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Differential Game Theory and Applications to Missile Guidance

Nomenclature

k:	is the epoch (in a discrete time game).
$\mathbb{P}$**:**	is the set of players in a game.
U:	is the set of strategies available to all the players.
$\mathbf{U^i}$**:**	is the set of strategies available to player **i**.
$\mathbf{J_{ij}}(\cdots)$**:**	is the objective function for players **i** and **j**.
$\mathbf{X_k}$**:**	is the set of current state of a game at epoch **k**.
$\mathbf{U_k}$**:**	is the set of strategies available to a player at epoch **k**.
$\underline{\mathbf{u}}_{\mathbf{ij}}(\mathbf{k})$**:**	is the strategy vector (input vector) available to player **i** against player **j** at epoch **k**.
$\mathbf{C_k}$**:**	is the set of constraints at epoch **k**.
$\mathbf{G_k}$**:**	is the set of elements of a discrete-time game.
t:	is the time in a continuous time (differential) game.
$\mathbf{X_t}$**:**	is the set of states of a game at time **t**.
$\mathbf{U_t}$**:**	is the set of strategies at time **t**.
$\underline{\mathbf{u}}_{\mathbf{ij}}(\mathbf{t})$**:**	is the strategy vector (input vector) available to player **i** against player **j** at time **t**.
$\mathbf{C_t}$**:**	is the set of constraints at time **t**.
$\mathbf{G_t}$**:**	is the set of elements of a continuous time (differential) game.
$\underline{\mathbf{x}}_{\mathbf{ij}}(\mathbf{t})$**:**	is the relative state vector of player **i** w.r.t. player **j** at time **t**.
$\underline{\mathbf{u}}_{\mathbf{i}}(\mathbf{t})$**:**	is the strategy vector (input vector) of player **i**.
F:	is the state coefficient matrix.
G:	is the input coefficient matrix.
Q:	is the PI weightings matrix on the current relative states.
S:	is the PI weightings matrix on the final relative states.
$\{\mathbf{R_i}, \mathbf{R_j}\}$**:**	are PI weightings matrices on inputs.

Differential Game Theory with Applications to Missiles and Autonomous Systems Guidance, First Edition.
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Abbreviations

APN: augmented PN
CF: cost function
LQPI: linear system quadratic performance index
OF: objective function
PI: performance index
PN: proportional navigation
UF: utility function
4-DOF: four degrees of freedom
w.r.t.: with respect to

1.1 Introduction

Over the last few decades a great deal of material has been published covering some of the major aspects of game theory. The well-known publications in this field include "Games and Economic Behaviour" by John von Neumann and Oskar Morgenstern.[1] Since then there has been a significant growth in publication on both the theoretical results and applications. A total of eight Nobel Prizes were given in Economic Sciences for work primarily in game theory, including the one given in 1994 to John Harsanyi, John Nash, and Reinhard Selten for their pioneering work in the analysis of non-cooperative games. In 2005, the Nobel Prizes in game theory went to Robert Aumann and Thomas Schelling for their work on conflict and cooperation through game-theory analysis. In 2007, Leonid Hurwicz, Eric Maskin, and Roger Myerson were awarded the Nobel Prize for having laid the foundations of mechanism design theory. These and other notable works on game theory are given in the references.[2–7]

Cooperative game theory application to autonomous systems with applications to surveillance and reconnaissance of potential threats, and persistent area denial have been studied by a number of authors; useful references on this and allied topics are given at the end of this chapter.[8–15] Usually, the (potential) targets and threats in a battlefield are intelligent and mobile, and they employ counter-strategies to avoid being detected, tracked, or destroyed. These action and counteraction behaviors can be formulated in a game setting, or more specifically, by pursuit/evasion differential games (with multiple players). It is noteworthy that application of differential games to combat systems can be considered to have been started by Rufus P. Isaacs when he investigated pursuit/evasion games.[8] However, most of the theoretical results focus on two-player games with a single pursuer and a single evader, which has since been extended to a multi-player scenarios.

1.1.1 Need for Missile Guidance—Past, Present, and Future

Guided missiles with the requirement to intercept a target (usually an aircraft) at a long range from the missile launch point have been in use since WWII. Guidance systems for missiles are needed in order to correct for initial aiming errors and to maintain intercept flight trajectory in the presence of atmospheric disturbances that may cause the missile to go off course. Traditionally, the use of the so-called proportional navigation (PN) guidance (law) provided the means to enable an attacking missile to maintain its intercept trajectory to its target. As aircraft became more agile and capable of high-g maneuvers, which they could use for evading an incoming threat, the PN guidance law was upgraded to the augmented PN (APN) guidance law that compensated for target maneuvers. Zarchan[24] gives a comprehensive explanation of PN and APN guidance implementation and performance. With advances in missile hardware and computer processing (on-board target tracking sensor and processors), most modern missiles now use the APN guidance. Rapid advances in autonomous system technologies have opened up the possibility that next generation aircraft will be pilotless and capable of performing "intelligent" high-g evasive maneuvers. This potential development has prompted missile guidance designers to look at techniques, such as game theory-based guidance and "intelligent" guidance to outwit potential adversaries.

Earlier reported research[16–27] on the application of game theory to the missile guidance problem has concentrated on engagement scenarios that involve two parties, comprising an attacking missile (pursuer) aimed against another missile or aircraft referred to as a target or an evader. In this book, the above approach is extended to a three-party engagement scenario that includes the situation where an attacking missile may have dual objectives—that is, to evade a defending missile and then continue its mission to engage its primary designated high-value target. The role of the defending missile is only to intercept the attacking missile; the attacking missile, on the other hand, must perform the dual role, that of evading the defending missile, as well as subsequently intercepting its primary target—the aircraft. Since participants in this type of engagement are three players (the aircraft target, the attacking missile, and the defending missile), involved in competition, we shall refer to this type of engagement scenario as a three-party game.

Game theory-based linear state feedback guidance laws are derived for the parties through the use of the well-known linear system quadratic performance index (LQPI) approach. Guidance commands generated are lateral accelerations that parties can implement in order either to intercept a target, or to evade an attacker. A missile/target engagement model has been developed, and feedback gain values are obtained by solving the matrix Riccati differential equation. Preliminary simulation results to demonstrate the characteristics of intercept and evasion strategies are included in Chapter 6. Simple (rule-based) intelligent strategies are also considered for enhancing evasion by a target or for improving the chances of intercept for an attacker.

1.2 Game Theoretic Concepts and Definitions

Game theory is concerned with studying and characterizing the dynamics of interactions between players involved in a collective and competitive activity or contest, where each player is required to make decisions regarding his/her strategy, and implement this strategy in order to gain an advantage. These decision makers will be referred to as players or parties. Each player's choice of the strategy, and the advantage gained by implementing this strategy, is defined through an objective function (OF), which that player tries to maximize. The OF in this case is also referred to as a utility function (UF), or pay-off. If a player sets out to minimize the objective function, it is referred to as a cost function (CF) or a loss function. The objective function of a player depends on the strategies (control or input variable) that a player implements in order to optimize the objective function. This involves action of at least one or more players involved in a game. The strategy that each party implements determines the strategies that the other players involved in a game are required to implement in order to achieve optimization of the objective function. This is particularly true of a competitive or non-cooperative game. In the case of a cooperative game, some or all of the parties may enter into a cooperative agreement so that the strategies selected provide collective advantage to parties in the coalition. A non-cooperative game is a non-zero-sum game if the sum of the player's objective function remains non-zero. If, however, the objective function can be made zero then the non-cooperative game will be called a zero-sum game. As far as the nature of the optimum solution is concerned it is a requirement that this solution be such that if all the players except one, and only one, execute optimum strategies, the pay-off for the player that deviates from the optimum would result in a disadvantage to this player. The optimum solution where none of the players can improve their pay-off by a unilateral move will be referred to as a non-cooperative equilibrium also known as the Nash equilibrium.[3]

A game can be either finite or infinite depending on the number of choices (moves) for the strategies available for the players. A finite game provides for a finite number of choices or alternatives in the strategy set for each player; however, if the choices in the strategy set are infinite then the game is an infinite game. For an infinite game, if the players' objective functions are continuous with respect to (w.r.t.) the action variables (strategies) of all players, then it is known as a continuous-time game. The evolution (transition or progression) of a game can be defined by the state variable (or the state of the game), which represents changes in the game environment as the players involved in the game implement their strategies. The state is a function of the prior state and the actions implemented that causes a change in the game environment. This functional relationship will be referred to as the game dynamics or the game dynamical model. We shall refer to a game as deterministic if the nature of the game dynamics model, the strategies (control variables), and the objective functions are such as to uniquely determine the outcome (the optimum solution). However, if the dynamics model, control variable, or the objective function associated with at least one of the players is defined via a probability function then the game will be referred to as a stochastic game. Stochastic games are not considered in this book. A dynamic game is said to be a differential dynamic game if the evolution of the states and of the decision process is defined through a continuous-time process, involving a set of differential equations. Where the evolution of the states and the decision occurs over discrete time intervals then the game is called a discrete-time game.

1.3 Game Theory Problem Examples

In order to develop a formal structure for the game theory problem and enable its subsequent solution in a manner that allows many of the control systems techniques to be used, we consider the following examples that have played a major role in the development of the game theory.

1.3.1 Prisoner's Dilemma

The prisoner's dilemma is a good example of a simple game that can be analyzed using the game theory principles. It was originally framed by Flood and Dresher in 1950[29] and later formalized by Tucker,[28] who named it the "prisoner's dilemma." We consider a situation where two prisoners A and B, are being interrogated, separately, about their role in a particular crime. Each prisoner is in solitary confinement with no means of communicating with the other. The interrogator, in order to induce A and B to betray each other and confess to their role in the crime, offers the following incentive to the prisoners:

- If A and B each betray the other, each of them serves two years in prison,
- If A betrays B but B remains silent, A will be set free and B will serve three years in prison (and vice versa),
- If A and B both remain silent then both of them will only serve one year in prison (on a lesser charge).

Using the above scenario, we can construct a strategy/pay-off table for each prisoner as shown in Table 1.3.1 below, with the pay-off shown as (A's pay-off, B's pay-off):

Table 1.3.1 Strategy Versus Pay-Off.

Strategies	A betrays B	A keeps silent
B betrays **A**	(2, 2)	(3, 0)
B keeps silent	(0, 3)	(1, 1)

Assuming that both the prisoners play an optimum strategy that minimizes each prisoner's pay-off then it follows that the "best strategy" from each player's perspective is to betray the other. Any other strategy would not necessarily lead to the minimum pay-off solution. For example, if B keeps silent hoping that A will also keep silent then this may not necessarily turn out to be the case, because A (motivated by self-interest) may decide to betray B and achieve a reprieve from imprisonment.

1.3.1.1 Observations and Generalization From the Above Example

We can make the following observations arising out of the above example regarding the elements of the game theory as follows.

(a) A game must have players (in the particular case of the above example: A and B)—in general, however, there could be more than two players; we shall therefore define the set of players in a game as the set:

$$P = \{p_i;\ i = 1, 2, \ldots, n\} \tag{1.3.1}$$

where

$p_i;\ i = 1, 2, \ldots, n$**:** are players involved in a game.

(b) A game must have strategies (in the particular case of the example above, there were two strategies available to each player: keep silent or betray the other player). In general for more than two players involved in a game, there could be a number of different strategies; we shall define the strategy set as:

$$\mathbf{U} = \{\mathbf{U}^{\mathbf{i}};\ \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\} \tag{1.3.2}$$

where

$\mathbf{U^i = \{\underline{u}_{ij}; j = 1, 2, \dots, j-1, j+1, \dots, n\}; \;\; i = 1, 2, \dots, n}$: is the set of strategies available to player **i** against player **j**.

Note that each strategy subset includes strategies of player **i** that he/she is able to exercise against another player **j**. In fact, player **i** can exercise multiple strategies against player **j**.

As we shall see later, when we consider the topic of missile guidance $\mathbf{\underline{u}_{ij}}$ can be either a scalar or a vector.

(c) A game has a cost function or a pay-off that the players either minimize or maximize in order to achieve their objectives. For this example the objective function (OF) may be written as:

$$\mathbf{J_{ij}(\cdots) = f(\underline{u}_{ij})} \tag{1.3.3}$$

That is, the OF is a function of players' strategies. We shall further qualify the nature of an OF later in this chapter, since in general the OF is also a function of the states of the game.

(d) The game considered in this example has only one move and will be referred to as a game with single epoch. In the next example we shall consider a game with multiple epochs.

For the particular example of the prisoner's dilemma it is seen that:

$\mathrm{P} = \{\mathrm{p}_1, \mathrm{p}_2\}$; $\begin{bmatrix} \mathbf{\underline{u}_{12}^T} \\ \mathbf{\underline{u}_{21}^T} \end{bmatrix} = \begin{bmatrix} \mathbf{u_{12}^1} & \mathbf{u_{12}^2} \\ \mathbf{u_{21}^1} & \mathbf{u_{21}^2} \end{bmatrix}$, here the superscripts $^{(1,2)}$ indicate the two components of a vector, representing two strategies.

Also $\mathbf{J_{12}}(\cdots) = \begin{bmatrix} \mathbf{(2,2)} & \mathbf{(3,0)} \\ \mathbf{(0,3)} & \mathbf{(1,1)} \end{bmatrix}$; $\mathbf{J_{21}}(\cdots) = \begin{bmatrix} \mathbf{(2,2)} & \mathbf{(0,3)} \\ \mathbf{(3,0)} & \mathbf{(1,1)} \end{bmatrix}$.

1.3.2 The Game of Tic-Tac-Toe

We now turn our attention to the well-known game of "Tic-tac-toe" (T3), which will enable us to introduce other aspects of the game theory. A generalization of this T3 game will give us a framework for formulating the game theory problem as a control systems problem, which will allow us to exploit the well-developed techniques of the optimal control. Tic-tac-toe (also known as Noughts and Crosses or **O**'s and **X**'s) is designed for two players, A and B, who take turns marking the spaces in a 3×3 grid. The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row (combinations) wins the game. Let us designate player A's move with a **O** and B's move with an **X**; and the move (i.e., the position on the 3×3 grid) that a player selects to make is designated by $\mathbf{(i, j)}$:$\mathbf{i = 1, 2, 3; j = 1, 2, 3}$ (see Figure 1.3.1). We shall assume that A moves first. We will also refer to a move position on the grid as the "state" of the game. The author of this book has encapsulated a possible strategy (algorithm) for the play (moves) as a game theory problem where each player makes a move so as to maximize the objective function defined by the following expression:

$$\mathbf{J(k) = J_A(k) = J_B(k) = \{n_1(k) + 2n_2(k) + 6n_3(k) + 12n_4(k)\}} \tag{1.3.4}$$

(a) States of the game		
(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)
(3, 1)	(3, 2)	(3, 3)

(b) Move: A; k=1; J=4		
	O	

Move: B; k=1; J=6		
X		
	O	

(d) Move: A; k=2; J=13		
X		**O**
	O	

(e) Move: B; k=2; J=23		
X		**O**
	O	
X		

Move: A; k=3; J=26		
X		**O**
O	**O**	
X		

Move: B; k=3; J=29		
X		**O**
O	**O**	**X**
X		

Tic-Tac-Toe Moves: Further moves can be made by inspection, e.g. the game outcome will be a draw with: A(k=4)→ **O**(1,2), B(k=4)→ **X**(3,2), A(k=5)→ **O**(3,3)

Figure 1.3.1 Moves for the Tic-Tac-Toe (T3) Game.

where

$\mathbf{n_1}(\cdots)$: is the total number of a player's own potential winning combinations with single entries, as a result of the move **k**.
$\mathbf{n_2}(\cdots)$: is the total number of opponent's potential winning combinations with single entries that are blocked by the move **k**.
$\mathbf{n_3}(\cdots)$: is the total number of player's own potential winning combinations with double entries, as a result of the move **k**.
$\mathbf{n_4}(\cdots)$: is the total number of opponent's potential winning combinations with double entries that are blocked by the move **k**.
$\mathbf{J_A}(\cdots)$; $\mathbf{J_B}(\cdots)$: are objective function values for players A and B respectively at epoch **k**.
k: is the move number or epoch.

It follows from (1.3.4) that maximizing the above OF for each move (epoch) is equivalent to maximizing the sum of the OF over all of the moves, that is the OF can also be written as:

$$\mathbf{J(N)} = \sum_{\mathbf{k=1}}^{\mathbf{N}} \{\mathbf{n_1(k) + 2n_2(k) + 6n_3(k) + 12n_4(k)}\} \tag{1.3.5}$$

1.3.2.1 Observations and Generalization From the Tic-Tac-Toe Example

A set of moves that occur through the maximization of the objective function (1.3.4) is shown in Figure 1.3.1; the values obtained from the OF are also given there. We make the following observations gleaned from the T3 game considered here:

(a) A T3 game may be regarded as a discrete (and a finite) game since each move is made at each epoch, which is not continuous.
(b) The value of the elements of the OF $\mathbf{n_i}$; $\mathbf{i = 1, 2, 3, 4}$ depend on the strategy (move) that a player utilizes. This, in turn, depends on the current state of the game and the current strategy; thus, we may regard the OF to be a function of the strategy and the current state of the game. That is, in general we may write:

$$\mathbf{J(N) = \sum_{k=1}^{N} f[X(k), U(k), k]} \tag{1.3.6}$$

where

X(k): is the set of states of a game at epoch **k**.
U(k): is the set of strategies available to players at epoch **k**.

(c) Obviously a player cannot make a move to a state (position on the grid) that is already occupied by its own previous moves or those of the opponent's moves. We will regard this as state constraints.

1.4 Game Theory Concepts Generalized

From observations made in the previous section we can now define the class of game theory problems that we shall consider in this book. The main theme in this book will be the continuous-time (differential) game theory and its application to missile guidance. We shall also give a formal definition of the discrete-time games for the sake of completeness.

1.4.1 Discrete-Time Game

(a) A discrete game has a set of **states** $\mathbf{X_k}$ defined as the set:

$$\mathbf{X_k = \{\underline{x}(k);\ k = 1, 2, \dots, N\}} \tag{1.4.1}$$

where

$\underline{\mathbf{x}}\mathbf{(k)}$**:** is the state vector of a game that depends on epoch **k**.
$\mathbf{k = 1, 2, \dots, N}$**:** are game epochs.

Note that in cases where we talk about relative states we adopt the notation: $\underline{\mathbf{x}}_{\mathbf{ij}}\mathbf{(k)}$.
(b) A discrete game has a set of players P given by:

$$P = \{p_i; i = 1, 2, \dots, n\} \tag{1.4.2}$$

(c) A discrete game has a set of strategies $\mathbf{U_k}$ given by:

$$\mathbf{U_k = \{\underline{u}_{ij}(k);\ k = 1, 2, \dots, N\};\ i, j = 1, 2, \dots, n;\ i \neq j;} \tag{1.4.3}$$

where

$\underline{\mathbf{u}}_{\mathbf{ij}}\mathbf{(k)}$**:** is the strategy vector (input vector) available to player **i** against player **j** in a game.

(d) A discrete game has an objective function $\mathbf{J}(\cdots)$ given by:

$$\mathbf{J_k(\cdots) = J[\underline{x}(k), \underline{u}_{ij}(k), k]} \tag{1.4.4}$$

(e) A discrete game can have rules or constraints $\mathbf{C_k}$ given by:

$$\mathbf{C_k = C[\underline{x}(k+1), \underline{x}(k), \underline{u}_{ij}(k), k] = 0} \tag{1.4.5}$$

Based on definitions (1.4.1) through (1.4.5), we may define a discrete-time game $\mathbf{G_k}$ as the set:

$$\mathbf{G_k = \{X_k, P, U_k, C_k, k\}} \tag{1.4.6}$$

A typical example of an OF and constraints for a discrete game may be written as follows:

$$\mathbf{J(\cdots) = \theta[\underline{x}(N)] + \sum_{k=1}^{N} \phi[\underline{x}(k), \underline{u}_{ij}(k), k]} \tag{1.4.7}$$

Where

$\boldsymbol{\theta}[\cdots]$; $\boldsymbol{\phi}[\cdots]$**:** are scalar cost functions.

The dynamic constraint is given by

$$\mathbf{\underline{x}(k+1) - \underline{\psi}[\underline{x}(k), \underline{u}_{ij}(k), k] = 0} \tag{1.4.8}$$

1.4.2 Continuous-Time Differential Game

A differential game is analogous to the discrete game with the exception that the game evolves in continuous time **t** and will be defined as follows:

(a) A differential game is assumed to have a set of states $\mathbf{X_t}$ defined as a set:

$$\mathbf{X_t = \{\underline{x}(t);\ t_0 \le t \le t_f\}} \tag{1.4.9}$$

where

$\underline{\mathbf{x}}\mathbf{(t)}$**:** is the state vector of a game, which is a function of time **t**; with start time $\mathbf{t_0}$ and end (final) time $\mathbf{t_f}$, with $\mathbf{t_0 \le t \le t_f}$.

(b) A differential game has a set of players **P** given by:

$$P = \{p_i; i = 1, 2, \ldots, n\} \tag{1.4.10}$$

(c) A differential game has a set of strategies $\mathbf{U_t}$ given by:

$$\mathbf{U_t = \{\underline{u}_{ij}(t);\ t_0 \le t \le t_f\}} \tag{1.4.11}$$

where

$\underline{\mathbf{u}}_{\mathbf{ij}}\mathbf{(t)}$**:** is the strategy vector (input vector) available to player **i** against player **j** in a game.

(d) A differential game has an objective function $\mathbf{J}(\cdots)$ given by:

$$\mathbf{J_t(\cdots) = J[\underline{x}(t), \underline{u}_{ij}(t), t]} \tag{1.4.12}$$

(e) A differential game can have rules or constraints $\mathbf{C}$ given by:

$$\mathbf{C_t = C(\dot{\underline{x}}(t),\ \underline{x}(t),\ \underline{u}_{ij}(t), t) = 0} \tag{1.4.13}$$

Based on definitions (1.4.9) through (1.4.13), we may define a differential game $\mathbf{G_t}$ as the set:

$$\mathbf{G_t = \{X_t, P, U_t, C_t, t\}} \tag{1.4.14}$$

A typical example of an OF and constraints for a differential game may be written as follows:

$$\mathbf{J(\cdots) = \theta[\underline{x}(t)] + \int_{t_0}^{t_f} \phi[\underline{x}(t),\ \underline{u}_{ij}(t),\ t]dt} \tag{1.4.15}$$

with the dynamic constraint given by:

$$\mathbf{\dot{\underline{x}}(t) - g[\underline{x}(t),\ \underline{u}_{ij}(t), t] = 0} \tag{1.4.16}$$

1.5 Differential Game Theory Application to Missile Guidance

The application of the differential game theory to the missile guidance problem requires describing the trajectory of a missile or missile dynamics as a set of differential equations of the type given in (1.4.16). The guidance objectives that a designer aims to meet can be expressed as an objective function of the type (1.4.15), which has to be optimized in order to determine guidance strategies (inputs) for missiles/aircraft involved in a given combat situation. Chapters 3, 4, and 6 are dedicated to developing the differential equations (also referred to as the system dynamics model) and the objective functions (also referred to as a performance index). In this book we shall confine ourselves to a linear system dynamical model and a performance index, which is a scalar quadratic function of system states and inputs and can be written, in a general form, respectively as:

$$\mathbf{\dot{\underline{x}}_{ij}(t) = F\underline{x}_{ij}(t) + G\underline{u}_i(t) - G\underline{u}_j(t)} \tag{1.5.1}$$

and

$$\mathbf{J(\cdots) = \frac{1}{2}\underline{x}_{ij}^T(t_f)S\underline{x}_{ij}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\underline{x}_{ij}^T Q\underline{x}_{ij} + \underline{u}_i^T R_i\underline{u}_i - \underline{u}_j^T R_j\underline{u}_j\right] dt} \tag{1.5.2}$$

where

$\mathbf{\underline{x}_{ij}(t) = \underline{x}_i(t) - \underline{x}_j(t)}$**:** is the relative state of player **i** w.r.t. player **j**.
$\mathbf{\underline{u}_i(t)}$**:** is the input of player **i**.
$\mathbf{\underline{u}_j(t)}$**:** is the input of player **j**.
F: is the state coefficient matrix.

G: is the input coefficient matrix.
Q: is the PI weightings matrix on the current relative states.
S: is the PI weightings matrix on the final relative states.
$\{\mathbf{R_i}, \mathbf{R_j}\}$**:** are PI weightings matrices on inputs.

The structure of the dynamical model (1.5.1) and that of the objective function (1.5.2) will be applicable to the game theory guidance problems considered in this book.

1.6 Two-Party and Three-Party Pursuit-Evasion Game

Consider a situation where a number of different parties are involved in a pursuit-evasion game, where each party endeavors, through the application of the game theory-based strategy to maximize its advantage as reflected in some pre-specified metric or pay-off. A typical example of a pursuer-evader game involves two parties $\{p_1, p_2\}$, where the objective of the pursuer p_1 is to catch up and intercept p_2, whereas the evader p_2 has the objective of avoiding the intercept. Clearly, one obvious objective function that both parties can use is the relative separation (projected miss-distance) between them. Let us further assume that both parties are moving w.r.t. each other in a given reference frame (e.g., the inertial frame). It can be assumed that both parties have the capability to change directions of their respective motions (maneuver capability), which they can exercise in order to achieve their objectives—p_1 tries to minimize the projected miss-distance, whereas p_2 tries to maximize it. This objective can be taken to be some (positive) function of the relative distance between the parties and the maneuver (input) capability, which each party can employ. One also needs to consider the extent of the manueverability of each party and the motion dynamics involved.

The above problem may be extended to a scenario where there are three or more parties involved in a pursuit-evasions game. Let us consider a situation involving three parties and specify these as $\{p_1, p_2, p_3\}$. An example of this type of game is the following (it is considered in some detail in later chapters of this book):

(a) $\{p_3, p_1\}$ to represent the engagement where p_3 is the pursuer and p_1 is the evader,
(b) $\{p_2, p_3\}$ to represent the engagement where p_2 is the pursuer and p_3 is the evader,
(c) $\{p_1, p_2\}$ to represent the engagement where p_1 and p_2 are coalition partners (i.e., neither party is a pursuer or an evader w.r.t. each other).

The particular ordering of the indices is immaterial, as long as there is no confusion as to which party is the pursuer and which one is the evader. In general, we may envisage a scenario in which the pair $\{p_i, p_j\}$ implies $\{p_i \text{ vs. } p_j\}$ and where each party can be considered to be employing dual strategies of pursuit as well as evasion. Problems of this type can be considered under the framework of the LQPI problem.

1.7 Book Chapter Summaries

In **Chapter 2**, the subject of optimum control is dealt with in some detail, and results that are important in many problems of practical interest are derived. Derivations considered in this chapter rely heavily on the calculus of variation and necessary and sufficient conditions for optimality are developed for a generalized scalar cost function subject to equality constraints defined by a non-linear dynamical system model. A simple scalar cost function involving system states and control input variables is used to introduce the reader to the steady-state (single-stage decision) optimization problem utilizing the Euler-Lagrange multiplier and the Hamiltonian. The dynamic optimum control problem is then considered, where the cost function is in the form of an integral over time, of a scalar function of system states and control (input) vectors plus a scalar function of the final system states. The optimum control problem involving a linear dynamical system model, where the cost function is a time integral of a scalar quadratic function of state and control vectors is also considered in this chapter. It is shown that the solution of this problem leads to the well-known matrix Riccati differential equation, which has to be solved backward in time. The application of the optimum control results to two-party and three-party game theory problems is considered, and conditions for optimality and convergence of the Riccati equation are given. The nature of the equilibrium point is investigated and conditions for the existence of a minimum, a maximum or a saddle point are derived. Extension of the differential game theory to multi-party (n-party) games is also described.

Chapter 3 considers the application of the differential game theory to the missile guidance problem. The scenario considered involves engagement between an attacker (interceptor/pursuer) and a target (evader), where the objective of the former is to execute a strategy (or maneuvers) so as to achieve intercept with the target, whereas the objective of the latter is to execute a strategy (maneuver) so as to evade the attacker and avoid or at least delay the intercept. Differential game approach enables guidance strategies to be derived for both the attacker and the target so that objectives of the parties are satisfied. Interceptor/target relative kinematics model for a 3-D engagement scenario is derived in state space form, suitable for implementing feedback guidance laws through minimization/maximization of the performance index (PI) incorporating the game theory based objectives. This PI is a generalization of those utilized by previous researchers in the field and includes, in addition to the miss-distance term, other terms involving interceptor/target relative velocity terms in the PI. This latter inclusion allows the designer to influence the engagement trajectories so as to aid both the intercept and evasion strategies. Closed-form expressions are derived for the matrix Riccati differential equations and the feedback gains that allow the guidance strategies of the interceptor and the target to be implemented. Links between the differential game theory based guidance, the optimal guidance, the proportional navigation (PN) and the augmented PN guidance are established. The game theory-based guidance technique proposed in this chapter provides a useful tool to study vulnerabilities of existing missile systems against current and future threats that may incorporate "intelligent" guidance. The technique can also be used for enhancing capabilities of future missile systems.

In **Chapter 4** we consider a three-party differential game scenario involving a target, an attacking missile, and a defending missile. We assume that this scenario involves an aircraft target that on becoming aware that it is being engaged by an attacking missile, fires a defending missile against this attacker, and itself performs a maneuver to escape the attacking missile. In order to engage the aircraft, the attacking missile performs both an evasive maneuver to defeat (evade) the defending missile and a pursuit maneuver to engage the aircraft target. A three-party game theoretic approach is considered for this scenario that uses a linear quadratic performance index optimization technique to obtain guidance strategies for the parties involved. The resulting guidance laws are then used in four degrees of freedom (4-DOF) engagement kinematics model simulation, to study the characteristics of the resulting intercept and evasion strategies. Simple (rule-based) AI techniques are also proposed in order to implement additional maneuvers to enable the parties to enhance their evasion/survival, or, in the case of the attacker, to evade the defender and subsequently achieve intercept with the target.

Chapter 5 is concerned with the development of the dynamics simulation model for performance analysis of guidance laws for missiles. This model uses a fixed-axis system convention under the assumption that the missile trajectory during an engagement can vary significantly from the collision course geometry. These models take into account autopilot lags and lateral acceleration limits, and while the guidance commands are computed in fixed axis, these are subsequently converted to body axis. This latter fact is particularly relevant in cases of engagements where the target implements evasive maneuvers, resulting in large variations of the engagement trajectory from that of a collision course. A linearized model is convenient for deriving the guidance laws (in analytical form); however, the study of their performance characteristics still requires a non-linear model that incorporates changes in body attitudes, and implements guidance commands in body axis rather than the fixed axis. In this chapter, a 4-DOF mathematical model for multi-party engagement kinematics is derived, suitable for developing, implementing, and testing modern missile guidance systems. The model developed here is suitable for both conventional and more advanced optimal intelligent guidance, particularly those based on the game theory guidance techniques. These models accommodate changes in vehicle body attitude and other non-linear effects (such as limits on lateral acceleration) and may be extended to include other aerodynamic affects.

Chapter 6 considers a simulation study of game theory-based missile guidance developed in Chapters 3, 4, and 5. The scenario considered involves an aircraft target, which is being engaged by a ground-launched missile and fires a defending missile against this attacker and itself performs a maneuver to escape the attacking missile. In order to engage the aircraft, the attacking missile first performs an evasive maneuver to defeat (evade) the defending missile and then an intercept maneuver to engage the aircraft target. Differential game approach is proposed that utilizes a linear quadratic performance index optimization technique to obtain guidance strategies for the parties; guidance strategies obtained are used in a 4-DOF simulation, to study the characteristics of the resulting intercept and evasion strategies. A simple (rule-based) AI technique is proposed for implementing additional maneuvers to enable the parties to enhance their evasion/survival or in the case of the attacker to achieve intercept.

The addendum of this chapter includes the MATLAB listing of the simulation program and a CD containing the *.m files: **faruqi_dgt_DEMO.m** and **kinematics3.m.**

1.7.1 A Note on the Terminology Used In the Book

In this book a missile has been referred to as an attacker or a defender, depending on role that it plays in an engagement; the generic term vehicle is also used for a missile or an aircraft. The term target is used to specify an aircraft target in a three-party game scenario considered; a missile can be a target if it plays this particular role. Also, while the book mostly talks about missiles, all the synthesis techniques considered in this text apply equally to “autonomous systems.” The term performance index (PI) is used to signify an objective function (OF). Terms such as a utility function (UF) and a cost function (CF) are also used, provided it is clear whether maximization or minimization of this function is considered. Terms such as kinematics model, dynamic model, or system model are used interchangeably. Terms such as input, control, or control inputs, guidance inputs/commands are used to mean the same; the term strategy is a generic term for these.

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