1

Introduction

This first chapter has the twofold objective of introducing the framework of input-output finite-time stability (IO-FTS), together with the notation that will be used throughout the book, and providing some useful background on the analysis of the behavior of dynamical systems.

In order to introduce the topics dealt with in this monograph, we first recall the concept of *state* FTS, and then we will extend it to the input-output case, both with zero and nonzero initial conditions. The former extension correspond to the concept of IO-FTS, while the latter represents a generalization of the finite-time boundedness (FTB) concept, namely IO-FTS with nonzero initial conditions (IO-FTS-NZIC).

Roughly speaking, FTS involves the behavior of the system state for an autonomous dynamical system with nonzero initial conditions, while IO-FTS looks at the input-output behavior of the system, with zero initial conditions. IO-FTS-NZIC mixes the two concepts, considering the input-output finite-time control problem with a nonzero initial condition. The common points to these definitions is that they are defined over a finite-time interval and that quantitative bounds are given for the admissible signals during this interval.

1.1 Finite-Time Stability (FTS)

The concept of finite-time stability (FTS) dates back to the fifties, when it was introduced in the Russian literature ([1–3]); later, during the sixties, this concept appeared for the first time in Western journals [4–6].

Given the dynamical system

$$\dot{x}(t) = f(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, we can give the following formal definition, which restates the original definition in a way consistent with the notation adopted in this monograph; in the following we consider the finite-time interval $\Omega := [t_0, t_0 + T]$, with $T > 0$.

Definition 1.1 (FTS, [2, 4, 8]) Given the time interval Ω, a set $\mathcal{X}_0 \subset \mathbb{R}^n$, and a family of sets $\mathcal{X}_t \subset \mathbb{R}^n$, system (1.1) is said to be *finite-time stable* with respect to (wrt) $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$ if

$$x_0 \in \mathcal{X}_0 \Rightarrow x(t) \in \mathcal{X}_t, \quad \forall\, t \in \Omega, \tag{1.2}$$

Finite-Time Stability: An Input-Output Approach, First Edition.
Francesco Amato, Gianmaria De Tommasi, and Alfredo Pironti.

where, with a slight abuse of notation, $x(\cdot)$ denotes the solution of (1.1) starting from x_0 at time t_0. ◇

Note that, in general, the set $\mathcal{X}_t$, called *outer (or trajectory) set*, possibly depends on time; obviously $\mathcal{X}_{t_0}$ must contain the *inner (or initial)* set $\mathcal{X}_0$, for well-posedness of Definition 1.1.

An issue that is important to clarify is why the property expressed by (1.2) is called FTS.

In order to answer this question, we recall the classical definition of Lyapunov stability (LS, [32, Ch. 4]; see also Appendix A.3). Let $\bar{x}$ be an equilibrium point for system (1.1), i.e., $f(t, \bar{x}) = 0$ for all $t \in \mathbb{R}_0^+$. The equilibrium point $\bar{x}$ is said to be *stable* in the sense of Lyapunov if for each $\varepsilon > 0$, there exists a positive scalar δ, possibly depending on t_0 and ε, such that $|x_0 - \bar{x}| < \delta(\varepsilon, t_0)$, implies

$$|x(t) - \bar{x}| < \varepsilon, \quad t \geq t_0,$$

and this holds for all $t_0 \in \mathbb{R}^+$.

The key points in the above definition are: an equilibrium point $\bar{x}$ is stable if, once an arbitrary value for ε has been fixed, which defines a *ball* centered in $\bar{x}$, then it must be possible to build an *inner* ball (of radius δ) such that, whenever the initial condition is inside such ball, the trajectory of the system starting from x_0 does not exit the *outer* ball (of radius ϵ). Moreover this property holds for an infinite time horizon, that is, for all t between t_0 and infinity.

Note that LS is a qualitative concept, that is, both the inner and the outer ball are not quantified; therefore, LS can be regarded as a structural property: a given equilibrium point $\bar{x}$ is either stable or it is not.

Now let us come back to Definition 1.1; even in this case there is an inner set $\mathcal{X}_0$, usually centered at an equilibrium point of system (1.1), and an outer set $\mathcal{X}_t$. FTS requires that, whenever the trajectory of (1.1) starts inside the inner set, it does not exit the outer set. From this point of view, Definition 1.1) mimics the one of LS, and this justifies the use of the term *stability*. However, differently from LS, this is only required over a *finite interval of time*, which should be possibly short with respect to steady state; i.e., FTS can be used to *shape* the behavior of the system during the transients.

Another important point is that FTS is a quantitative concept, since the inner and the outer set are specified once and for all. Therefore the same system can be finite-time stable for some choice of $\mathcal{X}_0$, $\mathcal{X}_t$, and Ω, and non-finite-time stable for a different choice of these parameters.

It is worth noting that, in principle, FTS does not necessarily requires that the inner set $\mathcal{X}_0$ contains any equilibrium point for system (1.2); however, this particular case will not be dealt with in this book, where we shall consider ellipsoidal sets centered at the origin of the state space.

A direct consequence of the discussion above is that FTS and LS are independent concepts; referring to a linear system, to simplify the terminology (see Appendix A.3), a system can be finite-time stable, despite not being stable in the sense of Lyapunov, and vice versa. While LS deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the state variables do not exceed a given threshold (for example to avoid saturation or the excitation of nonlinear dynamics) during the transients.

In the following, we shall focus on linear time-varying (LTV) systems

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{1.3}$$

with $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$ piecewise continuous; note that the assumption on piecewise continuity of $A(\cdot)$ guarantees existence and uniqueness of the solution of system (1.3) starting from x_0, at time t_0 (see Appendix A). Moreover, if we consider ellipsoidal state sets, i.e.,

$$\begin{aligned}\mathcal{X}_0 &:= \{x \in \mathbb{R}^n \text{ s.t. } x^T\Gamma_0 x \le 1, \text{with } \Gamma_0 > 0\},\\ \mathcal{X}_t &:= \{x \in \mathbb{R}^n \text{ s.t. } x^T\Gamma(t)x < 1, \text{with } \Gamma(t) > 0\, \forall\, t \in \Omega\},\end{aligned}$$

Definition 1.1 can be rewritten as follows.

Definition 1.2 (FTS for LTV Systems [12, 33, 34]) Given the time interval Ω, a positive definite matrix Γ_0, and a continuous, positive definite matrix-valued function $\Gamma(\cdot)$ defined over Ω, such that $\Gamma(t_0) < \Gamma_0$, system (1.3) is said to be *finite-time stable* wrt $(\Omega, \Gamma_0, \Gamma(\cdot))$ if

$$x_0^T\Gamma_0 x_0 \le 1 \Rightarrow x(t)^T\Gamma(t)x(t) < 1, \quad \forall\, t \in \Omega. \tag{1.4}$$

◊

As said above, the assumption that $\Gamma(t_0) < \Gamma_0$ in Definition 1.2 is needed to guarantee that the initial closed ellipsoid $\mathcal{X}_0$ is a proper subset of the open ellipsoid $\mathcal{X}_{t_0}$, hence guaranteeing the well-posedness of the definition itself.

A graphical explanation of the FTS concept is reported in Figure 1.1 for a second-order system with a constant matrix-valued function $\Gamma(t) = \Gamma$. In particular, if a system is FTS, then all the trajectories starting within the ellipse defined by Γ_0 should be like the one depicted in green in Figure 1.1. Conversely, two trajectories that are not FTS are reported in red.

In the following, we consider a numerical example.

Example 1.1 (Lyapunov stability and finite-time stability for LTI systems) This introductory example shows the difference between the two concepts of LS and FTS for a second-order linear time-invariant (LTI) system. To this aim, let us consider the following autonomous LTI system

$$\dot{x}(t) = A_1 x(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x(t), \quad x(0) = x_0. \tag{1.5}$$

System (1.5) is clearly Lyapunov stable, being negative the maximum real part of the eigenvalues of the matrix A_1.

While LS is a structural property of an LTI system, FTS it is not. Indeed, given the time interval $\Omega' = [0, 2]$ if we specify the two weighting matrices in Definition 1.2 as follows

$$\Gamma_0' = \begin{pmatrix} 10 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad \Gamma' = \begin{pmatrix} 4 & 0.4 \\ 0.4 & 0.65, \end{pmatrix}$$

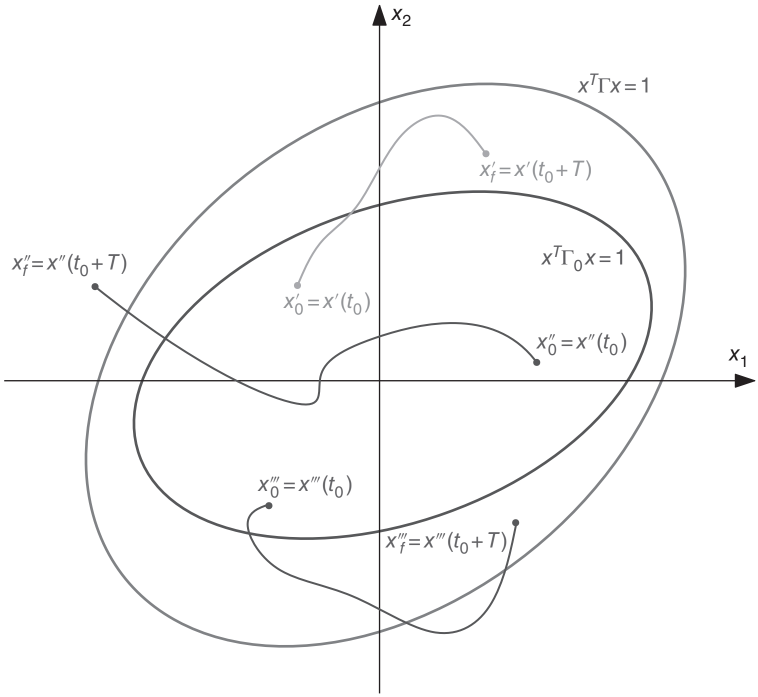


Figure 1.1 Given a time interval Ω, and the two ellipsoidal domains delimited by Γ_0 and by the constant matrix Γ, a second-order system is finite-time stable if all the trajectories over the considered time interval are like the one reported in light gray. Furthermore, in dark gray are reported two examples of trajectories that are not finite-time stable.

it turns out that system (1.5) is not FTS wrt $(\Omega', \Gamma_0', \Gamma')$, as it is clearly shown in Figure 1.2, since there is at least one state trajectory that starts within the initial domain defined by Γ_0' and that goes outside the ellipsoidal domain specified by Γ' during the time interval Ω'.

On the other hand, if we consider a different time interval for the FTS analysis, e.g., by letting $\Omega'' = [0, 0.25]$, systems (1.5) turns out to be FTS wrt $(\Omega'', \Gamma_0', \Gamma')$. Indeed, in the last case, it can be shown that all the state trajectories of (1.5) that start within the initial domain defined by Γ_0' remain within the target ellipsoidal domain defined by Γ' (one possible way to check FTS is to solve the feasibility problem reported in [35, Theorem 2.1-(v)]).

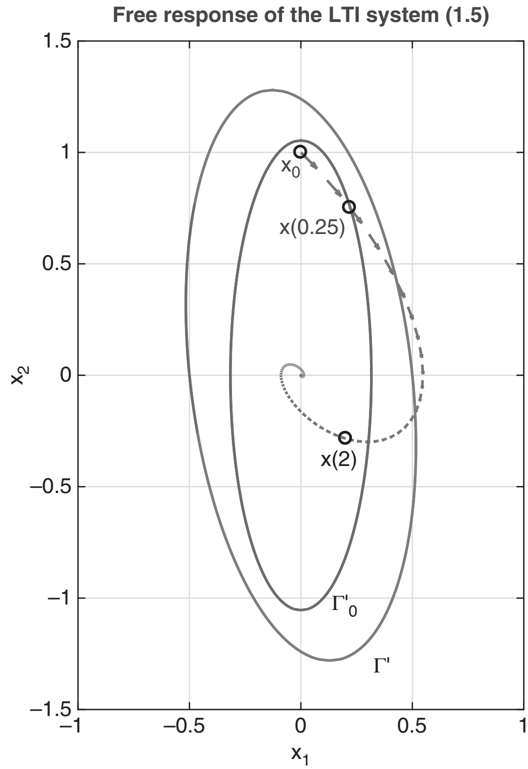
Let us now consider the following Lyapunov unstable system

$$\dot{x}(t) = A_2 x(t) = \begin{pmatrix} 0 & 1 \\ 0.9 & -0.1 \end{pmatrix} x(t), \quad x(0) = x_0. \tag{1.6}$$

Also in this case FTS is not related to LS, as it is shown in Figure 1.3, where the initial ellipsoidal domain Γ_0'' is given by

$$\Gamma_0'' = \begin{pmatrix} 0.45 & 0 \\ 0 & 1.2 \end{pmatrix},$$

Figure 1.2 Free response of the LS stable LTI system (1.5) when the initial state is set equal to $x(0) = (0 \quad 1)^T$. Although the considered LTI system is Lyapunov stable, the same system can be either FTS or not, depending on the FTS parameters.



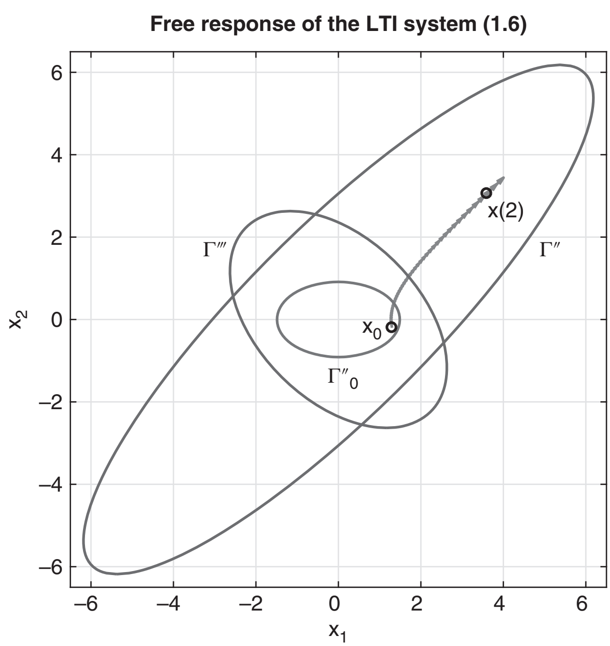


Figure 1.3 Free response of the LS unstable LTI system (1.6) when the initial state is set equal to $x(0) = (1.3 - 0.2)^T$. Even when Lyapunov unstable systems are considered, the finite-time stability depends on the chosen parameters.

the finite-time interval is taken equal to $\Omega''' = [0, 2]$, and two different time-invariant target domains, defined by

$$\Gamma'' = \begin{pmatrix} 0.1070 & -0.093 \\ -0.093 & 0.107 \end{pmatrix}, \quad \Gamma''' = \begin{pmatrix} 0.18 & 0.08 \\ 0.08 & 0.18 \end{pmatrix},$$

are chosen. It is straightforward to check that system (1.6) is not FTS wrt $(\Omega''', \Gamma_0'', \Gamma''')$, while it can be proven that the same unstable system (in the Lyapunov sense) is FTS wrt $(\Omega''', \Gamma_0'', \Gamma'')$. △

The pioneering works [1–6], although developing a nice theoretical framework, did not provide computationally tractable conditions for checking the FTS of a given dynamical system, unless simple cases were considered. Therefore, for a long period, this field of research was neglected by control scientists.

At the end of the last century, the development of the Linear Matrix Inequality theory (LMI,[7]) has fueled new interest in the field of finite-time control. In particular, FTS and stabilization have been investigated in the context of linear time-invariant systems, both continuous (e.g., [10, 13, 36–38]) and discrete-time [39, 40]. According to this modern approach to FTS, conditions for analysis and design are provided in terms of feasibility problems involving LMIs [15], by exploiting the properties of quadratic Lyapunov functions. A different approach, which looks to polyhedral Lyapunov functions, is presented in [41–44]; polyhedral Lyapunov functions are useful when the sets Γ_0 and Γ are polytopic rather than ellipsoidal. Finally, in [28, 29, 45] the concept of *annular* FTS has been introduced to take into account also a possible lower bound for the state variables.

A new impulse to the theory has been given by the use of *time-varying* quadratic Lyapunov functions, which, on the one hand has allowed to deal with the more general class of LTV systems, and more importantly, on the other hand has permitted to state non-conservative, i. e., necessary and sufficient, conditions for FTS and stabilization both for continuous an discrete-time systems [9, 11, 12, 23, 46–49]; such conditions require the solution of Differential Linear Matrix Inequalities (DLMIs, [15]), or Differential Lyapunov Equations (DLEs, [16]). In the recent papers [50, 51], the Separation Principle has been (partially) extended to the finite-time context.

An effort has been spent in order to extend the results obtained for linear systems to other contexts, such as uncertain linear systems [52], nonlinear systems [17, 18, 53], hybrid systems [19–23], and stochastic systems ([18, 23–29] among the others), and to consider mixed problems, such as FTS and pole placement [54], FTS with input constraints [55, 56], FTS and $\mathcal{H}_\infty$ control [57]. Most of the above results are collected in the monograph [35].

1.2 Input-Output Finite-Time Stability

IO-FTS represents the *natural* extension of the concept of FTS introduced in Section 1.1, to the case of non-autonomous dynamical systems.

Informally a system is said to be input-output finite-time stable if, for a given class of input signals, the output of the system does not exceed an assigned threshold during a specified time interval. As it is usual when dealing with input-output issues, the initial state of the system under consideration is assumed to be zero.

In order to formally define IO-FTS, let us consider the system

$$\dot{x}(t) = f(t, x, w), \quad x(t_0) = 0 \tag{1.7a}$$

$$y(t) = g(t, x, w), \tag{1.7b}$$

where $y(t) \in \mathbb{R}^p$ is the system output, and $w(t) \in \mathbb{R}^m$ the exogenous input, i.e., the non-manipulable input; we can give the following definition.

Definition 1.3 (IO-FTS, [30]) Given the time interval Ω, a family of sets $\mathcal{Y}_t \subset \mathbb{R}^p$, and a class of input signals $\mathcal{W}$ defined over Ω, system (1.7) is said to be *input-output finite-time stable* wrt $(\Omega, \mathcal{W}, \mathcal{Y}_t)$ if

$$w(\cdot) \in \mathcal{W} \;\Rightarrow\; y(t) \in \mathcal{Y}_t, \quad \forall\, t \in \Omega. \qquad \diamond$$

Similarly to what has been done between LS and *state* FTS, a parallelism can be traced between IO $\mathcal{L}_p$-stability, with particular reference to $\mathcal{L}_\infty$-stability (better known with the popular acronym of BIBO, Bounded–Input Bounded–Output, stability) and IO-FTS.

We recall that system (1.7) is said to be IO $\mathcal{L}_p$-stable [32] for any input of class $\mathcal{L}_p$ (the space of the p-integrable vector-valued functions, see Section 1.4.1), if the system exhibits a corresponding output that belongs to the same class. IO-stability of linear and nonlinear systems has been broadly studied since the early sixties [58–60]. Moreover, a number of results have been proposed in the literature to discuss robustness issues (see for example [61] and the bibliography therein).

As happened between *state* FTS and LS, also *classical* IO stability and IO-FTS differ because the latter involves signals defined over a finite-time interval and gives *quantitative* bounds on both inputs and outputs. Moreover, differently from classical IO stability, IO-FTS does not necessarily require inputs and outputs to belong to the same class of signals.

It turns out that also IO stability and IO-FTS are independent concepts. While IO stability deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, IO-FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the output variables do not exceed a given threshold during the transients, given a certain class of input signals.

Consider the non-autonomous LTV system

$$\dot{x}(t) = A(t)x(t) + F(t)w(t), \quad x(t_0) = 0 \tag{1.8a}$$

$$y(t) = C(t)x(t) + G(t)w(t), \tag{1.8b}$$

where $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$, $F(\cdot) : \Omega \mapsto \mathbb{R}^{n\times m}$, $C(\cdot) : \Omega \mapsto \mathbb{R}^{p\times n}$, and $G(\cdot) : \Omega \mapsto \mathbb{R}^{p\times m}$ are piecewise continuous matrix-valued functions that describe the system dynamics. Moreover, we assume the sets in the family $\mathcal{Y}_t$ to be ellipsoidal similarly to what has been done for *state* FTS.

Given these assumptions, Definition 1.3 can be refined as follows.

Definition 1.4 (IO-FTS for LTV Systems [30, 31]) Given the time interval Ω, a class of input signals $\mathcal{W}$ defined over Ω, and a continuous, positive definite matrix-valued function $Q(\cdot)$ defined over Ω, system (1.8) is said to be input-output finite-time stable wrt $(\Omega, \mathcal{W}, Q(\cdot))$ if

$$w(\cdot) \in \mathcal{W} \Rightarrow y^T(t)Q(t)y(t) < 1, \forall t \in \Omega. \qquad \diamond$$

The concept of IO-FTS has been introduced by the authors in the papers [30, 62], where sufficient conditions for a given linear time-varying (LTV) system to be IO finite-time stable and stabilizable have been provided.

In [31, 63, 64] necessary and sufficient conditions for finite-time stability and stabilization of LTV systems have been proposed, while an extension of these results to the case of impulsive dynamical linear systems (a special class of hybrid systems) has been considered in [65, 66].

It should be remarked that input–output stabilization of LTV systems on a finite time horizon is tackled also in [15]. However, as for *classic* IO stability, the concept of IO stability over a finite time horizon given in [15] does not give explicit bounds on input and output signals and does not allow the input and output to belong to different classes.

Example 1.2 (BIBO stability and input-output FTS for LTI systems) Similarly to what has been discussed in Example 1.1, by means of a simple numerical example, we now show that BIBO stability and IO-FTS are two independent concepts. Let us consider the following second-order LTI system, with one exogenous input and two outputs

$$\dot{x}(t) = Ax(t) + Fw(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t), \quad x(0) = 0 \tag{1.9a}$$

$$y(t) = Cx(t) = \begin{pmatrix} 1 & 1 \\ 1 & -0.5 \end{pmatrix} x(t). \tag{1.9b}$$

By simply looking at the eigenvalues of the A matrix in (1.9a), it readily follows that system (1.9) is Lyapunov stable; hence it is also BIBO stable.

We now consider the time interval $\Omega = [0, 2]$ and the class $\mathcal{W}_\infty$ of bounded signals over Ω, i.e., the class of signals such as $|w(t)| < 1$, $t \in \Omega$ (the reader can refer to Section 1.4.1 for a formal definition of this class of inputs).

System (1.9) can be either IO-FTS or not, depending on the choice of the weighting matrix-valued function $Q(\cdot)$ in Definition 1.2. In particular, let us consider the following two possible choices for a constant output weighting matrix

$$Q_1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 2 & 1 \\ 1 & 10 \end{pmatrix}.$$

By exploiting the results that will be presented in Chapter 2 it can be shown that system (1.9) is input-output finite-time stable wrt $(\Omega, \mathcal{W}_\infty, Q_1)$, while it is not input-output finite-time stable wrt $(\Omega, \mathcal{W}_\infty, Q_2)$.

As an example of response to a bounded disturbance (exogenous input), Figure 1.4 shows the time response of system (1.9) to the unitary step, i.e.,[1] to $w(t) = \delta_{-1}(t)$. It can be noticed that, when the matrix Q_2 is considered, then the weighted output $y^T(t)Q_2y(t)$ exceeds 1, hence system (1.9) is not IO-FTS for this choice of the output weighting matrix.

1 Here we denote with $\delta_{-1}(t)$ the unitary Heaviside step function.

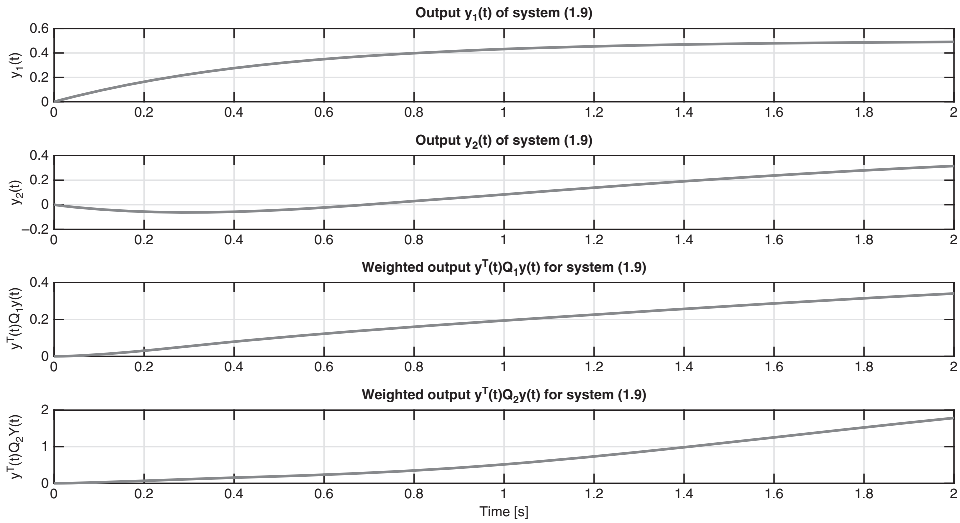


Figure 1.4 Time response of system (1.9) to the unitary step function. When the weighting matrix Q_2 is considered, then the weighted output exceeds 1; hence, the system is not IO-FTS. On the other hand, it can be proved that for all the exogenous inputs $w(t)$ belonging to the class of bounded signals in the time interval $\Omega = [0, 2]$, if the weighting matrix Q_1 is considered, then the weighted output never exceeds 1.

Similarly to what has been shown in Example 1.1, also in the case of IO-FTS it is possible to find an LTI system which is not BIBO stable but that can be either IO-FTS or not, depending on the chosen parameters. △

Combining the two concepts of *state* FTS and IO-FTS, it is possible to extend the definition of IO-FTS to the case of nonzero initial conditions (IO-FTS-NZIC). To this aim, in the following definition we consider system (1.8) when $x(t_0) = x_0 \neq 0$, i.e.,

$$\dot{x}(t) = A(t)x(t) + F(t)w(t), \quad x(t_0) = x_0 \tag{1.10a}$$
$$y(t) = C(t)x(t) + G(t)w(t), \tag{1.10b}$$

Definition 1.5 (IO-FTS-NZIC for LTV Systems [67]) Given the time interval Ω, a class of input signals $\mathcal{W}$ defined over Ω, a positive definite matrix Γ_0, and a continuous, positive definite matrix-valued function $Q(\cdot)$ defined over Ω, system (1.10) is said to be *input-output finite-time stable with nonzero initial conditions* wrt $(\Omega, \mathcal{W}, \Gamma_0, Q(\cdot))$, if

$$x_0^T \Gamma_0 x_0 \leq 1 \Rightarrow y^T(t)Q(t)y(t) < 1, \quad \forall t \in \Omega, \quad \forall w(\cdot) \in \mathcal{W}.$$

◇

It is worth to notice that the concept of IO-FTS-NZIC given in this book coincides with the definition of finite-time boundedness (FTB), when the output vector is taken equal to the state vector. The concept of FTB was introduced at the end of the last century in [8, 37, 38]; in the following years many papers have appeared in the literature dealing with the study of the FTB properties of various classes of systems (see, among others, [68–71] and the related applications [25, 72]).

1.3 FTS and Finite-Time Convergence

In the context of nonlinear systems an alternative definition of FTS has been given; essentially, this alternative definition of FTS coincides with a finite-time convergence property. For the sake of clarity, in this section we briefly discuss the main differences between the two existing finite-time frameworks.

FTS, in the sense of the convergence in finite-time of the state trajectory to an equilibrium point, is strictly related to LS and applies to autonomous systems (e.g., [73] for nonlinear continuous systems, and [74] for nonlinear impulsive systems). Hence, this alternative notion of FTS is unrelated to the notion of *state* FTS introduced in Section 1.1, since the former does not require to specify any bounding regions nor the time interval.

As in the case of the finite-time framework considered in this book, also finite-time convergence has been extended to the case of non-autonomous nonlinear systems. For example, the authors of [75] consider systems with a norm-bounded input signal over the interval $[0, +\infty]$, and a nonzero initial condition. In this case, the finite-time input-output stability is related to the property of a system to have a norm-bounded output that, after a finite time interval, does not depend anymore on the initial state. It follows that the IO-FTS considered in this book, and the extension of finite-time convergence to the input-output context are different concepts.

1.4 Background

This section introduces the notation adopted in the book, together with some useful preliminary definitions and results that will be exploited in the next chapters.

1.4.1 Vectors and signals

Let I denote the identity matrix; given a vector $v \in \mathbb{R}^n$ and a positive definite matrix $J \in \mathbb{R}^{n\times n}$, we will denote by $|v|_J$, $|v|_I =: |v|$, the Euclidean norm of v weighted by J, i.e.,

$$|v|_J = (v^T J v)^{\frac{1}{2}}.$$

Given the bounded time interval $\Omega = [t_0, t_0 + T]$. The symbol $\mathcal{L}_p(\Omega)$, denotes the space of vector-valued signals for which[2]

$$s(\cdot) \in \mathcal{L}_p(\Omega) \iff \left(\int_\Omega |s(\tau)|^p d\tau\right)^{\frac{1}{p}} < +\infty.$$

Given a symmetric positive definite, continuous matrix-valued function $R(\cdot)$, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left(\int_\Omega [s^T(\tau)R(\tau)s(\tau)]^{\frac{p}{2}} d\tau\right)^{\frac{1}{p}},$$

will be denoted by $\|s(\cdot)\|_{p,R}$. If $p = \infty$,

$$\|s(\cdot)\|_{\infty,R} = \operatorname*{ess\,sup}_{t\in\Omega} |s(t)|_{R(t)}.$$

2 For the sake of brevity, We denote by $\mathcal{L}_p$ the set $\mathcal{L}_p([0, +\infty))$.

When $R(\cdot) = I$, we will use the simplified notation $\|s(\cdot)\|_p$.

Given two vector-valued signals $u(\cdot) \in \mathcal{L}_p(\Omega)$, and $v(\cdot) \in \mathcal{L}_{p'}(\Omega)$, with $1/p + 1/p' = 1$, we define

$$\langle u(\cdot), v(\cdot)\rangle = \int_\Omega u^T(\tau)v(\tau)d\tau \ ; \tag{1.11}$$

when $p = p' = 2$, the operation $\langle u(\cdot), v(\cdot)\rangle$ coincides with the scalar product in $\mathcal{L}_2(\Omega)$.

Let p and p' such that $1/p + 1/p' = 1$; then the *Hölder* inequality (see [76], p. 33, [77], p. 548) states that, if $u(\cdot) \in \mathcal{L}_p(\Omega)$ and $v(\cdot) \in \mathcal{L}_{p'}(\Omega)$,

$$\int_\Omega |u^T(\tau)v(\tau)|d\tau \leq \|u(\cdot)\|_p \|v(\cdot)\|_{p'}. \tag{1.12}$$

For $p = p' = 2$, the Hölder inequality is also known as *Schwarz* inequality. Moreover, the following inequality holds

$$\|u(\cdot)\|_p = \sup_{\|v\|_{p'}=1} |\langle u(\cdot), v(\cdot)\rangle|. \tag{1.13}$$

Note that, given the notation introduced for the signals and vector norms, the definitions of IO-FTS and IO-FTS-NZIC for LTV systems and ellipsoidal outputs sets can be restated as follows.

Definition 1.6 Given the time interval Ω, a class of input signals $\mathcal{W}$ defined over Ω, and a continuous, positive definite matrix-valued function $Q(\cdot)$ defined over Ω, system (1.8) is said to be *input-output finite-time stable* wrt $(\Omega, \mathcal{W}, Q(\cdot))$, if

$$w(\cdot) \in \mathcal{W} \Rightarrow \|y(\cdot)\|_{\infty,Q} < 1.$$

◇

Definition 1.7 Given the time interval Ω, a class of input signals $\mathcal{W}$ defined over Ω, a positive definite matrix Γ_0, and a continuous, positive definite matrix-valued function $Q(\cdot)$ defined over Ω, system (1.8) is said to be *input-output finite-time stable with nonzero initial conditions* wrt $(\Omega, \mathcal{W}, \Gamma_0, Q(\cdot))$, if

$$|x_0|_{\Gamma_0} \leq 1 \Rightarrow \|y(\cdot)\|_{\infty,Q} < 1, \quad \forall\, w(\cdot) \in \mathcal{W}.$$

◇

Given the continuous, positive definite matrix $R(\cdot)$, throughout this book we will consider the following two classes of exogenous signals when dealing with IO-FTS:

i) the set of essentially bounded signals over Ω whose weighted norm is less than or equal to one

$$\mathcal{W}_\infty(\Omega, R(\cdot)) := \{w(\cdot) \in \mathcal{L}_\infty(\Omega) : \|w\|_{\infty,R} \leq 1\},$$

ii) the set of square integrable signals over Ω whose weighted norm is less than or equal to one

$$\mathcal{W}_2(\Omega, R(\cdot)) := \{w(\cdot) \in \mathcal{L}_2(\Omega) : \|w\|_{2,R} \leq 1\}.$$

In the rest of the book we will drop the dependency of the $\mathcal{W}$ sets on Ω and $R(\cdot)$, in order to simplify the notation.

1.4.2 Impulsive dynamical linear systems

Chapters 7–9 will deal with IO-FTS of a special class of hybrid systems, namely *Impulsive Dynamical Linear Systems (IDLSs).* IDLSs allow to model a wide range of real-world applications whose dynamical behavior includes both time-driven and event-driven dynamics. As an example, the automatic gear-box in cruise control falls in the category of hybrid systems that can be modeled as IDLS (for more details and further examples see [78, 79]).

This section introduces the definition of this class of dynamical systems, together with some related preliminary material.

The class of IDLSs is described by the equations

$$\mathcal{ILS}: \begin{cases} \dot{x}(t) = A(t)x(t) + F(t)w(t), \quad x(t_0) = 0, \quad t \notin \mathcal{T} & (1.14a) \\ x^+(t_i) = J(t_i)x(t_i), \quad t_i \in \mathcal{T} & (1.14b) \\ y(t) = C(t)x(t) + G(t)w(t), \quad t \in \Omega, & (1.14c) \end{cases}$$

where $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$, $F(\cdot) : \Omega \mapsto \mathbb{R}^{n\times m}$, $C(\cdot) : \Omega \mapsto \mathbb{R}^{p\times n}$, and $G(\cdot) : \Omega \mapsto \mathbb{R}^{p\times m}$ are piecewise continuous matrix-valued functions that describe the *continuous-time* dynamics of the system. On the other hand, $J(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$ is the matrix-valued function that describes the *resetting law* of the system. The elements of the set $\mathcal{T} = \{t_1, t_2, \dots, t_\nu\} \subset \Omega$ are called *resetting times*. The finiteness of the set $\mathcal{T}$ prevents the IDLS (1.14) from exhibiting Zeno behavior ; furthermore, we assume that the first resetting time $t_1 \in \mathcal{T}$ is such that $t_1 > t_0$, since we exclude the case of initial state $x(t_0) \neq 0$.

According to the continuous-time dynamics (1.14a) and the resetting law (1.14b), an IDLS presents a left-continuous trajectory with a finite jump from $x(t_i)$ to $x^+(t_i)$ at each resetting time $t_i \in \mathcal{T}$.

As we have done for LTV systems in Section A.4, we denote by $\Phi(\cdot,\cdot)$ the state transition matrix of the IDLS (1.14). It is straightforward to check the following properties for $\Phi(\cdot,\cdot)$

$$\Phi(t_0, t_0) = I, \tag{1.15a}$$

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A_c(t)\Phi(t, t_0), \quad t \notin \mathcal{T} \tag{1.15b}$$

$$\Phi^+(t_i, t_0) = J(t_i)\Phi(t_i, t_0), \quad t_i \in \mathcal{T}. \tag{1.15c}$$

Furthermore, for a given t in Ω, such that $t > t_k$ and $t < t_{k+1}$, with $t_k, t_{k+1} \in \mathcal{T}$, we have

$$\begin{aligned} \Phi(t, t_0) = \Phi_{k+1}(t, t_k)J(t_k)\Phi_k(t_k, t_{k-1})J(t_{k-1}) \times \cdots \\ \times J(t_2)\Phi_2(t_2, t_1)J(t_1)\Phi_1(t_1, t_0), \end{aligned} \tag{1.16}$$

where $\Phi_j(\cdot,\cdot)$ satisfies the matrix differential equation

$$\frac{\partial}{\partial t}\Phi_j(t, t_{j-1}) = A_c(t)\Phi_j(t, t_{j-1}), \quad t \in [t_{j-1}, t_j), \quad \Phi_j(t_{j-1}, t_{j-1}) = I,$$

with $j = 1, \dots, \nu + 1$,

Given (1.16), it is straightforward to verify that the impulsive response of the IDLS (1.14) is formally equal to (A.20), as in the case of LTV systems.

Moreover, also the Reachability Gramian of (1.14) can be recursively defined (see [80, 81] for more details), and the following lemma holds.

Lemma 1.1 ([80, 81]) The reachability Gramian $W_r(\cdot,\cdot)$ of the IDLS (1.14) is the unique positive semidefinite solution to the Difference-DLE (D/DLE)

$$\dot{W}_r(t,t_0) = A_c(t)W_r(t,t_0) + W_r(t,t_0)A^T(t) + F(t)F^T(t), \quad t \notin \mathcal{T} \tag{1.17a}$$

$$W_r^+(t_i,t_0) = J(t_i)W_r(t_i,t_0)J^T(t_i), \quad t_i \in \mathcal{T} \tag{1.17b}$$

$$W_r(t_0,t_0) = 0 \tag{1.17c}$$

▲

The definition of IO-FTS for IDLSs is the same as the one given for LTV systems; therefore, we shall refer to Definition 1.6, when considering IDLSs.

Before concluding this section, we would like to remark that IDLSs can be used to model the class of switching linear systems (SLS, [82]).

Indeed, IDLSs can also be seen as a special case of SLSs. Given a right-continuous *switching signal* σ, i.e. a piecewise constant function $\sigma(\cdot) : \mathbb{R}_0^+ \mapsto \mathcal{P} \subset \mathbb{N}$, whose discontinuities correspond to the resetting times, and the family of linear systems

$$\dot{x}(t) = A_p(t)x(t) + F_p(t)w(t), \tag{1.18a}$$

$$y(t) = C_p(t)x(t) + G_p(t)w(t), \tag{1.18b}$$

where $p \in \mathcal{P} = \{1, \dots, l\}$, the class of SLSs is given by

$$\dot{x}(t) = A_{\sigma(t)}(t)x(t) + F_{\sigma(t)}(t)w(t), \quad x(t_0) = 0, \quad t \notin \mathcal{T} \tag{1.19a}$$

$$x(t_i^+) = J(t_i)x(t_i), \quad t_i \in \mathcal{T} \tag{1.19b}$$

$$y(t) = C_{\sigma(t)}(t)x(t) + G_{\sigma(t)}w(t), \quad t \in \Omega. \tag{1.19c}$$

It follows that IDLSs (1.14) can also be seen as SLSs when the special case of a single dynamic with discontinuities in correspondence of the resetting times is considered. Hence, the two definitions are equivalent unless the linear systems in the family (1.18) have different orders.

1.5 Book Organization

After the introductory Chapter 1 (this chapter), in Chapters 2 and 3 both the analysis and design of continuous-time LTV systems in the form (1.8) are considered. We focus on the two classes of inputs, $\mathcal{W}_2$ and $\mathcal{W}_\infty$, introduced in Section 1.4.1, and some conditions guaranteeing IO-FTS and finite-time stabilization will be presented for LTV systems. More precisely, the proposed approach will lead to necessary and sufficient conditions ($\mathcal{W}_2$ case) and sufficient conditions ($\mathcal{W}_\infty$ case) for analysis and synthesis, all based on feasibility problems involving DLMIs or DLEs.

In Chapters 4 and 5 some extensions of the original definition of IO-FTS are considered. In particular, in Chapter 4 we consider the case in which the initial state is nonzero; this leads to the definition of IO-FTS-NZIC, for which some sufficient conditions are derived. In Chapter 5 we consider the usual situation where there are some amplitude constraints on the control inputs, introducing the concept of *structured* IO-FTS.

In Chapter 6 the robustness issues are considered; this represents the starting point for considering the mixed $\mathcal{H}_\infty/FTS$ control problem.

In Chapter 7, the FTS analysis for IDLS in the form (1.14) is considered; Chapter 8 deals with the design problem for the same class of systems, while in Chapter 9, the case in which the resetting times of the IDLS (1.14) are uncertain is considered.

Finally, in Chapter 10, we illustrate a hybrid architecture, where the controller is implemented by both finite-time control techniques and the classical robust control (infinite horizon) approach. This application shows that the IO-FTS approach is useful to refine the system behavior during the transients, while classical IO Lyapunov stability is a fundamental requirement to guarantee the correct behavior at steady state.

The book is equipped with five appendices. Appendix A provides some fundamental results on LTV systems; Appendix B recalls some properties of Schur Complements, which are often used in our book; Appendix C illustrates some numerical techniques to solve DLMIs and D/DLMIs, while Appendix D presents some examples of MATLAB® code used to solve optimization problems with this type of constraints. Finally, Appendix E discusses some real-world examples where the IO-FTS approach can be exploited.

There are some issues that are not investigated in this book. For example, we do not discuss the extension of the IO-FTS theory to nonlinear and/or stochastic systems (see for example [83]), systems with delays [84–86], 2D-systems [87]. IDLSs are only considered from the deterministic point of view, while there is a growing interest for impulsive and switching systems regulated by stochastic phenomena (see [88] and the bibliography therein).

For self-containedness purposes, the proofs of all the main theorems are provided; also, a reference is made to the paper where the theorem has been originally stated. Moreover each chapter is equipped with a summary, which recalls the main topics we have dealt with in the chapter itself.

All numerical computations done in the examples have been performed within the MATLAB® environment using the YALMIP parser [89] to specify the optimization problems, and by solving them either using the LMI Toolbox® [90] or other optimization solvers, such as SeDuMi [91].