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Introduction to Finite Element Model Updating

1.1 Introduction

Finite element model updating methods are intended to correct and improve a numerical model to match the dynamic behaviour of real structures (Marwala, 2010). Modern computers, with their ability to process large matrices at high speed, have facilitated the formulation of many large and complicated numerical models, including the boundary element method, the finite difference method and the finite element models. This book deals with the finite element model that was first applied in solving complex elasticity and structural analysis problems in aeronautical, mechanical and civil engineering. Finite element modelling was proposed by Hrennikoff (1941) and Courant and Robbins (1941). Courant applied the Ritz technique and variational calculus to solve vibration problems in structures (Hastings *et al*., 1985). Despite the fact that the approaches used by these researchers were different from conventional formulations, some important lessons are still relevant. These differences include mesh discretisation into elements (Babuska *et al*., 2004).

The Cooley–Turkey algorithms, which are used to speedily obtain Fourier transformations, have facilitated the development of complex techniques in vibration and experimental modal analysis. Conversely, the finite element model ordinarily predicts results that are different from the results obtained from experimental investigation. Among reasons for the discrepancy between finite element model prediction and experimentally measured data are as the following (Friswell and Mottershead, 1995; Marwala, 2010; Dhandole and Modak, 2011):

- model structure errors resulting from the difficulty in modelling damping and complex shapes such as joints, welds and edges;
- model order errors resulting from the difficulty in modelling non-linearity and often assuming linearity;

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- model parameter errors resulting in difficulty in identifying the correct material properties;
- errors in measurements and signal processing.

In finite element model updating, it is assumed that the measurements are correct within certain limits of uncertainty and, for that reason, a finite element model under consideration will need to be updated to better reflect the measured data. Additionally, finite element model updating assumes that the difficulty in modelling joints and other complicated boundary conditions can be compensated for by adjusting the material properties of the relevant elements. In this book, it is also assumed that a finite element model is linear and that damping is sufficiently low not to warrant complex modelling (Mottershead and Friswell, 1993; Friswell and Mottershead, 1995). Using data from experimental measurements, the initial finite element model is updated by correcting uncertain parameters so that the model is close to the measured data. Alternatively, finite element model updating is an inverse problem and the goal is to identify the system that generated the measured data (Brincker *et al*., 2001; Dhandole and Modak, 2010; Zhang *et al.*, 2011; Boulkaibet, 2014; Fuellekrug *et al*., 2008; Cheung and Beck, 2009; Mottershead *et al*., 2000).

There are two main approaches to finite element model updating, namely, maximum likelihood and Bayesian methods (Marwala, 2010; Mottershead *et al.*, 2011). In this book, we apply a Bayesian approach to finite element model updating.

1.2 Finite Element Modelling

Finite element models have been applied to aerospace, electrical, civil and mechanical engineering in designing and developing products such as aircraft wings and turbo-machinery. Some of the applications of finite element modelling are (Marwala, 2010): thermal problems, electromagnetic problems, fluid problems and structural modelling. Finite element modelling typically entails choosing elements and basis functions (Chandrupatla and Belegudu, 2002; Marwala, 2010). Generally, there are two types of finite element analysis that are used: two-dimensional and three-dimensional modelling (Solin *et al*., 2004; Marwala, 2010).

Two-dimensional modelling is simple and computationally efficient. Three-dimensional modelling, on the other hand, is more accurate, though computationally expensive. Finite element analysis can be formulated in a linear or non-linear fashion. Linear formulation is simple and usually does not consider plastic deformation, which non-linear formulation does consider. This book only deals with linear finite element modelling, in the form of a second-order ordinary differential equation of relations between mass, damping and stiffness matrices. A finite element model has *nodes*, with a grid called a *mesh*, as shown in Figure 1.1 (Marwala, 2001). The mesh has material and structural properties with particular loading and boundary conditions. Figure 1.1 shows the dynamics of a cylinder, and the mode shape of the first natural frequency occurring at 433 Hz.

These loaded nodes are assigned a specific density all over the material, in accordance with the expected stress levels of that area (Baran, 1988). Sections which undergo more stress will then have a higher node density than those which experience less or no stress. Points of stress concentration may have fracture points of previously tested materials, joints, welds and high-stress areas. The mesh may be imagined as a spider's web so that, from each node, a mesh

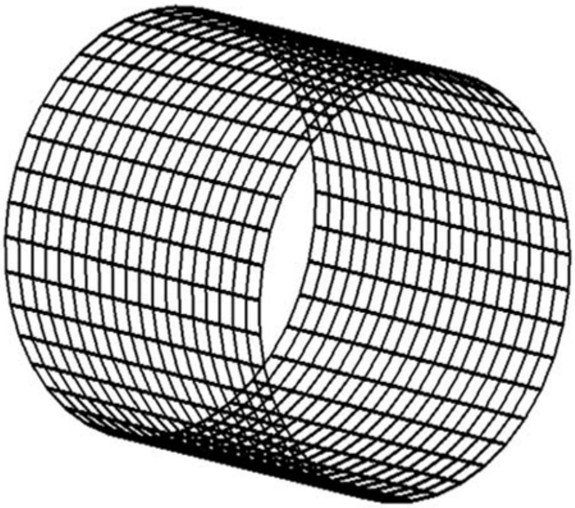

Figure 1.1 A finite element model of a cylindrical shell

element extends to each of the neighbouring nodes. This web of vectors has the material properties of the object, resulting in a study of many elements.

On implementing finite element modelling, a choice of elements needs to be made and these include beam, plate, shell elements or solid elements. A question that needs to be answered when applying finite element analysis is whether the material is isotropic (identical throughout the material), orthotropic (only identical at 90°) or anisotropic (different throughout the material) (Irons and Shrive, 1983; Zienkiewicz, 1986; Marwala, 2010).

Finite element analysis has been applied to model the following problems (Zienkiewicz, 1986; Marwala, 2010):

- vibration analysis for testing a structure for random vibrations, impact and shock;
- fatigue analysis to approximate the life cycle of a material or a structure due to cyclical loading;
- heat transfer analysis to model conductivity or thermal fluid dynamics of the material or structure.

Hlilou *et al.* (2009) successfully applied finite element analysis in softening material behaviour, while Zhang and Teo (2008) successfully applied it in the treatment of a lumbar degenerative disc disease. White *et al*. (2008) successfully applied finite element analysis for shallow-water modelling, while Pepper and Wang (2007) successfully applied it in wind energy assessment of renewable energy in Nevada. Miao *et al.* (2009) successfully applied a three-dimensional finite element analysis model in the simulation of shot peening. Bürg and Nazarov (2015) successfully applied goal-oriented adaptive finite element methods in elliptic problems, while Amini *et al*. (2015) successfully applied finite element modelling in functionally graded piezoelectric harvesters. Haldar *et al.* (2015) successfully applied finite element modelling in the study of the flexural behaviour of singly curved sandwich composite structures, while Millar and Mora (2015) successfully applied finite element methods to study the buckling in simply supported Kirchhoff plates. Jung *et al.* (2015) successfully used finite element models and computed tomography to estimate cross-sectional constants of composite blades, while Evans and Miller (2015) successfully applied a finite element model to predict the failure of pressure vessels. Other successful applications of finite element analysis are in the areas of metal powder compaction processing (Rahman *et al*., 2009), ferroelectric materials (Schrade *et al.*, 2007), rock mechanics (Chen *et al*., 2009), orthopaedics (Easley *et al.*, 2007), carbon nanotubes (Zuberi and Esat, 2015), nuclear reactors (Wadsworth *et al*., 2015) and elastic wave propagation (Gao *et al.*, 2015; Gravenkamp *et al*., 2015).

1.3 Vibration Analysis

An important aspect to consider when implementing finite element analysis is the kind of data that the model is supposed to predict. It can predict data in many domains, such as the time, modal, frequency and time–frequency domains (Marwala, 2001, 2010). This book is concerned with constructing finite element models to predict measured data more accurately. Ideally, a finite element model is supposed to predict measured data irrespective of the domain in which the data are presented. However, this is not necessarily the case because models updated in the time domain will not necessarily predict data in the modal domain as accurately as they will for data in the time domain. To deal with this issue, Marwala and Heyns (1998) used data in the modal and frequency domains simultaneously to update the finite element model in a multi-criteria optimisation fashion. Again, whichever domain is used, the updated model performs less well on data in a different domain than those used in the updating process. In this book, we use data in the modal domain. Raw data are measured in the time domain and Fourier analysis techniques transform the data into the frequency domain. Modal analysis is applied to transform the data from the frequency domain to the modal domain. All of these domains include similar information, but each domain reveals different data representations.

1.3.1 Modal Domain Data

The modal domain expresses data as natural frequencies, damping ratios and mode shapes. The technique used for extracting the modal properties is a process called *modal analysis* (Ewins, 1995). Natural frequencies are basic characteristics of a system and can be extracted by exciting the structure and analysing the vibration response. Cawley and Adams (1979) used changes in the natural frequencies to identify damage in composite materials. Farrar *et al.* (1994) successfully used the shifts in natural frequencies to identify damage on an I-40 bridge. Other successful applications of natural frequencies include damage detection in tabular steel offshore platforms (Messina *et al*., 1996, 1998), spot welding (Wang *et al.*, 2008) and beam-like structures (Zhong and Oyadiji, 2008; Zhong *et al*., 2008).

A mode shape represents the curvature of a system vibrating at a given mode and a particular natural frequency. West (1982) successfully applied the modal assurance criterion for damage on a Space Shuttle orbiter body flap, while Kim *et al*. (1992) successfully used the coordinate modal assurance criterion of Lieven and Ewins (1988) for damage detection in structures. Further applications of mode shapes include composite laminated plates (Araújo dos Santos *et al.*, 2006; Qiao *et al.*, 2007), linear structures (Fang and Perera, 2009), beam-type structures (Qiao and Cao, 2008; Sazonov and Klinkhachorn, 2005), optical sensor configuration (Chang and Pakzad, 2015), multishell quantum dots (Vanmaekelbergh *et al.*, 2015) and creep characterisation (Hao *et al*., 2015).

1.3.2 Frequency Domain Data

The measured excitation and response of the structure are converted into the frequency domain using Fourier transforms (Ewins, 1995; Maia and Silva, 1997), and from these the *frequency response function* is extracted. Frequency response functions have, in general, been used to identify faults (Sestieri and D'Ambrogio, 1989; Faverjon and Sinou, 2009). D'Ambrogio and Zobel (1994) used frequency response functions to identify the presence of faults in a truss structure, while Imregun *et al*. (1995) used frequency response functions for damage detection. Lyon (1995) and Schultz *et al*. (1996) used measured frequency response functions for structural diagnostics. Other direct applications of the frequency response functions include the work of Shone *et al*. (2009), Ni *et al*. (2006), X. Liu *et al*. (2009), White *et al.* (2009) and Todorovska and Trifunac (2008). Additional applications include missing-data estimation (Ugryumova *et al*., 2015), identification of a non-commensurable fractional transfer (Valério and Tejado, 2015), as well as damage detection (Link and Zimmerman, 2015).

1.4 Finite Element Model Updating

In real life, it turns out that the predictions of the finite element model are quite different from the measurements. As an example, for a finite element model of a simply suspended beam, the differences between the model-predicticted natural frequencies and the measured frequencies are shown in Table 1.1 (Marwala and Sibisi, 2005; Marwala, 2010). These results are for a fairly easy structure to model, and they demonstrate that the finite element model's data are different from the measured data. Finite element model updating has been studied quite extensively (Friswell and Mottershead, 1995; Mottershead and Friswell, 1993; Maia and Silva, 1997; Marwala, 2010). There are three approaches used in finite element model updating: direct methods, iterative deterministic and uncertainty quantification methods. Direct approaches are computationally inexpensive, but reproduce modal properties that are physically unrealistic.

Although the finite element model can predict measured quantities, the updated model is limited in that it loses the connectivity of nodes, results in populated matrices and in loss of

Table 1.1 Comparison of finite element model and real measurements

Mode number	Finite element frequencies (Hz)	Measured frequencies (Hz)
1	42.30	41.50
2	117.0	114.5
3	227.3	224.5
4	376.9	371.6

matrix symmetry. All these factors are physically unrealistic. Iterative techniques use changes in physical parameters to update the finite element models and, thereby, generate models that are physically realistic (Marwala, 2010). However, since they are based on optimisation techniques, the problems of global versus local optimum solution and over-fitting the measured data, these methods still produce unrealistic results. Esfandiari *et al.* (2009) used the sensitivity approach, frequency response functions and natural frequencies for model updating in structures, while Wang *et al*. (2009) used the Zernike moment descriptor for finite element model updating. Yuan and Dai (2009) updated a finite element model of damped gyroscopic systems, while Kozak *et al*. (2009) used a miscorrelation index for model updating. Arora *et al*. (2009) proposed a finite element model updating approach that used damping matrices, while Schlune *et al*. (2009) implemented finite element model updating to improve bridge evaluation. Yang *et al.* (2009) investigated several objective functions for finite element model updating in structures, while Bayraktar *et al.* (2009) applied modal properties for updating a finite element model of a bridge. Li and Du (2009) used the most sensitive design variable for finite element model updating of stay-cables, while Steenackers *et al*. (2007) successfully applied transmissibility for finite element model updating. Xu Yuan and Ping Yu (2015) proposed finite element model updating of damped structures, while Shabbir and Omenzetter (2015) applied particle swarm optimisation for finite element model updating. The uncertainty quantification techniques, however, include the uncertainties related to the modelled structure (or systems) during the updating procedure. The uncertainty quantification approaches that treat uncertain parameters as random parameters with joint distribution functions are called the probabilistic techniques and these comprise Bayesian and perturbation methods, whereas the non-probabilistic (possibilistic) approaches use the interval method or membership functions (fuzzy technique) to define uncertain parameters. In this book, only the Bayesian approach is used to update structures.

Other successful implementations of finite element model updating methods include applications in bridges (Huang and Zhu, 2008; Jaishi *et al.*, 2007; Niu *et al*., 2015), composite structures (Pavic *et al*., 2007), helicopters (Shahverdi *et al.*, 2006), atomic force microscopes (Chen, 2006), footbridges (Živanović *et al*., 2007), estimating constituent-level elastic parameters of plates (Mishra and Chakraborty, 2015) and identifying temperature-dependent thermal-structural properties (Sun *et al*., 2015). The process of finite element updating is illustrated in Figure 1.2 (Boulkaibet, 2014).

1.5 Finite Element Model Updating and Bounded Rationality

As illustrated in Figure 1.2, optimisation involves minimising the distance between measurements and the model output, whichever way the model is defined, whether deterministically or probabilistically. The minimisation process gives either a local optimum solution or a global one, and one is never sure whether the solution is global or local, particularly for complex problems. Furthermore, the data to be used should be universally represented, meaning that all the domain representations must be used, and this is not possible. A definition of 'rational solution' implies that the solution is optimised, all information is used and an optimal objective function for optimisation is used. In finite element model updating, this is not possible.

In rational theory, the limited availability of information required in making a rational decision, and the limitations of devices for making sense of incomplete decisions, are covered by

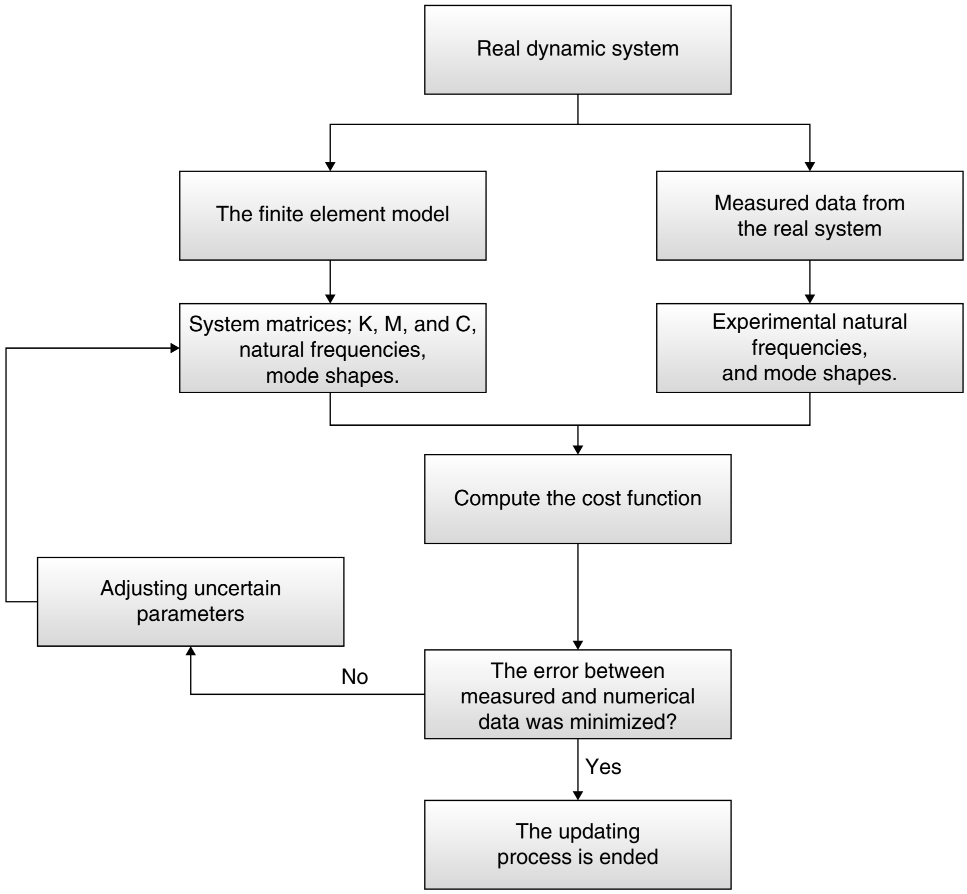

Figure 1.2 Finite element model updating procedure

the theory of bounded rationality, and it was proposed by Herbert Simon (Simon, 1957, 1990, 1991; Tisdell, 1996). The theory of bounded rationality has been used in modelling by researchers such as Lee (2013), Gama (2013), Jiang *et al*. (2013), Stanciu-Viziteu (2012), Aviad and Roy (2012) and Murata *et al*. (2012). Herbert Simon coined the term *satisficing*, by combining the terms 'satisfying' and 'sufficing', which is the concept of making optimised decisions under the limitations that the data used in making such decisions are imperfect and incomplete, while the model used to make such decisions is inconsistent and imperfect. In the same vein, the finite element model updating problem is a satisficing problem, not a process of seeking the correct model.

1.6 Finite Element Model Updating Methods

This section reviews methods that have been used for finite element model updating. They are grouped into classes, and more details on these may be found in Marwala (2010). There are three categories of finite element model updating techniques: direct methods; iterative methods; and uncertainty quantification methods.

1.6.1 Direct Methods

Direct methods (Friswell and Mottershead, 1995; Marwala, 2010) are one of the earliest strategies used for finite element model updating. They possess the ability to reproduce the exact experimental data and without using iterations, which makes these algorithms computationally efficient. These methods are still used for finite element model updating, and modern instruments and sensors that have lately been used in experiments allow these methods to overcome some of their disadvantages, such as lack of node connectivity and the need for a large amount of data to reproduce the exact experimental matrices. In this subsection, several direct methods – the matrix update methods, the Lagrange multiplier method, the optimal matrix methods and the eigenstructure assignment method – are briefly described.

1.6.1.1 Matrix Update Methods

Matrix update methods operate by modifying structural model matrices, that is, the mass, stiffness and damping matrices (Baruch, 1978). These are obtained by minimising the distance between analytical and measured matrices as follows (Friswell and Mottershead, 1995; Marwala, 2010):

$$\mathbf{E}_i = \left(-\omega_i^2\mathbf{M} + j\omega_i\mathbf{C} + \mathbf{K}\right)\phi_i, \tag{1.1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix of the structure, $\mathbf{E}_i$ is the error vector (also known as the residual force), $j = \sqrt{-1}$, ω_i is the ith natural frequency and ϕ_i is the ith mode shape. The residual force is the harmonic force with which the unupdated model will have to be excited at a frequency of ω_i so that the structure will respond with the mode shape ϕ_i. The Euclidean norm of $\mathbf{E}_i$ is minimised by updating physical parameters in the model (Ewins, 1995; Marwala, 2010) and choosing an optimisation routine. These techniques are classified as *iterative* since they are employed by iteratively changing the relevant parameters until the error is minimised. Ojalvo and Pilon (1988) minimised the Euclidean norm of the residual force for the ith mode of the structure by using the modal properties. The residual force in the equation of motion in the frequency domain may be minimised as (Friswell and Mottershead, 1995):

$$\mathbf{E} = \left(-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}\right)\mathbf{X}^m - \mathbf{F}^m, \tag{1.2}$$

where $\mathbf{X}^m$ and $\mathbf{F}^m$ are the Fourier-transformed displacement and force matrices, respectively. Each column of the matrix corresponds to a measured frequency point. The Euclidean norm of the error matrix $\mathbf{E}$ is minimised by updating physical parameters in the model. The methods described in this subsection are computationally expensive. In addition, it is challenging to identify a global minimum because of multiple stationary points, which are caused by the non-unique nature of inverse problems (Janter and Sas, 1990; Mares and Surace, 1996; Friswell *et al*., 1994; Dunn, 1998).

1.6.1.2 Lagrange Multiplier Method

The Lagrange multiplier method is an optimisation technique that deals with the objective function and constraints of an optimisation equation (Rad, 1997). It is implemented by minimising a constrained objective function, where the constraints are imposed by Lagrange multipliers (Marwala, 2010; Minas and Inman, 1988; Heylen and Sas, 1987).

1.6.1.3 Optimal Matrix Methods

Optimal matrix methods employ analytical rather than numerical solutions to obtain matrices from the damaged systems. They are formulated using Lagrange multipliers and perturbation matrices, and the optimisation problem is posed to minimise (Friswell and Mottershead, 1995)

$$E(\Delta\mathbf{M}, \Delta\mathbf{C}, \Delta\mathbf{K}) + \lambda R(\Delta\mathbf{M}, \Delta\mathbf{C}, \Delta\mathbf{K}), \tag{1.3}$$

where E is the objective function, λ is the Lagrange multiplier, R is the constraint of the equation and Δ denotes the perturbation of the system matrices. Different permutations of perturbations are tried until the difference between the finite element model results and the measured results is minimised. Baruch and Bar Itzhack (1978), Berman and Nagy (1983) and Kabe (1985) formulated Equation 1.3by minimising the Frobenius norm of the error, while maintaining the symmetry of the matrices. McGowan *et al*. (1990) introduced an extra constraint that maintained the connectivity of the structure and used measured mode shapes to update the stiffness matrix to locate structural damage. Zimmerman *et al*. (1995) used a partitioning method for matrix perturbations as sums of element or substructural perturbation matrices to reduce the rank of unknown perturbation matrices. Carvalho *et al.* (2007) successfully applied a direct method for model updating with incomplete measured modal data. A limitation of these approaches is that the updated model is physically unrealistic.

1.6.1.4 Eigenstructure Assignment Methods

Eigenstructure assignment approaches are based on control system theory, and the system under consideration is made to respond in a predetermined configuration. An updated finite element model is that with eigenstructure which is obtained from measured data. Zimmerman and Kaouk (1992) applied this approach successfully to update a finite element model of a cantilevered beam based on modal properties, while Schultz *et al*. (1996) updated a finite element model using the measured frequency response functions. The limitation of this technique that the number of sensor locations is less than the degrees of freedom in the finite element model. To deal with this limitation, either the mode shapes and frequency response functions are expanded to the size of the finite element model or the mass and stiffness matrices of the finite element model are reduced to the size of the measured data. The reduction/expansion approaches that are applied are static reduction (Guyan, 1965; Gysin, 1990; Imregun and

Ewins, 1993), dynamic reduction (Paz, 1984), improved reduced systems (O'Callahan, 1989) and the system-equivalent reduction process (O'Callahan *et al.*, 1989).

1.6.2 Iterative Methods

Iterative methods (Friswell and Mottershead, 1995; Marwala, 2010) were developed to overcome the weakness of the direct methods and to update finite element models of complex systems. These methods use non-linear equations to deal with the non-convex optimisation problem which arises when a complex system is updated. In these methods, a set of parameters are iteratively adjusted to minimise an objective function (also called a penalty function), where most of the objective functions used in model updating contain only modal and/or response functions data. In this subsection, two popular iterative methods are briefly discussed.

1.6.2.1 Sensitivity Methods

Sensitivity approaches work on the premise that experimentally measured data are perturbations of design data around a finite element model. Therefore, experimentally measured data ought to be approximately equal to data predicted by the finite element model for this approach to work. These approaches uses the derivatives of either the modal properties or the frequency response functions as a basis for finite element model updating. Many procedures have been developed to calculate the derivative of the modal properties and frequency response functions, including Fox and Kapoor (1968), Norris and Meirovitch (1989), Haug and Choi (1984), Chen and Garba (1980) and Adhikari and Friswell (2001). Ben-Haim and Prells (1993) used frequency response function sensitivity to update a finite element model, while Lin *et al.* (1995) used modal sensitivity for finite element model updating and Hemez (1993) used element sensitivity for finite element updating. Alvin (1997) improved the convergence rate by using statistical confidence measurements in finite element model updating.

1.6.2.2 Optimisation Methods

Huang and Zhu (2008) applied optimisation methods for the finite element model updating of bridge structures. The optimisation method was augmented by a sensitivity analysis. Schwarz *et al.* (2007) updated a finite element model which minimised the difference between the modes of a finite element model and those from the experiment. Bakir *et al.* (2007) applied sensitivity approaches for finite element model updating. They used a constrained optimisation method to minimise the differences between the natural frequencies and mode shape.

Jaishi and Ren (2007) applied a multi-objective optimisation approach for finite element model updating. Their multi-objective cost function was based on the differences between eigenvalues and strain energy. Liu *et al.* (2006) updated a finite element model of a 14-bay beam with semi-rigid joints and a boundary using a hybrid optimisation method. Zhang and Huang (2008) applied a gradient descent optimisation method for the finite element model updating of bridge structures. The objective function was formulated as the summation of the frequency difference and modal shapes. Parameter alteration was guided by engineering judgement.

1.6.3 Artificial Intelligence Methods

Finite element modelling updating can be achieved through the use of artificial intelligence techniques. Artificial intelligence techniques are computational tools that are inspired by the way nature and biological systems work. Within the context of finite element model updating, some of the techniques that have been applied are genetic algorithms, particle swarm optimisation, fuzzy logic, neural networks, and support vector machines. A genetic algorithm simulates natural evolution, where the law of the survival of the fittest is applied to a population of individuals. This natural optimisation method is used for optimising a function (Mitchell, 1998). Particle swarm optimisation is an evolutionary optimisation method that was developed by Kennedy and Eberhart (1995), inspired by algorithms that model the flocking behaviour seen in birds. The response surface method is a procedure that functions by generating a response for a given input and then constructs an approximation to a complicated model such as a finite element model (Kamrani *et al.*, 2009).

Finite element models are computationally expensive methods. To manage the computational load, some form of emulator to approximate the finite element model can be implemented. Y. Liu *et al*. (2009) used fuzzy theory, while Jung and Kim (2009) employed the hybrid genetic algorithm for finite element model updating. Tan *et al*. (2009) used support vector machines and wavelet data for finite element model updating in structures, while Zapico *et al.* (2008) applied neural networks. Further successful applications of artificial intelligence methods to finite element model updating include Tu and Lu (2008) and Yan *et al*. (2007), as well as Fei *et al*. (2006) who applied genetic algorithms. Feng *et al*. (2006) applied a hybrid of a genetic algorithm and simulated annealing, and He *et al*. (2008) applied a hybrid of a genetic algorithm and neural networks.

Marwala (2010) used the particle swarm optimisation technique for finite element model updating, and the results were compared to those obtained from the genetic algorithm. Furthermore, simulated annealing was also introduced and applied to finite element model updating, and the results were compared to those from particle swarm optimisation. To deal with the issue of computational efficiency, a response surface method that combines the multi-layer perceptron and particle swarm optimisation was introduced and applied to finite element model updating. The results were compared to those from the genetic algorithm, particle swarm optimisation and simulated annealing.

1.6.4 Uncertainty Quantification Methods

Due to the numerical and experimental uncertainties associated with the updated models, formulating the updating problems as iterative optimisation with constraints may not produce stable and accurate results. Modelling uncertainties are caused by predictions used to model the systems, especially when the physical components used to model the systems are complex and not sufficiently well understood. On the other hand, experimental uncertainties are caused by noise resulting from the measurements or by the variability of the system parameters (Der Kiureghian and Ditlevsen, 2009; Soize, 2010; Walker *et al.*, 2003). In this subsection, the perturbation method, minimum variance method and Bayesian approach are briefly described.

1.6.4.1 Perturbation Method

The perturbation technique uses a Taylor series to extend the terms in model updating equations around a predefined point and then to estimate the mean and variance of the updated parameters (Khodaparast, 2010; Hua *et al.*, 2008; Khodaparast *et al.*, 2008). One type of perturbation technique uses the least-squares method for stochastic finite element model updating, by assuming that the measured data and updating parameters are statistically independent. Another perturbation technique was developed by Hua *et al.* (2008) and assumes that the measured vector $\mathbf{Z}_X$ can be obtained by adding a random component ($\Delta\mathbf{Z}_X$) to a deterministic component (mean value) as follows (Khodaparast, 2010; Hua *et al.*, 2008; Boulkaibet, 2014):

$$\mathbf{Z}_X = \hat{\mathbf{Z}}_X + \Delta\mathbf{Z}_X, \tag{1.4}$$

where the perturbation vector ($\Delta\mathbf{Z}_X$) has zero mean and represents the uncertainty in the measured data. The structural parameters $\boldsymbol{\theta}$, the sensitivity matrix $\mathbf{S}$ and the predictions $\mathbf{Z}$ are defined around the mean value of these vectors and/or matrices as follows (Friswell and Mottershead, 1995; Khodaparast, 2010; Boulkaibet, 2014):

$$\boldsymbol{\theta} = \hat{\boldsymbol{\theta}} + \sum_{i=1}^{n} \frac{\partial \boldsymbol{\theta}}{\partial \Delta\mathbf{Z}_X^i} \Delta\mathbf{Z}_X^i, \tag{1.5}$$

$$\mathbf{S} = \hat{\mathbf{S}} + \sum_{i=1}^{n} \frac{\partial \mathbf{S}}{\partial \Delta\mathbf{Z}_X^i} \Delta\mathbf{Z}_X^i, \tag{1.6}$$

$$\mathbf{Z} = \hat{\mathbf{Z}} + \sum_{i=1}^{n} \frac{\partial \mathbf{Z}}{\partial \Delta\mathbf{Z}_X^i} \Delta\mathbf{Z}_X^i. \tag{1.7}$$

With subscript j denoting iteration number, we obtain (Friswell and Mottershead, 1995; Boulkaibet, 2014):

$$\hat{\mathbf{Z}}_X = \hat{\mathbf{Z}}_j + \hat{\mathbf{S}}_j \left(\hat{\boldsymbol{\theta}}_{j+1} - \hat{\boldsymbol{\theta}}_j \right) \tag{1.8}$$

$$\hat{\mathbf{S}}_j \frac{\partial \boldsymbol{\theta}_{j+1}}{\partial \Delta\mathbf{Z}_X^i} = \hat{\mathbf{S}}_j \frac{\partial \boldsymbol{\theta}_j}{\partial \Delta\mathbf{Z}_X^i} + \left(e - \frac{\partial \mathbf{Z}_j}{\partial \Delta\mathbf{Z}_X^i} - \frac{\partial \mathbf{S}_j}{\partial \Delta\mathbf{Z}_X^i} \right) \left(\hat{\boldsymbol{\theta}}_{j+1} - \hat{\boldsymbol{\theta}}_j \right). \tag{1.9}$$

Here, the vector $\mathbf{e} = [0 \cdots 0\,1\,0 \cdots 0]$ has all components equal to zero except for a 1 in ith position. Subtracted from it are (Friswell and Mottershead, 1995):

$$\frac{\partial \mathbf{Z}_j}{\partial \Delta\mathbf{Z}_X^i} = \mathbf{S}_j \frac{\partial \boldsymbol{\theta}_j}{\partial \Delta\mathbf{Z}_X^i}, \tag{1.10}$$

$$\frac{\partial \mathbf{S}_j}{\partial \Delta\mathbf{Z}_X^i} = \sum_{k=1}^{p} \frac{\partial \mathbf{S}_j}{\partial \boldsymbol{\theta}_k} \cdot \frac{\partial \boldsymbol{\theta}_k}{\partial \Delta\mathbf{Z}_X^i}. \tag{1.11}$$

Equation 1.10 defines the approximated mean of the uncertain parameters, while Equation 1.11 defines the covariance matrix that is obtained through (Boulkaibet, 2014):

$$\mathbf{V}_{\boldsymbol{\theta}_j} = \boldsymbol{\Theta}_{j,\Delta\mathbf{Z}_X} \mathbf{V}_{\mathbf{Z}_X} \boldsymbol{\Theta}^T_{j,\Delta\mathbf{Z}_X}, \tag{1.12}$$

where $\mathbf{V}_{\mathbf{Z}_X}$ denotes the covariance of the measured data and

$$\boldsymbol{\Theta}_{j,\Delta\mathbf{Z}_X} = \begin{bmatrix} \frac{\partial\boldsymbol{\theta}_j^1}{\partial\Delta\mathbf{Z}_X^1} & \frac{\partial\boldsymbol{\theta}_j^1}{\partial\Delta\mathbf{Z}_X^2} & \cdots & \frac{\partial\boldsymbol{\theta}_j^1}{\partial\Delta\mathbf{Z}_X^n} \\ \frac{\partial\boldsymbol{\theta}_j^2}{\partial\Delta\mathbf{Z}_X^1} & \frac{\partial\boldsymbol{\theta}_j^2}{\partial\Delta\mathbf{Z}_X^2} & & \frac{\partial\boldsymbol{\theta}_j^2}{\partial\Delta\mathbf{Z}_X^n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial\boldsymbol{\theta}_j^p}{\partial\Delta\mathbf{Z}_X^1} & \frac{\partial\boldsymbol{\theta}_j^p}{\partial\Delta\mathbf{Z}_X^2} & \cdots & \frac{\partial\boldsymbol{\theta}_j^p}{\partial\Delta\mathbf{Z}_X^n} \end{bmatrix}. \tag{1.13}$$

1.6.4.2 Minimum Variance Method

The minimum variance approach is an iterative procedure that takes into account the parameter variability and the uncertainties related to constructing the finite element model (Friswell and Mottershead, 1995; Boulkaibet, 2014). This technique minimises the variance of the uncertain parameters, at each iteration, during the updating process. Suppose $\boldsymbol{\theta}_i$ is the vector of uncertain parameters at the ith iteration of the updating procedure. Then the variance matrix of the parameters at the ith iteration is $E\left(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^T\right) = \mathbf{V}_i$. Subtracting the finite element predicted output $\mathbf{Z}_i$ at the ith iteration from the measurement data $\mathbf{Z}_X$ yields (Friswell and Mottershead, 1995; Boulkaibet, 2014)

$$\delta\mathbf{Z} = \mathbf{Z}_X - \mathbf{Z}_i = \mathbf{S}(\boldsymbol{\theta} - \boldsymbol{\theta}_i). \tag{1.14}$$

Then the approximated uncertain parameter vector at the $(i+1)$th iteration, $\boldsymbol{\theta}_{i+1}$, is written as follows (Friswell and Mottershead, 1995):

$$\boldsymbol{\theta}_{i+1} - \boldsymbol{\theta}_i = \mathbf{T}(\mathbf{Z}_X - \mathbf{Z}_i), \tag{1.15}$$

where $\mathbf{T}$ represents an unknown transformation matrix. The new variance of the estimated parameters, $\boldsymbol{\theta}_{i+1}$, for the $(i+1)$th iteration is given by (Chen, 2001)

$$\mathbf{V}_{i+1} = E\left(\boldsymbol{\theta}_{i+1}\boldsymbol{\theta}_{i+1}^T\right) = \mathbf{V}_i + \left(\mathbf{D}_i - \mathbf{V}_i\mathbf{S}_i^T\right)^T\mathbf{T}^T + \mathbf{T}\left(\mathbf{D}_i^T - V_i.S_i\right) + \mathbf{T}\mathbf{V}_{zi}\mathbf{T}^T, \tag{1.16}$$

where $\mathbf{D}_i = E\left(\boldsymbol{\theta}_i\mathbf{Z}_X^T\right)$ is the correlation between the parameter approximation and the measurement noise. The output error variance is (Chen, 2001)

$$\mathbf{V}_{zi} = \mathbf{S}_i \mathbf{V}_i S_i^T - \mathbf{S}_i \mathbf{D}_i - \mathbf{D}_i^T \mathbf{S}_i^T + \mathbf{V}_e, \tag{1.17}$$

where $\mathbf{V}_e = E\left(\mathbf{Z}_X \mathbf{Z}_X^T\right)$ and the transformation matrix is achieved by minimising the variance at the $(i+1)$th iteration as follows (Chen, 2001):

$$\mathbf{T} = \left(\mathbf{V}_i \mathbf{S}_i^T - \mathbf{D}_i\right) \mathbf{V}_{zi}^{-1}. \tag{1.18}$$

The updated parameters $\boldsymbol{\theta}_{i+1}$, V_{i+1} and D_{i+1} are obtained as (Chen, 2001):

$$\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i + \left(\mathbf{V}_i \mathbf{S}_i^T - \mathbf{D}_i\right) \mathbf{V}_{zi}^{-1} (\mathbf{Z}_X - \mathbf{Z}_i), \tag{1.19}$$

$$\mathbf{V}_{i+1} = \mathbf{V}_i - \left(\mathbf{V}_i \mathbf{S}_i^T - \mathbf{D}_i\right) \mathbf{V}_{zi}^{-1} \left(\mathbf{V}_i \mathbf{S}_i^T - \mathbf{D}_i\right)^T, \tag{1.20}$$

$$D_{i+1} = \mathbf{D}_i - \left(\mathbf{V}_i \mathbf{S}_i^T - \mathbf{D}_i\right) \mathbf{V}_{zi}^{-1} (\mathbf{S}_i \mathbf{D}_i - \mathbf{V}_e). \tag{1.21}$$

1.6.4.3 Bayesian Approaches

The Bayesian method is a technique based on Bayes' theorem for making statistical inference by using the evidence (observations) to update the probability that a hypothesis is true (Marwala, 2009, 2010). Wong *et al*. (2006) used Bayesian methods to update a bridge model using sensor data, while Marwala and Sibisi (2005) conducted finite element updating in beam structures. Mares *et al*. (2006) used the Monte Carlo method for stochastic model updating, while Lindholm and West (1995) applied a Bayesian parameter approximation for finite element model updating and used this to model experimental dynamic response data. Hemez and Doebling (1999) successfully used a Bayesian approach for finite element model updating and applied this to linear dynamics, while Zheng *et al.* (2009) used a Bayesian approach for finite element model updating of a sky-bridge.

1.7 Bayesian Approach *versus* Maximum Likelihood Method

Finite element model updating is essentially an optimisation problem, where the design variables are the parameters of a finite element model that needs updating. There are two ways to approach this problem: the maximum likelihood technique (also known as the frequentist approach) and the Bayesian approach. The maximum likelihood approach defines an objective function, which is usually some distance between the model and the measured data. Then an optimisation method is applied to identify the optimal design variables. The problem with this approach is that it often overfits the data and it does not offer a probabilistic view of the finite element model updating problem.

Another technique which offers a probabilistic view of the finite element model updating problem is the Bayesian approach. The Bayesian framework is represented mathematically as follows (Bishop, 1995):

$$P(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta)P(\theta)}{P(\mathcal{D})}, \tag{1.22}$$

where $P(\theta)$ is the probability distribution function of the design space in the absence of any data, also called the *prior distribution function* and $\mathcal{D} = (y_1, \ldots, y_N)$ is a matrix containing the data. The expression $P(\theta|\mathcal{D})$ is the *posterior probability distribution function* after the data have been observed, $P(\mathcal{D}|\theta)$ is the likelihood function, and $P(\mathcal{D})$ is the normalisation function (also known as the evidence). This book takes the probability view on finite element model updating.

1.8 Outline of the Book

As stated earlier, finite element models are widely used to model the dynamic behaviour of many systems, including electrical, aerospace and mechanical engineering systems. This book is about probabilistic finite element model updating, which is achieved using Bayesian statistics. The aim of finite element model updating is to ensure that the finite element model better reflects the measured data. The finite element model updating process is limited by the theory of bounded rationality, as the data that could possibly be used for the updating problem are infinite and the number of resulting models that could possibly be identified is infinite because of the infinite starting points in the optimisation of the updating process, and the infinite ways of formulating the updating problem. In this book, the Bayesian framework is employed to estimate the probabilistic finite element models which take into account the uncertainties in the measurements and the modelling procedure. The Bayesian formulation achieves this by setting up the finite element model as a posterior distribution of the model, given the measured data. The data are estimated from the likelihood distribution function, the prior distribution function and the evidence. The finite element model updating posterior distribution function is complex and therefore, even for a fairly simple problem, cannot be estimated analytically. This book describes various sampling techniques based on the Markov chain Monte Carlo (MCMC) method that estimate the posterior probability distribution function of the finite element model updating problem. MCMC is a computational procedure based on the random walk and Markov process. The sampling methods described in this book are slice sampling, nested sampling, the Metropolis–Hastings algorithm, hybrid Monte Carlo (HMC) and shadow hybrid Monte Carlo (SHMC). These sampling methods are applied to estimate the posterior probability distribution function of the finite element model updating problem and are applied to mechanical and aeronautical structures.

This book explains the use of computational statistic techniques in aeronautical and mechanical engineering, a subject that will be of interest and useful to researchers, graduates and postgraduate students.

Chapter 2 discusses model selection in finite element model updating problems. It introduces various methods that can be used to select the best finite element model. A good model satisfies the principle of Occam's razor, which states that the simplest model that describes the observed data is the best one. Furthermore, the chapter studies criteria for model selection: the Akaike information criterion, optimal design, statistical hypothesis testing, Occam's razor, the Bayes factor, structural risk minimisation, cross-validation and the Bayesian information criterion. These techniques are described within the context of finite element model updating. Nested sampling, cross-validation and regularisation techniques are applied for model selection in structures.

Chapter 3 describes Bayesian statistics in structural mechanics. It introduces the concept of Bayesian statistics within the context of structural mechanics. Bayesian statistics basically states that the probability of an event A happening, given that event B has happened (also called the posterior probability), is equal to the product of the probability of event B happening given that event A has happened (also called the likelihood function), and the probability of event A happening (also called the prior), divided by the probability of event B happening (also called the evidence). A mass and spring system with a single degree of freedom is used to estimate the distribution of the stiffness given the distribution of the measured natural frequency and the mass.

In Chapter 4, the MCMC, which is a statistical procedure for computationally sampling a probability distribution function based on the Markov process, random walk and Monte Carlo simulation, is used for finite element model updating. Two approaches are used to update a finite element model of a mechanical structure: the Metropolis–Hastings approach and slice sampling. Slice sampling is a simple method that offers an adaptive step size, which is automatically adjusted to match the characteristics of the posterior distribution function.

In Chapter 5, Monte Carlo dynamically weighted importance sampling (MCDWIS) is applied for finite element model updating. An aeronautical structure application is presented. The motivation for applying MCDWIS is in the complexity of computing normalising constants in higher-dimensional or multimodal systems. MCDWIS accounts for this intractability by analytically computing importance sampling estimates at each time step of the algorithm, thus removing the need for perfect sampling. In addition, a dynamic weighting step with an adaptive pruned-enriched population control scheme allows for further control over weighted samples and population size. The performance of the MCDWIS simulation is graphically illustrated for all algorithm dependent parameters and shows unbiased, stable sample estimates. MCDWIS is then compared to the Metropolis–Hastings technique.

In Chapter 6, the adaptive Metropolis–Hastings (AMH) algorithm and Bayesian statistics are used for finite element model updating. In the AMH method the Gaussian proposal distribution is adapted using the full information gathered hitherto; because of the adaptive characteristics of the method, this technique is non-Markovian but also possesses full ergodic properties. The AMH method is implemented to update a finite element model of a cantilevered beam, an H-shaped structure, as well as an aircraft structure and the results are compared to the results from the MCDWIS method.

In Chapter 7, HMC and Bayesian finite element model updating is discussed. MCMC basically operates by moving from one state to another, through the random walk procedure, where the transition between one state and another is determined using the Markov chain and the acceptance or rejection of a state is decided using the Metropolis–Hastings method. HMC improves the search by using the gradient information to move from one state to another. In this way, the acceptance rate is greatly improved. Formally, HMC is implemented by calculating the Hamiltonian, which is the sum of the potential energy (position) and the kinetic energy (velocity).

In Chapter 8, a shadow HMC is applied for finite element model updating. To deal with this constraint, that the HMC acceptance rate is influenced by the system size and the time step used to estimate the molecular dynamics trajectory, the SHMC algorithm is used. The SHMC algorithm improves sampling for large system sizes and time steps by sampling from a modified Hamiltonian function instead of the normal Hamiltonian function. The SHMC is implemented to update a finite element model of an aircraft structure.

In Chapter 9, the separable shadow hybrid Monte Carlo (S2HMC) method is implemented for finite element model updating. The HMC method is a powerful sampling method for solving higher-dimensional complex problems. It uses the molecular dynamics (MD) as a global Monte Carlo move to reach areas of high probability. However, the HMC acceptance rate is sensitive to the system size, as well as the time step used to evaluate the MD trajectory. To overcome this, we propose the use of the S2HMC method. This method generates samples from a separable shadow Hamiltonian. The accuracy and the efficiency of this sampling method are tested on the updating of an aeronautical structure.

In Chapter 10, an evolutionary method for sampling a posterior probability density function for updating finite element models is discussed. The evolutionary sampling algorithm hybridises the concepts of genetic algorithms, simulated annealing and MCMC methods and, accordingly, these techniques are described in this chapter. The evolutionary sampling method uses concepts such as reproduction, mutation and crossover to construct the Markov chain to obtain samples. This method is then tested on the updating of a truss structure.

In Chapter 11 the adaptive hybrid Monte Carlo method is used for finite element model updating. The convergence rate of the HMC algorithm is high compared to the Metropolis–Hastings method because its trajectory is augmented by the derivative of the posterior probability distribution function. Nevertheless, the performance of the HMC method deteriorates when sampling from the posterior probability functions of high dimension and exhibits strong correlations between the uncertain parameters. The adaptive hybrid Monte Carlo approach facilitates efficient sampling from complex posterior distribution functions in high dimensions. The performance of the adaptive hybrid Monte Carlo method is tested for finite element model updating.

In Chapter 12, various issues associated with Bayesian sampling are discussed, including the formulation of the posterior probability distribution function. Sampling methods, nested sampling, Metropolis–Hastings, HMC, SHMC and adaptive hybrid Monte Carlo are discussed and compared and conclusions are drawn. Outstanding issues with regard to the application of Bayesian statistics for finite element model updating are extensively discussed. In particular, reversible jump Monte Carlo, the Dirichlet distribution, the expectation–maximisation algorithm and the distribution of optimal posterior probability models are described and proposed for future studies of finite element model updating.

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