]^{-q}} \equiv \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \geq a, \tag{1.4} $$

It is important to note that this is the *key definition* of the fractional integral and is used by most investigators. Miller and Ross [95] provide four separate developments of this important equation. It can be shown that whenever q is a positive integer, this equation becomes a standard integer-order multiple integral. The Riemann–Liouville fractional derivative is defined as the integer-order derivative of a fractional integral

$$ {}_a d_t^{q} x(t) \equiv \frac{d^m}{dt^m} ({}_a d_t^{q-m} x(t)), \quad t \geq a, \tag{1.5} $$

where m is typically chosen as the smallest integer such that $q-m$ is negative, and the integer-order derivatives are those as defined in the traditional calculus. These equations define the *uninitialized* fractional integral and derivative.

For most engineering problems, system components, by virtue of their histories, are placed into some initial configuration or are initialized. Using mechanical systems as an example, the initial conditions are often mass positions and velocities at time zero. Fractional-order components, however, require a time-varying initialization Lorenzo [77] and Hartley [85], as they inherently have a fading infinite memory. Considering the aforementioned fractional-order integral, we assume that the fractional-order integration was started in the past, beginning at some time a, while the given problem begins at time $c > a$, where c is usually taken to be zero. Consider two fractional integrals of the same order acting on $x(t)$, where $x(t)$ and all of its derivatives are zero for all $t < a$. If the integral starting at c is to continue the integral starting at a, we must add an initialization ψ, thus

$$ \begin{aligned} {}_a d_t^{-q} x(t) = {}_c d_t^{-q} x(t) + \psi \Rightarrow \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau \\ = \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau + \psi, \quad t \geq c, \quad q > 0, \end{aligned} \tag{1.6} $$

therefore

$$ \psi = \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau - \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau = \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \geq c,\ q > 0, \tag{1.7} $$

clearly a time-varying function. This term represents the historical effect (Lorenzo and Hartley [68, 71]) or the initialization required for the fractional integral. The *initialized* fractional-order integration operator then is defined as

$$ {}_cD_t^{-q}\,x(t) \equiv {}_cd_t^{-q}\,x(t) + \psi(x_i, -q, a, c, t), \quad t \geq c, \tag{1.8}$$

where

$$\psi(x_i, -q, a, c, t) \equiv \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x_i(\tau)\,d\tau, \quad t \geq c. \tag{1.9}$$

$\psi(x_i, -q, a, c, t)$ is called the initialization function and is generally a time-varying function that must be added to the fractional-order operator to account for the effects of the past. This is a generalization of the constant of integration that is usually added to the normal order-one integral. The subscript i is appended to x to indicate that x_i is not necessarily the same as x. Clearly then, ${}_cD_t^{-q}\,x(t) = {}_ad_t^{-q}\,x(t), \quad t \geq c$. The initialization function is a time-varying function and is required to properly bring the historical effects of the fractional-order integral into the future. Similar considerations also apply for fractional-order derivatives [68, 71], that is, for any real value of q. Again, for convenience, $c = 0$ is typically chosen.

1.2.3 The Nature of the Fractional-Order Operator

The important properties of integer-order integration and differentiation have been shown to hold for initialized fractional-order operators (Lorenzo and Hartley [68] and [71]), including linearity and the index law. Physical insight into the nature of the fractional operators may be found in Hartley and Lorenzo [44, 47]. The fractional differintegral operator is a linear operator, and all the properties associated with linear operators hold for them. Also of considerable importance is the index law; that is, ${}_ad_t^{u+v}x(t) = {}_ad_t^u\,{}_ad_t^v x(t)$. The index law essentially allows us to state, for example, that the half-derivative of the half-derivative of a function is the same as the first-derivative of that function.

Laplace transforms are standard tools for integer-order operators and can still be readily used for fractional-order operators. In this regard, the Laplace transform of the initialized fractional-order differintegral is shown in Lorenzo and Hartley [68, 71] to be

$$L\left\{{}_0D_t^q x(t)\right\} = L\left\{{}_0d_t^q x(t) + \psi(x, q, a, 0, t)\right\} = s^q X(s) + L\left\{\psi(x, q, a, 0, t)\right\} \quad \text{for all real } q. \tag{1.10}$$

It is important to note that $L\{{}_0d_t^q x(t)\} = s^q X(s)$, for all q, as this is the generalization of the derivative rule for the integer-order situation. Also, note that $L^{-1}\{s^{-q}\} = t^{q-1}/\Gamma(q), q > 0$, which leads directly to the Riemann–Liouville definition via convolution

$${}_0d_t^{-q}\,x(t) \Leftrightarrow s^{-q}X(s) \;\Leftrightarrow\; \int_0^t \frac{(t)^{q-1}}{\Gamma(q)} x(t-\tau)\,d\tau = \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\,d\tau. \tag{1.11}$$

The Laplace transform for the fractional integral is given [78] as

$$\begin{aligned} L\left\{{}_0D_t^{-q}h(t)\right\} &= L\left\{{}_0d_t^{-q}f(t)\right\} + L\left\{\psi\left(f_i, -q, -a, 0, t\right)\right\} \\ &= \frac{1}{s^q}L\{f(t)\} + \frac{1}{s^q\Gamma(q)}\int_{-a}^{0} e^{-\tau s}\Gamma(q+1, -\tau s) f_i(\tau)d\tau. \quad q \geq 0, \end{aligned} \tag{1.12}$$

where $q \geq 0$ and

$$h(t) = \begin{cases} f_i(t) & -a < t \leq 0, \\ f(t) & 0 < t, \end{cases}$$

and where f_i may differ from f during the initialization period. More detailed forms are presented in Ref. [78].

The transform for the fractional derivative of order u, where $u = n - q$, is given by

$$L\left\{{}_0D_t^u f(t)\right\} = s^{n-q}L\left\{f(t)\right\} - \sum_{j=0}^{n-1} s^{n-1-j}\psi^{(j)}(f_i, -q, -a, 0, t)|_{t=0+} + s^n L\left\{\psi(f_i, -q, -a, 0, t)\right\}, \tag{1.13}$$

where $u = n - q \geq 0$, $n = 1, 2, 3, \ldots$, $q \geq 0$, $f_i(t) = 0, \forall t < -a$, and

$$s^n L\{\psi(f_i, -q, -a, 0, t)\} = \frac{s^{n-q-1}}{\Gamma(q+1)}\left[e^{as}\Gamma(q+1, as)f_i(-a) - \Gamma(q+1)f_i(0) + \int_{-a}^{0} e^{-\tau s}\,\Gamma(q+1, -\tau s)f_i'(\tau)d\tau\right]. \tag{1.14}$$

In this relationship, $\psi(f_i, -q, -a, 0, t)$ is the initialization function for the fractional integral part of the operator. An alternative form of equation (1.14) where the integration is based on $f_i(t)$ rather than $f_i'(t)$ is given by

$$L\left\{{}_0D_t^u f(t)\right\} = s^{n-q}L\{f(t)\} - \sum_{j=0}^{n-1} s^{n-1-j}\psi^{(j)}(f_i, -q, -a, 0, t)|_{t=0+} + \frac{s^{n-q}}{\Gamma(q)}\int_{-a}^{0} e^{-\tau s}\Gamma(q, -\tau s)f_i(\tau)d\tau, \tag{1.15}$$

where $u = n - q \geq 0, \quad n = 1, 2, 3, \ldots, \quad q \geq 0, f_i(t) = 0, \quad \forall t < -a.$

These forms simplify for constant initialization [78], that is, when f_{i} = constant = b,

$$L\{{}_0D_t^u f(t)\} = s^{n-q}L\{f(t)\} + b\,s^{n-q-1}\left[\frac{e^{as}\Gamma(q-n+1, as)}{\Gamma(q-n+1)} - 1\right],$$
$$q \text{ not integer}, \quad 0 \leq u = (n-q) \leq n, \quad n = 1, 2, 3, \ldots. \tag{1.16}$$

1.3 The Traditional Trigonometry

The application of the traditional integer-order trigonometry to analysis as well as engineering and science goes well beyond the calculation of triangles and triangulation. The applications include Fourier analysis, spectral analysis, solutions to ordinary and partial differential equations, and more. The trigonometric functions are found in nearly every branch of mathematics. The traditional trigonometry was originated for the solution of plane triangles. However, an additional way of interpreting the integer-order trigonometry is based on its relationship to the exponential function. The connections between the trigonometric functions and the exponential functions are very close. These relationships center on the Euler equation; that is, for $z = x + iy$

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y), \tag{1.17}$$

as well as the definitions

$$\cos(t) \equiv \frac{e^{it} + e^{-it}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \tag{1.18}$$

and

$$\sin(t) \equiv \frac{e^{it} - e^{-it}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \tag{1.19}$$

for the sine and cosine functions. In fact, the exponential and trigonometric functions are fundamental to complex numbers and complex computation.

The hyperbolic functions are also based on the exponential function; these are given in the following relationships:

$$\cosh(t) \equiv \frac{e^t + e^{-t}}{2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}, \tag{1.20}$$

and

$$\sinh(t) \equiv \frac{e^t - e^{-t}}{2} = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}. \tag{1.21}$$

The development of the fractional calculus has involved new functions that generalized the common exponential function. These functions allow the opportunity to generalize both the hyperbolic functions and the trigonometric functions to "fractional" or "generalized" versions. Two of these functions, to be detailed later in the book, are the F-function (Hartley and Lorenzo [45]), which is the solution of the fundamental fractional differential equation

$${}_cD_t^q x(t) = -ax(t) + bu(t) \tag{1.22}$$

and its generalization, the R-function (Lorenzo and Hartley [69, 70]). They are defined as

$$F_q(a,t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1}}{\Gamma((n+1)q)}, \quad t > 0 \tag{1.23}$$

and its νth differintegral

$$R_{q,\nu}(a,t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)}, \quad t > 0. \tag{1.24}$$

The Laplace transforms of these functions are determined in Ref. [69] as

$$L\left\{F_q(a,t)\right\} = \frac{1}{s^q - a} \quad \text{and} \quad L\left\{R_{q,\nu}(a,t)\right\} = \frac{s^\nu}{s^q - a}, \quad \mathrm{Re}(q-\nu) \geq 0. \tag{1.25}$$

It can be seen from the series definitions of these functions that they contain the exponential function

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots = \sum_{0}^{\infty} \frac{(at)^n}{\Gamma(n+1)} \tag{1.26}$$

as the $q = 1$, $\nu = 0$ special case. This result and the fact that the F- and R-functions are eigenfunctions for the qth-order derivative are powerful drivers toward a new generalized trigonometry based on the fractional (or generalized) exponential function, that is, the F- or the R-function. The expectation and hope is that such a trigonometry will lead also to new generalizations of all the products of the integer-order trigonometry, a situation that will be broadly useful. These expectations and more derive from the usefulness of the ordinary trigonometry.

It is well known, and follows from equation (1.26), that

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \tag{1.27}$$

$$e^{it} = \left\{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right\} + i\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right\}. \tag{1.28}$$

These series are, of course, recognized as representing the circular functions giving the well-known Euler equation

$$e^{it} = \cos(t) + i\sin(t). \tag{1.29}$$

It is important to note that $\cos(t)$ is a summation of terms that are simultaneously both the real part of e^{it} and are even powers of t. Also, that $\sin(t)$ is a summation of terms that are simultaneously both the imaginary part of e^{it} and are odd powers of t. This observation will prove important in the development to follow. Not all of the new fractional trigonometric functions will share this property.

The R-function, since it includes within it the fractional differintegrals of the F-function, and is a representation of the fractional eigenfunction, is used as the generalizing basis of the exponential function. Based on the R-function, parallels with the integer-order trigonometry are used to generate related fractional trigonometries. The properties of these new trigonometries and identities flowing from the definitions are then developed.

The trigonometries derived from these generalizations will be jointly termed "The Fractional Trigonometry." The definitions for the fractional trigonometries can be based on several different parallels between various properties of the integer-order trigonometry and the proposed fractional-order trigonometries. For example, parallels based on equations (1.17)–(1.19) each provide a basis for definitions. Laplace transforms of the new functions are determined. Fractional differential equations for which the functions are solutions and various intra- and interrelationships of the new trigonometric functions are studied.

1.4 Previous Efforts

There have been previous definitions offered for fractional trigonometric functions. These efforts, each amounting to a page or so of definitions, have been based on the Mittag-Leffler function and are discussed in Appendix A. In all cases, the definitions are subsets of those to be presented here.

1.5 Expectations of a Generalized Trigonometry and Hyperboletry

There are some characteristics that a generalized trigonometry should have and additional characteristics that may be desirable. We require that any fractional trigonometry should

contain the traditional, integer-order, plane trigonometry as a special case,
have an eigenfunction basis,
exhibit series compatibility between defined functions and generalized exponentials, and
form a basis for the solution of fractional-order linear differential equations.

These requirements are essentially self-explanatory. The first requires backward compatibility to the ordinary trigonometry. The second and fourth requirements are a way of saying that the new generalized trigonometry should be closely coupled to the solution of fractional differential equations and that the solutions should be expressible as linear combinations of the functions. The expectation flowing from this is that we expect insight into the solutions of fractional differential equations from the fractional trigonometry to be similar to that obtained from the trigonometric solutions associated with the solutions of ordinary differential equations.

In general, our requirements and expectations for the generalized hyperbolic functions parallel those listed for the fractional trigonometry. For example, we require backward compatibility with the traditional hyperbolic functions, and so on. In addition, we expect to maintain or generalize the relationships between the traditional integer-order trigonometric functions and the traditional integer-order hyperbolic functions when the fractional versions are defined.