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INTRODUCTION TO N-DIMENSIONAL GEOMETRY

1.2 POINTS, VECTORS, AND PARALLEL LINES

1.2.5 Problems

A remark about the exercises is necessary. Certain questions are phrased as statements to avoid the incessant use of “prove that”. See Problem 1, for example. Such statements are supposed to be proved. Other questions have a “true–false” or “yes–no” quality. The point of such questions is not to guess, but to justify your answer. Questions marked with $*$ are considered to be more challenging. Hints are given for some problems. Of course, a hint may contain statements that must be proved.

1. Let S be a nonempty set in $\mathbb{R}^n$. If every three points of S are collinear, then S is collinear.

   Solution. Let x_1 and x_2 be two distinct points in S, then there is a unique line ℓ passing through these two points. Now let x be an arbitrary point in S, from the hypothesis, x_1, x_2, and x must be on some line ℓ', and since x_1 and x_2 uniquely determine the line ℓ, we must have $\ell' = \ell$. Therefore, every point x in S is on the line ℓ. □

3. Given that the line L has the linear equation

$$\mu_1 x_1 + \mu_2 x_2 = \delta,$$

Solutions Manual to Accompany Geometry of Convex Sets, First Edition. I. E. Leonard and J. E. Lewis.

show that the point

$$\left(\frac{\mu_1\delta}{\mu_1^2+\mu_2^2}, \frac{\mu_2\delta}{\mu_1^2+\mu_2^2}\right)$$

is on the line, and that the vector $(-\mu_2, \mu_1)$ is parallel to the line.
Hint. If p is on the line and if $p+v$ is also on the line, then v must be parallel to the line.

Solution. Substituting the coordinates of this point into the linear equation for L, we see that

$$\mu_1 \cdot \frac{\mu_1\delta}{\mu_1^2+\mu_2^2} + \mu_2 \cdot \frac{\mu_2\delta}{\mu_1^2+\mu_2^2} = \frac{(\mu_1^2+\mu_2^2)\delta}{\mu_1^2+\mu_2^2} = \delta,$$

so that the given point is on L.

Since not both μ_1 and μ_2 are 0, we may assume that $\mu_1 \neq 0$, and let

$$p_1 = \left(\frac{\mu_1\delta}{\mu_1^2+\mu_2^2}, \frac{\mu_2\delta}{\mu_1^2+\mu_2^2}\right) \quad \text{and} \quad p_2 = \left(\frac{\delta}{\mu_1}, 0\right),$$

then both p_1 and p_2 are on the line L, and therefore $w = p_1 - p_2$ is parallel to L. However,

$$w = \frac{\mu_2\delta}{\mu_1(\mu_1^2+\mu_2^2)}\,(-\mu_2, \mu_1),$$

so the vector $v = (-\mu_2, \mu_1)$ is parallel to L.

Note that this follows immediately from the fact that the vector

$$v_\perp = (\mu_1, \mu_2)$$

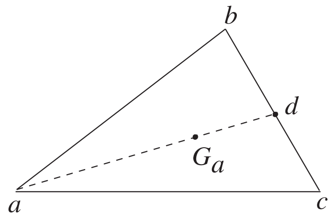
is the normal vector to the line L. □

5. The centroid of three noncollinear points a, b, and c in $\mathbb{R}^n$ is defined to be

$$G = \frac{1}{3}(a+b+c).$$

Show that this definition of the centroid yields the synthetic definition of the centroid of the triangle with vertices a, b, c, namely, the point at which the three medians of the triangle intersect. Prove also that the medians do indeed intersect at a common point.

Solution. Given a triangle with vertices $a, b, c \in \mathbb{R}^n$, let $d \in \mathbb{R}^n$ be the midpoint of the segment $[b, c]$ and let $G_a \in \mathbb{R}^n$ be the point along the median ad which is $\frac{2}{3}$ the distance from a to d.



We have $G_a = a + \frac{2}{3}(d - a)$, and since $d = \frac{1}{2}(b + c)$, then

$$G_a = a + \frac{2}{3} \cdot \frac{1}{2}(b + c) - \frac{2}{3}a = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c.$$

If we define G_b and G_c similarly, then we see that

$$G_a = G_b = G_c = \frac{1}{3}(a + b + c),$$

so that the point $\frac{1}{3}(a + b + c)$ lies on each of the three medians. Thus, this is the synthetic definition of the centroid and the medians intersect at a single point. □

1.4 INNER PRODUCT AND ORTHOGONALITY

1.4.3 Problems

In the following exercises, assume that "distance" means "Euclidean distance" unless otherwise stated.

1. (a) The ***unit cube*** in $\mathbb{R}^n$ is the set of points

$$\{x = (\alpha_1, \alpha_2, \ldots, \alpha_n) : |\alpha_i| \le 1, i = 1, 2, \ldots, n\}.$$

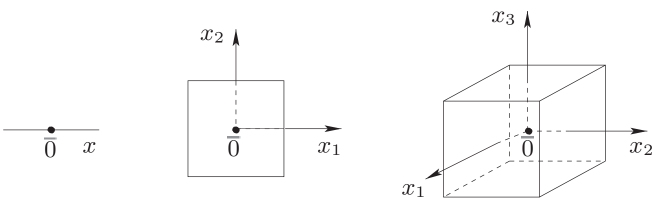
Draw the unit cube in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

(b) What is the length of the longest line segment that you can place in the unit cube of $\mathbb{R}^n$?

(c) What is the radius of the smallest Euclidean ball that contains the unit cube of $\mathbb{R}^n$?

Solution.

(a) The unit cubes are sketched below.



(b) If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are points in the unit cube in $\mathbb{R}^n$, then the Euclidean distance between a and b is

$$d(a,b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

where $|a_k| \leq 1$ and $|b_k| \leq 1$ for $k = 1, 2, \ldots, n$.
The maximum distance will occur when $|a_k| = |b_k| = 1$ and $b_k = -a_k$ for $k = 1, 2, \ldots, n$, that is, when a and b are vertices of the cube that are diagonally opposite. In this case, the maximum distance is

$$d_{\max} = \sqrt{2^2 + 2^2 + \cdots + 2^2} = 2\sqrt{n}.$$

(c) The smallest Euclidean ball that contains the unit cube is one that has diameter equal to $d_{\max} = 2\sqrt{n}$, the length of the longest line segment in the cube. The ball is

$$\overline{B}(\overline{0}, R) = \{x \in \mathbb{R}^n \,:\, \|x\| \leq \sqrt{n}\}$$

and has radius $R = \sqrt{n}$. □

3. Find the distance between the points $(1, -2)$ and $(-2, 3)$ using
   (a) the ℓ_1 metric,
   (b) the “sup” metric,
   (c) the Euclidean metric.

Solution. If $x = (1, -2)$ and $y = (-2, 3)$, then

$$|x_1 - y_1| = |1 - (-2)| = 3 \quad \text{and} \quad |x_2 - y_2| = |(-2) - 3| = 5.$$

Therefore,

(a) With the ℓ_1 metric,

$$\|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2| = 3 + 5 = 8.$$

(b) With the “sup” metric,

$$\|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{3, 5\} = 5.$$

(c) With the Euclidean metric,

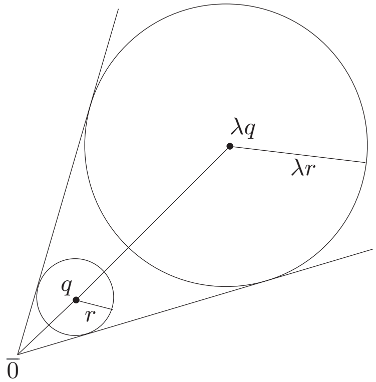
$$\|x - y\|_2 = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{1/2} = (3^2 + 5^2)^{1/2} = \sqrt{34}.$$

□

5. Show that a positive homothet of a closed ball is a closed ball.

Solution. Let $\lambda > 0, q \in \mathbb{R}^n$, and $r > 0$, we will show that

$$\lambda \overline{B}(q, r) = \overline{B}(\lambda q, \lambda r).$$



Let $x \in \lambda \overline{B}(q, r)$, then $x = \lambda z$ where $z \in \overline{B}(q, r)$, and therefore

$$\|x - \lambda q\| = \|\lambda z - \lambda q\| = \lambda \|z - q\| \leq \lambda r,$$

so that $x \in \overline{B}(\lambda q, \lambda r)$, and

$$\lambda \overline{B}(q, r) \subseteq \overline{B}(\lambda q, \lambda r).$$

Conversely, if $x \in \overline{B}(\lambda q, \lambda r)$, then letting $z = \frac{1}{\lambda} x$, we have

$$\|z - q\|_2 = \frac{1}{\lambda} \|x - \lambda q\|_2 \leq \frac{1}{\lambda} \cdot \lambda r = r,$$

that is

$$\frac{1}{\lambda} x = z \in \overline{B}(q, r),$$

so that $x = \lambda z \in \lambda \overline{B}(q, r)$, and

$$\overline{B}(\lambda q, \lambda r) \subseteq \lambda \overline{B}(q, r).$$

□

1.6 HYPERPLANES AND LINEAR FUNCTIONALS

1.6.3 Problems

In the following exercises, unless otherwise stated, assume that the closed unit ball is the closed unit ball in the Euclidean norm.

*1. Find a hyperplane $H = f^{-1}(1)$ in $\mathbb{R}^4$ that is tangent to the unit cube at the point $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$. Verify your answer.

Solution. Let f be the linear functional represented by $p = (1, 0, 0, 0) \in \mathbb{R}^4$, then the hyperplane

$$H = \{x \in \mathbb{R}^4 : f(x) = \langle p, x \rangle = 1\}$$

is tangent to the unit cube at $q = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$.

To verify this, note that the point q is in H, since

$$f(q) = \langle p, q \rangle = 1 \cdot 1 + 0 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{4} = 1.$$

Also, for any point $x = (x_1, x_2, x_3, x_4)$ in the cube, $|x_i| \leq 1$ for $1 \leq i \leq 4$, so that $f(x) = x_1$ and $|f(x)| \leq 1$. □

3. Find an equation for the hyperplane of
   (a) Problem 2 (a),
   (b) Problem 2 (b).

Solution.

(a) The hyperplane through the point $(1, 3)$ is perpendicular to the line through through the points $(0, 0)$ and $(1, 3)$. Therefore,

$$H = \{x \in \mathbb{R}^2 : f(x) = \beta\} = \{(x_1, x_2) : x_1 + 3x_2 = \beta\}.$$

Since $(1, 3)$ is on H, we have

$$f(1, 3) = \beta = 1 + 3 \cdot 3 = 10,$$

and the equation of the hyperplane is

$$H = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 3x_2 = 10\}.$$

(b) The hyperplane tangent to the unit sphere at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is perpendicular to the line through the points $(0, 0)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore,

$$H = \{x \in \mathbb{R}^2 : f(x) = \beta\} = \{(x_1, x_2) : \tfrac{\sqrt{3}}{2}x_1 + \tfrac{1}{2}x_2 = \beta\}.$$

Since $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on H, we have

$$f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \beta,$$

so that

$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \beta,$$

that is, $\beta = \frac{3}{4} + \frac{1}{4} = 1$. The equation of the hyperplane is

$$H = \{(x_1, x_2) \in \mathbb{R}^2 : \tfrac{\sqrt{3}}{2} x_1 + \tfrac{1}{2} x_2 = 1\}.$$

□

5. Given the linear functional $f(x_1, x_2) = 4x_1 - 3x_2$, find
   (a) the point x on the closed unit ball where $f(x)$ is a maximum,
   (b) the point x in the hyperplane $f^{-1}(2)$ that is closest to the origin,
   (c) the point x in the hyperplane $f^{-1}(3)$ that is closest to the origin.

   Solution.

   (a) If $x = (x_1, x_2) \in \mathbb{R}^2$, then from the Cauchy-Schwarz inequality we have

$$|f(x)| = |4x_1 - 3x_2| \le (4^2 + 3^2)^{1/2} (x_1^2 + x_2^2)^{1/2} = 5\|x\|_2,$$

   and so $|f(x)| \le 5$ for all $x \in \overline{B}(\overline{0}, 1)$.
   Now let $x_0 = \left(\frac{4}{5}, -\frac{3}{5}\right)$, then

$$\|x_0\|_2 = \left(\left(\tfrac{4}{5}\right)^2 + \left(-\tfrac{3}{5}\right)^2\right)^{1/2} = \left(\tfrac{16}{25} + \tfrac{9}{25}\right)^{1/2} = 1,$$

   so that $x_0 \in \overline{B}(\overline{0}, 1)$, and

$$f(x_0) = 4\left(\tfrac{4}{5}\right) - 3\left(-\tfrac{3}{5}\right) = \tfrac{16+9}{5} = 5.$$

   Therefore, f attains its maximum on the closed unit ball $\overline{B}(\overline{0}, 1)$ at x_0.

   (b) The hyperplane $H = f^{-1}(2)$ has equation

$$H = \{(x_1, x_2) : 4x_1 - 3x_2 = 2\},$$

   and the line L through the points $(0, 0)$ and $(4, -3)$ is perpendicular to H and has parametric equations

$$x_1 = 0 + 4\lambda = 4\lambda$$
$$x_2 = 0 - 3\lambda = -3\lambda$$

   for $-\infty < \lambda < \infty$. The point $x = (x_1, x_2)$ on the hyperplane H with minimum Euclidean norm; that is, the point closed to the origin, is the point where the line L intersects H. Therefore,

$$f(x) = f(4\lambda, -3\lambda) = (4^2 + 3^2)\lambda = 2,$$

   so that $\lambda = \frac{2}{25}$, and the closest point on H to $\overline{0}$ is $x = \left(\frac{8}{25}, -\frac{6}{25}\right)$.

(c) Analogous to part (b), the point $x = (x_1, x_2)$ on the hyperplane

$$H = f^{-1}(3) = \{(x_1, x_2) : 4x_1 - 3x_2 = 3\}$$

closest to the origin is the point $x = (4\lambda, -3\lambda)$ with $f(4\lambda, -3\lambda) = 3$. Thus, we want

$$4(4\lambda) - 3(-3\lambda) = 3,$$

that is, $\lambda = \frac{3}{25}$.
The point on $f^{-1}(3)$ closest to the origin is $x = \left(\frac{12}{25}, -\frac{9}{25}\right)$. □

7. Let f be the linear functional on $\mathbb{R}^3$ represented by the vector $p = (3, -2, -3)$ and let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(1, 1, -2), (-3, 4, 1), (60, 10, 15), (-8, -2, 4), (0, 1, 1)\}.$$

(a) Determine which points of $\mathcal{S}$ are on the same side of $f^{-1}(0)$.
(b) Which point or points of $\mathcal{S}$ are closest to $f^{-1}(0)$?
(c) Which points of $\mathcal{S}$ are on the same side of $f^{-1}(8)$ as the origin?
(d) Find the point or points of $\mathcal{S}$ that are closest to $f^{-1}(8)$.

Solution. For each $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then value of f is given by

$$f(x) = \langle p, x\rangle = 3x_1 - 2x_2 - 3x_3,$$

and for $x \in S$, we have

$$f(1, 1, -2) = 7, \quad f(-3, 4, 1) = -20, \quad f(60, 10, 15) = 55,$$
$$f(-8, -2, 4) = -32, \quad f(0, 1, 1) = -5.$$

(a) The points $(1, 1, -2)$, and $(60, 10, 15)$, are on one side of $f^{-1}(0)$; that is, in the halfspace

$$H_+ = \{x \in \mathbb{R}^3 : f(x) > 0\},$$

while the points $(-3, 4, 1)$, $(-8, -2, 4)$, and $(0, 1, 1)$ are on the other side; that is, in the halfspace

$$H_- = \{x \in \mathbb{R}^3 : f(x) < 0\}.$$

(b) Since $|f(0, 1, 1)| = 5$, the point of S closest to $f^{-1}(0)$ is the point $(0, 1, 1)$.

(c) Only the point $(60, 10, 15)$ is in the halfspace

$$K_+ = \{x \in \mathbb{R}^3 : f(x) > 8\},$$

while the points $(1, 1, -2)$, $(-3, 4, 1)$, $(-8, -2, 4)$, and $(0, 1, 1)$ are in the halfspace

$$K_- = \{x \in \mathbb{R}^3 : f(x) < 8\}.$$

(d) Since $|f(1, 1, -2) - 8| = 1$, while $|f(x) - 8| > 1$ for all other points $x \in S$, the point closest to $f^{-1}(8)$ is $(1, 1, -2)$. $\square$

9. Given that H is the hyperplane $f^{-1}(2)$, and given that $g = 4f$, find β such that $g^{-1}(\beta)$ is exactly the same as H.

Solution. Note that the point $(x, y) \in g^{-1}(\beta)$ if and only if $g(x, y) = \beta$, that is, if and only if $4f(x, y) = \beta$.

This last equation is true if and only if $f(x, y) = \frac{\beta}{4}$, that is, if and only if $(x, y) \in f^{-1}\left(\frac{\beta}{4}\right)$.

Taking $\frac{\beta}{4} = 2$, then $g^{-1}(\beta) = f^{-1}(2)$, so that $f^{-1}(2) = H = g^{-1}(\beta)$, and therefore $\beta = 8$. $\square$

11. Let L be the line

$$L = \{x \in \mathbb{R}^n : x = \mu p + (1 - \mu)q, -\infty < \mu < \infty\}$$

where p and q are distinct points in $\mathbb{R}^n$, and let f be a linear functional on $\mathbb{R}^n$ such that $f(p) = 6$ and $f(q) = 1$. Find

(a) the point where L intersects the hyperplane $f^{-1}(-2)$,

(b) the scalar β such that the hyperplane $f^{-1}(\beta)$ passes through the midpoint of the line segment joining p and q.

Solution.

(a) If $z = \mu p + (1 - \mu)q$ is on L, then

$$f(z) = \mu f(p) + (1 - \mu)f(q) = 6\mu + 1 - \mu = 5\mu + 1 = -2,$$

so that $\mu = -\frac{3}{5}$, and the hyperplane $f^{-1}(2)$ intersects the line L at the point $z = -\frac{3}{5}p + \frac{8}{5}q$.

(b) We want

$$\beta = f\left(\frac{p}{2} + \frac{q}{2}\right) = \frac{1}{2}f(p) + \frac{1}{2}f(q) = \frac{6}{2} + \frac{1}{2} = \frac{7}{2},$$

so that $\beta = \frac{7}{2}$. $\square$

13. Given that $H = f^{-1}(1)$, where the linear functional f on $\mathbb{R}^4$ is represented by the vector $p = (1, 0, 1, -1)$, find
    (a) a line L_1 through $\overline{0}$ that intersects H in exactly one point,
    (b) a line L_2 through $\overline{0}$ that misses H.

    Solution.

    (a) The vector $p = (1, 0, 1, -1)$ is perpendicular to the hyperplane H, and Therefore the line L_1 through the origin in the direction of p intersects H in exactly one point.
    (b) Since $f(\overline{0}) = 0 < 1$, then $\overline{0} \notin H$. We take the vector $q = (1, 0, -1, 0)$ and note that

$$\langle p, q \rangle = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) + (-1) \cdot 0 = 0$$

    and the vector q is perpendicular to p. Thus, $\overline{0}$ and q are in $f^{-1}(0)$, so the line L_2 through $\overline{0}$ and q lies entirely in $f^{-1}(0)$ and so misses $H = f^{-1}(1)$. □

15. If the hyperplane $H = f^{-1}(\alpha)$ intersects the straight line L in exactly one point, then for every scalar β, the hyperplane $H_\beta = f^{-1}(\beta)$
    intersects L in exactly one point.

    Hint. Conclude that this must happen because of what we know from Problems 12 and 14.

    Solution. If the line L misses H_β, then L lies in a hyperplane parallel to H_β, which is parallel to H. This contradicts the fact that L intersects H in exactly one point. If the line L intersects H_β in more than one point, then L lies in H_β, which is parallel to H, again, a contradiction. Therefore, L intersects H_β in exactly one point. □

17. Prove Theorem 1.6.3.

    Solution. The theorem states that if f and g are linear functionals on $\mathbb{R}^n$ represented, respectively, by the vectors a and b, then $f + g$ and λf are represented, respectively, by the vectors $a + b$ and λa.

    Clearly, for $x \in \mathbb{R}^n$, we have

$$(f + g)(x) = f(x) + g(x) = \langle a, x \rangle + \langle b, x \rangle = \langle a + b, x \rangle,$$

    so $f + g$ is represented by the vector $a + b$.

    Similarly, for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have

$$(\lambda f)(x) = \lambda f(x) = \lambda \langle a, x \rangle = \langle \lambda a, x \rangle,$$

    so λf is represented by the vector λa. □

19. Show that a hyperplane in $\mathbb{R}^n$ has a unique point of minimum norm.

Solution. Let H be the hyperplane

$$H = \{x \in \mathbb{R}^n : \langle p, x \rangle = \alpha\}$$

where $p \in \mathbb{R}^n$, with $p \neq \overline{0}$ and $\alpha \in \mathbb{R}$. We may assume that $\alpha > 0$, otherwise, replace p by $-p$.

Let L be the line

$$L = \{x \in \mathbb{R}^n : x = \lambda p, -\infty < \lambda < \infty\},$$

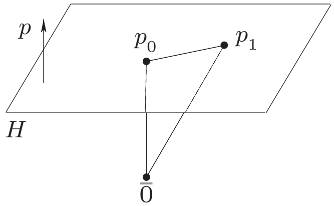
and let p_0 be the point where L intersects H.

Since $p_0 \in H$, we have $\langle p, p_0 \rangle = \alpha$, and since $p_0 \in L$, we have $p_0 = \lambda p$ for some $\lambda \in \mathbb{R}$, so that

$$\langle p, p_0 \rangle = \lambda \langle p, p \rangle = \lambda \|p\|_2^2.$$

Therefore, $\lambda = \frac{\alpha}{\|p\|_2^2}$, and $p_0 = \frac{\alpha}{\|p\|_2^2} \cdot p$.

We claim that p_0 is the unique point of minimum norm in H. To see this, suppose that p_1 is any point of H with $p_1 \neq p_0$.



Since the vector $p_1 - p_0 \in H$ is orthogonal to the vector p_0, from the Pythagorean Theorem, we have

$$\|p_1\|_2^2 = \|p_1 - p_0\|_2^2 + \|p_0\|_2^2,$$

and since $p_1 \neq p_0$, then $\|p_1 - p_0\|_2 > 0$, so that

$$\|p_1\|_2^2 > \|p_0\|_2^2.$$

□

21. Show that if f is a linear functional on $\mathbb{R}^n$ and f is represented by the vector $p \in \mathbb{R}^n$, then

$$\|p\| = \max\{f(x) : x \in B\} = \max\{\langle p, x \rangle : x \in B\},$$

where B is the closed unit ball in $\mathbb{R}^n$.

Solution. If $x \in \mathbb{R}^n$, then from the Cauchy-Schwarz inequality we have

$$f(x) = \langle p, x \rangle \leq \|p\| \, \|x\|,$$

and therefore

$$f(x) \leq \|p\|$$

for all $x \in B$.

Now let $x_0 = \frac{p}{\|p\|}$, so that $\|x_0\| = 1$ and $x_0 \in B$. However,

$$f(x_0) = \langle x_0, p \rangle = \frac{1}{\|p\|} \langle p, p \rangle = \frac{\|p\|^2}{\|p\|} = \|p\|,$$

so that f attains its maximum value on the closed unit ball B at x_0, and

$$\|p\| = \max\{f(x) : x \in B\}.$$

□

23. Develop a general formula for the point q on the hyperplane

$$H_\beta = \{x \in \mathbb{R}^n : \langle p, x \rangle = \beta\}$$

that is closest to the point x_0. Assume that $p \neq \overline{0}$.

Solution. From the previous problem the procedure is clear. Since the vector p is orthogonal to H_β, we only have to find the point where the line L through x_0 parallel to p intersects H_β. Thus, we want to find $q \in L \cap H_\beta$.

Now, q is on L if and only if $q = x_0 + \lambda\, p$ for some scalar λ, and $q \in H$ if and only if $\langle p, q \rangle = \beta$, that is, if and only if

$$\langle p, x_0 + \lambda\, p \rangle = \beta.$$

Thus, we want

$$\langle p, x_0 \rangle + \lambda \, \langle p, p \rangle = \beta,$$

that is,

$$\lambda = \frac{\beta - \langle p, x_0 \rangle}{\|p\|^2}.$$

Therefore, the point q on the hyperplane H_β which is closest to the point x_0 is

$$q = x_0 + \frac{\beta - \langle p, x_0 \rangle}{\|p\|^2} \, p.$$

□