1

PLASMA FUNDAMENTALS

INTRODUCTION

Plasmas are the most common state of matter both by mass and by volume, although most of them are rarefied and intergalactic; nevertheless, plasma is one of the four fundamental states of matter. Langmuir (1928) first coined the word *plasma* in 1928. Plasma is an electrically conducting medium and has the basic tendency toward global electrical neutrality. The most fundamental definition of this state of matter is that the pairs of closely positioned positively and negatively charges in the plasma shield each other from an externally applied electric potential. This characteristic charge separation distance is the Debye shielding length and is the smallest dimension compared with all other macroscopic scales. Within this shielding sphere, the paired charged particles are never free from each other but interact permanently.

Plasma does exist not only in gas and liquid but also in a solid conductor. The electrons are bound but still can move within atomic or molecular structures between collisions. Although mass motion of the charges does not generally take place, when an external electromagnetic field is applied, the dynamic effect can always be observed in conduction, Hall Effect, and polarization.

Plasmas, like gases, have a negligible shear stress and therefore do not have a definite shape or volume. However, the electric-conducting feature makes it respond to electromagnetic fields, so plasma can be confined either by a solid container or by a magnetic field. Plasma also appears in a wide range of formations such as filaments, microdischarges, multiple layers in the presence of shock waves, and cellular structure

Computational Electromagnetic-Aerodynamics, First Edition. Joseph J.S. Shang.

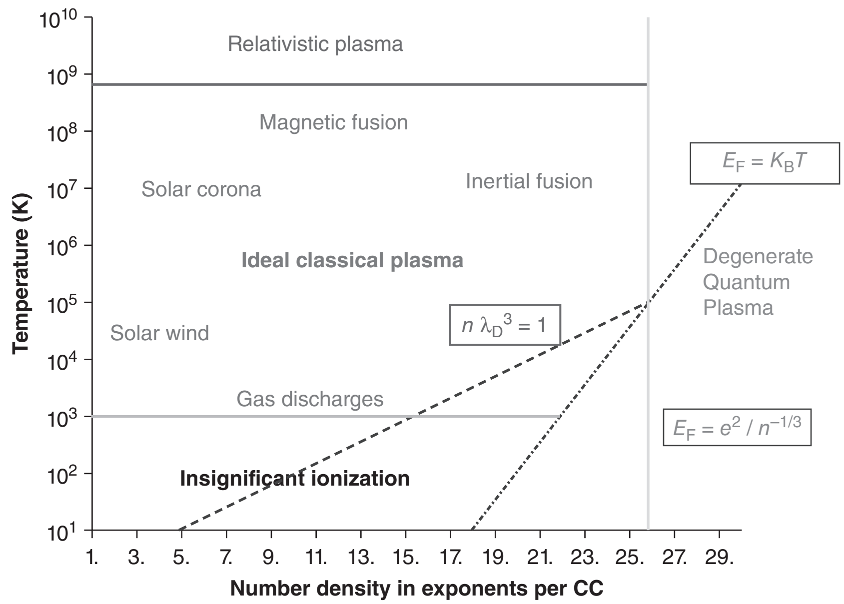

FIGURE 1.1 Classification for types of plasma.

in space. The spontaneous formations of unique spatial features on a wide range of length scales manifest the complexity of plasma. Because the electric conductivity is dependent on its charged particle number density and thermodynamic condition, the classification of plasma is usually by the electron number density and temperature in electron volts or the static temperature (Figure 1.1).

The topics of degenerated quantum plasma and relativistic plasma are beyond the scope of computational magneto-aerodynamics. For this reason, the present discussion is focused on the plasma domain that exists around atmospheric pressure, and on mixed thermal and nonthermal plasma. According to convention, the thermal condition of plasma is based on the relative temperatures of the electron, ion, and neutral components. By nonthermal plasma is meant that the ions and neutral particles have a much lower temperature than the electrons. For most gas discharges generated by electric currents, the electron temperature is at most 3 eV (around 10^5 K), while the ions and neutrals are at near room temperature. When ionization is achieved by a strong shock compression in high temperature or earth reentry, the temperatures of the plasma composition are on the same order, around 10,000 K or higher, but are not always equilibrated so as to remain in thermodynamic nonequilibrium (Jackson, 1975). Therefore, the plasma of interest is generally limited to an electron number density up to $10^{16}/\text{cm}^3$ and an overall temperature lower than 10^6 K.

From the discussion given earlier, plasma is an electric conducting medium but is globally neutral and characterized by a minimum macroscopic length scale between charges of opposite polarity known as the Debye shielding length. A strong electrostatic force exists between the paired charges; any small perturbation to the charges separation distance will trigger a high-frequency oscillation by the restoring force. This oscillatory motion is referred to as the plasma frequency that is distinguished from the lower frequency oscillation involving mass motion. In the presence of a sufficient number of charged particles, the oscillation also leads to a number of unique collective behaviors that are exemplified by the strong response to electromagnetic fields. Plasmas do not naturally conform to their surroundings but will respond to local and distant conditions by the Coulomb force and Lorentz acceleration. The other unique feature of quasi neutrality is that plasma stores the inductive energy and contributes to resistance and inductance when the current circuit forms a closed loop. Finally, the disturbance communication in plasma not only is through collision processes but also conveys by transverse wave with phase velocities greater than the speed of sound (Spitzer, 1956; Celexoix, 1960).

All physical phenomena of plasma are governing by the Maxwell and Navier-Stokes equations to become an interdisciplinary subject in macroscopic scales. Once the plasma is treated as a continuum, only the global thermodynamic and kinematic properties are needed to describe its behavior through a link between microscopic and macroscopic dynamics. This approach is accurate when the microscopic structure can be linked approximately to macroscopic motion by the Maxwell–Boltzmann distribution. However, the detailed wave–particle interaction will be unresolved.

1.1 ELECTROMAGNETIC FIELD

An electromagnetic field of plasma always consists of coexisted electric and magnetic fields. Charged particles interacting among themselves and with externally applied fields create induced components; the total field is a sum of applied and induced components. The magnitude of induced components can vary greatly in comparison with applied fields.

According to Coulomb's law, the electrostatic force between charged particles is a collinear force along the unit space vector between two charges separated by a distance r_{ij},

$$F_{ij} = \frac{1}{2\pi\varepsilon_o}\frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij} \tag{1.1}$$

The magnitude of the force is directly proportional to the product of the two charges and inversely proportional to the square of the distance between them. In SI units, the electric permittivity ε_o has a value of 8.85×10^{-12} Farad/m (Krause, 1953). The charges q_i and q_j are measured in coulombs, which attract to each other of opposite polarity but repulse if the same. The resulting force per unit charge is defined as the electric field intensity E. Through this definition, the electrostatic force can be given as

$$F_i = Eq_i \tag{1.2}$$

The SI unit of electric field intensity is the newton per coulomb. However, in practical application, an equivalent unit is often given in volt per meter. In the case of multiple point charges, the principle of superposition determines the force on a particular charge. The entire electric field of a charged space is the vector sum of the field from all the individual source charges,

$$E_i = \frac{1}{4\pi\varepsilon_o}\Sigma\frac{q_i}{r_{ij}^2}\hat{r}_{ij} \tag{1.3}$$

In an electrically conducting medium, the charge particle movement induces current and exerts additional force on each other. In an isolator, the molecules of the dielectric material will polarize to reduce the net local field intensity. This effect is described by a dielectric constant κ and Coulomb's law becomes

$$F_{ij} = \frac{1}{4\pi\varepsilon_o\kappa}\frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij} \tag{1.4}$$

Electric charge in motion constitutes a conductive electric current. In metallic conductors, the charge is carried by electrons with an elementary negative charge of 1.61×10^{-19} coulombs. In liquid conductors, such as electrolytes, the charge is carried by both positive and negative ions. The electric field compels the free charges into continuous motion and results in an electric current, which can be defined in terms of the electric flux vector per unit area.

The conductive current density considered in here is different from the convective and displacement current density. The convective current does not involve conductors and consequently does not satisfy Ohm's law. The current flows through an isolating medium such as the liquid, rarified gas, or vacuum, and the best example is an electron beam within a vacuum tube (Jahn, 1968). The displacement current arises from the time-varying electric field and is introduced by Maxwell to account for the generation of a magnetic field when the conductive current is zero. Without the concept of the displacement of current, the electromagnetic wave propagation would be impossible.

The conductive electric current density J has a physical dimension of coulomb per second or ampere,

$$J = \Sigma n_i q_i u_i = \Sigma \rho_i u_i \tag{1.5}$$

where n_i denotes the charge particle number density, u_i is the averaged velocity of the charged particles, and the current density ρ_i is simply the sum of the total electric charges per unit volume of species i.

A magnetic field is generated by an electric current and the orientation of this field is defined by the right-hand rule. The current may be due to electromagnetic fields by a permanent magnet, an electron beam, or a conductive current in current-carrying wire. The differential magnetic field intensity is governed by the Biot–Savart law for magnetostatics, which is the counter part of Coulomb's law for electrostatics. The field intensity is proportional to the product of a differential current element Idl and

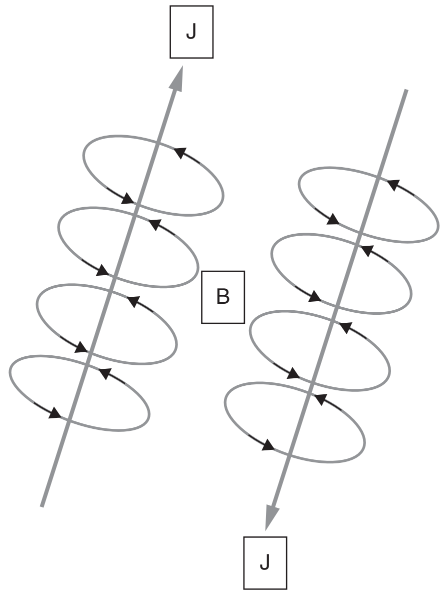

FIGURE 1.2 Induced magnetic field by electric current.

sine of the angle between the element and the line connecting the point of interest and is inversely proportional to the square of the mutual distance. The induced magnetic field line is continuous. In vector form, the Biot–Savart's law gives (Figure 1.2)

$$dH = \frac{Idl \times \hat{r}_{ij}}{4\pi r_{ij}^2} \tag{1.6a}$$

A current-carrying wire produces a magnetic field perpendicular to the electric current, and the magnetic flux density B has the physical unit of newton per amp-meter. In analogy to electrostatics, the intermediary B field is the magnetic flux density or is also called the magnetic induction. The connection between the magnetic field strength H and the magnetic flux density B in an isotropic medium is the constitute relation, $B = \mu B$, and μ is referred to as the magnetic permeability. In free space, it has the dimensional value of $\mu = 4\pi \times 10^{-1}$ hennery/meter.

If two current-carrying wires are brought into the vicinity of each other, each wire is surrounded by two individual magnetic fields leading to a force that acts on the wires. When the wires are carrying current in the same direction, the wires are attracted to each other, while wires carrying current move in opposite directions, the wires are repelled. The incremental force existing between the elements of the current path L_i and L_j is

$$dF_{ij} = \frac{\mu}{4\pi} \frac{J_i J_j}{r_{ij}^2} [dl_i \times (dl_j \times \hat{r}_{ij})] \tag{1.6b}$$

By definition of the magnetic induction is

$$dB_{ij} = (\mu J_j/4\pi)\left(dl_j \times \hat{r}_{ij}/r_{ij}^2\right) \tag{1.6c}$$

Analogous to that of electrostatics, the force on a current element becomes

$$dF_{ij} = J_i dl_i \times dB_{ij} \tag{1.7}$$

Basically, there are but two modes of the electromagnetic body force in an electrically charged medium that can exert on the charged particles. Figure 1.4 is the interaction of the electric field with the free charge density of the medium $F_e = \rho_e E$, which is known as the electrostatic force. The interaction of an externally applied magnetic field with the electric current that is driven by a force within the medium is $F_m = J \times B$ and it is the well-known Lorentz force.

As a consequence of a moving charge q in a steady and uniform magnetic field with an applied magnetic flux density B will be pushed by a force normal to B, namely,

$$F = qu \times B \tag{1.8}$$

Under the circumstance, the normal velocity component of the charge particles is always restricted in the plane perpendicular to a constant and steady magnetic B. The velocity component $u_{\|}$, which is parallel to B, will not be affected by the magnetic field but the velocity component u normal to B will be generated. Therefore, the charge particle motion in the normal plane to the magnetic field must follow a circular path and the charged particles and will execute a spiral motion. The gyro radius can be determined easily by the balance of the centrifugal acceleration and the electromagnetic force exerted on a singly charged particle, $eu_{\perp}B = mu_{\perp}^2\big/r_b$. The radius of the circular motion of a group of charged particles in the plane perpendicular the magnetic field is $r_b = mu_{\perp}/qB$, with an angular gyro velocity $\omega_b = qB/m$. The gyro radius is often referred to as the Larmor or cyclotron radius and can be evaluated on the basis of a single electron, $r_b = mu_{\perp}/eB$. Similarly, the angular gyro velocity has also been widely referred to as the Larmor or cyclotron frequency. The charge particle trajectory in three-dimensional electromagnetic field is then a helix with its axis parallel to B and a pitch of $2\pi u_{\|}\big/\omega B$.

The gyro frequency follows directly from its single-particle motion and can be given as

$$\omega_b = \frac{eB}{m} = 1.76 \times 10^{11} B\,(\text{rad/s}) \tag{1.9}$$

where the magnetic flux density B needs to be in the SI unit of webers/m^2.

The equation of motion for a charged particle in a uniform electromagnetic field is then

$$F = q(E + u \times B) = m\frac{du}{dt} \tag{1.10}$$

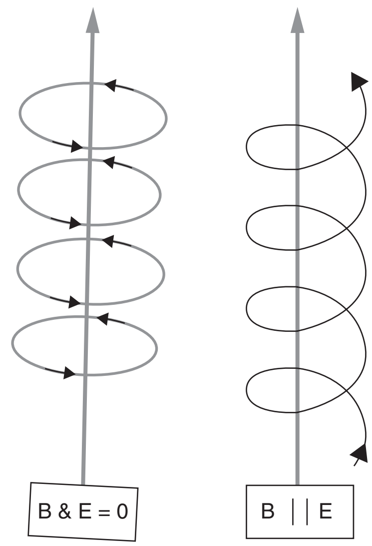

FIGURE 1.3 Trajectory of charge in electromagnetic field.

The electromagnetic field of moving charged particles therefore is always associated with a combination of electric and magnetic fields. A rectilinear acceleration is aligned with the externally applied electric field and a gyration intrinsically revolves around an applied magnetic field vector. In the absence of an electric field, the angular acceleration is restricted in the plane perpendicular to the magnetic field. When the electric field vector is applied in addition to the magnetic field, the charged particles will execute a persisted spiral trajectory as shown in Figure 1.3. The orientation of the resultant helical trajectory is dictated by the direction of the applied electric field intensity.

It becomes clear that the stationary charges generate an electrostatic field; in fact, the remotely acting Coulomb force is the genesis of the intrinsic characteristics of plasma in the Debye shielding length and the plasma frequency. While the direct current produces magnetostatic field, the dynamics of the electromagnetic field must more often be accompanied by a time-varying electric current density. In short, within a static electromagnetic field, the electric and magnetic fields are independent from each other whereas in dynamic electromagnetic fields the two fields are interdependent.

1.2 DEBYE LENGTH

To understand and to characterize the macroscopic behavior of a collection of charged particles, two fundamental parameters that associate with the electric properties of the ionized gas, namely the Debye length and plasma frequency, are paramount. The basic property of plasma is its tendency in maintaining electric neutrality (Langmuir, 1929).

This particular state requires an enormously large electrostatic force between an electron and an ion. A quantitative estimate of this dimension over which a deviation from charge neutrality may occur can be obtained by the Gauss law for electric field $\nabla \cdot D = \rho$. The electric displacement is related to the electric field intensity by a constitutive relationship $D = \varepsilon E$, where ε is the electric permittivity by the fact that the electric field intensity is inversely proportional to the gradient of the electric potential $E = -\nabla\varphi$. The resultant equation is known as the Poisson equation of plasmadynamics

$$\varepsilon \nabla^2 \varphi(r) = -\rho_e = -e(n_i - n_e) \tag{1.11a}$$

In a thermodynamic equilibrium condition, the number densities of electrons and ions are given by the Boltzmann distributions,

$$n_e = ne^{e\varphi(r)/\kappa T} \quad \text{and} \quad n_i = ne^{-e\varphi(r)/\kappa T} \tag{1.11b}$$

Substituting these expressions into the Poisson equation and reducing the equation to one dimension in space yield

$$\varepsilon \frac{d^2\varphi}{dx^2} = en_\infty [\mathrm{e}^{(e\varphi/\kappa T_e)} - 1] \tag{1.12}$$

Expanding the exponential function in Taylor series in ascending order of $e\varphi/\kappa T$ yields

$$\varepsilon \frac{d^2\varphi}{dx^2} = en_\infty \left[\frac{e\varphi}{\kappa T_e} + \frac{1}{2}\left(\frac{e\varphi}{\kappa T_e}\right)^2 + \cdots \right] \tag{1.13}$$

Although the magnitude of $e\varphi/\kappa T$ may not be necessarily negligible over an entire domain between charged particles, the potential is known to drop rapidly with respect to the separation distance. Therefore, it is permissible to keep only the leading term, then

$$\varepsilon \frac{d^2\varphi}{dx^2} = \frac{e^2 n_\infty}{\kappa T_e}\varphi \tag{1.14}$$

This leads to the definition of the Debye shielding length

$$\lambda_d = \left[\frac{\varepsilon \kappa T_e}{e^2 n}\right]^{1/2} \tag{1.15}$$

The electric potential has been found to exhibit an exponentially decay behavior as a function of distance between charged particles,

$$\varphi = \varphi_o \exp(-|x|/\lambda_d) \tag{1.16a}$$

In practical applications, the Debye length can be calculated easily by

$$\lambda_d = 69(T/n)^{1/2}\text{m}, T(\text{K}); \quad \lambda_d = 7430(\kappa T/n)^{1/2}\text{m}, T(\text{ev}) \tag{1.16b}$$

These simplified results for the Debye length are evaluated by the following constants: electron volt (ev) 1.1604×10^4 K, Boltzmann constant 1.3807×10^{-16} erg/K, elementary charge 4.8032×10^{-10} coulombs, and electric permittivity of free space 8.8542×10^{-12} farad/m.

The typical value of the Debye length in a magnetohydrodynamics generator is in the order of magnitude of the electron mean free path O(10^{-6}–10^{-7} m) at a temperature of 2500 K and charge number density of 10^{20}/m^3. In plasma generated by electronic impact at standard atmospheric conditions, the Debye length is 1.14×10^{-7} m.

Another way to estimate the separation distance between charged particles Δx is by recognizing that this position is the neutral state of the balanced average kinetic energy of the electron and the electric potential generated by the charged particles. The averaged separation distance can be obtained by integrating the one-dimensional Poisson equation of plasmadynamics

$$\frac{d^2\varphi}{dx^2} = -\frac{q}{\varepsilon} = \frac{n_e e}{\varepsilon} \tag{1.17}$$

Integrate consecutively with respect to x and set the farfield electric field to 0 ($E_o = 0$) to yield

$$\begin{aligned} \frac{d\varphi}{dx} &= \frac{n_e e}{\varepsilon} x + E_o \\ \Delta\varphi &= \frac{n_e e}{2\varepsilon} \Delta x^2 \end{aligned} \tag{1.18}$$

The average energy needed to separate the charged particles by a distance Δx must overcome the total thermal potential. The mean separation distance between the charged particles is then

$$\begin{aligned} \frac{\kappa T_e}{2} &= \frac{n_e e}{2\varepsilon} \Delta x \\ \Delta x &= \left[\frac{\varepsilon \kappa T_e}{n_e e^2}\right]^{1/2} \end{aligned} \tag{1.19}$$

This charge separation distance is exactly the Debye length derived previously. The temperature of the free-electron component of the gas T_e is not necessarily equilibrated with the ions or neutrals. The Debye length has always been adopted as an index of the typical charge separation distance in plasmas that are sustained by the random thermal energy of the electrons. For this reason, the Debye length is often

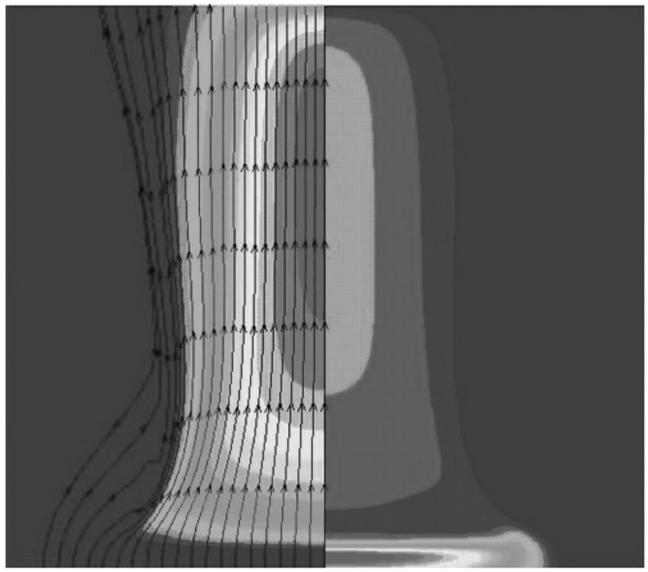

FIGURE 1.4 Cathode layer in a direct current discharge.

interpreted as a physical dimension for shielding the plasma from the effects of an externally applied electric field.

A most illustrative phenomenon of Debye shielding is the plasma sheath. The charge neutrality of plasma does not prevail in the immediate adjacent region to a solid surface, especially over electrodes of an applied electric field, and is the so-called plasma sheath. The Debye length gives the order of magnitude of the thickness of the actual sheath that separates the mass of the neutral plasma from the solid wall. This physical characteristic is easily identifiable in the cathode and anode layers of a glow discharge that is generated by a strong electric field (Figure 1.4). A major contributing factor to this behavior is the recombination mechanisms of chemical kinetics by depleting the charges of ionized gas when striking the surface. The electron absorption and emission properties of the solid surface also play an important role. This phenomenon is closely related to the catalyticity of the surface material.

1.3 PLASMA FREQUENCY

Since plasma has a tendency to be macroscopically neutral, it will always tend to return to its neural equilibrium state after a local perturbation. Unavoidably in this process, the inertia of electrons will overshoot and oscillate around the equilibrium condition with a characteristic rate known as the plasma frequency. The plasma frequency is a measure of the time scale of restoring forces that exerts on a displaced electron returning within Debye length to an ion and keep plasma to an electrically neutral state. In the absence of magnetic field and without externally applied thermal perturbation, the coupled ion is considered in a fixed location in uniform plasma that extends to infinite (Mitchner et al., 1973). The conservation equations of electron motion are

$$\begin{aligned} &\frac{\partial n_e}{\partial t} + \frac{d(n_e u_e)}{dx} = 0 \\ &mn_e \left[\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} \right] = -en_e E \end{aligned} \tag{1.20}$$

The compatible electric field E can be determined by the classic Poisson equation of plasma dynamics

$$\varepsilon \frac{\partial E}{\partial x} = e(n_i - n_e) \tag{1.21}$$

The system of equations can be solved easily with a linearized approach by setting the known initial conditions and considering the displaced electron as a small perturbation,

$$n = n_o + \varepsilon n_1, \quad u_e = u_e + \varepsilon u_1, \quad E = E_o + \varepsilon E_1, \quad \varepsilon \ll 1 \tag{1.22a}$$

The resultant linearized governing equations to O(ε) become

$$\begin{aligned} &\frac{\partial n_1}{\partial t} + u_1 \frac{dn_1}{dx} = 0 \\ &m\frac{\partial u_1}{\partial t} + n_o \frac{\partial u_1}{\partial x} = 0 \\ &\varepsilon \frac{\partial E_1}{\partial x} = -en_1 \end{aligned} \tag{1.22b}$$

The perturbed oscillating variables are naturally definable by simple harmonic functions

$$u_1 = u_1 e^{i(\kappa x - \omega t)}, \qquad n_1 = n_1 e^{i(\kappa x - \omega t)}, \qquad E_1 = E_1 e^{i(\kappa x - \omega t)} \tag{1.22c}$$

Substituting these into the governing equations and eliminating n_1 and E_1 yields

$$-im\omega u_1 = -i\frac{n_o e^2}{\varepsilon \omega} \tag{1.23}$$

From which $\omega^2 = \frac{n_o e^2}{m\varepsilon}$. Thus the plasma frequency is

$$\omega_p = \left[\frac{n_o e^2}{m\varepsilon}\right]^{1/2} \quad \text{rad/s} \tag{1.24}$$

The practical approximate formula of the corresponding circular frequency in SI units is

$$v_p = \frac{\omega_p}{2\pi} \approx 8.97\sqrt{n_e}\ (\text{rad/s}) \tag{1.25}$$

The pertinent parameters for the plasma frequency calculation are the electric permittivity in free space of 8.8542×10^{-12} farad/m, the elementary charge $e = 1.6022 \times 10^{-19}$ coulombs, and the electron mass m of 9.1095×10^{-25} g (Raizer, 1991).

As an example, the electron number density in an electron-impact discharge has a value of around $10^{18}/\text{m}^3$ or $10^{12}/\text{cm}^3$. The plasma frequency yields a value of 8.7×10^9 Hz, which is within the H and X bands of the microwave frequency with the corresponding wavelength range between 2.4 and 4.25 cm. The frequency, however, is lower than the typical average electron collision frequency within a plasma generator of 2×10^{11}/s.

1.4 POISSON EQUATION OF PLASMADYNAMICS

The electrons are in thermodynamic equilibrium and, according to the statistic mechanics, have the Boltzmann distribution, $n_e = ne^{e\varphi(r)/\kappa T}$. The potential energy of an electron at a position r is $-e\varphi(r)$. Similarly, under known equilibrium condition, the ions retain the Boltzmann distribution, $n_i = ne^{-e\varphi(r)/\kappa T}$. The potential energy distribution for plasma is governed in general by the Poisson equation. Therefore, this equation of plasmadynamics is a fundamental formula in describing the collective behavior of plasma in a continuum. In this formulation, the individual charged particles are no longer recognized but to be described by the charge number density ρ_e to have a physical unit of coulomb per cubic meter. The sum of charges in an elementary volume can be given as

$$\rho_e = \overset{\lim}{\Delta v \rightarrow \varepsilon} \frac{\sum_i ze}{\Delta v} = e \sum_i n_i z_i \tag{1.26}$$

where n_i is the number density of the charge species and z_i denotes the number of elementary charges carried by the particles.

The total electric field generated by a group of charges can be evaluated by the principle of superposition. The integrated surface electric field over control surface must be balanced by the total number of charges within the control volume. Thus

$$\iint_i E(r)_{\mathrm{i}} ds = \frac{1}{\varepsilon} \iiint \rho_e dv \tag{1.27}$$

By means of Stokes' divergence theorem, we have

$$\iiint \nabla \cdot E dv = \frac{1}{\varepsilon} \iiint \rho_e dv \tag{1.28}$$

This equality shall be held to an infinitesimal volume; in fact, it is one of the two Gauss laws of electromagnetics.

$$\nabla \cdot E = \frac{1}{\varepsilon} \rho_e \tag{1.29}$$

If the electric field intensity can be derived from the potential function $E = -\nabla\varphi$, which is the most prevailing condition in free space away from any solid surface, then the following equation becomes the well-known Poisson equation of plasma dynamics:

$$\nabla^2 \varphi = -\frac{\rho_e}{\varepsilon} \tag{1.30}$$

This equation can also be easily derived from Gauss' law, $\nabla \cdot D = \rho_e$ for the electric field of the Maxwell equations. The relationship between electric displacement D and electric field intensity E is given by a constitution coefficient ε known as the electric permittivity. The electric field intensity just equals to the negative gradient of electric potential φ.

1.5 ELECTRIC CONDUCTIVITY

The electric and magnetic fields in plasma always produce an electric current that is related to the diffusion velocities of the various charged particle species. Therefore, it is necessary to evaluate the electrical conductivity as microscopic properties together with the thermodynamic state of the plasma. Between collisions, the individual particles move in accordance with the $E \times B$ drift and diffusion velocities that are the driving mechanisms for the charged particle motions. Now the electrical resistivity is the resultant phenomenon of collision process. By introducing a mean or drift motion of the electrons in the gas, the electric current is

$$J = nqu = \frac{nq^2}{m\nu_e}E \tag{1.31}$$

In this equation, m is the mass of the electron and ν_c is the average electron collision frequency that is related to the atomic scale momentum transfer cross section and the random thermal motions of the individual particles. The current density is related empirically to the applied electric and magnetic fields by a bulk electric conductivity $J = \sigma(E + u \times B)$. The proportional constant is known as the electrical conductivity,

$$\sigma = \frac{nq^2}{m\nu_c} \tag{1.32}$$

In fact, it is the reciprocal of the electric resistance that has the SI unit of ohm-meter and is a function of the medium temperature. The electrical conductivity is expressed in reciprocal of ohm-meter or mho. For example, the fused quartz has the conductivity of 10^{-17}, the glass of 10^{-12}, sea water of 4, and silver of 6.1×10^7 mho/m. Note that Ohm's law at a point is $J = \sigma E$ and that J and E have the same direction in an isotropic medium and in the absence of a magnetic field. Under this special circumstance, electrical conductivity is a scalar quantity.

In the presence of a magnetic field, collisions play a similar role in determining the drift velocity of a charged particle. The magnetic field through the Lorentz acceleration generates a gyro motion of the charged particle with a gyro radius proportional to the velocity component perpendicular to and inversely proportional to the magnetic flux density B. The angular velocity of the gyro motion has been introduced, which has earlier been known as the gyro frequency. It may be clear the that helical trajectory of the charged particles has a wide and varying range of orientations in collision; the electrical conductivity is no longer a scalar quantity but is a tensor of rank 2.

The trajectory of the drift motion of a charged particle in an electromagnetic field is fascinating and complex. In the presence of both electric and magnetic fields, the charged particle in any inertial frame will execute a helical motion. The electric force usually is decomposed to components parallel and perpendicular to the magnetic field. The relationship between the transformed (reference) frame moving with the mean velocity of plasma and the laboratory (fixed) frame is $E_{\mathrm{tr}} = E + u \times B$. In the nonrelastivistic approximation, the transformed magnetic flux density is $B_{\mathrm{tr}} = B - u \times E/c^2$. For the case when coordinates system is moving with a velocity perpendicular to both the electric and magnetic field or on a laboratory frame, the helix trajectory will become a prolate cycloid with loops or the curtate cycloid (Jahn, 1968).

The ratio of the gyro frequency to the collision frequency of the charged particle is known as the Hall parameter ω_b/ν_c. This parameter characterizes the charged particle's motion as it responds to the applied E and B fields. The $E \times B$ component of the current is the so-called Hall current that defines the gyro motion. When $\omega_b/\nu_c \gg 1$, the charged particle rarely can execute one cycle of its motion before collision and hence cannot move in the helical trajectory. Thus, the primary component of the electric current is parallel to E. On the opposite limit, $\omega_b/\nu_c \ll 1$, the charged particles complete multiple gyro motions between collisions. As a consequence, the major component of the electric current will be in the direction of $E \times B$. Finally under the condition $\omega_b/\nu_c \sim 1$, the current will have comparable components parallel and perpendicular to E.

The Hall parameter depends on the charge-to-mass ratio and therefore has different value for various charged species. A singly charged ion and an electron have substantially different masses by a ratio of around 1836 to 1. This disparity complicates the formulation for the overall electrical conductivity. In practical applications in the presence of an applied magnetic field, the electrical conductivity has been formulated directly by using the plasma frequency, or by describing the modified electric field in parallel and normal component, equation (1.14) with respect to an applied magnetic flux density B_o. The derivation for the nonscalar electric conductivity is tedious. For purpose of illustration that the electric conductivity is really a tensor, an abbreviated derivation by Jahn is presented for the electric conductivity in a steady magnetic field (Jahn, 1968). Three distinct current components have been identified by the component, which lies along the electric field E and follows the drift $E \times B_o$, and the third component aligns with B_o but in proportional to the component of E parallel to B_o.

If B_o is aligned in the z coordinate of a Cartesian system, the conductivity appears as a tensor of rank 2,

$$\frac{\sigma}{\varepsilon} = \omega_p^2 \begin{bmatrix} b_{xx} & b_{xy} & 0 \\ b_{yx} & b_{yy} & 0 \\ 0 & 0 & b_{zz} \end{bmatrix} \tag{1.33}$$

In this matrix formulation,

$$\begin{aligned} b_{xx} = b_{yy} &\rightarrow \frac{\nu_c}{w_b^2 + \nu_c^2} \\ b_{zz} &\rightarrow \frac{1}{\nu_c} \\ b_{xy} = -b_{yx} &\rightarrow \frac{\omega_b}{\omega_b^2 + \nu_c^2} \end{aligned} \tag{1.34}$$

The gyro frequency is a rotational vector that requires a sign to distinguish the clockwise and counterclockwise rotations. Again, because of the substantial difference in the mass of an electron and an ion, the electrical conductivity tensors are qualitatively different from each other.

In partially ionized plasma, the electric conductivity is complicated by the huge disparity in mass of the electrons and ions. Therefore, a systematic approximation must be applied in the derivation. From the basic characterization of an electromagnetic field, it can be recognized that the B field plays no role in determining the component of the motion of particles parallel to it. Therefore, if one lets the magnetic field to be aligned with the Z coordinate, the particle motion can be decomposed into components parallel and perpendicular to the magnetic field. The effective electron conductivity may be written accordingly as

$$\sigma_{e,\perp} = \frac{\sigma_e}{1 + \beta_e^2}, \qquad \sigma_{e,\parallel} = \frac{\beta_e \sigma_e}{1 + \beta_e^2} \tag{1.35}$$

Introducing the slip factor $s = \beta_e \beta_i = \mu_e \mu_i B^2$ for weakly ionized plasma, the components of the electrical conductivity can be approximated as (Mitchner et al., 1973)

$$\sigma_\perp \approx \frac{\sigma(1+s)}{(1+s)^2 + \beta_e^2}, \qquad \sigma_\parallel \approx \frac{\beta_e \sigma}{(1+s)^2 + \beta_e^2} \tag{1.36}$$

The electric conductivity in Cartesian tensor form appears as

$$\sigma = \begin{bmatrix} \sigma_\perp & -\sigma_\parallel & 0 \\ \sigma_\parallel & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\perp \end{bmatrix} \tag{1.37}$$

Ohm's law subjected to constant driving fields E and B can be constructed by assigning a simplified scalar electrical conductivity as

$$E = \frac{1}{\sigma}[J + \beta_e(J \times B) + sB \times (J \times B)] \tag{1.38}$$

In this equation, the second term is immediately identifiable as the Hall current and the third term is often referred to as the slipping current unique to weakly ionized plasma.

1.6 GENERALIZED OHM'S LAW

A simplified relationship between electric current and electric field intensity that has been widely used in classic magnetohydrodynamics is the generalized Ohm's law. This approximation has the distinct advantage in compact formulation to bypass the complex and detailed description of electrical conductivity. The derivation and approximations of the law are based on the single-fluid, quasi-neutral plasma with single charged ions. First, the mass density, averaged mass velocity, and current density are defined as follows:

$$\begin{aligned} &\rho = Mn_i + mn_e \approx n_e(M = N) \\ &u = \frac{1}{\rho}(n_i M u_i + n_e m u_e) \approx \frac{Mu_i + mu_e}{M + m} \\ &J = e(n_i u_i + n_e u_e) \approx n_e(u_i - u_e) \end{aligned} \tag{1.39}$$

The equation of charged particles' motion in an applied steady electric and magnetic fields can be written as equation (1.39). Additional forces such as the gravitational or any nonelectromagnetic body force exerting on the plasma can also be included. But, first, the convective terms $(u \cdot \nabla)u$ or the divergence of the dyadic tensor $\nabla \cdot (\rho uu)$ for both electrons and ions is neglected, because the velocities of the mass-averaged organized motion for electrons and ions are assumed to be small. The shear stress is neglected for simplicity and the error is limited as long as the Larmor radius is much smaller than the other characteristic length scales of charged particles motion, the equations of charged particles motion can given as

$$\begin{aligned} Mn\frac{\partial u_i}{\partial t} &= en(E + u_i \times B) - \nabla p_i + Mng + P_{ie} \\ mn\frac{\partial u_e}{\partial t} &= -en(E + u_e \times B) - \nabla p_e + mng + P_{ei} \end{aligned} \tag{1.40}$$

The two equations describe the motion of the ions and electron, respectively. In equation (1.40), symbols M and m denote unit mass of ion and electron, respectively, and n is the number density of charged species. The sum of the above equations is

$$n\partial(Mu_i + mu_e)\big/\partial t = en(u_i - u_e) \times B - \nabla P + n(M + n)g \tag{1.41a}$$

In this equation, p_e and p_i are the partial pressures of the electrons and ions, respectively. The total momentum transferred to ions or electrons by the collision process are P_{ie} and P_{ei}, and they follow from Newton's third law, $P_{ei} = -P_{ie}$. The equation of motion of the single fluid in an electromagnetic field is then

$$\rho\partial u/\partial t = J \times B - \nabla p + \rho g. \tag{1.41b}$$

Multiply the equation of motion for ions by m, the equation of motion for electrons by M, and subtract the latter from the former to obtain

$$Mmn\frac{\partial(u_i - u_e)}{\partial t} = en(M + n)E + en(mu_i + Mu_e) \times B - m\nabla p_i + M\nabla p_e - (M + m)P_{ei} \tag{1.42}$$

By invoking the definitions of number density, mass-averaged velocity, and current density,

$P_{ei} = n^2e^2(u_i - u_e)/\sigma$, the above equation becomes

$$\frac{Mmn}{e}\frac{\partial}{\partial t}\left(\frac{J}{n}\right) = e\rho E - (M+m)ne\left(\frac{J}{\sigma}\right) - m\nabla p_i + M\nabla p_e + en(mu_i + Mu_e)\times B \tag{1.43a}$$

The last term can be rearranged as

$$\begin{aligned} mu_i + Mu_e &= Mu_i + mu_e + M(u_e - u_i) + m(u_i - u_e) \\ &= \frac{\rho}{n}u - (M-m)\frac{J}{en} \end{aligned} \tag{1.43b}$$

Dividing equation (1.43) by ρe yields

$$E + u\times B - J/\sigma = \frac{1}{\rho e}\left[\frac{Mmn}{e}\frac{\partial}{\partial t}\left(\frac{J}{n}\right) + (M-m)J\times B + m\nabla p_i - M\nabla p_e\right] \tag{1.44}$$

Further simplifications are possible by considering the temporal variation of the organized motion to be negligible and the relative lower mass of electrons to ions and neutrals. The resultant approximation is known as the generalized Ohm's Law, which describes the electric properties in an electrically conducting medium,

$$\sigma(E + u\times B) = J + \frac{1}{en}(J\times B - \nabla p_e) \tag{1.45}$$

In the classic formulation of the magnetohydrodynamics governing equation, the following expression is frequently adopted in magnetohydrodynamic formulation:

$$J = \sigma(E + u\times B) \tag{1.46a}$$

The physical interpretation of this equation is that the current density is driven by an effective electric field that is composed of the applied field E and an induced field $u\times B$.

The generalized Ohm's Law can be modified to include the Hall current as (Mitchner and Kruger, 1973)

$$J = \sigma[E + u\times B - \beta(J\times B)] \tag{1.46b}$$

where σ is the scalar electric conductivity, and hence the Hall current and frequency have been defined previously as $J\times B$ and $\beta = 1/en = B/\omega_c n$. Again the cyclotron or Larmor frequency is given as $\omega_c = eB/m$.

1.7 MAXWELL'S EQUATIONS

Maxwell's equations form the basic laws governing all phenomena of an electromagnetic field in free space and in media. These fundamental equations were established by James Maxwell in 1873 and verified experimentally by Heinrich Hertz in 1888. In 1905, Albert Einstein's special theory of relativity has further affirmed the rigorousness of the Maxwell's theory (Kong, 1986). This set of equations has been given in integral form, but for the present purpose, the differential form is presented as

$$\frac{\partial B}{\partial t} + \nabla \times E = 0 \tag{1.47a}$$

$$\frac{\partial D}{\partial t} - \nabla \times H + J = 0 \tag{1.47b}$$

$$\nabla \cdot B = 0 \tag{1.47c}$$

$$\nabla \cdot D = \rho_e \tag{1.47d}$$

The first equation is Farady's law, which relates the time rate of change of the magnetic flux density B (weber/m^2) to the curl of the electric field intensity E (volt/m). The second equation is the generalized Ampere's circuit law relating the time rate of the electric displacement D (coulomb/m^2) to the curl of the magnetic field strength H (ampere/m) and electric current density J (ampere/m^2). The last two equations are Gauss' laws for magnetic and electric fields, respectively. The first law simply requires that the divergence of B must vanish in any electromagnetic field to preclude the existent of a magnetic dipole. The second law defines the direct relationship between the divergence of the electric displacement D and the electric charge density ρ_e(coulomb/m^3).

From a pure mathematical point of view, Maxwell's equations have a closure issue, namely, the number of dependent variables is unbalanced by the number of unknowns. Meanwhile, for problem solving, the material media needs more characterization. This dilemma is removed by the constitutive relations. In an isotropic media, the required properties are

$$\begin{aligned} D &= \varepsilon E \\ B &= \mu H \end{aligned} \tag{1.48}$$

where ε denotes the electric permittivity in free space that it has a value of $4\pi \times 10^{-7}$ henry/m, and the symbol μ is the magnetic permeability in free space that has a value of approximately 8.85×10^{-12} farad/m.

It is learned that the two Gauss' laws, equations (1.14c) and (1.47d), are not independent equations. A conservation law for electric charge and current density has been regarded as a fundamental equation. This equation can be derived from the Faraday's and generalized Ampere's laws by taking the divergence of the latter and introducing the Gauss' divergence law for electric displacement, yielding

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot J = 0 \tag{1.49}$$

The charge conservation equation above has been demonstrated to be invaluable for verifying the global behavior for computational electro-aerodynamic simulations.

In summary, Maxwell's equations constitute a hyperbolic partial differential equation system and thus must be supplemented with boundary conditions and initial values wherever derivatives do not exist such as at the media interface. It is noted that in a single medium the electromagnetic field is continuous. However, at the boundary between two different media, the field may change abruptly in both magnitude and direction. It is convenient to describe the boundary condition by the tangential and normal components on the media interface. On a stationary boundary interface, the boundary conditions are

$$\begin{aligned} n \times (E_1 - E_2) &= 0 \\ n \cdot (B_1 - B_2) &= 0 \\ n \cdot (D_1 - D_2) &= \rho_\mathrm{s} \\ n \times (H_1 - H_2) &= J_\mathrm{s} \end{aligned} \tag{1.50a}$$

The first boundary condition indicates that the tangential electric field E is continuous across the interface boundary. Conversely, the normal component of the magnetic flux density B is also permitted to pass uninterrupted through the media interface. According to the third equation, the difference in normal component of the electric displacement D must be balanced by the surface charge density. It is important to note that the surface charge is the actual electric charge separated by a finite distance between charged particles but not the surface charge due to polarization. Finally, the discontinuity of the tangential components of the magnetic field strength H is equal to the surface current density J_s.

On a moving boundary, because the partial temporal derivatives are no longer commuted with the surface integral, the boundary conditions are required to formulate by the integral form of the Maxwell equations. Additional components via the normal velocity are required to append on the tangential formulation of the electric and magnetic field strength (Kong, 1986).

$$\begin{aligned} &n \times (E_1 - E_2) - (n \cdot u_b)(B_1 - B_2) = 0 \\ &n \cdot (B_1 - B_2) = 0 \\ &n \cdot (D_1 - D_2) = \rho_s \\ &n \times (H_1 - H_2) + (n \cdot u_b)(D_1 - D_2) = J_\mathrm{s} \end{aligned} \tag{1.50b}$$

1.8 WAVES IN PLASMA

An ionized gas can generate a wide variety of oscillatory motions. Three types of waves in plasma have been classified as electromagnetic, hydromagnetic, and electrostatic (Spitzer, 1956; Celeroix, 1960). However, it is most unlikely that all of these idealized waves can exist in their pure form. The electromagnetic waves in the simplest cases are only transverse waves that are associated with the transverse motion of lines of magnetic induction. The tension in the lines of force tends to restore deformation back to the original line. These waves are exemplified by electromagnetic waves propagating in vacuum (Friedrichs et al., 1958). The phase velocity is on the order of the speed of light. Hydromagnetic or magnetohydrodynamic waves are similar to acoustic waves that are low-amplitude longitudinal, and compression waves motions are induced by the electromagnetic coupling of the electrons and ions. The wave propagation speed is governed by the speeds of sound. Finally, electrostatic waves are also a longitudinal wave and propagate because of thermal agitation (Landau et al., 1960).

Maxwell's equations govern the propagation of an electromagnetic wave. In order to describe wave motion for the purpose of better understanding, the plasma is considered as an unbound isotropic and stationary medium. The wave equation in plasma is derived by taking the temporal derivative of the generalized Ampere circuit law to get

$$\frac{\partial^2(\varepsilon E)}{\partial t^2} = \frac{1}{\mu}\nabla\times\frac{\partial B}{\partial t} + \frac{\partial J}{\partial t} \tag{1.51a}$$

then replaces the time derivative of the magnetic field strength H by Faraday's induction law

$$\frac{\partial^2(\varepsilon E)}{\partial t^2} = -\frac{1}{\mu}\nabla\times(\nabla\times E) + \frac{\partial J}{\partial t} \tag{1.51b}$$

and finally invokes a vector identity and substitutes into equation (1.51a).

$$\nabla\times(\nabla\times E) \equiv \nabla(\nabla\cdot E) - \nabla^2 E \tag{1.51c}$$

For an isotropic medium $D = \varepsilon E$ and $B = \mu H$ the basic electromagnetic wave equation appears as

$$\varepsilon\mu\frac{\partial^2 E}{\partial t^2} - \mu\frac{\partial J}{\partial t} = \nabla^2 E - \nabla(\nabla\cdot E) \tag{1.51d}$$

The electric permittivity and magnetic permeability are designated as ε and μ, respectively. The phase velocity is $(\varepsilon\mu)^{-1/2}$ and is the speed of light, which is $c = 2.998 \times 10^8$ m/s (De Hoffman et al., 1950). In discussing the wave motion, the orientation of the particle motion is relevant. From an earlier derivation for the generalized Ohm's law, the equation of charged particle motion has been described previously as equation (1.41) and is used as the reference for discussion

$$\rho\partial u/\partial t = J\times B - \nabla p + \rho g \tag{1.52}$$

If the electric intensity E and the current density J are parallel to the direction of wave propagation, only the electrostatic restoring force is presented. The longitudinal wave is the electrostatic wave. In this wave, the frequency is very high and the ions are unaffected by the electric field and the electrons retain the Boltzmann distribution.

The electric charge in the electrostatic wave is due to the divergence of the current density. There are two types of oscillations, namely the electron and ion oscillations. How the electron oscillations are excited in plasma is not clear, but these waves are damped as the wavelength approaches the Debye shielding length. The positively charged ion oscillations are waves of relatively low frequency. Electrical neutrality is preserved and the electrostatic wave usually behaves as an acoustic wave.

The hydromagnetic wave appears only in the presence of a magnetic field and at frequencies lower than the cyclotron frequency of the ions. The driving mechanism is the Lorentz acceleration $J \times B$. The oscillation is conserved to be waves in the line of force. The basic form is known as the Alfven (1950) wave, which is a purely magnetohydrodynamic phenomenon depending on the magnetic field and inertia of the medium. The wave velocity and the induced magnetic flux density are parallel to each other, while the mean magnetic field is normal to them. In the Alfven wave, the divergence of the medium velocity vanishes and the pressure gradient term becomes zero. For these reasons, the pressure and density do not vary in such transverse waves. Therefore, transverse wave motion is possible in an incompressible medium.

The phase velocity of these waves is the Alfven speed that is based on the component of the applied magnetic field in the direction of the wave propagation. From this view point, the Alfven wave is the vibration of a line of force. Its phase velocity is

$$u_p = B_o \big/ \sqrt{\rho\mu} \tag{1.53a}$$

The normal stress component associated with lines of force is the well-known Maxwell tensile stress *BB*.

When the electric field intensity is perpendicular to the wave front, it will lead to an electromagnetic wave. The electrons will interfere with these transverse waves and increase the wave speed. When the electron number density exceeds a critical value with increasing plasma frequency, the electromagnetic wave cannot propagate through the plasma in the presence of an electromagnetic field. Another characteristic of wave motion in plasma is that the waves can propagate by different phase speeds. Since transverse waves can exist in both a compressible and incompressible medium, the electromagnetic waves in compressible medium have been referred to as magnetoacoustic waves (Sutton et al., 1965). Two phase velocities u_{f} and u_{s}, corresponding to the fast and slow waves, are

$$u_{\mathrm{f,s}} = \frac{1}{2}\left\{\sqrt{c^2 + 2c[B\big/(\rho\mu)^{1/2}] + B^2\big/(\rho\mu)} \pm \sqrt{c^2 - 2c[B\big/(\rho\mu)^{1/2}] + B^2\big/(\rho\mu)}\right\} \tag{1.53b}$$

The phase velocities are dependent on different angles between the applied magnetic field and the direction of wave propagation. The velocity of the fast wave is always greater than the Alfven wave speed, and that of the slow wave is always less than it. The phase velocities of the electromagnetic waves degenerate to the speed of sound when the transverse magnetic field vanishes.

1.9 ELECTROMAGNETIC WAVES PROPAGATION

Both longitudinal compression and transverse waves can coexist in plasma. The transverse wave is associated with the magnetic field, and the mechanism of this wave is often interpreted as the tension of the magnetic lines of force that restores any disturbance to a straight line, thereby leading to a transverse oscillation. In this sense, the transverse wave always propagates in the perpendicular direction to a magnetic field and will cease when the magnetic field vanishes.

One of the most dramatic manifestations of plasma is the properties of an electromagnetic wave propagation. The intensity of an incident wave always attenuates in a weakly ionized gas, because it is a semiconductive medium or lossy material. The complete reflection of the electromagnetic wave at the interface of media also can occur when the frequency of the incident wave is lower than the plasma frequency and becomes as the cutoff frequency. Another unique feature of wave motion in plasma is that it is not solely depended on the collision mechanism as in acoustic wave. Multiple modes of the transverse wave also exist such as the Alfven wave associated with ions and the Whistler wave related to electrons. Both transverse waves are generated by an initial disturbance perpendicular to magnetic field. For this reason, these transverse waves are often regarded as the vibration of a line of magnetic force. All the direction-dependent disturbance propagations lead to a very complex wave system and introduce fascinating physics in resonance and energy damping. A systematic discussion of the plasma wave dynamics will be deferred until the governing equations of the classic magnetohydrodynamics are presented later.

The equation of electromagnetic wave propagation is

$$\varepsilon\mu\frac{\partial^2 E}{\partial t^2} - \mu\frac{\partial J}{\partial t} = \nabla^2 E - \nabla(\nabla\cdot E) \tag{1.54a}$$

The general solutions of a transverse plane wave are then

$$\begin{aligned} E &= E_o e^{i(\omega t \pm kr)} \\ J &= J_o e^{i(\omega t \pm kr)} \end{aligned} \tag{1.54b}$$

where $\kappa = (\varepsilon\mu\omega^2 - i\mu\sigma\omega)^{1/2}$ is a complex wave number where the electric conductivity was defined previously as $\sigma = n_e e^2 / m_e v_c$, and v_c characterizes the averaged electron collision frequency. In the absence of an external applied magnetic field, the magnitude of the induced magnetic flux density is on the order of $|E|/c$. If the effect of the magnetic flux density B is neglected in the generalized Ohm's law, the temporal behavior of the electric current density can be approximated as

$$\frac{1}{v_c}\frac{\partial J}{\partial t} \approx \sigma E - J \tag{1.55}$$

By satisfying the plasma wave equation and the above approximation to establish the relation between the electric field strength and current density, a dispersive relationship for a transverse plane wave was obtained by Mitchner and Kruger (1973),

$$\left(\frac{c\kappa}{\omega}\right)^2 = \left[1 + \frac{(\varepsilon - \varepsilon_o)}{\varepsilon} - \frac{(\omega_p/\omega)^2}{1 + (\nu_c/\omega)^2}\right] + i\left[\frac{(\omega_p/\omega)^2(\nu_c/\omega)}{1 + (\nu_c/\omega)^2}\right] \quad (1.56)$$

In this equation, $\omega_p = (n_e e^2/\varepsilon m_e)^{1/2}$ is the plasma frequency. The small difference between the electric permittivity in free space and plasma is ignored. The complex wave number can be expressed as $\kappa = \alpha + i\beta$ and then the electric field intensity becomes

$$E = E_o e^{-\alpha x} e^{i(\beta x - \omega t)} \quad (1.57a)$$

The real number α is the attenuation constant, and β is the phase constant and related to the wavelength $\beta = 2\pi/\lambda$ and phase velocity of the wave $u_{\text{ph}} = \omega/\beta$. It has also been pointed out by Mitchner et al. that $(\omega_p/\omega)^2/[1+(\nu_c/\omega)^2] \gg R$, where $R = (\varepsilon - \varepsilon_o)/\varepsilon_o$ is the so-called refractivity of the medium. In the limiting condition $\omega \gg \nu_c$, collisions are negligible and the dispersion relation becomes

$$\left(\frac{c\kappa}{\omega}\right)^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \quad (1.57b)$$

If $\omega > \omega_p$, κ is a real number and the wave propagates through the plasma without attenuation. However, if $\omega < \omega_p$ κ becomes an imaginary number and the solution $E = E_o \exp -(\alpha x + i\omega t)$ can no longer describe a propagating wave. It is referred to as an evanescent wave and on average it does not transport energy and will be reflected from the media interface. From the simplified analysis, for electromagnetic waves with frequencies lower than the plasma frequency, the incident wave will be reflected at the interface of plasma as exhibit in Figures 1.5 and 1.6.

The attenuation of an incident electromagnetic wave in plasma is depicted by computational results at a frequency lower than the plasma frequency $\omega_p = (n_e e^2/\varepsilon m_e)^{1/2}$. The numerical results describe the rapid dissipation of wave amplitude in terms of the penetration distance versus the ratio of frequencies in decibels and the phase shift.

In fact, the dissipation is a manifestation of the Debye shielding of charges within plasma. While collisions tend to modify the attenuation of the electromagnetic wave, but the general behavior of the cutoff frequency is still similar. By meaning of a critical electron number density, the wave propagation pattern is established.

$$n_{ec} = \frac{\varepsilon m_e \omega^2}{e^2} \quad (1.58a)$$

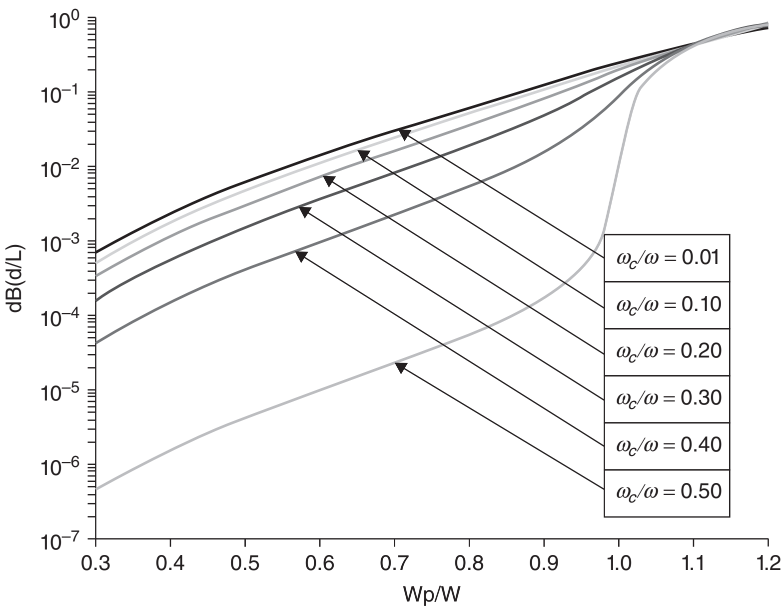

FIGURE 1.5 Incident wave attenuation.

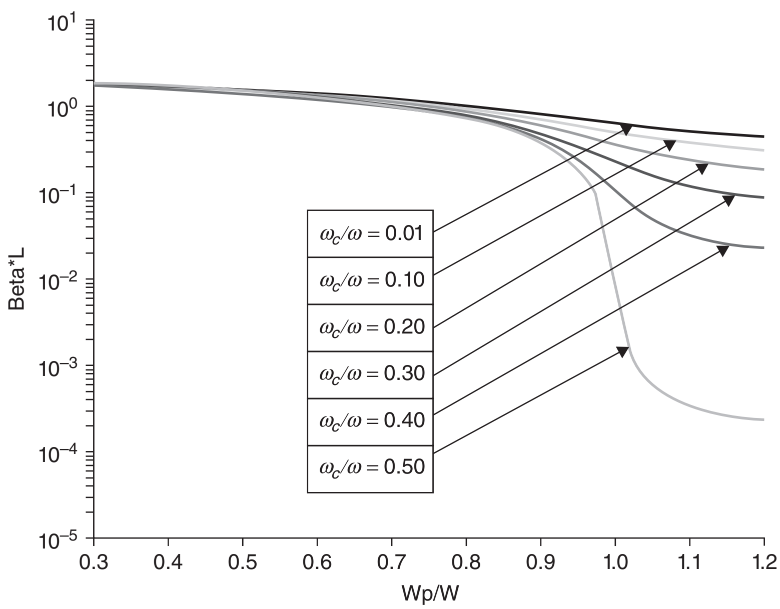

FIGURE 1.6 Incident wave phase relationship.

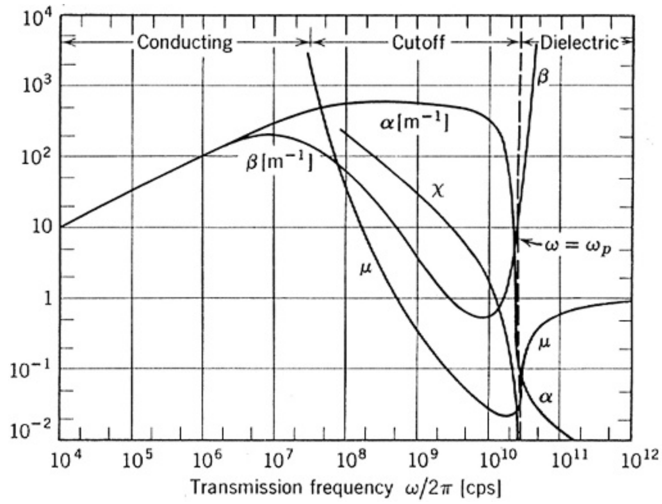

FIGURE 1.7 Amplitude-frequency relation of electromagnetic wave. From Head, M., and Wharton, C. *Plasma Diagnostics with Microwaves*. Used with permission from John Wiley & Sons, 1965.

If $n_e < n_{ec}$ $(\omega > \omega_p)$, the electromagnetic waves can propagate through a collision-free plasma, but the wave amplitude will be attenuated by collisions. If $n_e > n_{ec}$ $(\omega < \omega_p)$, the electromagnetic waves now may still be able to propagate through the plasma but are strongly attenuated. The $1/e$ depth of penetration is inversely proportional to the square root of product of the frequency of incident wave, the electric permeability and conductivity of the medium $1\big/\sqrt{\pi\omega\mu\sigma}$. Both the wavelength and the wave attenuation rate are implicit functions of the free electron density and the collision frequency. These properties determine the ability of an electromagnetic wave to penetrate the plasma by describing the phase change and reflection coefficients at the interface of the medium. Once the real and imaginary wave numbers are known, the electron number density and average plasma collision frequency can be found.

As an illustration of the peculiar behavior of electromagnetic wave attenuation in plasma is the gist of plasma diagnostics and the communication blackout in the earth reentry. The propagation constants as a function of frequency in typical laboratory hydrogen plasma are given by Heald and Wharton (Heald et al., 1965) as shown in Figure 1.7. The data is collected at an electron number density of $10^{13}/\text{cm}^3$. Three frequency domains are labeled on the top of the graph. In the intermediate frequency spectrum,$\nu < \omega < \omega_p$, an electromagnetic wave will not propagate through the plasma that is similar to that of the cutoff frequency of waveguides. When an incident microwave has a frequency lower the cutoff frequency, the wave will not able to propagate through the electrically conducting medium.

The refractive index and attenuation are related to the complex propagation constants via the attenuation and phase coefficients,

$$\alpha = \chi\omega/c \quad \text{and} \quad \beta = \mu\omega/c \tag{1.58b}$$

In the absence of a magnetic field, these indexes can be obtained by assuming the Lorentz conductivity that is accurate when the collision frequency is independent of electron velocity as the limiting condition.

The most interesting electromagnetic wave propagation phenomenon occurs in the intermediate frequency spectrum where the frequency of the incident wave lies in between the collision frequency of momentum transfer and the plasma frequencies, $\nu < \omega < \omega_p$. The real refractive index and the attenuation indexes of the incident wave can be approximated as

$$\mu \approx (\nu\omega_p \big/ 2\omega^2)(1 - 5\nu^2 \big/ 8\omega^2 + \omega^2 \big/ 2\omega_p^2)\ \chi \approx (\omega_p \big/ \omega)(1 - 3\nu^2 \big/ 8\omega^2 - \omega^2 \big/ 2\omega_p^2) \tag{1.58c}$$

And the plasma penetration depth can be given as

$$\delta \approx (c \big/ \omega_p)(1 + 3\nu \big/ 8\omega^2 + \omega^2 \big/ 2\omega_p^2) \tag{1.58d}$$

From the above approximations, the penetration depth of the evanescent wave is practically constant in the interior of the frequency region. The physical dimension of the depth is comparable to the wavelength of the plasma frequency to have a value of a few millimeters.

At the high-frequency domain, the incident electromagnetic wave has a greater frequency than the plasma frequency, $\omega > \omega_p$. Under this circumstance, the plasma behaves as a dielectric in which the electromagnetic wave propagates through with a very low attenuation similar to that of a relatively low loss medium.

1.10 JOULE HEATING

The Joule heating is also known as Ohmic and resistive heating. It is the process by which the passage of an electric current through a medium releases heat. In other words, the electromagnetic field does work to overcome the resistive force that equals to the dissipated heat in response to resistance. The amount of energy released is proportional to the square of the current. This relationship is known as Joule's first law and the IS unit of the energy was subsequently named after Joule by the symbol J (J = 1 Newton-meter, J = 10^7 erg).

Joule heating is the consequence of the interaction of a charged particle in an electric circuit accelerated by an electric field but must give up some of their kinetic energy every time they collide with ions of the conductor. The increase in kinetic or vibrational energy of the ions manifests itself as heat and an increase in the temperature of the conductor.

The most general and fundamental formula for Joule heating is

$$q = E \cdot J = \frac{I^2}{\sigma} = \sigma E^2 \tag{1.59a}$$

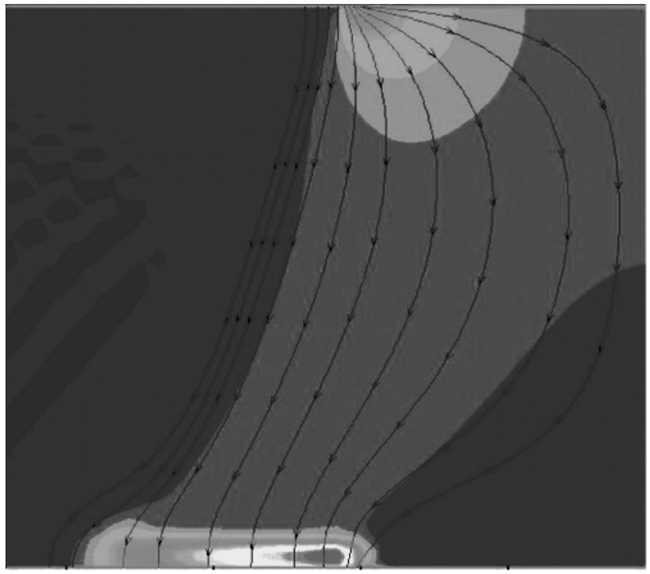

FIGURE 1.8 Concentrated regions of Joule heating in a direct current discharge.

The electric current in a circuit is the volumetric integral of the electric current density

$$I = \iiint J dv \tag{1.59b}$$

The total energy dissipated in the conductor in time is then $W = \int E \cdot I dt$ and known as Joule's law. An identical formula can also be used for AC power, except that the parameters are averaged over one or more cycles. In these cases of a phase difference between I and E, a multiplier $\cos\phi$ must be included.

The composite graph by the computational electric current density contours together with the electric field vector traces is given in Figure 1.8, and from these data the Joule heating can be evaluated. For the computational simulation of a direct current discharge, the anode is placed at the upper border of the figure and cathode is located in the opposite lower boundary. The gap distance between electrodes is 5 cm. As shown from the figure, the electric field intensifies at the sharp edge of the anode. The electron number density has a maximum value of $1.7 \times 10^{10}/\text{cm}^3$ locally. This value is 3.4 times greater than the infinite parallel electrode counterpart. At the connecting edge of the anode, the anode layer is completely distorted by the stronger local electric field. From the contour presentation, the cathode layer can be clearly discerned even through the discharge structure over the cathode and is no longer uniform. The current density is clustered locally. Thus, the Joule heating is concentrated along the outer edge of cathode closest to the anode, at an applied direct current electric field of 840 volt and the calculated total current of 4.88 mA. In the graphic example, the Joule heating of the glow discharge at the pressure of 5 Torr is 4.10 J (Shang et al., 2009; Shang et al., 2012).

1.11 TRANSPORT PROPERTIES

Diffusion is one of the major complications in the analysis of plasma dynamics. The diffusive mass flux is generated by three driving mechanisms due to the gradients of species concentration, temperature, and external forces exerted on the different species. From the kinetic theory of dilute gases, the description of diffusion does not extend to the presence of internal freedoms in molecules. In an electromagnetic field, additional random motions by the electric statistical force and by Lorentz acceleration are recognized as new mechanisms for the forced diffusion. In a diffusing mixture, the velocities of individual species can be significantly different from each. However, the peculiarity of the Debye length of plasma creates a unique constraint known as ambipolar diffusion (Howatson, 1975). The species diffusion velocity u_i is defined by a relative velocity component with respect to a local mass-averaged velocity of the gas mixture,

$$u = \frac{\sum_i \rho_i u_i}{\sum_i \rho_i}. \tag{1.60}$$

Here ρ_i is the density of species i. The diffusion velocity of species i is defined by the difference of the mass-averaged value and the species diffusion velocity on a stationary frame of reference. By definition, the sum of mass diffusion of all species is identical to zero. $\Sigma \rho_i u_i = 0$.

A landmark of the kinetic theory of a diluted gas mixture is its ability to describe the transport properties of any gaseous mixture. To be consistent with the kinetic model of the internal structure of a gas, the transport properties of the gas mixture for thermal diffusion, viscosity, and thermal conductivity shall be calculated from Boltzmann equation by using the Chapman–Enskog expansion (Chapman et al., 1950). The thermal diffusivity is

$$D_{i,j} = 1.858 \times 10^{-3} \sqrt{T^3 \frac{M_i + M_j}{M_i M_j}} / \sigma_{i,j}^2 \Omega^{(1,1)} \tag{1.61}$$

The molecular viscosity is given as

$$\mu = 2.67 \times 10^{-5} \sqrt{M_i T} / \sigma_i^2 \Omega^{(2,2)} \tag{1.62}$$

and the thermal conductivities are

$$\kappa_{i,m} = 1.989 \times 10^{-4} \sqrt{\frac{T}{M_i}} / \sigma_i^2 \Omega^{(2,2)} \tag{1.63}$$

$$\kappa_{i,p} = 2.519 \times 10^{-4} \sqrt{\frac{T}{M_i}} / \sigma_i^2 \Omega^{(2,2)} \tag{1.64}$$

for a monatomic and a polyatomic gas, respectively (Bird et al., 1960; Hirshfelder et al., 1954).

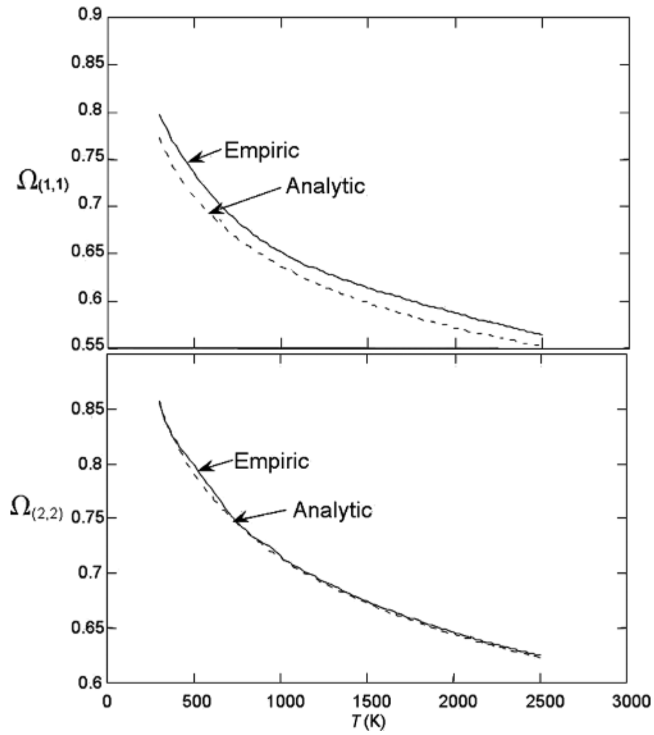

FIGURE 1.9 Comparison of collision integrals for transport properties.

All collision integrals and cross sections can be obtained from either the Lenard–Jones potential for nonpolar molecules or a polarizability model for ion-neutral nonresonant collisions by the more recent works by Capitelli et al. (2000) and Levin and Wright (2004). For the purpose of understanding, a very wide range of temperatures from 200 to 28,000 K by the 11 species of ionized air is calculated. According to the study of Capitelli et al., the calculated collision integrals are not dramatically dependent on the specific form of the potential models. The collision integral is a dimensionless parameter that is generated by a triple integral process whose value depends on the dynamics of binary collisions, and thus is controlled by the intermolecular force law.

The comparison of collision integrals by analytic results and an existent empirical database is presented in Figure 1.9. The graph depicts a comparative study of the integral $\Omega^{(1,1)}$ from two different approaches for molecular hydrogen over a temperature range up to 2500 K. The analytic result using the Lenard–Jones potential is consistently 3.1% lower than the polynomial fitted data; therefore, the diffusion coefficient of hydrogen is anticipated to be proportionally greater than data at an identical thermodynamic state. On the contrary, the reduced collision integral $\Omega^{(2,2)}$ has the nearly identical value to that of an empirical database. The depicted comparison, in a sense, is a study of the state-of-the-art status in transport properties calculations.

The agreement between analytic results and empirical database reflects an impressive accumulation of knowledge for transport properties, but the database does not include ionized gases. The analytic approach for transport properties evaluation must be used for electromagnetic–aerodynamics interaction.

The variation of these products of collision integral and cross section is less pronounced in the high-temperature region to reflect a linear dependence to the local temperature and molecular weight of the species. Although there is substantial progress in the kinetic theory of gases for calculating transport properties, the specific comparison of the calculated transport properties still yields a difference of 16.3%–24.2% in viscosity and thermal conductivity for high-temperature air using different sets of collision integrals in the temperature range from 300 to 30,000 K.

The Chapman–Enskog theory gives the viscosity and thermal conductivity for a gas mixture. These formulas are also known as Wilke's mixing rule

$$\begin{aligned} \mu_{i,j} &= \sum \frac{x_i \mu_i}{\sum x_i \phi_{i,j}} \\ \phi_{i,j} &= \frac{1}{\sqrt{8}} \left(1 + \frac{M_i}{M_j}\right)^{-1/2} \left[1 + \left(\frac{\mu_i}{\mu_j}\right)^{1/2} \left(\frac{M_j}{M_i}\right)^{1/4}\right]^2 \end{aligned} \tag{1.65}$$

and

$$\begin{aligned} \kappa_{i,j} &= \sum \frac{x_i \kappa_i}{\sum x_i \phi_i} \\ \phi_{i,j} &= \left[1 + \left(\frac{\kappa_i}{\kappa_j}\right) \left(\frac{M_j}{M_i}\right)^{1/4}\right]^2 \Big/ \left[8 \left(1 + \frac{M_i}{M_j}\right)\right]^{1/2} \end{aligned} \tag{1.66}$$

In these equations, the symbol x_i denotes the molar fraction of the components for the gas mixture. The transport properties of the present analysis are derived from kinetic theory via the collision integrals and collision cross sections.

For the purpose of understanding, a very wide range of temperatures from 200 to 28,000 K by the 11 species of air mixture is calculated. According to the study of Capitelli et al. (2000), the calculated collision integrals are not dramatically dependent on the specific form of the potential models. The collision integral is a dimensionless parameter that is generated by a triple integral process whose value depends on the dynamics of binary collisions and thus is controlled by the intermolecular force law.

The variation of these products of collision integral and cross-section is less pronounced in the high-temperature region to reflect a linear dependence to the local temperature and molecular weight of the species. Although there is substantial progress in the kinetic theory of gases for calculating transport properties, the specific comparison of the calculated transport properties still yields a difference of 16.3%–24.2% in viscosity and thermal conductivity for high-temperature air using different sets of collision integrals in the temperature range from 300 to 30,000 K.

1.12 AMBIPOLAR DIFFUSION

In an ionized gas, the random or thermal motion of a charged particle is the result of mutual collisions plus forces acting on the particle by an electromagnetic field. The acceleration of an electron has a magnitude of eE/m in the negative direction of E. After numerous collisions, the average kinetic energy reaches a constant value and the force diffusion has directional average velocity. This motion is the well-known drift velocity. A similar drift velocity also is attained by the positively charged ion but in the positive E direction, and, due to the greater mass of the ions, its drift velocity is much lower than that of the electron.

The averaged energy gain between collisions is dependent on the ratio of E/p. The proportionality constant between the drift velocity and the electric field $u_e = -\mu_e E$ is called mobility,

$$\mu_e = \frac{u_e}{E} = \frac{e}{m_e \nu} \tag{1.67a}$$

In this equation, ν is the averaged collision frequency for momentum transfer, and the mobility of the drift velocity of ion can also be given as

$$\mu_i = e/m_i \nu \tag{1.67b}$$

The self-diffusion or ordinary diffusion is proportional to the concentration gradient of the number particle density ∇n. The diffusion coefficient determined by the elementary kinetic theory of gases is proportional to the mean random velocity and the mean-free path between collisions. The rate of flow particles per unit area is

$$\Gamma = -D\nabla n; \quad D \sim \lambda u \tag{1.68}$$

From the viewpoint of the conservation of number particle density, the mass flux density is the Fick's second law of diffusion. This relationship is valid for both electrons and ions. Since the mean-free path of electrons is greater than that of ions, the electron's random velocity is much greater; it follows that $D_e > D_i$.

The diffusion of charged particles is related to mobility; both arise from the random motion and unbalanced collision force. In one-dimensional motion of an ion,

$$u_i = \frac{\Gamma_i}{n_i} = \frac{1}{n_i} D_i \frac{\partial n_i}{\partial x} = \frac{1}{p_i} D_i \frac{\partial p_i}{\partial x} \tag{1.69}$$

The gradient of the partial pressure $\partial p_i/\partial x$ must be balanced by the total force acting on the ion. For the present consideration, the force is exerted by the electric field, thus $eEn_i = \partial p_i/\partial x$. By the definition of mobility, we have

$$\frac{u_i}{\mu_i} = E = \frac{1}{en_i} \frac{\partial p_i}{\partial x} = \frac{1}{en_i} \frac{u_i p_i}{D_i} \tag{1.70}$$

or

$$\frac{D_i}{\mu_i} = \frac{p_i}{en_i} = \frac{\kappa T}{e} \tag{1.71}$$

This relationship between mobility and diffusion is known as the Einstein relation (Surzhikov et al., 2004).

The often encountered charge separation is the result of the disparity in the diffusion of electrons and ions. The electromagnetic field augments the drift velocity of the ions and retards the electrons. When this process reaches a state of local equilibrium for the drift velocity, this resultant process is called ambipolar diffusion.

Again, by a simple one-dimensional analysis, the limiting and reasonable estimated value of ambipolar diffusion coefficient can be found. Since

$$-D_e\frac{\partial n_e}{\partial x} - n_e\mu_e E = -D_i\frac{\partial n_i}{\partial x} + n_i\mu_i E \tag{1.72a}$$

and $n_i = n_e$ for a globally neural plasma.

The ambipolar diffusion coefficient can be written as

$$D_a\frac{\partial n}{\partial x} = -D_e\frac{\partial n}{\partial x} - n\mu_e E = -D_i\frac{\partial n}{\partial x} + n\mu_i E \tag{1.72b}$$

Eliminating E to yield

$$D_a = \frac{D_e\mu_i + D_i\mu_e}{\mu_e + \mu_i} \tag{1.72c}$$

In general, $\mu_e \gg \mu_i$, and the ambipolar diffusion coefficient can be approximated as

$$D_a = \frac{D_e\mu_i + D_i\mu_e}{\mu_e} \tag{1.73}$$

There are two conditions commonly encountered in practical applications. First, at the same temperature for electron and ion by thermal ionization,

$$D_e\mu_e = D_i\mu_i \tag{1.74a}$$

We then have

$$D_a = 2D_i \tag{1.74b}$$

Second, in electron impact ionization, $T_e \gg T_i$ thus $D_e\mu_e \gg D_i\mu_i$. The ambipolar diffusion coefficient can then be approximated as

$$D_a = D_e(\mu_i/\mu_e) = (\kappa T_e/e)\mu_i \tag{1.74c}$$

The diffusion by a magnetic field can be deduced by the equation of electron motion,

$$n_e m v u = n_e e u \times B - \nabla p_e \tag{1.75}$$

The effects of the magnetic field on the diffusion are twofold by the peculiarity of the biased orientation of an externally applied magnetic field and the interaction of the Hall current $eu \times B$.

The gradient of the partial pressure gradient transverse to B corresponds to a diffusion coefficient perpendicular to the magnetic field

$$D_{e\perp} = \frac{D_e}{1 + (\omega_b / v_c)^2} \tag{1.76}$$

The diffusion components paralleled to B are altered by both the partial pressure gradients and magnetic field to yield,

$$D_{e\parallel} = D_e \frac{\omega_b / v_c}{1 + (\omega_b / v_c)^2} \tag{1.77}$$

Recall the Larmor frequency is $\omega_b = eB/m$ and v is the averaged collision frequency for the momentum transfer.

REFERENCES

Alfven, H., *Cosmical Electrodynamics*, Clarendon, Press, Oxford, 1950.

Bird, R.B., Stewart, W.E., and Lightfoot, E.N., *Transport Phenomena*, John Wiley & Sons, New York, 1960.

Capitelli, M., Gorse, C., and Longo, S., Thermophysics and heat transfer, *J. Collision Integrals High-Temperature Air Species*, vol. 14 no. 2, 2000, pp. 259–268.

Celcroix, J.L., *Introduction to the Theory of Ionized Gases*, Interscience Publishers, New York, 1960.

Chapman, S., and Cowling, T.G., *The Mathematical Theory of Non-Uniform Gases*, 2nd ed., Cambridge University Press, 1964.

De Hoffmann, F., and Teller, E., Magneto-hydrodynamic shocks, *Phys. Rev.*, vol. 80, no. 4, pp. 692–703, 1950.

Friedrichs, K.O., and Kranzer, H., Nonlinear wave motion, Notes on magnetohydrodynamics, VIII, N.Y. University Rept. NYO 6486, July 31, 1958.

Heald, M.A., and Wharton, C.B., *Plasma Diagnostics with Microwaves*, John Wiley & Sons, New York, 1965.

Hirschfelder, J.O., Curtiss, C.F., and Bird, B.B., *Molecular Theory of Gases and Liquids*, 2nd ed., John Wiley & Sons, New York, 1954.

Howatson, A.M., *An Introduction to Gas Discharge*, Pergamon Press, Oxford, 1975.

Jackson, J.D., *Classical Electrodynamics*, 2nd ed., John Wiley & Sons, New York, 1975.

Jahn R.G., *Physics of Electric Propulsion*, McGraw-Hill, New York, 1968.

Kong, J.A., *Electromagnetic Wave Theory*, John Wiley & Sons, New York, 1986.

Krause, J.D., *Electromagnetics*, 1st ed., McGraw-Hill, New York, 1953.

Landau, L.D., and Lifshitz, E.M., *Electrodynamics of Continuous Media*, Pergamon Press, New York, 1960.

Langmuir, L. Oscillations in ionized gases, *Proc. Natl. Acad. Sci.*, vol. 14, no. 8, 1928.

Langmuir, L. The interaction of electron and positive ion space charge in cathode sheath, *Phys. Rev.*, vol. 33, no. 6, 1929, p. 954.

Levin, E., and Wright, M.J., Collision integrals for ion–neutral interactions of nitrogen and oxygen, *J. Thermophys. Heat Transfer*, vol. 18, no. 1, 2004, pp. 143–147.

Mitchner, M., and Kruger, C.H., *Partially Ionized Gases*, John Wiley & Sons, New York, 1973.

Raizer, Y.P., *Gas Discharge Physics*, Springer-Verlag, Berlin, 1991.

Shang, J.S., Huang, P.G., Yan, H., and Surzhikov, S.T., Computational simulation of direct current discharge, *J. Appl. Phys.*, vol. 105, 2009, pp. 023303-1-14.

Shang, J.S., Surzhikov, S.T., and Yan, H., Hypersonic nonequilibrium flow simulation based on kinetics models, *Frontiers Aerospace Eng.*, vol. 1, no. 1, November 2012, pp. 1–12.

Spitzer, L., *Physics of Fully Ionized Gases*, Interscience Publishers, New York, 1956.

Surzhikov, S.T., and Shang, J.S., Two-component plasma model for two-dimensional glow discharge in magnetic field, *J. Comput. Phys.*, vol. 199, no. 2, September 2004, pp. 437–464.

Sutton, G.W., and Sherman, A., *Engineering Magnetohydrodynamics*, McGraw-Hill, New York, 1965.