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Matrix Algebra

The method used in this textbook to formulate computational models is characterised by the use of matrices. The different quantities – load, section force, stiffness and displacement – are separated and gathered into groups of numbers. All load values are gathered in a load matrix and all stiffnesses in a stiffness matrix. This is one of the primary strengths of the method. With a matrix formulation, the formulae describing the relations between quantities are compact and easy to view. Physical mechanisms and underlying principles become clear. We begin with a short summary of the matrix algebra and the notations that are used.

1.1 Definitions

A matrix consists of a set of *matrix elements* ordered in *rows* and *columns*. If the matrix consists of only one column it is referred to as a *column matrix* and if it has only one row it is referred to as a *row matrix*. Such matrices are *one-dimensional* and may also be referred to as *vectors*. A vector is denoted by a lower case letter set in bold:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1.1}$$

where a_1, a_2 and a_3 are the components of the vector. A *two-dimensional* matrix is denoted by a capital letter set in bold:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{1.2}$$

where A_{11}, A_{12} and so on are elements of the matrix **A**. An arbitrary component of a matrix is denoted A_{ij}, where the first index refers to the row number and the second index to the column number. The matrix **A** in (1.2) has the *dimensions* 4×3 and the matrix **B** has the dimensions 3×3.

Structural Mechanics: Modelling and Analysis of Frames and Trusses, First Edition.
Karl-Gunnar Olsson and Ola Dahlblom.

Since the number of rows and columns in **B** are equal, it is a *square matrix*. If it is only the *diagonal elements* B_{ii} that are different from 0, the matrix is a *diagonal matrix*. A diagonal matrix where all the diagonal elements are equal to 1 is an *identity matrix* and is usually denoted **I**. The *transposed matrix* $\mathbf{A}^T$ of a matrix **A** is formed by letting the rows of **A** become columns of $\mathbf{A}^T$, that is the *transpose* of **A** in (1.2) is

$$\mathbf{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \end{bmatrix} \tag{1.3}$$

A matrix **A** is *symmetric* if $\mathbf{A} = \mathbf{A}^T$. Only square matrices can be symmetric. A matrix with all elements equal to 0 is referred to as a *zero matrix* and is usually denoted **0**.

1.2 Addition and Subtraction

Matrices of equal dimensions can be added and subtracted. The result is a new matrix of the same dimensions, where each element is the sum of or the difference between the corresponding elements of the two matrices. If

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{1.4}$$

the sum of **A** and **B** is given by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.5}$$

where

$$\mathbf{C} = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \\ A_{31} + B_{31} & A_{32} + B_{32} & A_{33} + B_{33} \end{bmatrix} \tag{1.6}$$

and the difference between **A** and **B** is given by

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \tag{1.7}$$

where

$$\mathbf{D} = \begin{bmatrix} A_{11} - B_{11} & A_{12} - B_{12} & A_{13} - B_{13} \\ A_{21} - B_{21} & A_{22} - B_{22} & A_{23} - B_{23} \\ A_{31} - B_{31} & A_{32} - B_{32} & A_{33} - B_{33} \end{bmatrix} \tag{1.8}$$

1.3 Multiplication

Multiplying a matrix **A** with a scalar c results in a matrix with the same dimensions as **A** and where each element is the corresponding element of **A** multiplied by c, that is

$$c\mathbf{A} = \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ cA_{21} & cA_{22} & cA_{23} \\ cA_{31} & cA_{32} & cA_{33} \end{bmatrix} \tag{1.9}$$

Multiplication between two matrices

$$\mathbf{C} = \mathbf{AB} \tag{1.10}$$

can be performed only if the number of columns in **A** equals the number of rows in **B**. The element C_{ij} is then computed according to

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \tag{1.11}$$

For

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{1.12}$$

the product of the matrices, $\mathbf{C} = \mathbf{AB}$, is obtained from

$$\mathbf{C} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \tag{1.13}$$

In general,

$$\mathbf{BA} \neq \mathbf{AB} \tag{1.14}$$

1.4 Determinant

For every quadratic matrix **A** ($n \times n$), it is possible to compute a scalar value called a *determinant*. For $n = 1$,

$$\det \mathbf{A} = A_{11} \tag{1.15}$$

For $n > 1$, the determinant $\det \mathbf{A}$ is computed according to the expression

$$\det \mathbf{A} = \sum_{k=1}^{n} (-1)^{i+k} A_{ik} \det M_{ik} \tag{1.16}$$

where i is an arbitrary row number and $\det M_{ik}$ is the determinant of the matrix obtained when the ith row and the kth column is deleted from the matrix **A**. For $n = 2$, this results in

$$\det \mathbf{A} = A_{11}A_{22} - A_{12}A_{21} \tag{1.17}$$

and for $n = 3$

$$\det \mathbf{A} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} \tag{1.18}$$

1.5 Inverse Matrix

The quadratic matrix **A** is *invertible* if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{1.19}$$

The matrix $\mathbf{A}^{-1}$ is then the *inverse* of $\mathbf{A}$. For the inverse $\mathbf{A}^{-1}$ to exist, it is necessary that $\det \mathbf{A} \neq 0$. If

$$\mathbf{A}^{-1} = \mathbf{A}^T \tag{1.20}$$

the matrix $\mathbf{A}$ is *orthogonal* and then

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I} \tag{1.21}$$

1.6 Counting Rules

The following counting rules apply to matrices (under the condition that the dimensions of the matrices included are such that the operations are defined).

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.22}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{1.23}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{1.24}$$

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T \tag{1.25}$$

$$\mathbf{I}\mathbf{A} = \mathbf{A} \tag{1.26}$$

$$c(\mathbf{A}\mathbf{B}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B}) \tag{1.27}$$

$$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A} \tag{1.28}$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \tag{1.29}$$

$$(\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C}) \tag{1.30}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C} \tag{1.31}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C} \tag{1.32}$$

$$\det \mathbf{A}\mathbf{B} = \det \mathbf{A} \det \mathbf{B} \tag{1.33}$$

$$\det \mathbf{A}^{-1} = 1/ \det \mathbf{A} \tag{1.34}$$

$$\det c\mathbf{A} = c^n \det \mathbf{A} \tag{1.35}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{1.36}$$

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{1.37}$$

1.7 Systems of Equations

A *linear system of equations* with n equations and p unknowns can be written in matrix form as

$$\mathbf{K}\,\mathbf{a} = \mathbf{f} \tag{1.38}$$

where $\mathbf{K}$ has the dimensions $n \times p$, $\mathbf{a}$ the dimensions $p \times 1$ and $\mathbf{f}$ the dimensions $n \times 1$. Usually, the coefficients in $\mathbf{K}$ are known, while the coefficients in $\mathbf{a}$ and $\mathbf{f}$ can be known as well as unknown. For the case when all the components of **a** are unknown and all the components of **f** are known, there are three types of systems of equations:

- $n = p$, the number of equations equals the number of unknowns. The matrix **K** is quadratic. Depending on the contents of **K** and **f**, four different characteristic cases can be recognised. These are often indications of different states or behaviours that may be important to notice: If $\det \mathbf{K} \neq 0$, there is a *unique solution*.
  - For $\mathbf{f} = \mathbf{0}$, this solution is the trivial one, $\mathbf{a} = \mathbf{0}$.
  - For $\mathbf{f} \neq \mathbf{0}$, there is a unique solution, $\mathbf{a} \neq \mathbf{0}$. In general, this is an indication of a functioning physical model.

  If $\det \mathbf{K} = 0$, there is no unique solution. This may be an indication of an, in some way, unstable physical model.
  - For $\mathbf{f} = 0$, there are infinitely many solutions. This is the case for eigenvalue problems, which, for example, can be a method to gain knowledge about unstable states of the model.
  - For $\mathbf{f} \neq 0$, there is either none or infinitely many solutions; there may be elements missing in the model or the set of boundary conditions may be incomplete.
- $n < p$, the number of equations is less than the number of unknowns. The system is underdetermined. There are infinitely many solutions.
- $n > p$, the number of equations exceeds the number of unknowns. The system is overdetermined. In general, there is no solution.

In the following symmetric matrices, **K** and **A** are considered which are common in the forthcoming applications.

1.7.1 Systems of Equations with Only Unknown Components in the Vector **a**

For the case when $\det \mathbf{K} \neq 0$ and $\mathbf{f} \neq \mathbf{0}$, the unknowns in the vector **a** can be determined by Gaussian elimination. This is shown in the following example.

Example 1.1 Solving a system of equations with only unknown components in the vector a
We are looking for a solution to the system of equations

$$\begin{bmatrix} 8 & -4 & -2 \\ -4 & 10 & -4 \\ -2 & -4 & 10 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 18 \\ 6 \end{bmatrix} \tag{1}$$

The unknowns are determined by Gaussian elimination. In this procedure, all elements different from 0 are eliminated below the diagonal: let the first row remain unchanged. From row 2 we subtract row 1 multiplied by the quotient $K_{21}/K_{11} = -4/8 = -0.5$. From row 3 we subtract row 1 multiplied by the quotient $K_{31}/K_{11} = -2/8 = -0.25$. In this way, we obtain

$$\begin{bmatrix} 8 & -4 & -2 \\ 0 & 8 & -5 \\ 0 & -5 & 9.5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \\ 4 \end{bmatrix} \tag{2}$$

In the next step, we let the rows 1 and 2 remain. From row 3 we subtract row 2 multiplied by the quotient $K_{32}/K_{22} = -5/8 = -0.625$. We have triangularised the coefficient matrix $\mathbf{K}$ and obtain

$$\begin{bmatrix} 8 & -4 & -2 \\ 0 & 8 & -5 \\ 0 & 0 & 6.375 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \\ 12.75 \end{bmatrix} \tag{3}$$

With the system of equations in this form, we can determine a_3, a_2 and a_1 by back-substitution

$$a_3 = \frac{12.75}{6.375} = 2; \quad a_2 = \frac{14 - (-5)a_3}{8} = 3;$$

$$a_1 = \frac{-8 - (-4)a_2 - (-2)a_3}{8} = 1 \tag{4}$$

and with that, we have the solution

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \tag{5}$$

To check the results, we can substitute the solution into the original system of equations and carry out the matrix multiplication

$$\begin{bmatrix} 8 & -4 & -2 \\ -4 & 10 & -4 \\ -2 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{which gives} \quad \begin{bmatrix} -8 \\ 18 \\ 6 \end{bmatrix} \tag{6}$$

This is equal to the original right-hand side of the system of equations, that is the solution found is correct.

1.7.2 Systems of Equations with Known and Unknown Components in the Vector **a**

The systems of equations that we consider, in general, has a square matrix $\mathbf{K}$, initially with $\det \mathbf{K} = 0$, and a vector $\mathbf{f} \neq \mathbf{0}$. Moreover, it is usually the case that some components of $\mathbf{a}$ are known and the corresponding components of $\mathbf{f}$ are unknown. One systematic way to solve such a system of equations begins with a *partition* of the matrices, which means that they are divided into *submatrices*

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \tilde{\mathbf{K}} \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} \mathbf{g} \\ \tilde{\mathbf{a}} \end{bmatrix}; \quad \mathbf{f} = \begin{bmatrix} \mathbf{r} \\ \tilde{\mathbf{f}} \end{bmatrix} \tag{1.39}$$

where the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \tilde{\mathbf{K}}, \mathbf{g}$ and $\tilde{\mathbf{f}}$ contain known quantities, while $\tilde{\mathbf{a}}$ and $\mathbf{r}$ are unknown. With use of these submatrices, the system of equations (1.38) can be expressed as

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \tilde{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \tilde{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \tilde{\mathbf{f}} \end{bmatrix} \tag{1.40}$$

The system of equations can be divided into two parts and then be written as

$$\mathbf{A}_1\mathbf{g} + \mathbf{A}_2\tilde{\mathbf{a}} = \mathbf{r} \tag{1.41}$$

$$\mathbf{A}_3\mathbf{g} + \tilde{\mathbf{K}}\tilde{\mathbf{a}} = \tilde{\mathbf{f}} \tag{1.42}$$

or

$$\tilde{\mathbf{K}}\tilde{\mathbf{a}} = \tilde{\mathbf{f}} - \mathbf{A}_3\mathbf{g} \tag{1.43}$$

$$\mathbf{r} = \mathbf{A}_1\mathbf{g} + \mathbf{A}_2\tilde{\mathbf{a}} \tag{1.44}$$

where the right-hand side of the equation (1.43) consists of known quantities. The purpose of the partition of the system of equations is to, within the original system of equations, find a sub-system with det $\tilde{\mathbf{K}} \neq 0$, that is a system with a unique solution. The unknowns in $\tilde{\mathbf{a}}$ can then be computed from (1.43). One way to perform this computation is to use Gaussian elimination. Once $\tilde{\mathbf{a}}$ has been determined, $\mathbf{r}$ can be computed from (1.44).

Example 1.2 Solving a system of equations with both known and unknown components in the vector a

In the system of equations

$$\left[\begin{array}{cccc|cc} 20 & 0 & 0 & 0 & -20 & 0 \\ 0 & 15 & 0 & -15 & 0 & 0 \\ 0 & 0 & 16 & 12 & -16 & -12 \\ 0 & -15 & 12 & 24 & -12 & -9 \\ \hline -20 & 0 & -16 & -12 & 36 & 12 \\ 0 & 0 & -12 & -9 & 12 & 9 \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ -3 \\ 0 \\ \hline a_5 \\ a_6 \end{array}\right] = \left[\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \hline 0 \\ -15 \end{array}\right] \tag{1}$$

the vector **a** has known and unknown components. The solution can then be systematised using partitioning (1.40). The auxiliary lines show this partition. The system of equations is partitioned into two parts according to (1.41) and (1.42):

$$\begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 15 & 0 & -15 \\ 0 & 0 & 16 & 12 \\ 0 & -15 & 12 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -20 & 0 \\ 0 & 0 \\ -16 & -12 \\ -12 & -9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} -20 & 0 & -16 & -12 \\ 0 & 0 & -12 & -9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 36 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -15 \end{bmatrix} \tag{3}$$

In the lower system of equations, there are two equations and two unknowns. If the known terms of the system are gathered on the right-hand side of the equal sign, cf. (1.43), we obtain

$$\begin{bmatrix} 36 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -15 \end{bmatrix} - \begin{bmatrix} -20 & 0 & -16 & -12 \\ 0 & 0 & -12 & -9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} \tag{4}$$

or

$$\begin{bmatrix} 36 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -48 \\ -51 \end{bmatrix} \tag{5}$$

From this system of equations, the unknown elements can be determined by Gaussian elimination: the first row remains unchanged. From row 2 we subtract row 1 multiplied by the quotient $K_{21}/K_{11} = 12/36 = 0.33333$. In this way, we obtain

$$\begin{bmatrix} 36 & 12 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -48 \\ -35 \end{bmatrix} \tag{6}$$

and the unknown a_5 and a_6 can be determined by back-substitution

$$a_6 = \frac{-35}{5} = -7; \quad a_5 = \frac{-48 - 12a_6}{36} = 1 \tag{7}$$

$$\begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix} \tag{8}$$

With a_5 and a_6 being known, the unknown coefficients in $\mathbf{f}$ can be determined using the upper system of equations obtained from the partition, cf. (1.44),

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 15 & 0 & -15 \\ 0 & 0 & 16 & 12 \\ 0 & -15 & 12 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -20 & 0 \\ 0 & 0 \\ -16 & -12 \\ -12 & -9 \end{bmatrix} \begin{bmatrix} 1 \\ -7 \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ 20 \\ 15 \end{bmatrix} \tag{9}$$

and with that, all the unknowns are determined.

1.7.3 Eigenvalue Problems

At times it is of interest to study the case when $\det \mathbf{K} = 0$ and $\mathbf{f} = \mathbf{0}$. Mainly, two different types of problems appear. A system of equations in the form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0} \tag{1.45}$$

is referred to as an *eigenvalue problem* or sometimes *standard eigenvalue problem*. For a solution to exist, it is required that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{1.46}$$

A system of equations in the form

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{a} = \mathbf{0} \tag{1.47}$$

is referred to as a *generalised eigenvalue problem* and for a solution to exist it is required that

$$\det(\mathbf{A} - \lambda\mathbf{B}) = 0 \tag{1.48}$$

Solving an eigenvalue problem means that the values of λ, which fulfil Equations (1.46) and (1.48) are determined, that is the eigenvalues λ_i are computed. The number of eigenvalues

λ_i is equal to the number of unknowns in the system of equations. Two or more eigenvalues may coincide. A symmetric matrix **K** with real elements has only real eigenvalues. For each eigenvalue λ_i there is an eigenvector $\mathbf{a}_i$. The unknowns in the eigenvector $\mathbf{a}_i$ cannot be uniquely determined, but their relative magnitude can be computed.

If the product of two vectors $\mathbf{b}^T\mathbf{c} = 0$, then the vectors **b** and **c** are orthogonal. For eigenvectors, we have $\mathbf{a}_i^T\mathbf{a}_j = 0$ for $i \neq j$, that is any two eigenvectors are always orthogonal.

The following example shows how an eigenvalue problem is solved:

Example 1.3 Solving an eigenvalue problem

We want to find a solution to the eigenvalue problem

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0} \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \tag{2}$$

The determinant of $(\mathbf{A} - \lambda\mathbf{I})$ can be computed as

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{bmatrix} = (5-\lambda)(8-\lambda) - 4 = \lambda^2 - 13\lambda + 36 \tag{3}$$

When this expression is set to zero, the equation

$$\lambda^2 - 13\lambda + 36 = 0 \tag{4}$$

is obtained. The solutions to this equation are the eigenvalues

$$\lambda_1 = 4; \quad \lambda_2 = 9 \tag{5}$$

By substituting the computed eigenvalues into the first equation in the original system of equations we obtain

$$(5-4)a_1 - 2a_2 = 0; \quad \mathbf{a}_1 = t_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{6}$$

and

$$(5-9)a_1 - 2a_2 = 0; \quad \mathbf{a}_2 = t_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{7}$$

where t_1 and t_2 are arbitrary scalar multipliers, $t_1 \neq 0$, $t_2 \neq 0$. Had we substituted the eigenvalues into the second equation instead, the results would be the same. Computation of the product of the two eigenvectors yields

$$\mathbf{a}_1^T\mathbf{a}_2 = t_1 t_2 \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0 \tag{8}$$

The fact that the product is 0 means that the eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal.

Exercises

1.1 Begin with the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 8 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 & -2 & 4 \\ 1 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

and perform the following matrix operations manually.
(a) $\mathbf{A} + \mathbf{B}$
(b) $\mathbf{A}\mathbf{B}^T$
(c) $\mathbf{B}^T\mathbf{A}$
(d) $\mathbf{A}\mathbf{C}$
(e) $\det \mathbf{C}$

1.2 Introduce the matrices

$\mathbf{A}$ with dimensions 4×3

$\mathbf{B}$ with dimensions 3×6

$\mathbf{C}$ with dimensions 1×8

$\mathbf{D}$ with dimensions 6×1

Which of the following operations are possible to perform? For the possible operations, give the dimensions of $\mathbf{E}$
(a) $\mathbf{E} = \mathbf{A}\mathbf{B}$
(b) $\mathbf{E} = \mathbf{B}\mathbf{D}$
(c) $\mathbf{E} = \mathbf{A}\mathbf{B}\mathbf{C}\mathbf{D}$
(d) $\mathbf{E} = \mathbf{A}\mathbf{B}\mathbf{D}\mathbf{C}$
(e) $\mathbf{E} = \mathbf{B}^T\mathbf{A}^T$

1.3 Solve the following system of equations manually. Check the solution.

$$\begin{bmatrix} 20 & 1 & -10 \\ -10 & 3 & 10 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 9 \end{bmatrix}$$

1.4 Solve the following systems of equations manually and check the solutions.

(a) $$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 10 \end{bmatrix}$$

(b) $$\begin{bmatrix} 6 & -4 & -2 \\ -4 & 12 & -8 \\ -2 & -8 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ 16 \\ -6 \end{bmatrix}$$

(c) $$\begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 7 & -2 & -1 & 0 \\ 0 & -2 & 5 & -3 & 0 \\ 0 & -1 & -3 & 7 & -3 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ a_2 \\ 0 \\ a_4 \\ 3 \end{bmatrix} = \begin{bmatrix} f_1 \\ 4 \\ f_3 \\ -1 \\ f_5 \end{bmatrix}$$

1.5 Begin with the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 6 & 4 & 1 & -2 \\ 0 & 3 & 4 & 1 \\ 1 & 2 & -4 & 6 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 3 & 4 & 1 & -2 \\ 6 & 8 & 1 & 0 \\ 2 & 2 & 3 & -2 \\ 1 & 4 & 0 & 4 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} -4 \\ 2 \\ 3 \\ 1 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 1 & 4 & -3 & 6 \end{bmatrix}$$

and perform the following matrix operations with CALFEM. For the sub-exercises with more than one matrix operation, compare and comment on the results.

(a) $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$
(b) $\mathbf{AB}$ and $\mathbf{BA}$
(c) $(\mathbf{AB})^T$, $(\mathbf{BA})^T$ and $\mathbf{B}^T\mathbf{A}^T$
(d) $\mathbf{CD}$ and $\mathbf{DC}$
(e) $\mathbf{C}^T\mathbf{AC}$
(f) $\det \mathbf{A}$, $\mathbf{A}^{-1}$ and $\mathbf{AA}^{-1}$

1.6 Compute the determinant of the matrices in the following systems of equations with CALFEM. If possible, solve the systems of equations and check the solutions. If any of the systems is unsolvable, explain why.

(a) $$\begin{bmatrix} -4 & 3 & 0 & 1 \\ 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -3 \\ 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 8 & -3 & -5 \\ -3 & 5 & -2 \\ -5 & -2 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$$

1.7 Consider the eigenvalue problem $(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix}$$

(a) Compute the eigenvalues.
(b) Compute the eigenvectors and check that they are orthogonal.